Ould M. ABDERRAHMANE

Stratification theory from the Newton polyhedron point of view

<http://aif.cedram.org/item?id=AIF_2004__54_2_235_0>
A stratification of a variety $V$ is an expression of $V$ as the disjoint union of a locally finite set of connected analytic manifolds, called strata, such that the frontier of each stratum is the union of a set of lower-dimensional strata. The most important notion in stratification theory is the regularity condition between strata. The notion of ($w$)-regularity introduced by Verdier in [15] plays a very important role in the study of algebraic and analytic varieties. Moreover, he showed that the ($w$)-regularity condition implies the Whitney ($b$)-regularity condition. The ($c$)-regularity, defined by K. Bekka in [2], is weaker than the Whitney ($b$)-regularity, and he showed that the ($c$)-regularity condition implies topological triviality. In this paper, we will investigate these regularity conditions relative to a Newton filtration in terms of the defining equations of the strata. The article is organized as follows. In Section 1 we present a characterization for Bekka's ($c$)-regularity condition. Next we give a criterion for regularity conditions in terms of the defining equations of the strata, following [1] we introduce a pseudo-metric adapted to the Newton polyhedron in Section 2. Using this construction we obtain versions relative to the Newton filtration of the Fukui-Paunescu Theorem (Theorem 4 below). In this approach it is possible to consider a version relative to a Newton filtration of the ($w$)-regularity condition. We show that this

* This research was supported by the Japan Society for the Promotion of Science.
* Keywords: Stratification -- Regularity condition -- Newton polyhedron.
condition implies the (c)-regularity condition. In Section 3, using the
criterion of the regularity condition given in Section 2, we prove that the
J. Damon and T. Gaffney condition in ([5], Theorem 1) implies the (w)-
regularity condition related to the Newton polyhedron.

Since complex varieties can be considered as real varieties, we shall
only consider the real case.

**Notation.** — To simplify the notation, we will adopt the following
conventions: for a function $g(x, t)$, we denote by $\partial g$ the gradient of $g$
and by $\partial_x g$ the gradient of $g$ with respect to the variables $x$. For a non
zero vector $v$ of $\mathbb{R}^n$, we denote by $L(v)$ the line spanned by $v$. Also, let
$\mathbb{R}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n, \text{ each } x_i \geq 0, i = 1, \ldots, n \}$ and $\mathbb{Q}^n_+ = \mathbb{Q}^n \cap \mathbb{R}^n_+$,
$\mathbb{Z}^n_+ = \mathbb{Z}^n \cap \mathbb{Q}^n_+$.

Let $\varphi, \psi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be two functions. We say that $|\varphi(x)| \lesssim |\psi(x)|$ if there exists a constant $C$ such that $|\varphi(x)| \leq C|\psi(x)|$. We write
$|\varphi| \sim |\psi|$ if $|\varphi(x)| \lesssim |\psi(x)|$ and $|\psi(x)| \lesssim |\varphi(x)|$. Finally, $|\varphi(x)| \ll |\psi(x)|$
when $x$ tends to $x_0$ means $\lim_{x \rightarrow x_0} \frac{\varphi(x)}{\psi(x)} = 0$.

1. Stratification.

In this section, we recall some definitions about stratification. The
stratification theory has been introduced by H. Whitney [16] and R. Thom
[13].

Let $M$ be a smooth manifold, and let $X, Y$ be smooth submanifolds
of $M$ such that $Y \subseteq \overline{X}$ and $X \cap Y = \emptyset$.

(i) (Whitney (a)-regularity)
$(X, Y)$ is (a)-regular at $y_0 \in Y$ if:
for each sequence of points $\{x_i\}$ which tends to $y_0$ such that the
sequence of tangent spaces $\{T_{x_i}X\}$ tends in the Grassman space of
$(\dim X)$-planes to some plane $\tau$, then $T_{y_0}Y \subset \tau$. We say $(X, Y)$ is
(a)-regular if it is (a)-regular at any point $y_0 \in Y$.

(ii) (Bekka (c)-regularity)
Let $\rho$ be a smooth non-negative function such that $\rho^{-1}(0) = Y$. $(X, Y)$
is (c)-regular at $y_0 \in Y$ for the control function $\rho$ if:
for each sequence of points $\{x_i\}$ which tends to $y_0$ such that the
sequence of tangent spaces $\{\text{Ker} \rho(x_i) \cap T_{x_i}X\}$ tends in the Grassman
space of $(\dim X - 1)$-planes to some plane $\tau$, then $T_{y_0}Y \subset \tau$. $(X, Y)$
is (c)-regular at $y_0$ if it is (c)-regular for some control function $p$. We say $(X,Y)$ is (c)-regular if it is (c)-regular at any point $y_0 \in Y$.

