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AXIOMATIC THEORY OF HARMONIC FUNCTIONS. NON-NEGATIVE SUPERHARMONIC FUNCTIONS

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In the most recent investigations on the axiomatic theory of harmonic functions two axiomatic systems have been mostly used: Brelot's axiomatic theory [4], [5], which is especially adequate to the theory of linear partial differential equations of elliptic type, and Bauer's axiomatic theory [1], which may be applied in addition to some linear partial differential equations of parabolic type.

The aim of the present paper is to extend some results from Brelot's theory, obtained by R.-M. Hervé [8], to a more general axiomatic theory [2]. This axiomatic theory starts like Brelot's and Bauer's theory with a sheaf of linear spaces of real continuous functions on a locally compact space for which the regular sets form a basis (axiom H_1). It is assumed that for any point there exists a positive harmonic function defined on a neighbourhood of this point (axiom H_0), that the set of open sets for which the minimum principle for hyperharmonic functions is valid forms a covering of the space (axiom H_2) and that Bauer's convergence axiom K_1 is satisfied (axiom H_3). It isn't required that either the space is non-compact or has a countable basis. It will be proved in theorem 1.1 that the axioms H_0 and H_3 imply the local connection of the space. Any space on which Bauer's axioms are locally fulfilled, satisfies the above axioms.

This paper is divided in three chapters. The first one is devoted to the introduction of the axioms and to a survey of some preliminary results. Here the hyperharmonic functions, the nearly-hyperharmonic functions and the specific order are introduced and studied. It is proved that the hyperharmonic functions form a conditionally complete ordered set with respect to the specific order and if the locally bounded non-negative hyperharmonic functions have no zero in common any non-negative hyperharmonic function is the least upper bound with respect to the natural order of its continuous finite hyperharmonic minorants. In the second chapter the non-negative superharmonic functions are studied. A superharmonic function is a hyperharmonic function which is finite on a dense set. It is proved that the set of differences of non-negative superharmonic functions is a conditionally complete vectorlattice with respect to the specific order. In order to define a carrier for the non-negative hyperharmonic functions we had to introduce a sheaf $\tilde{\mathscr{H}}^+$ of non-negative superharmonic functions, which coincides with the sheaf of non-negative harmonic functions if and only if Doob's convergence axiom is fulfilled. The carrier of a non-negative superharmonic function is the complementary set of the greatest open set to which the restriction of this function belongs to $\tilde{\mathscr{H}}^+$. This notion of carrier enables us to carry over R.-M. Hervé's construction of the measures associated with a non-negative superharmonic function. In the last chapter we show that a finite non-negative superharmonic function is the specific least upper bound of its specific minorants whose restrictions to their carriers are continuous. Then we introduce some interesting closed ideals (with respect to the specific order) of non-negative superharmonic functions: the closed ideal of substractible non-negative superharmonic functions, the closed ideal of non-negative quasicontinuous superharmonic functions and the closed ideal of non-negative superharmonic functions which satisfy, in a certain sense, Brelot's axiom D. If the non-negative locally bounded superharmonic functions have no zero in common and if the minimum principle for hyperharmonic functions is valid on any relatively compact open set then any function of this last ideal, which is orthogonal to $\tilde{\mathscr{H}}^+$, is the specific least upper bound of its continuous finite specific minorants and therefore quasicontinuous. If moreover Brelot's axiom D is fulfilled, then any non-negative hyperharmonic function is quasicontinuous.

I. PRELIMINARIES.

1. Axioms and definitions.

Let X be a locally compact space and $\mathscr{H}$ a sheaf on X of real vector spaces of real continuous functions called *harmonic functions.*

An open relatively compact set U of X is called *regular* if it has non-empty boundary ∂U and any real continuous function f on ∂U possesses a unique continuous extension to $\bar{\mathrm{U}}$, whose restriction $\mathrm{H}_f^{\mathrm{U}}$

to U is harmonic, non-negative if f is non-negative. For any regular set U and any $x \in \mathrm{U}$ the map $f \to \mathrm{H}_f^{\mathrm{U}}(x)$ is a linear non-negative functional on the space of real continuous functions on $\partial \mathrm{U}$; we denote by ω_x^{U} the measure on $\partial \mathrm{U}$ associated with this functional and we call it *harmonic measure.*

A numerical function[1] s on an open set U is called *hyperharmonic* if:

a) it does not take the value $-\infty$,

b) it is lower semicontinuous,

c) any point $x \in \mathrm{U}$ possesses a neighbourhood $\mathrm{U}_s(x) \subset \mathrm{U}$ such that for every regular set V, $\bar{V} \subset \mathrm{U}_s(x)$, and any $y \in \mathrm{V}$

$$s(y) \geqslant \int^* s\, d\omega_y^{\mathrm{V}}.$$

An open set U is called an MP-*set* if any hyperharmonic function s on U is non-negative if there exists a compact subset K_s of X such that s is non-negative on $\mathrm{U} - \mathrm{K}_s$ and for any boundary point x of U

$$\liminf_{y \to x} s(y) \geqslant 0.$$

It follows immediately from this definition that any hyperharmonic function on a compact MP-set is non-negative.

We shall suppose that the sheaf $\mathscr{H}$ satisfies the following axioms:

H_0. *For any point $x \in \mathrm{X}$ there exists a harmonic function on a neighbourhood of x, positive at x.*

H_1. *The regular sets form a basis of* X.

H_2. *The* MP-*sets form a covering of* X.

H_3. *For any open set* U *the least upper bound of any upper directed non-empty set of equally bounded harmonic functions on* U *is harmonic.*

2. First consequences.

THEOREM 1.1. — *The space* X *is locally connected.*

Let x be a point of X and U be an open neighbourhood of x on which there exists a positive bounded harmonic function u. For any open set V, $x \in \mathrm{V} \subset \mathrm{U}$, such that $\mathrm{U} - \mathrm{V}$ is open, we denote by u_{V} the harmonic function on U equal to zero on V and equal to u on $\mathrm{U} - \mathrm{V}$. The least upper bound v of the family $(u_{\mathrm{V}})_{\mathrm{V}}$ is harmonic

[1] Real (resp. numerical) function is a map into the real axis (resp. real axis completed with $+\infty$ and $-\infty$).

on U and is equal either to zero or to u at any point of U. Hence the set W on which v vanishes is open and therefore connected. Since it contains x, X is locally connected.

COROLLARY 1.1. — *The regular domains form a basis of* X.

Let U be an open subset of X and f a numerical function on ∂U. We denote by $\mathcal{S}_f^{\mathrm{U}}$ the set of lower bounded hyperharmonic functions s on U such that

$$\liminf_{y \to x} s(y) \geqslant f(x)$$

at any $x \in \partial \mathrm{U}$ and non-negative on $\mathrm{U} - \mathrm{K}_s$, where K_s is a compact subset of X, and by $\bar{\mathrm{H}}_f^{\mathrm{U}}$ the function on U

$$\bar{\mathrm{H}}_f^{\mathrm{U}}(x) = \inf_{s \in \mathcal{S}_f^{\mathrm{U}}} s(x).$$

We shall denote, as usual, for any non-negative hyperharmonic function s on X and any set $\mathrm{A} \subset \mathrm{X}$, by $\mathrm{R}_s^{\mathrm{A}}$ the greatest lower bound of the set of non-negative hyperharmonic functions on X which dominate s on A.

LEMMA 1.1. — *Let* U *be a regular* MP-*set and f a lower bounded numerical function on* ∂U. *Then* $\bar{\mathrm{H}}_f^{\mathrm{U}}$ *is hyperharmonic on* U *and for* $x \in \mathrm{U}$

$$\bar{\mathrm{H}}_f^{\mathrm{U}}(x) = \int^* f \, d\omega_x^{\mathrm{U}}.$$

If f is bounded $\bar{\mathrm{H}}_f^{\mathrm{U}}$ *is harmonic on* U.

The assertion follows from the axiom H_3 and from the fact that if f is lower semicontinuous, the function

$$x \to \int^* f \, d\omega_x^{\mathrm{U}}$$

is hyperharmonic and belongs to $\mathcal{S}_f^{\mathrm{U}}$.

THEOREM 1.2. — *Let* U_1, U_2 *be two open sets and for any* $i = 1, 2$ *let* s_i *be a hyperharmonic function on* U_i. *If the function s defined on* $\mathrm{U}_1 \cup \mathrm{U}_2$ *by*

$$s(x) = \inf_{\mathrm{U}_i \ni x} s_i(x)$$

is lower semicontinuous then it is hyperharmonic.

It is sufficient to show that s is hyperharmonic on a neighbourhood of a point $x \in \mathrm{U}_1 \cap \partial \mathrm{U}_2$. Let W be an MP-set containing x and

U be an open neighbourhood of x on which there exists a positive harmonic function u. We shall prove that we can take as $U_s(x)$ (required in the definition of hyperharmonic functions) the set $U \cap W \cap U_{s_1}(x)$. Indeed let V be a regular set, $\bar{V} \subset U_s(x)$, and f a real continuous function on ∂V, $f \leqslant s$. Obviously $s \geqslant H_f^V$ on $V - U_2$. For any $\varepsilon > 0$ we denote by s_ε the function on W equal to $\inf(s + \varepsilon u - H_f^V, 0)$ on $V \cap U_2$ and equal to zero on $W - V \cap U_2$. Since it is hyperharmonic on $V \cap U_2$ and equal to zero outside a compact subset of $V \cap U_2$ it is hyperharmonic. W being an MP-set it is non-negative. Hence, ε being arbitrary, $s \geqslant H_f^V$ also on $V \cap U_2$. It follows immediately that for any $y \in V$

$$s(y) \geqslant \int^* s \, d\omega_y^V.$$

COROLLARY 1.2. — *Any open subset of an* MP-*set is also an* MP-*set.*

COROLLARY 1.3. — *The regular* MP-*domains form a basis of* X.

COROLLARY 1.4. — *Let* U *be a regular* MP-*set and s a hyperharmonic function on* X. *The function s' on* X *equal to s on* $X - U$ *and equal to* $\bar{H}_s^U$ *on* U *is hyperharmonic and not greater than s.*

The assertion follows immediately from the Lemma 1.1 and from the theorem since s' is obviously lower semicontinuous.

Remark. — *It follows from this corollary that for any hyperharmonic function s on an open set* U *and for any* $x \in U$ *one may take in the definition of hyperharmonic functions any* MP-*set containing x and contained in* U *in the place of* $U_s(x)$, *this means independently of s.*

From Theorem 1.2 *it follows also that for any non-negative hyperharmonic function s on* X *and any open set* U *we have* $\bar{H}_s^U = R_s^{X-U}$ *on* U. *Hence if* s_1, s_2 *are non-negative hyperharmonic functions on* X *then* [3] (*Theorem* 3.2) $\bar{H}_{s_1+s_2}^U = \bar{H}_{s_1}^U + \bar{H}_{s_2}^U$(2).

3. Nearly hyperharmonic functions

A numerical function s on an open set U is called *nearly hyperharmonic* if it is locally lower bounded and for any regular MP-set V, $\bar{V} \subset U$, and for any $x \in V$ we have

$$s(x) \geqslant \int^* s \, d\omega_x^V.$$

(2) This equality will be used only in lemma 3.3.

The greatest lower bound of a locally equally lower bounded set of nearly hyperharmonic functions is also nearly hyperharmonic.

LEMMA 1.2. — *Let s be a nearly hyperharmonic function on* U. *The function $\hat{s}$ equal to*

$$\liminf_{y \to x} s(y)$$

at any $x \in$ U is hyperharmonic and

$$\hat{s}(x) = \sup_{x \in V} \int^* s \, d\omega_x^V = \lim_{V, \mathscr{F}_x} \int^* s \, d\omega_x^V,$$

where V *is a regular* MP-set *and $\mathscr{F}_x$ is the filter of the sections on the set of all regular* MP-*sets containing x ordered by the converse inclusion relation.*

Let V be a regular MP-set, $\bar{V} \subset$ U. From the Lemma 1.1 it follows

$$\hat{s}(y) \geqslant \int^* s \, d\omega_y^V \geqslant \int^* \hat{s} \, d\omega_y^V$$

for any $y \in$ V. Hence $\hat{s}$ is hyperharmonic.

Let $x \in$ U, u be a harmonic function on a neighbourhood of x, $u(x) = 1$, and α be a real number, $\alpha < \hat{s}(x)$. For a sufficiently small regular MP-neighbourhood V of x we have

$$\alpha u < s$$

on V. The last assertions of the lemma follow from

$$\alpha = \int \alpha u \, d\omega_x^V \leqslant \int^* s \, d\omega_x^V \leqslant \hat{s}(x).$$

LEMMA 1.3. — *Let s_1, s_2 be nearly hyperharmonic functions. Then $s_1 + s_2$ is nearly hyperharmonic and*

$$\widehat{s_1 + s_2} = \hat{s}_1 + \hat{s}_2.$$

If $(s_n)_{n \in N}$ is an increasing sequence of nearly hyperharmonic functions then $s = \lim_{n \to \infty} s_n$ is also nearly hyperharmonic and

$$\hat{s} = \lim_{n \to \infty} \hat{s}_n.$$

For any family $\mathscr{S} = (s_\iota)_{\iota \in I}$ of hyperharmonic functions we denote by

$$\vee \mathscr{S} \quad \text{or} \quad \bigvee_{\iota \in I} s_\iota \qquad (\text{resp. } \wedge \mathscr{S} \quad \text{or} \quad \bigwedge_{\iota \in I} s_\iota)$$

the least upper bound (resp. the greatest lower bound) of $\mathscr{S}$ in the set of hyperharmonic functions if it exists.

LEMMA 1.4. — *For any upper directed (resp. locally equally lower bounded) family $\mathscr{S} = (s_\iota)_{\iota\in I}$ of hyperharmonic functions $\vee\mathscr{S}$ (resp. $\wedge\mathscr{S}$) exists and*

$$\vee\mathscr{S} = \sup_{\iota\in I} s_\iota \qquad (\text{resp. } \wedge\mathscr{S} = \widehat{\inf_{\iota\in I} s_\iota}).$$

For any hyperharmonic function s we have

$$s + \bigvee_{\iota\in I} s_\iota = \bigvee_{\iota\in I}(s + s_\iota) \qquad (\text{resp. } s + \bigwedge_{\iota\in I} s_\iota = \bigwedge_{\iota\in I}(s + s_\iota)).$$

Obviously $\inf\limits_{\iota\in I} s_\iota$ is a nearly hyperharmonic function. The equality

$$s + \bigwedge_{\iota\in I} s_\iota = \bigwedge_{\iota\in I}(s + s_\iota)$$

follows from

$$s(x) + \inf_{\iota\in I} s_\iota(x) = \inf_{\iota\in I}(s + s_\iota)(x)$$

and Lemma 1.3.

Remark. — *The set of hyperharmonic function is a conditionally complete lattice with respect to the order relation $\leqslant$.*

LEMMA 1.5. — *Let s_1, s_2 be hyperharmonic functions on* X, $s_1 \geqslant s_2$, *such that for any regular* MP-*sets* V *and any* $x \in$ V

$$s_1(x) + \int^* s_2\, d\omega_x^V \geqslant s_2(x) + \int^* s_1\, d\omega_x^V.$$

The function s on X *equal to $s_1 - s_2$ where s_2 is finite and equal to $+\infty$ where s_2 is infinite, is nearly hyperharmonic and*

$$s_1 = s_2 + \hat{s}.$$

For any V and any $x \in$ V at which s is finite we have

$$s(x) = s_1(x) - s_2(x) \geqslant \int^* s_1\, d\omega_x^V - \int^* s_2\, d\omega_x^V = \int^* s\, d\omega_x^V.$$

Let s_1, s_2 be hyperharmonic functions on X. The relation:

« *there exists a non-negative hyperharmonic function s on* X *such that $s_2 = s_1 + s$* »

is an order relation. It is called *specific order* (M. Brelot, R.-M. Hervé) and we denote it

$$s_1 \preccurlyeq s_2.$$

Obviously $s_1 \preccurlyeq s_2 \Rightarrow s_1 \leqslant s_2$. For any family $\mathscr{S} = (s_\iota)_{\iota \in I}$ of hyperharmonic functions we denote

$$\curlyvee \mathscr{S} \quad \text{or} \quad \curlyvee_{\iota \in I} s_\iota \qquad (\text{resp. } \curlywedge \mathscr{S} \quad \text{or} \quad \curlywedge_{\iota \in I} s_\iota)$$

the least upper (resp. greatest lower) bound with respect to the specific order if it exists.

LEMMA 1.6. — *Let s be a hyperharmonic function on* X *and*

$$A = \{x \in X | s(x) < \infty\}.$$

For any $x \in \bar{A}$ and any regular MP-*neighbourhood* V *of x we have*

$$\omega_x^V(X - A) = 0.$$

Since the function

$$y \to \int^* s \, d\omega_y^V$$

is finite on $A \cap V$ and s is infinite on $X - A$ the function

$$y \to \omega_y^V(X - A)$$

vanishes on $A \cap V$. Being harmonic it vanishes on $\bar{A} \cap V$.

Remark. — *It follows from this lemma that the function on* X *equal to* 0 *on $\bar{A}$ and equal to $+\infty$ on $X - \bar{A}$ is hyperharmonic. Hence, by Theorem* 1.2, *the function on* X *equal to* 0 *on $\bar{A}$ and equal to a non-negative hyperharmonic function on $X - \bar{A}$ is hyperharmonic.*

LEMMA 1.7. — *Let s, s_1, s_2 be hyperharmonic functions on* X *and*

$$A = \{x \in X | s(x) < \infty\}.$$

If $s_1 \leqslant s_2$ on A *then $s_1 \leqslant s_2$ on $\bar{A}$.*

Let $x \in \bar{A}$ and V be a regular MP-neighbourhood of x. By the preceding lemma we have

$$\int^* s_1 \, d\omega_x^V \leqslant \int^* s_2 \, d\omega_x^V.$$

The assertion follows now from Lemma 1.2.