1.1. A criterion for (c)-regularity.

We suppose now that $M = \mathbb{R}^{n+m}$ and $0 \in Y \subset \overline{X} - X$ (the regularity conditions are defined locally). Modulo an analytic transformation of $\mathbb{R}^{n+m}$ near 0, if necessary, we may assume that $Y$ coincides with its tangent space $T_0Y$. Let $(x,t) = (x_1, \ldots, x_n, t_1, \ldots, t_m)$ denote a system of coordinates of $\mathbb{R}^{n+m}$. For notational convenience we also use $x_{n+s} = t_s$. We assume that

$$Y = \{(x,t) \in \mathbb{R}^{n+m} \mid x_1 = \ldots = x_n = 0\}.$$

Then we can characterize (c)-regularity as follows:

**Theorem 1.** The pair $(X,Y)$ is (c)-regular at 0 for the control function $p$ if and only if $(X,Y)$ is (a)-regular at 0 and $|\partial_t(p|_X)(x,t)| \ll |\text{grad}(p|_X)(x,t)|$ as $(x,t) \in X$ and $(x,t) \to 0$.

The following proof is inspired by the proof of Bekka-Koike ([3], Theorem 2.4)

**Proof.** At first, we have the following equality:

$$T_{(x,t)}X = (\text{Ker } dp(x,t) \cap T_{(x,t)}X) \oplus K_{(x,t)},$$

where $K_{(x,t)} = (\text{Ker } dp(x,t) \cap T_{(x,t)}X)^\perp \cap T_{(x,t)}X = L(\partial(p|_X)(x,t))$ i.e., a line spanned by the gradient of the function $p|_X$.

$(\Rightarrow)$ Let $(x_i, t_i)$ be a sequence of points $X$ which tends to 0 such that $T_{(x_i, t_i)}X$ tends to some $(\dim X)$-dimensional space $\tau$. Taking a subsequence if necessary we can suppose that $\text{Ker } dp(x_i, t_i) \cap T_{(x_i, t_i)}X$ tends to some $(\dim X - 1)$-dimensional space $\tau'$ and $K_{(x_i, t_i)}$ tends to some one-dimensional space $L$. By Bekka (c)-regularity $\{0\} \times \mathbb{R}^m \subset \tau'$. Since $\text{Ker } dp(x_i, t_i) \cap T_{(x_i, t_i)}X \subset T_{(x_i, t_i)}X$ and $K_{(x_i, t_i)}$ is orthogonal to $\text{Ker } dp(x_i, t_i) \cap T_{(x_i, t_i)}X$, we have $\{0\} \times \mathbb{R}^m \subset \tau$ and $L$ is orthogonal to $\{0\} \times \mathbb{R}^m$ which means $(X,Y)$ is (a)-regular at 0 and $|\partial_t(p|_X)(x_i, t_i)| \ll |\partial(p|_X)(x_i, t_i)|$.

$(\Leftarrow)$ Let $(x_i, t_i)$ be a sequence of points $X$ which tends to 0 such that $\text{Ker } dp(x_i, t_i) \cap T_{(x_i, t_i)}X$ tends to some $(\dim X - 1)$-dimensional space $\tau$. 

TOME 54 (2004), FASCICULE 2
When passing to a subsequence one can suppose that all the $T_{(x,t),X}$ have the same dimension (dim $X$), and that this sequence of space converges to some space $\tau'$ and $K_{(x,t),X}$ tends to some one-dimensional space $L$. By the Whitney (a)-regularity $\{0\} \times \mathbb{R}^m \subset \tau'$. Since $|\partial_t(p_{|X})(x,t)| \ll |\partial(p_{|X})(x,t)|$, which implies $L \subset \mathbb{R}^n \times \{0\}$, $L$ is orthogonal to $\{0\} \times \mathbb{R}^m$. Hence we have $\{0\} \times \mathbb{R}^m \subset \tau$.

This completes the proof of the theorem. 

1.2. Ratio test conditions and $(w)$-regularity.

For $X$, $Y$ as above, we say $X$ is $(r)$-regular (resp. $(w)$-regular) over $Y$ at 0, if for any unit vector $v$ tangent to $Y$

$$|\pi_p(v)||\langle x, t \rangle| \ll |x| \quad \text{as} \quad p = (x, t) \in X \quad \text{and} \quad (x, t) \to 0$$

(resp. $|\pi_p(v)| \lesssim |x|$ when $p = (x, t) \in X$ near 0) where $\pi_p$ denotes the orthogonal projection of $\mathbb{R}^{n+m}$ to the normal space of $X$ at $p \in X$. We can find a lot of information about this in [6, 8, 14].