LEMMA 1.8.(3) — *Let s_1, s_2, s_3 be hyperharmonic functions on* X *such that*

$$s_1 \preccurlyeq s_2 \preccurlyeq s_3$$

and $\mathscr{S}_{ij}$ $(i, j, = 1, 2, 3, i < j)$ be the set of non-negative hyperharmonic functions s on X *such that*

$$s_i + s = s_j$$

If we denote $s_{ij} = \wedge \mathscr{S}_{ij}$, then $s_{ij} \in \mathscr{S}_{ij}$ and for any $s \in \mathscr{S}_{13}$

$$s_{12} + s_{23} \preccurlyeq s.$$

For any $s \in \mathscr{S}_{ij}$ we have

$$s_j = s_i + s.$$

Hence, by Lemma 1.4,

$$s_j = s_i + s_{ij}, \qquad s_{ij} \in \mathscr{S}_{ij}.$$

We denote

$$\mathrm{A} = \{x \in \mathrm{X} | s_1(x) < \infty\}.$$

Obviously $s_{ij} = \mathrm{R}^{\mathrm{A}}_{s_{ij}}$ and for any $s \in \mathscr{S}_{13}$

$$s = s_{12} + s_{23}$$

on A. Hence [3] (Theorem 3.2)

$$s \geqslant \mathrm{R}^{\mathrm{A}}_{s} = \mathrm{R}^{\mathrm{A}}_{s_{12}+s_{23}} = \mathrm{R}^{\mathrm{A}}_{s_{12}} + \mathrm{R}^{\mathrm{A}}_{s_{23}} = s_{12} + s_{23}.$$

Since for any regular MP-set V, $\bar{\mathrm{V}} \cap \bar{\mathrm{A}} = \emptyset$, and any $x \in \mathrm{V}$ we have

$$s_{ij}(x) = \int^{*} s_{ij} \, d\omega^{\mathrm{V}}_{x},$$

there exists by Lemma 1.5 a non-negative hyperharmonic function t on $\mathrm{X} - \bar{A}$ such that

$$s = s_{12} + s_{23} + t.$$

The function t' on X equal to 0 on $\bar{A}$ and equal to t on $\mathrm{X} - \bar{A}$ is a non-negative hyperharmonic function and by Lemma 1.7

$$s = s_{12} + s_{23} + t', \qquad s_{12} + s_{23} \preccurlyeq s.$$

(3) This lemma is used only to show that in lemma 1.10 $\wedge \mathscr{S}_i$ exists and belongs to $\mathscr{S}_i$. These assertions, however, are not used in the rest of this paper.

LEMMA 1.9.—*Let $\mathscr{S} = (s_\iota)_{\iota\in I}$ be a specifically upper directed (resp. specifically lower directed and locally equally lower bounded) family of hyperharmonic functions on* X. *Then* $\curlyvee\mathscr{S}$ *(resp.* $\curlywedge\mathscr{S}$*) exists,*

$$\curlyvee\mathscr{S} = \vee\mathscr{S} \qquad (\text{resp. } \curlywedge\mathscr{S} = \wedge\mathscr{S})$$

and for any $x \in X$ *(resp.* $x \in X$ *such that* $\inf_{\iota\in I} s_\iota(x) < +\infty$*)*

$$(\curlyvee\mathscr{S})(x) = \sup_{\iota\in I} s_\iota(x) \qquad (\text{resp. } (\curlywedge\mathscr{S})(x) = \inf_{\iota\in I} s_\iota(x)).$$

Let $\kappa \in I$. For any $\iota \in I$ such that $s_\iota \succcurlyeq s_\kappa$ (resp. $s_\iota \preccurlyeq s_\kappa$) we denote by t_ι the nearly hyperharmonic function on X equal to $s_\iota - s_\kappa$ (resp. $s_\kappa - s_\iota$) wherever s_κ (resp. s_ι) is finite and equal to $+\infty$ elsewhere. By Lemma 1.3

$$s_\iota = s_\kappa + \hat{t}_\iota \qquad (\text{resp. } s_\kappa = s_\iota + \hat{t}_\iota)$$

the family $(\hat{t}_\iota)_{\iota\in I}$ being upper directed we get by Lemma 1.4

$$\sup_{\iota\in I} s_\iota = s_\kappa + \sup_{\iota\in I} \hat{t}_\iota, \qquad \bigvee_{\iota\in I} s_\iota = s_\kappa + \bigvee_{\iota\in I} \hat{t}_\iota$$

$$(\text{resp. } s_\kappa = \inf_{\iota\in I} s_\iota + \sup_{\iota\in I} \hat{t}_\iota, \qquad s_\kappa = \bigwedge_{\iota\in I} s_\iota + \bigvee_{\iota\in I} \hat{t}_\iota).$$

Hence $\vee\,\mathscr{S}$ (resp. $\wedge\,\mathscr{S}$) is a specific majorant (resp. minorant) of $\mathscr{S}$. It can be proved similarly that any specific majorant (resp. minorant) of $\mathscr{S}$ is a specific majorant (resp. minorant) of $\vee\,\mathscr{S}$ (resp. $\wedge\,\mathscr{S}$).

LEMMA 1.10.—(R.-M. Hervé). *Let* $(U_\iota)_{\iota\in I}$ *be a family of open sets on* X *and for any* $\iota \in I$ *let* t_ι *be a hyperharmonic function on* U_ι *such that if* $U_\iota \neq X$ *any hyperharmonic function on* U_ι *dominating* $-t_\iota$ *is non-negative. Let* $\mathscr{S}$ *be the set of hyperharmonic functions on* X *whose restrictions to* U_ι *are specific majorants of* t_ι *for any* $\iota \in I$, *and for any* $\iota \in I$ *let* $\mathscr{S}_\iota$ *be the set of non-negative hyperharmonic functions* s_ι *on* U_ι *such that* $t_\iota + s_\iota$ *is a restriction to* U_ι *of an element of* $\mathscr{S}$. *If* $\mathscr{S}$ *is locally equally lower bounded and contains only non-negative elements when at least an* U_ι *is different from* X, *then* $\mathscr{S}$ *and* $\mathscr{S}_\iota$ *possess a specific greatest lower bound and*

$$\curlywedge\mathscr{S} = t_\iota + \curlywedge\mathscr{S}_\iota \qquad on \qquad U_\iota, \qquad \iota \in I.$$

For any $\iota \in I$ *and any regular* MP-*set* V, $\bar{V} \subset U_\iota$, $V \cap \left(\bigcup_{\kappa\neq\iota} U_\kappa\right) = \emptyset$, *we have*

$$(\curlywedge\mathscr{S}_\iota)(x) = \int^* (\curlywedge\mathscr{S}_\iota)\, d\omega_x^V, \qquad x \in V.$$

We denote $s = \wedge \mathscr{S}$, $s_\iota = \wedge \mathscr{S}_\iota$. By Lemma 1.4 we have on U_ι $s = t_\iota + s_\iota$.

Let $s' \in \mathscr{S}$, $s'_\iota \in \mathscr{S}_\iota$, such that $s' = t_\iota + s'_\iota$ on U_ι, V be a regular MP-set and f be a real continuous function on ∂V, $f \leqslant s'$. The function s^* on X (resp. s^*_ι on U_ι) equal to s on $X - V$ (resp. s_ι on $U_\iota - V$) and equal to

$$\inf((s' + \bar{H}^V_s - H^V_f), s)$$
$$(\text{resp. } \inf((s'_\iota + \bar{H}^V_s - H^V_f), s_\iota))$$

on V (resp. $V \cap U_\iota$) is lower semicontinuous and therefore by Theorem 1.2 hyperharmonic. s^*_ι is non-negative and $s^* = t_\iota + s^*_\iota$ on U_ι for any $\iota \in I$. Hence $s^* \in \mathscr{S}$, $s^*_\iota \in \mathscr{S}_\iota$, $s^* \geqslant s$,

$$s' + \bar{H}^V_s \geqslant s + \bar{H}^V_{s'}$$

By Lemma 1.5 we get $s' \succcurlyeq s$, $s = \curlywedge\mathscr{S}$. By Lemma 1.8 $s_\iota = \curlywedge\mathscr{S}_\iota$, taking t_ι, s, s' in the role of s_1, s_2, s_3 respectively. Hence

$$\curlywedge\mathscr{S} = t_\iota + \curlywedge\mathscr{S}_\iota \text{ on } U_\iota.$$

Let $\iota \in I$ and V be a regular MP-set, $\bar{V} \subset U_\iota$, $V \cap \left(\bigcup_{\kappa \neq \iota} U_\kappa\right) = \emptyset$. We denote by s^* (resp. s^*_ι) the hyperharmonic function on X (resp. U_ι) equal to s on $X - V$ (resp. s_ι on $U_\iota - V$) and equal to

$$t_\iota + \bar{H}^V_{s_\iota} \qquad (\text{resp. } \bar{H}^V_{s_\iota})$$

on V. Then

$$s^* = t_\iota + s^*_\iota, \qquad s^* \in \mathscr{S}, \qquad s^*_\iota \in \mathscr{S}_\iota, \qquad s^*_\iota = s_\iota.$$

Remark.—The set of hyperharmonic functions is conditionally complete with respect to the order relation $\preccurlyeq$.