Let $F: (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)$ be an analytic map-germ. We denote by $V_F$ the variety of the zero locus of $F$. One can note that $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m\}$ gives a stratification of $V_F$ around $\{0\} \times \mathbb{R}^m$. Hereafter, we will assume that

$$X = F^{-1}(0) - \{0\} \times \mathbb{R}^m \quad \text{and} \quad Y = \{0\} \times \mathbb{R}^m.$$ 

Setting $F := (F_1, \ldots, F_p)$, assume that the Jacobi matrix of $F$ has rank $k$ on $X$ near 0, where $k \leq p$ is the codimension of $X$ in $\mathbb{R}^{n+m}$. We note that the normal space to $X$ is generated by the gradient of the functions $F_j$ ($j = 1, \ldots, p$) at each $P \in X$ near 0. Let us recall some definitions and notations, used by Fukui and Paunescu in [6].

Let $j_1, \ldots, j_k$ be integers with $1 \leq j_1 < \cdots < j_k \leq p$. We set $J = \{j_1, \ldots, j_k\}, F_J = (F_{j_1}, \ldots, F_{j_k})$ and

$$dF_J = dF_{j_1} \wedge \cdots \wedge dF_{j_k}, \quad \text{where} \quad dF_j = \sum_{i=1}^{n+m} \frac{\partial F_j}{\partial x_i} dx_i,$$

$$d_x F_J = d_x F_{j_1} \wedge \cdots \wedge d_x F_{j_k}, \quad \text{where} \quad d_x F_j = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i} dx_i,$$

and we define $d^x F_J$ by $dF_J = d_x F + d^x F_J$. 

ANNALES DE L’INSTITUT FOURIER
For \( I \subset \{1, \ldots, n\} \), \( S \subset \{1, \ldots, m\} \), \( J \subset \{1, \ldots, p\} \) with \( \#I + \#S = \#J = k \), we set \( \frac{\partial F_j}{\partial(x_i, t_s)} \) to be the Jacobian of \( F_j \) with respect to the variables \( x_i \) (\( i \in I \)), and \( t_s \) (\( s \in S \)). When \( S = \emptyset \), we simply denote it by \( \frac{\partial F_j}{\partial x_i} \). We then define \( \|dF\|, \|d_x F\| \) and \( \|d^x F\| \) by the following formulae:

\[
\|dF\|^2 = \sum_j \|dF_j\|^2 \quad \text{where} \quad \|dF_j\|^2 = \sum_{I,S} \left| \frac{\partial F_j}{\partial(x_I, t_S)} \right|^2, \\
\|d_x F\|^2 = \sum_j \|d_x F_j\|^2 \quad \text{where} \quad \|d_x F_j\|^2 = \sum_{I} \left| \frac{\partial F_j}{\partial x_I} \right|^2, \\
\|d^x F\|^2 = \sum_j \|d^x F_j\|^2 \quad \text{where} \quad \|d^x F_j\|^2 = \sum_{I,S: S \neq \emptyset} \left| \frac{\partial F_j}{\partial(x_I, t_S)} \right|^2.
\]

For a matrix \( M \) we denote by \( |M| \) the absolute value of its determinant.

Then we have a simple criterion for the regularity conditions of \( \Sigma(V_F) \) as follows:

**Theorem 2.** — For \( X, Y \) as above, we have the following equivalences

(i) \((X, Y)\) is \((a)\)-regular at 0 if and only if \( \|d^x F\| \ll \|dF\| \) when \((x, t) \to 0\) on \( X \).

(ii) \((X, Y)\) is \((r)\)-regular at 0 if and only if \( \|d^x F\| \ll |x| \|d_x F\| \) when \((x, t) \to 0\) on \( X \).

(iii) \((X, Y)\) is \((w)\)-regular at 0 if and only if \( \|d^x F\| \ll |x| \|d_x F\| \) holds on \( X \) near 0.

(iv) \((X, Y)\) is \((c)\)-regular at 0 for the function \( \rho \) if and only if \( \|d^x F\| \ll \|dF\| \) and \( \|\partial_t \rho \|_X \ll \frac{\|d^x F \wedge d\rho\|}{\|dF\|} \) as \((x, t) \in X, (x, t) \to 0\).

Here, \( \|dF \wedge d\rho\|^2 = \sum_j \|dF_j \wedge d\rho\|^2 \).