LEMMA 1.11.—*Let f be a lower semicontinuous function on X and $\mathscr{S}$ be the set of hyperharmonic majorants of f. If the function s_0*

$$x \to \inf_{s \in \mathscr{S}} s(x), \qquad x \in X,$$

is locally lower bounded it is hyperharmonic. If s_0 is bounded in a neighbourhood of a point x and

$$\limsup_{y \to x} f(y) < s_0(x)$$
$$(\text{resp. } \limsup_{y \to x} f(y) \leqslant s_0(x)),$$

then s_0 *is harmonic on a neighbourhood of* x (*resp.* s_0 *is continuous at* x).

For the proof see [6](Theorem 3), [3] (Propositions 3.1, 3.2, and 3.3).

THEOREM 1.3. — *If for any point* $x \in X$ *there exists a non-negative locally bounded hyperharmonic function on* X *positive at* x *then any hyperharmonic function which is minorated by a continuous finite function* f, $f \leqslant 0$ *outside a compact set, is the least upper bound of its continuous finite hyperharmonic minorants which dominate* f.

II. SUPERHARMONIC FUNCTIONS.

1. Specific order.

A hyperharmonic function on an open set is called *superharmonic* if it is finite on a dense subset. A subset A of an open set U is called *negligible on U* or simply *negligible* if for any regular• MP-set V, $\bar{V} \subset U$, we have $\omega_x^V(A \cap \partial V) = 0$ for any $x \in V$.

LEMMA 2.1. — *Let* s *be a hyperharmonic function on* X. *The following conditions are equivalent:*

a) s *is superharmonic;*

b) s *is finite outside a negligible set;*

c) *for any open set* U *and any hyperharmonic functions* s_1, s_2 *on* U *such that*

$$s + s_1 \leqslant s + s_2$$

it follows $s_1 \leqslant s_2$.

$a \Rightarrow b$ follows from Lemma 1.6.

$b \Rightarrow c$ follows from Lemma 1.7.

$c \Rightarrow a$ is trivial.

We shall denote by $\mathscr{S}^+(X)$ the set of non-negative superharmonic functions on X. The relation $(s_1, s_2) \sim (s'_1, s'_2)$ defined by

$$s_1 + s'_2 = s_2 + s'_1$$

is an equivalence relation on $\mathscr{S}^+(X) \times \mathscr{S}^+(X)$ according to the Lemma 2.1. We denote by $[\mathscr{S}^+](X)$ the quotient set of the set $\mathscr{S}^+(X) \times \mathscr{S}^+(X)$ with respect to this equivalence relation and by $[s_1, s_2]$ the equivalence class of the element (s_1, s_2). Again by Lemma 2.1 the relation $[s_1, s_2] \preccurlyeq [s'_1, s'_2]$ defined by

$$s_1 + s'_2 \preccurlyeq s_2 + s'_1$$

is an order relation on $[\mathscr{S}^+](X)$ called *specific order* (M. Brelot, R. -M. Hervé). Defining the sum of two elements of $[\mathscr{S}^+](X)$ by

$$[s_1, s_2] + [s'_1, s'_2] = [s_1 + s'_1, s_2 + s'_2]$$

and the multiplication with real numbers by

$$\alpha[s_1, s_2] = [\alpha s_1, \alpha s_2],$$
$$-\alpha[s_1, s_2] = [\alpha s_2, \alpha s_1]$$

for $\alpha \geqslant 0$, $[\mathscr{S}^+](X)$ becomes an ordered real vector space.

The map

$$s \rightarrow [s, 0]$$

is an isomorphism of $\mathscr{S}^+(X)$ on the set of non-negative elements of $[\mathscr{S}^+](X)$. We shall identify $\mathscr{S}^+(X)$ with this set.

THEOREM 2.1. *$[\mathscr{S}^+](X)$ is a conditionally complete vectorlattice* (4).

The assertion follows from the above remarks and from Lemma 1.10.

We shall denote by $\curlyvee$ (resp. $\curlywedge$) the join (resp. meet) operation with respect to the specific order in $[\mathscr{S}^+](X)$.

2. Carriers of non-negative superharmonic functions.

Two non-negative elements of a vectorlattice are called *orthogonal* if their meet is zero. Two subsets of non-negative elements of a vector lattice are called *orthogonal* if any element of one set is orthogonal to any element of the other set. A subset $\mathscr{T}$ of non-negative elements of a vectorlattice is called *positive ideal* if the sum of two non-negative elements belongs to $\mathscr{T}$ if and only if both elements belong to $\mathscr{T}$. $\mathscr{T}$ is called *closed positive ideal* if moreover it contains the least upper bounds of its subsets, whenever they exist. If the vectorlattice is conditionally complete then for any non-negative element s and for any closed positive ideal $\mathscr{T}$ there exists a unique decomposition

$$s = s_1 + s_2,$$

where $s_1 \in \mathscr{T}$ and s_2 is orthogonal to $\mathscr{T}$; we shall call this decomposition the *Riesz decomposition* of s with respect to $\mathscr{T}$ and s_1 the *component of* s in $\mathscr{T}$.

(4) Espace de Riesz complètement réticulé (in the terminology of N. Bourbaki).

$U \to \mathscr{S}^+(U)$ is a sheaf. We denote by ψ the set of sheaves $\mathscr{F}$ on X such that, for any U, $\mathscr{F}(U)$ is a closed positive ideal of $[\mathscr{S}^+](U)$ and contains $\mathscr{H}^+(U)$, where $\mathscr{H}^+(U)$ stands for $\mathscr{H}(U) \cap \mathscr{S}^+(U)$. Obviously the sheaf $\mathscr{S}^+$ belongs to ψ. We denote

$$\tilde{\mathscr{H}}^+(U) = \bigcap_{\mathscr{F} \in \psi} \mathscr{F}(U).$$

Obviously $\tilde{\mathscr{H}}^+ \in \psi$. Any non-negative superharmonic function s on U for which there exists a covering $\mathfrak{B}$ of U of regular MP-sets, such that for any $V \in \mathfrak{B}$ and $x \in V$

$$s(x) = \int^* s \, d\omega_x^V,$$

belongs to $\tilde{\mathscr{H}}^+(U)$.

Doob's convergence axiom [7] (the least upper bound of an upper directed set of harmonic functions is harmonic if it is finite on a dense set) is equivalent to $\tilde{\mathscr{H}}^+ = \mathscr{H}^+$.

LEMMA 2.2. — *Let s be a non-negative superharmonic function on X orthogonal to $\tilde{\mathscr{H}}^+(X)$. Then any hyperharmonic majorant of $-s$ is non-negative.*

Let s_0 be a hyperharmonic majorant of $-s$. We denote by $\mathscr{S}$ the set of non-negative superharmonic functions on X dominating $-s_0$. We have

$$s_0 + \wedge \mathscr{S} = \bigwedge_{s' \in \mathscr{S}} (s_0 + s') \geqslant 0$$

and therefore

$$s_0' = \wedge \mathscr{S} \in \mathscr{S}.$$

For any regular MP-set V the function on X equal to s_0' on $X - V$ and equal to $\bar{H}_{s_0'}^V$ on V belongs to $\mathscr{S}$ and therefore

$$s_0' = \bar{H}_{s_0'}^V.$$

Since $s_0' \leqslant s$ we get, by Lemma 1.5, $s_0' \preccurlyeq s$. Hence $s_0' = 0$ because it is an element of $\tilde{\mathscr{H}}^+(X)$.

Let s be a non-negative superharmonic function on X and X_0 Alexandroff's compactification of X. We shall call *carrier of* s, and we shall denote it by Carr s, the set of points $x \in X_0$ such that, for any neighborhood U of x, the restriction of s to $U \cap X$ does not belong to $\tilde{\mathscr{H}}^+(U \cap X)$ to which we add the Alexandroff point if s possesses a non-identically zero minorant from $\tilde{\mathscr{H}}^+(X)$. Obviously Carr s is closed.

THEOREM 2.2. — *The map*

$$s \to \operatorname{Carr} s$$

possesses the following properties:

a) $s = 0 \Leftrightarrow \operatorname{Carr} s = \emptyset$;

b) $s \preccurlyeq s' \Rightarrow \operatorname{Carr} s \subset \operatorname{Carr} s'$;

c) for any $s \in \mathscr{S}^+(X)$ *and for any two compact subsets* K_1, K_2 *of* X_0, $K_1 \cup K_2 = X_0$, *there exists* $s_1, s_2 \in \mathscr{S}^+(X)$ *such that*

$$s = s_1 + s_2, \qquad \operatorname{Carr} s_i \subset K_i \qquad (i = 1, 2).$$

The properties *a)* and *b)* are trivial. Suppose that the Alexandroff point belongs to K_1 and let us denote $U = X - K_1$. Let $h + p$ be the Riesz decomposition of the restriction of s to U with respect to $\tilde{\mathscr{H}}^+(U)$, $h \in \tilde{\mathscr{H}}^+(U)$. According to Lemma 2.2 and Lemma 1.10 there exists a non-negative superharmonic function s_2 on X such that

$$\begin{aligned} &s_2 \preccurlyeq s, \\ &s_2 = p + s'_2 \text{ on } U, \qquad s'_2 \in \mathscr{S}^+(U), \\ &\operatorname{Carr} s_2 \subset \bar{U} \subset K_2. \end{aligned}$$

Let s_1 be the non-negative superharmonic function on X defined by

$$s = s_1 + s_2.$$

We have on U

$$\begin{aligned} p + h &= s_1 + p + s'_2, \\ h &= s_1 + s'_2. \end{aligned}$$

Hence the restriction of s_1 to U belongs to $\tilde{\mathscr{H}}^+(U)$ and therefore

$$\operatorname{Carr} s_1 \subset K_1.$$

3. Abstract carriers.

Let Y be a compact space and $\mathscr{L}$ be a vectorlattice. An *abstract carrier* on $(\mathscr{L}, Y)$ is a map $s \to K(s)$ of the set of non-negative elements of $\mathscr{L}$ into the set of compact subsets of Y which satisfies the following axioms:

a) $s = 0 \Leftrightarrow K(s) = \emptyset$;

b) $s \preccurlyeq s' \Rightarrow K(s) \subset K(s')$;

c) for any $s \in \mathscr{L}^+$ and for any two open subsets G_1, G_2 of Y, $G_1 \cup G_2 = Y$, there exist $s_1, s_2 \in \mathscr{L}^+$ such that

$$s = s_1 + s_2, \qquad K(s_i) \subset G_i \qquad (i = 1, 2).$$

K(s) will be called *the abstract carrier* of s.

It is easy to verify that the map

$$s \to \operatorname{Carr} s$$

is an abstract carrier on $([\mathscr{S}^+](X), X_0)$.

We have

$$K(s_1 + s_2) = K(s_1 \curlyvee s_2) = K(s_1) \cup K(s_2),$$

$$K(s_1 \curlywedge s_2) \subset K(s_1) \cap K(s_2),$$

$$K(s_1) \cap K(s_2) = \emptyset \Rightarrow s_1, s_2 \text{ are orthogonal},$$

and for any finite open covering $(G_i)_{1 \leqslant i \leqslant n}$ of Y and for any $s \in \mathscr{L}^+$ there exist $(s_i)_{1 \leqslant i \leqslant n}$, $s_i \in \mathscr{L}^+$, such that

$$s = \sum_{i=1}^{n} s_i, \qquad K(s_i) \subset G_i \qquad (i = 1, 2, \dots n).$$

Let $s \in \mathscr{L}^+$. A *division of* s is a finite family of elements of $\mathscr{L}^+$ whose sum is equal to s. We shall denote by Δ the set of all divisions of s. A division $(s_i)_{i \in I}$ of s is called finer than a division $(t_j)_{j \in J}$ of s if there exists a decomposition $(I_j)_{j \in J}$ of I such that

$$t_j = \sum_{i \in I_j} s_i$$

for any $j \in J$. This relation is an upper directed preorder relation on Δ. For any real continuous function f on Y and any $\delta \in \Delta$, $\delta = (s_i)_{i \in I}$, we denote

$$\delta^*(f) = \sum_{i \in I} [\sup_{x \in K(s_i)} f(x)] s_i,$$

$$\delta_*(f) = \sum_{i \in I} [\inf_{x \in K(s_i)} f(x)] s_i,$$

$$\varepsilon(\delta, f) = \sup_{i \in I} (\sup_{x \in K(s_i)} f(x) - \inf_{x \in K(s_i)} f(x)) (^5)$$

We have

$$\delta^*(f) \preccurlyeq \delta_*(f) + \varepsilon(\delta, f) s$$

and, if δ' is finer than δ,

$$\delta_*(f) \preccurlyeq \delta'_*(f) \preccurlyeq \delta'^*(f) \preccurlyeq \delta^*(f).$$

(5) We make, in these definitions, the conventions

$$\sup_{x \in \phi} f(x) = \inf_{x \in \phi} f(x) = 0.$$

Since for any finite open covering $(G_i)_{i\in I}$ of Y there exists a division $\delta = (s_i)_{i\in I}$ of s such that for any $i \in I$, $K(s_i) \subset G_i$, we may find for any positive number ε a division δ of s such that

$$\varepsilon(\delta, f) < \varepsilon.$$

Hence if $\bigwedge_{\varepsilon>0} \varepsilon s = 0$ we have

$$\bigwedge_{\delta\in\Delta} \delta^*(f) = \bigvee_{\delta\in\Delta} \delta_*(f)$$

whenever one of these two elements exists.

In general, if $\bigwedge_{\delta\in\Delta} \delta^*(f)$, $\bigvee_{\delta\in\Delta} \delta_*(f)$ exists and are equal we denote by $f.s$ their common value. We have $1 . s = s$.

LEMMA 2.3.—*If $f.s$ is defined for an $f \geqslant 0$ then*

$$K(f.s) \subset K(s) \cap \operatorname{Supp} f\ (^6),$$
$$K(s) \subset K(f.s) \cup \{x \in Y | f(x) = 0\}.$$

Since, for a sufficiently great natural number n,

$$f.s \preccurlyeq ns$$

we have $K(f.s) \subset K(s)$. Let (s_1, s_2) be a division of s such that $K(s_1) \cap \operatorname{Supp} f = \emptyset$ (resp. $K(s_1) \cap (K(f.s) \cup \{x \in Y | f(x) = 0\}) = \emptyset$) and $K(s_2) \subset G$, where G is an arbitrary open neighborhood of $\operatorname{Supp} f$ (resp. $K(f.s) \cup \{x \in Y | f(x) = 0\}$). Since

$$f.s \preccurlyeq [\sup_{x\in K(s_2)} f(x)]s_2 \qquad (\text{resp.}\ f.s \succcurlyeq [\inf_{x\in K(s_1)} f(x)]s_1)$$

we have

$$K(f.s) \subset K(s_2) \subset G \qquad (\text{resp.}\ K(s_1) \subset K(f.s)$$

hence

$$s_1 = 0, \qquad K(s) = K(s_2) \subset G).$$

LEMMA 2.4.—*Let f, g be real continuous functions on* Y *and $s, t \in \mathscr{L}^+$ such that $f.s$, $g.s$, $f.t$ are defined. Then:*

a) $(f - g).s$ *is defined and* $(f - g).s = f.s - g.s$;

b) $f.(s + t)$ *is defined and* $f.(s + t) = f.s + f.t$;

c) $\inf(f, g).s$ *and* $\sup(f, g).s$ *are defined and*

$$\inf(f, g).s = (f.s) \curlywedge (g.s), \quad \sup(f, g).s = (f.s) \curlyvee (g.s);$$

d) if $f \geqslant 0$ then $f.(s \curlywedge t)$ and $f.(s \curlyvee t)$ are defined and

(6) $\operatorname{Supp} f = \overline{\{x \in Y | f(x) \neq 0\}}$.

$f.(s \curlywedge t) = (f.s) \curlywedge (f.t), f.(s \curlyvee t) = (f.s) \curlyvee (f.t)$.

a) follows from the inequalities

$$\delta_*(f) - \delta^*(g) \preccurlyeq \delta_*(f-g) \preccurlyeq \delta^*(f-g) \preccurlyeq \delta^*(f) - \delta_*(g)$$

satisfied by any division δ of s.

b) is trivial.

c) We set

$$f' = f - \inf(f, g),$$
$$g' = g - \inf(f, g).$$

For any division δ of s we have

$$\delta_*(f') \curlywedge \delta_*(g') = 0,$$
$$\delta_*(\inf(f, g)) \preccurlyeq \delta_*(f) \preccurlyeq \delta_*(f') + \delta^*(\inf(f, g)),$$
$$\delta_*(\inf(f, g)) \preccurlyeq \delta_*(g) \preccurlyeq \delta_*(g') + \delta^*(\inf(f, g)),$$
$$\delta^*(\inf(f, g)) - \delta_*(\inf(f, g)) \preccurlyeq \delta^*(f) - \delta_*(f) + \delta^*(g) - \delta_*(g).$$

Hence, if δ' is a division of s finer than δ, we have

$$\delta_*(\inf(f, g)) \preccurlyeq \delta_*(f) \curlywedge \delta_*(g) \preccurlyeq \delta'_*(f) \curlywedge \delta'_*(g) \preccurlyeq$$
$$\preccurlyeq \delta'^*(\inf(f, g)) \preccurlyeq \delta^*(\inf(f, g)),$$
$$\delta_*(\inf(f, g)) \preccurlyeq f.s \curlywedge g.s \preccurlyeq \delta^*(\inf(f, g)) \preccurlyeq$$
$$\preccurlyeq \delta_*(\inf(f, g)) + \delta^*(f) - \delta_*(f) + \delta^*(g) - \delta_*(g).$$

The assertion for $\sup(f, g)$ follows from the relation

$$\sup(f, g) = f + g - \inf(f, g).$$

d) Let δ_0(resp. δ_1, δ_2) be a division of $s \curlywedge t$ (resp. $s - s \curlywedge t$, $t - s \curlywedge t$)

$$\delta_j = (s_i^j)_{i \in \mathrm{I}_j} (j = 0, 1, 2), \qquad \mathrm{I}_0 \cap \mathrm{I}_1 = \mathrm{I}_0 \cap \mathrm{I}_2 = \emptyset.$$