**Proof.** — Since (i), (ii) and (iii) have already been obtained in [6], we only have to prove (iv). Indeed, following ([6], lemma 1.4), one get that the orthogonal projection \( \pi \) of \( v \in T_{(x,t)} M \) to the tangent space \( T_{(x,t)} X \) is expressed by the following form:

\[
\pi(v) = \sum_{i=1}^{n+m} \sum_j \frac{(dF_j \wedge dx_i, dF_j \wedge v)}{\|dF\|^2} \frac{\partial}{\partial x_i}.
\]
Since $\partial \rho_{|X} = \pi(\partial \rho)$, we can easily see that $\langle \partial \rho_{|X}, \partial \rho \rangle = \frac{\|dF \wedge \partial \rho\|^2}{\|dF\|^2}$, but $
abla = \partial \rho_{|X} + \partial \rho_{|N}$ (where $N$ denotes the normal space to $X$), which implies

\begin{equation}
|\partial \rho_{|X}|^2 = \langle \partial \rho_{|X}, \partial \rho \rangle = \frac{\|dF \wedge \partial \rho\|^2}{\|dF\|^2}.
\end{equation}

Hence, we can deduce from Theorem 1 that (iv) holds. \square

We next state one sufficient condition for $(c)$-regularity.

**Corollary 3.** Suppose that $\partial_t \rho = 0$, then $X$ is $(c)$-regular over $Y$ at 0, if

\begin{equation}
\|d^xF\| \ll \frac{\|dF \wedge \partial \rho\|}{|\partial \rho|} \quad \text{as } (x, t) \in X, \ (x, t) \to 0.
\end{equation}

Note that when $p = k = 1$, this inequality is a necessary condition for $(c)$-regularity.

**Proof.** It is trivial that (1.3) implies $(X, Y)$ is $(a)$-regular at 0. We first remark, by (1.1) the following equality:

$$
\partial_{t_j} \rho_{|X} = \sum_j \frac{\langle dF_j \wedge dt_j, dF_j \wedge \partial \rho \rangle}{\|dF\|^2} \frac{\partial}{\partial t_j} = \sum_{i=1}^n \frac{\partial \rho}{\partial x_i} \sum_j \frac{\langle dF_j \wedge dt_j, dF_j \wedge dx_i \rangle}{\|dF\|^2} \frac{\partial}{\partial t_j}.
$$

Then, by Cauchy-Schwartz inequality, we have

$$
|\partial_{t_j} \rho_{|X}| \lesssim \frac{|\partial \rho| \|d^xF\|}{\|dF\|} \quad \text{for } j = 1, \ldots, m.
$$

We now assume (1.3). We then have $|\partial_t \rho_{|X}| \ll \frac{\|dF \wedge \partial \rho\|}{\|dF\|}$ as $(x, t) \in X, \ (x, t) \to 0$. It follows from the equivalence in (iv) of Theorem 2 that $(X, Y)$ is $(c)$-regular at 0. \square

2. $(w)$-regularity and $(c)$-regularity relative to the Newton filtration.

Let us recall some basic definitions and properties of the Newton filtration (see [1, 5, 7] for details). Let $A \subset \mathbb{Q}^*_+$. A Newton polyhedron

ANNALES DE L’INSTITUT FOURIER
\( \Gamma_+ (A) \subset \mathbb{R}^n \) is defined by \{the convex closure of \( A + \mathbb{R}_+^n \}\}. The Newton boundary of \( A \), \( \Gamma(A) \) is the union of the compact faces of \( \Gamma_+ (A) \). We let \( \mathcal{F}(A) \) denote the union of the top dimensional faces of \( \Gamma(A) \). The Newton vertex \( \text{Ver}(A) \) is defined by \{\( \alpha : \alpha \) is vertex of \( \Gamma(A) \)\}. \( A \) is called convenient if the intersection of \( \Gamma_+ (A) \) with each coordinate axis is non-empty. Throughout, we suppose that \( A \) is convenient.

From the Newton polyhedron, we construct the Newton filtration. We first observe that by the convenience assumption on \( A \), any face \( F \in \mathcal{F}(A) \), \( \dim F = n - 1 \). So let \( w^F \) be the unique vector of \( \mathbb{Q}_+^n \) such that \( F = \{ b \in \Gamma_+ (A) : \langle b, w^F \rangle = 1 \} \). We can suppose that the vertices of \( A \) are sufficiently close to the origin so that all the \( w^F \in \mathbb{Z}_+^n \). We will suppose henceforth that \( A \) satisfies this property. Then, we construct the following map \( \phi : \mathbb{R}_+^n \to \mathbb{R}_+ \). The restriction of \( \phi \) to each cone \( C(F) \) (where \( C(F) \) denotes the cone of half-rays emanating from 0 and passing through \( F \)) is defined as follows:
\[
\phi|_{C(F)} (\alpha) = \langle \alpha, w^F \rangle, \quad \text{for all } \alpha \in C(F).
\]