We denote by $\sigma_j (j = 1, 2)$ the family defined on $\mathrm{I}_0 \cup \mathrm{I}_j$ equal to δ_0 on I_0 and equal to δ_j on I_j; σ_1(resp. σ_2) is a division of s (resp. t). We have

$$\sigma_j^*(f) = \delta_j^*(f) + \delta_0^*(f),$$
$$\sigma_1^*(f) \curlywedge \sigma_2^*(f) = \delta_0^*(f) + \delta_1^*(f) \curlywedge \delta_2^*(f).$$

Since any s_i^1 is orthogonal to any s_k^2 we have $\delta_1^*(f) \curlywedge \delta_2^*(f) = 0$ and $\sigma_1^*(f) \curlywedge \sigma_2^*(f) = \delta_0^*(f)$. Similarly we get $\sigma_{1*}(f) \curlywedge \sigma_{2*}(f) = \delta_{0*}(f)$. Hence

$$(f \,.\, s) \curlywedge (f \,.\, t) = f \,.\, (s \curlywedge t).$$

For the last assertion we remark that

$$s \curlyvee t + s \curlywedge t = s + t.$$

Let $\delta = (s_i)_{i\in I}$(resp. $\sigma = (t_j)_{j\in J}$) be a division of $s \curlyvee t$ (resp. $s \curlywedge t$), $I \cap J = \emptyset$, and τ the division of $s + t$ defined on $I \cup J$ equal to δ on I and equal to σ on J. We have

$$\delta^*(f) + \sigma^*(f) = \tau^*(f), \qquad \delta_*(f) + \sigma_*(f) = \tau_*(f).$$

From here we see that $f \,.\, (s \curlyvee t)$ is defined and

$$f \,.\, (s \curlyvee t) + f \,.\, (s \curlywedge t) = f \,.\, (s + t),$$
$$f \,.\, (s \curlyvee t) + (f \,.\, s) \curlywedge (f \,.\, t) = f \,.\, s + f \,.\, t.$$

LEMMA 2.5. — *Let l be a linear map of the set of real continuous functions on* Y *into* $\mathscr{L}$ *such that for any* $f \geqslant 0$

$$l(f) \succcurlyeq 0, \qquad K(l(f)) \subset \operatorname{Supp} f.$$

If $\underset{\varepsilon}{\curlywedge} (\varepsilon \,.\, l(1)) = t$ *then for any real continuous function* g *on* Y, $g \,.\, l(1)$ *is defined and*

$$g \,.\, l(1) = l(g).$$

Let $(\psi_i)_{i\in I}$ be a finite family of non-negative real continuous functions defined on the real axis such that $\sum\limits_{i\in I} \psi_i = 1$. For $i \in I$ we denote $g_i = \psi_i \circ g$. The family $\delta = (l(g_i))_{i\in I}$ is a division of $l(1)$ and we have

$$\sum_{i\in I} \alpha_i l(g_i) \preccurlyeq \delta_*(g) \preccurlyeq \delta^*(g) \preccurlyeq \sum_{i\in I} \beta_i l(g_i),$$
$$\sum_{i\in I} \alpha_i l(g_i) \preccurlyeq l(g) \preccurlyeq \sum_{i\in I} \beta_i l(g_i),$$

where

$$\alpha_i = \inf_{x\in \operatorname{Supp} g_i} g(x), \qquad \beta_i = \sup_{x\in \operatorname{Supp} g_i} g(x).$$

If the vectorlattice $\mathscr{L}$ is conditionally complete then $f.s$ is defined for any $s \in \mathscr{L}^+$ and any f. In this case we define for any $s \in \mathscr{L}$ and any f

$$f.s = f.(s \curlyvee 0) + f.((-s) \curlyvee 0).$$

THEOREM 2.3. — *Let $\mathscr{L}$ be a conditionally complete vectorlattice, Y be a compact space, K be an abstract carrier on $(\mathscr{L}, Y)$ and $\mathscr{C}$ be the ring of real continuous functions on Y. Then $\mathscr{L}$ is a $\mathscr{C}$-modulus with respect to the composition law*

$$(f, s) \to f.s$$

and

a) if $s \succcurlyeq 0$ then

$$\sup(f, g).s = (f.s) \curlyvee (g.s), \qquad \inf(f, g).s = (f.s) \curlywedge (g.s)$$

b) if $f \geqslant 0$ then

$$f.(\curlyvee_{\iota \in I} s_\iota) = \curlyvee_{\iota \in I} (f.s_\iota), \qquad f.(\curlywedge_{\iota \in I} s_\iota) = \curlywedge_{\iota \in I} (f.s_\iota).$$

The theorem follows from the Lemmas 2.4 and 2.5.

Remark. — *The set of elements of $\mathscr{L}^+$ whose abstract carriers are contained in a given compact set K form a closed positive ideal. For any $s \in \mathscr{L}^+$ we shall denote by s_K the component of s in this ideal.*

III. CLOSED IDEALS.

1. Substractible functions.

LEMMA 3.1. — *We suppose that for any $x \in X$ there exists a non-negative locally bounded superharmonic function on X positive at x. Then the following assertions are equivalent:*

a) for any x and any regular MP-neighbourhood V of x there exists a non-negative superharmonic function s on X such that

$$s(x) > \int^* s \, d\omega_x^V ;$$

b) for any x, $y \in X$, $x \neq y$, there exists two non-negative superharmonic functions s, t on X finite at these points such that

$$s(x)t(y) - s(y)t(x) \neq 0.$$

a) $\Rightarrow$ *b).* Let x, $y \in X$, $x \neq y$, V be a regular MP-set, $x \in V$, $y \notin V$, and s be a non-negative superharmonic function on X such that

$$s(x) > \int^* s \, d\omega_x^V.$$

We may suppose s locally bounded and $s(y) \neq 0$. Let t be the superharmonic function on X equal to s on $X - V$ and equal to $\bar{H}_s^V$ on V. Obviously

$$s(x)t(y) - s(y)t(x) \neq 0.$$

b) $\Rightarrow$ *a).* Let V be an MP-neighbourhood of x and y a point of the carrier of the measure ω_x^V. Let further s, t be two non-negative superharmonic functions on X finite at x and y and such that

$$s(x)t(y) - s(y)t(x) \neq 0.$$

By Theorem 1.3 we may suppose that s and t are continuous. Obviously $s(x) \neq 0$, $t(x) \neq 0$ and therefore we may suppose $s(x) = t(x)$. Then

$$(s \wedge t)(x) > \int^* (s \wedge t) \, d\omega_x^V.$$

We say that a non-negative superharmonic function s on an open set U is *substractible on an open subset* V of U if

$$s \leqslant s' \Rightarrow s \preccurlyeq s'$$

for any superharmonic function s' on V.

LEMMA 3.2. — *Let s be a non-negative superharmonic function on an open set* U.

a) If s is substractible on any element of a family of open subsets of U *then s is substractible on their join.*

b) Let V *be a regular* MP*-set,* $\bar{V} \subset U$, *t a non-negative superharmonic function on* U *and* W *the open subset of* V *where the function*

$$\bigwedge_f (s + t - H_f^V)$$

is positive, where f is a real continuous minorant of $s + t$. If s is substractible on U *then s is substractible on* W.

c) If V *is an open subset of* U *and* t *is a non-negative superharmonic function on* U *infinite on* U − V *then if* s *is substractible on* V *it is substractible on* U.

a) follows from the fact that the non-negative superharmonic functions form a sheaf, using Lemma 2.1.

b) Let s' be a superharmonic function on W, $s \leqslant s'$. We set

$$u = \bigvee_f \mathrm{H}_f^{\mathrm{V}}, \qquad p = \bigwedge_f (s + t - \mathrm{H}_f^{\mathrm{V}}),$$

where f is a real continuous minorant of $s + t$. For any natural number n the function equal to $n(s + t)$ on U − W and equal to $nu + (s' \wedge np)$ on W is superharmonic on U and dominates s. Hence it is a specific majorant of s and we get

$$s + s_n = nu + (s' \wedge np)$$

on W, where s_n is a non-negative superharmonic function on W. We put

$$v = \bigvee_g \mathrm{H}_g^{\mathrm{V}}, \qquad q = \bigwedge_g (s - \mathrm{H}_g^{\mathrm{V}}),$$

where g is a real continuous minorant of s and we have

$$q(x) + \int^* (s' \wedge np)\, d\omega_x^{\mathrm{V}'} \leqslant (s' \wedge np)(x) + \int^* q\, d\omega_x^{\mathrm{V}'}$$

for any regular MP-neighbourhood V′ of x, $\bar{\mathrm{V}}' \subset \mathrm{W}$. Hence, by Lemma 1.5, $q \preccurlyeq s' \wedge np$, $q \preccurlyeq s'$, $s' = q + s''$, $v \leqslant s''$, where s'' is a non-negative superharmonic function on W. From

$$s'' = \bigvee_g \mathrm{H}_g^{\mathrm{V}} + \bigwedge_g (s'' - \mathrm{H}_g^{\mathrm{V}}),$$

where g is a real continuous minorant of s, we have $v \preccurlyeq s''$.

c) Let s' be a non-negative superharmonic function on U, $s \leqslant s'$. Then there exists a non-negative superharmonic function s'' on V such that

$$s' = s + s''.$$

For any $\varepsilon > 0$ the function s_ε on U equal to $s'' + \varepsilon t$ on V and infinite on U − V is superharmonic. We have

$$s + \bigwedge_\varepsilon s_\varepsilon = s'$$

on V and therefore on U since U − V is negligible.

Remark. — *If* s *is a substractible non-negative superharmonic function on* U *it is not always substractible on any open subset of* U.

This is nevertheless true if the assertion a) from the Lemma 3.1 *is satisfied and a fortiori if Bauer's « Trennungsaxiom » is fulfilled.*

We denote for an open set U by $\mathscr{E}(U)$ the set of non-negative superharmonic functions on U which are substractible on any open subset of U.

THEOREM 3.1. — *a)* $U \to \mathscr{E}(U)$ *is a sheaf ;*

b) for any U, $\mathscr{E}(U)$ *is a closed positive ideal of* $[\mathscr{S}^+](U)$;

c) $\tilde{\mathscr{H}}^+(U) \subset \mathscr{E}(U)$;

d) any locally bounded element of $\mathscr{E}(U)$ *is harmonic.*

a) follows from the definition and from Lemma 3.2. *b)* Let s_1, s_2 be non-negative superharmonic functions on U, V be an open subset of U and s' be a non-negative superharmonic function on V. If $s_1 + s_2 \in \mathscr{E}(U)$ then

$$s_1 \leqslant s' \Rightarrow s_1 + s_2 \leqslant s' + s_2 \Rightarrow s_1 + s_2 \preccurlyeq s' + s_2 \Rightarrow s_1 \preccurlyeq s'.$$

Hence $s_1 \in \mathscr{E}(U)$. If $s_1, s_2 \in \mathscr{E}(U)$ then

$$s_1 + s_2 \leqslant s' \Rightarrow s_1 \leqslant s' \Rightarrow s_1 + s = s'$$
$$\Rightarrow s_2 \leqslant s \Rightarrow s_2 \preccurlyeq s \Rightarrow s_1 + s_2 \preccurlyeq s',$$

where s is a non-negative superharmonic function on V. Hence $s_1 + s_2 \in \mathscr{E}(U)$. Let $\mathscr{S} = (s_\iota)_{\iota \in I}$ be a specific upper directed family of elements of $\mathscr{E}(U)$. Then, since the restriction of $\curlyvee \mathscr{S}$ to V is the specific least upper bound of the restrictions of s_ι to V,

$$\curlyvee \mathscr{S} \leqslant s' \Rightarrow s_\iota \leqslant s' \Rightarrow s_\iota \preccurlyeq s' \Rightarrow \curlyvee \mathscr{S} \preccurlyeq s'.$$

Hence $\curlyvee \mathscr{S} \in \mathscr{E}(U)$. *c)* follows from *a), b)* and $\mathscr{H}^+(U) \subset \mathscr{E}(U)$. *d)* Any locally bounded element of $\mathscr{E}(U)$ is harmonic since it is locally specifically dominated by a harmonic function.

THEOREM 3.2. — *If s is a non-negative superharmonic function on* X *such that there exists a dense subset* A *of* X, $A \cap s^{-1}(\{\infty\}) = \phi$, *such that for any point* $x \in A$ *there exists a non-negative superharmonic function* s_x *on* X *infinite almost everywhere with respect to the measure on* X

$$f \to (f \,.\, s)(x),$$

f real continuous with compact carrier on X, *then* $s \in \mathscr{E}(X)$.

It is sufficient to prove that s vanishes if it is orthogonal to $\mathscr{E}(X)$. Let $x \in A$ and K be a compact subset of $s_x^{-1}(\{\infty\})$. The component s_K of s in the positive closed ideal of non-negative superharmonic functions with carriers contained in K belongs to $\mathcal{E}(X)$ by Lemma 3.2 and therefore it vanishes. We get

$$s(x) = \sup_{K} s_K(x) = 0, \qquad s = 0.$$

2. Closed ideals.

This paragraph is consecrated to the study of some interesting closed positive ideals of $[\mathcal{S}^+](X)$.

THEOREM 3.3. — *The closed positive ideal generated by the set of non-negative superharmonic functions on* X *whose restrictions to their carriers* (7) *are finite continuous contains any non-negative finite superharmonic function on* X.

Let s be a non-negative finite superharmonic function on X, orthogonal to any non-negative superharmonic function whose restriction to its carrier is continuous and finite. In particular s is orthogonal to $\tilde{\mathcal{H}}^+(X)$. For any $x \in X$ the map

$$f \to (f \,.\, s)(x)$$

is a measure μ on X_0. Let K be a compact subset of X such that the restriction of s to K is continuous. Then $s_K = 0$. We get by Lemma 1.9

$$\mu(K) = \inf_f \mu(f) = \inf_f (f \,.\, s)(x) = s_K(x) = 0,$$

where f runs the set of non-negative continuous finite functions on X_0 greater than 1 on K. By Lusin's property μ vanishes on any compact subset of X and, since $\mu(X_0 - X) = 0$,

$$s(x) = \mu(X_0) = 0.$$

We denote by $\mathcal{Q}$ the set of non-negative superharmonic functions s on X such that for any relatively compact open set U

$$\wedge \mathcal{Q}_s^U = 0,$$

where $\mathcal{Q}_s^U$ is the set of non-negative superharmonic functions s' on U for which there exists an upper semicontinuous real function f on U such that

$$f \leqslant s \leqslant s' + f.$$

We denote by $\mathcal{M}$ (resp. $\mathcal{M}'$) the set of non-negative superharmonic functions s on X such that for any relatively compact MP-set U, $\bar{H}_s^U$ is the component of the restriction of s to U on $\tilde{\mathcal{H}}^+(U)$(resp. $\mathcal{E}(U)$).

(7) More "precisely" to the trace on X of their carriers.

LEMMA 3.3. — *$\mathcal{Q}$, $\mathcal{M}$, $\mathcal{M}'$ are closed positive ideals in $[\mathcal{S}^+](X)$ and*

$$\mathcal{Q} \subset \mathcal{M}' \subset \mathcal{M}.$$

Let s_1, s_2 be two non-negative superharmonic functions on X. Then

$$\mathcal{Q}^U_{s_1} + \mathcal{Q}^U_{s_2} \subset \mathcal{Q}^U_{s_1+s_2} \subset \mathcal{Q}^U_{s_1} \cap \mathcal{Q}^U_{s_2}$$

for any relatively compact open set U. Hence

$$s_1 + s_2 \in \mathcal{Q} \Leftrightarrow s_1, s_2 \in \mathcal{Q}.$$

For any relatively compact MP-set U we have [3] (Theorem 3.2)

$$\bar{H}^U_{s_1+s_2} = \bar{H}^U_{s_1} + \bar{H}^U_{s_2}.$$

Hence

$$s_1 + s_2 \in \mathcal{M} \text{ (resp. } \mathcal{M}') \Leftrightarrow s_1, s_2 \in \mathcal{M} \text{ (resp. } \mathcal{M}').$$

Let $\mathcal{S} = (s_\iota)_{\iota \in I}$ be a specifically upper directed family of elements of $\mathcal{Q}$ (resp. $\mathcal{M}$, $\mathcal{M}'$), $s = \curlyvee \mathcal{S}$, and for any $\iota \in I$, t_ι be a non-negative superharmonic function on X such that $s = s_\iota + t_\iota$. Then for any relatively compact set (resp. MP-set) U we have

$$\mathcal{Q}^U_{s_\iota} + t_\iota \subset \mathcal{Q}^U_s \qquad (\text{resp. } \bar{H}^U_{s_\iota} \leqslant \bar{H}^U_s).$$

Hence $s \in \mathcal{Q}$ (resp. $\mathcal{M}$, $\mathcal{M}'$).

Obviously $\mathcal{M}' \subset \mathcal{M}$. Let $s \in \mathcal{Q}$, U be a relatively compact MP-set, V be an open relatively compact set, $\bar{U} \subset V$, s' be a non-negative superharmonic function on V and f be a real upper semicontinuous function on V such that

$$f \leqslant s \leqslant s' + f.$$

Let further t be the component of the restriction of s to U in $\mathcal{E}(U)$. For any $x \in \bar{U}$ let u_x be a harmonic function on a neighbourhood W_x of x, $u_x \geqslant f$. Since $t \leqslant s \leqslant s' + u_x$ on $W_x \cap U$ we have

$$t + t_x = s' + u_x,$$

where t_x is a non-negative superharmonic function on $W_x \cap U$. Hence there exists a superharmonic function t' on U equal to $t_x - u_x$ on $W_x \cap U$ for any $x \in \bar{U}$. We have

$$t + t' = s', \qquad t' \geqslant -f, \qquad t' \geqslant \bar{H}^U_{-f} \geqslant \bar{H}^U_{-s} \geqslant -\bar{H}^U_s.$$

Hence, since $\bar{H}^U_s$ is a superharmonic function,

$$t \leqslant s' + \bar{H}^U_s, \qquad t \leqslant \bar{H}^U_s.$$

Since $\bar{H}^U_s \in \tilde{\mathcal{H}}^+(U)$, $t = \bar{H}^U_s$.

LEMMA 3.4. — *Let* $s \in \mathcal{M}$.

a) At any point $x \in \operatorname{Carr} s$

$$\limsup_{y \to x} s(y)$$

is either infinite or equal to

$$\limsup_{y \to x} s(y), \qquad y \in \operatorname{Carr} s.$$

b) If any relatively compact open set is an MP-*set and*

$$\bigwedge_{U} \hat{R}_s^{X-U} = 0,$$

where U *is a relatively compact open set, then for any non-negative superharmonic function* s' *we have*

$$s' \geqslant s \text{ on } \operatorname{Carr} s \Rightarrow s' \geqslant s \text{ on } X.$$

b) Let U be a relatively compact MP-set. Then,

$$s = \bar{H}_s^{U-K} \leqslant s' + \hat{R}_s^{X-U} \qquad \text{on} \qquad U - K,$$

where we have put $K = \operatorname{Carr} s$. Hence, U being arbitrary,

$$s \leqslant s' \qquad \text{on} \qquad X.$$

a) Suppose

$$\limsup_{y \to x} s(y) < +\infty.$$

Since the property has a local character we may suppose that X is an MP-set, s is bounded and there exists a positive harmonic function u on X, $u(x) = 1$. For any $\varepsilon > 0$ there exists a compact neighbourhood V of x such that

$$\sup_{y \in V \cap \operatorname{Carr} s} \frac{s_V(y)}{u(y)} \leqslant \limsup_{\operatorname{Carr} s \ni y \to x} s_V(y) + \varepsilon = \beta_\varepsilon,$$

where s_V is the component of s on the closed positive ideal of non-negative superharmonic functions whose carriers lies in V. By *b)*

$$s_V \leqslant \beta_\varepsilon u,$$

$$\limsup_{y \to x} s(y) - \limsup_{\operatorname{Carr} s \ni y \to x} s(y) = \limsup_{y \to x} s_V(x) - \limsup_{\operatorname{Carr} s \ni y \to x} s_V(y) \leqslant \varepsilon.$$

THEOREM 3.4.— *Suppose that any relatively compact open subset of* X *is an* MP-*set and for any* $x \in X$ *there exists a non-negative locally bounded superharmonic function on* X *positive at* x. *For a* *non-negative superharmonic function s on X whose component in $\tilde{\mathscr{H}}^+(X)$ is harmonic the following assertions are equivalent:*

a) $s \in \mathcal{Q}$;

b) $s \in \mathscr{M}'$;

c) $s \in \mathscr{M}$;

d) s is the specific least upper bound of its continuous finite specific minorants.

d) ⇒ *a)* ⇒ *b)* ⇒ *c)* follows from Lemma 3.3.

c) ⇒ *d)*. It is sufficient to suppose that the carrier of s is a compact subset of X. Let K be a compact subset of X such that s is infinite on K. By the preceding lemma *b)*

$$\varepsilon s \geqslant s_K$$

for any $\varepsilon > 0$. Hence

$$s_K = 0.$$

We may assume therefore that s is bounded on its carrier. It follows then, again by the preceding lemma *b)*, that s is locally bounded. Hence by Theorem 3.3 we may take further s continuous on its carrier. By Lemma 3.4*a)* s is finite and continuous.

COROLLARY 3.1.— $\mathscr{E}(X) \cap \mathscr{M} \subset \tilde{\mathscr{H}}^+(X)$.

For some further developments of the theory it is necessary to require the following condition: *any locally bounded non-negative superharmonic function belongs to $\mathscr{M}$*. This condition coincides with Brelot's axiom D if $\mathscr{H}$ satisfies Brelot's axiom 3. We shall call it also *axiom D*.

3. Quasicontinuity.

Let s be a non-negative hyperharmonic function on X and for any subset A of X let $\mathcal{Q}_s'^{A,X} = \mathcal{Q}_s'^{A}$ be the set of non-negative hyperharmonic functions s' on X such that the restriction of s to

$$\{x \in A \,|\, s'(x) \leqslant 1\}$$

is continuous. We say that s is *quasicontinuous on* A if

$$\wedge \mathcal{Q}_s'^{A} = 0.$$

LEMMA 3.5. — *Let $(\mathscr{S}_n)_{n\in N}$ be a sequence of families of non-negative hyperharmonic functions such that*

$$\wedge \mathscr{S}_n = 0, \qquad n \in N.$$

Then

$$\wedge \mathcal{S} = 0,$$

where $\mathcal{S}$ *is the family of hyperharmonic functions of the form*

$$\sum_{n=1}^{\infty} s_n, \, s_n \in \mathcal{S}_n.$$

The assertion follows from the fact that $\inf_{s \in \mathcal{S}_n} s(x)$ is equal to zero outside a negligible set and from the fact that a countable union of negligible sets is negligible.

LEMMA 3.6.— *Let* A *be a relatively compact subset of* X *and let* $\mathcal{S} = (s_\iota)_{\iota \in \mathrm{I}}$ *be an upper directed family of non-negative hyperharmonic functions on* X *such that*

$$\vee \mathcal{S} = +\infty.$$

on $\bar{\mathrm{A}}$. *If* s *is a non-negative hyperharmonic function on* X *such that for any* $\iota \in \mathrm{I}$, $s \wedge s_\iota$ *is quasicontinuous on* A, *then* s *is quasicontinuous on* A.

Let $(\iota_n)_{n \in \mathrm{N}}$ be a sequence in I such that $s_{\iota_n} > n$ on $\bar{\mathrm{A}}$. Then

$$s'_{\iota_n} \in \mathcal{Q}'^{\mathrm{A}}_{s \wedge s_{\iota_n}} \Rightarrow \sum_{n \in \mathrm{N}} s'_{\iota_n} \in \mathcal{Q}'^{\mathrm{A}}_{s}.$$

Hence, by the preceding lemma,

$$\wedge \, \mathcal{Q}'^{\mathrm{A}}_{s} = 0.$$

THEOREM 3.5. — *a) For any subset* A *of* X *the non-negative superharmonic functions quasicontinuous on* A *form a closed positive ideal of* $[\mathcal{S}^+](\mathrm{X})$.

b) The restriction of any element of $\mathcal{Q}$ *to a relatively compact open set* U *is quasicontinuous on* U.

c) Any non-negative locally bounded superharmonic function quasicontinuous on any compact subset of X *belongs to* $\mathcal{Q}$.

a) Let s_1, s_2 be non-negative superharmonic functions on X. Then

$$\mathcal{Q}'^{\mathrm{A}}_{s_1} + \mathcal{Q}'^{\mathrm{A}}_{s_2} \subset \mathcal{Q}'^{\mathrm{A}}_{s_1+s_2}, \, \mathcal{Q}'^{\mathrm{A}}_{s_1+s_2} + \varepsilon(s_1 + s_2) \subset \mathcal{Q}'^{\mathrm{A}}_{s_1} \cap \mathcal{Q}'^{\mathrm{A}}_{s_2}.$$

for any $\varepsilon > 0$. Hence $s_1 + s_2$ is quasicontinuous on A if and only if s_1 and s_2 are quasicontinuous on A. Let $\mathcal{S} = (s_\iota)_{\iota \in \mathrm{I}}$ be a specifically upper directed family of non-negative superharmonic functions quasicontinuous on A and

$$s = \curlyvee \mathcal{S}.$$

For any $\iota \in \mathrm{I}$ let t_ι be the non-negative superharmonic function on X such that

$$s = s_\iota + t_\iota .$$

For any infinite subset J of I we have

$$s'_\iota \in \mathcal{C}'^{\mathrm{A}}_{s_\iota} \, (\iota \in \mathrm{J}) \Rightarrow \sum_{\iota \in \mathrm{J}} (s'_\iota + t_\iota) \in \mathcal{C}'^{\mathrm{A}}_s .$$

Hence if J is countable

$$\wedge \, \mathcal{C}'^{\mathrm{A}}_s \leqslant \sum_{\iota \in \mathrm{J}} t_\iota$$

and therefore

$$\wedge \, \mathcal{C}'^{\mathrm{A}}_s = 0.$$

b) Let $s \in \mathcal{C}$ and U be a relatively compact open subset of X. Then for any sequence $(s_n)_{n \in \mathrm{N}}$, $s_n \in \mathcal{C}^{\mathrm{U}}_s$,

$$\sum_{n=1}^{\infty} s_n \in \mathcal{C}'^{\mathrm{U},\mathrm{U}}_s$$

and therefore

$$\wedge \, \mathcal{C}'^{\mathrm{U},\mathrm{U}}_s = 0.$$

c) Let s be a non-negative locally bounded superharmonic function on X quasicontinuous on any compact subset of X and U be a relatively compact open subset of X. We have

$$s' \in \mathcal{C}'^{\mathrm{U}}_s \Rightarrow \alpha s' \in \mathcal{C}^{\mathrm{U}}_s$$

where $\alpha = \sup_{x \in \mathrm{U}} s(x)$.

THEOREM 3.6. — *Assume that any relatively compact open set is an* MP-*set and that the non-negative locally bounded superharmonic functions on* X *have no zero in common. If axiom* D *is fulfilled then any non-negative hyperharmonic function on* X *is quasicontinuous on any compact subset of* X.

Let s be a non-negative hyperharmonic function on X. For any non-negative locally bounded superharmonic function s' on X, $s \wedge s'$ is quasicontinuous on any compact subset of X by Theorems 3.4 and 3.5*a*). Hence, by Lemma 3.6, s is quasicontinuous on any compact subset of X.

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