We extend this map to \( \mathbb{R}_+^n \) as follows:
\[
(2.1) \quad \phi(\alpha) = \min \{ \langle \alpha, w^F \rangle : F \in \mathcal{F}(A) \}, \quad \text{for all } \alpha \in \mathbb{R}_+^n.
\]

The map \( \phi \) is linear on each cone \( C(F) \) (where \( F \in \mathcal{F}(A) \)), and the value of \( \phi \) along each point over \( \Gamma(A) \) is equal to 1 and \( \phi(\mathbb{Z}_+^n) \subset \mathbb{Z}_+ \). This is called the Newton filtration induced by \( A \).

For any monomial \( x^\alpha \), we define \( \text{fil}(x^\alpha) = \phi(\alpha) \). This extends to a filtration on the ring \( \mathcal{C}_n \) of analytic function germs : \((\mathbb{R}^n, 0) \to (\mathbb{R}, 0)\) (via Taylor expansion) by defining
\[
(2.2) \quad \text{fil} \left( \sum c_\alpha x^\alpha \right) = \min \{ \phi(\alpha) : c_\alpha \neq 0 \}.
\]

We denote the set of \( g \) with \( \text{fil}(g) \geq l \) in \( \mathcal{C}_n \) by \( \mathcal{A}_l \). The number \( \text{fil}(g) \) will be also called the level of \( g \) with respect to \( A \).

Now we introduce the control functions associated to \( A \) as follows:
\[
(2.3) \quad \rho(x) = \left( \sum_{\alpha \in \text{Ver}(A)} x^{2p\alpha} \right)^{\frac{1}{2p}} \quad \text{and} \quad \overline{\rho}(x) = \sum_{\alpha \in \text{Ver}(A)} x^{2p\alpha},
\]

where \( p \) a positive integer. Moreover if \( p \) is big enough (it suffices, for example, that \( p\alpha \in \mathbb{Z}_+^n \)), \( \overline{\rho} \) will be \( C^w \).
Note that for an element \( g = \sum c_\alpha x^\alpha \in C_n \), the support of \( g \) is \( \text{supp}(g) = \{ \alpha : c_\alpha \neq 0 \} \); it is clear that \( g \in A_l \) if and only if \( \text{supp}(g) \subset \Gamma_+(lA) \) which is also equivalent to \( |g| \lesssim \rho^l \) (see [1, 5] for details). Thus \( A_l \) can be written as

\[
A_l = \{ g \in C_n : \text{supp}(g) \subset \Gamma_+(lA) \} = \{ g \in C_n : |g| \lesssim \rho^l \}.
\]

We say that an analytic function germ \( g \in C_n \) is an \( A \)-form of degree \( d \) if \( \text{supp}(g) \subset \Gamma(dA) \) (i.e., \( g \in A_d \setminus A_{d+1} \)). Furthermore, for \( f \in C_n \), we denote the Taylor expansion of \( f(x) \) at the origin by \( \sum_\nu c_\nu x^\nu \). Setting

\[
H_j(x) = \sum_{\nu \in \Gamma(jA)} c_\nu x^\nu, \quad j \in \mathbb{Z}_+,
\]

we can write \( f(x) = \sum_j H_j(x) \) (Newton filtration), where \( H_j \) is \( A \)-form of degree \( j \). Also if \( \#F(A) = 1 \), we can replace the Newton filtration associated with \( A \) by the weighted filtration associated to \( w^F \). Moreover, if \( w^F = (1, \ldots, 1) \), this Newton filtration coincides with the usual filtration.

### 2.1. Compensation factor.

Let \( \rho_i : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be a continuous function. We say that \( \rho_i \) is the \( i \)-th compensation factor associated with \( A \) if for each \( g \in C_n \), we have that \( |\rho_i \partial_{x_i} g| \lesssim \rho^{\hat{hi}(g)} \). Next we give some examples of compensation factors associated with \( A \).

(i) Here, we have the trivial example for the compensation factors, given by

\[
\rho_i(x) = x_i \quad \text{for} \quad i = 1, \ldots, n.
\]

(ii) Let \( L_j = L(x_j) \) denote the \( x_j \)-axis. We then put \( \alpha^j = L_j \cap \Gamma(A) \) for \( j = 1, \ldots, n \) (the axial vertices of \( \Gamma(A) \)). We define the weight of the variable \( x_i, A(i) = A(x_i) = \max\{ w^F_i : F \in F(A) \} \). We may introduce the compensation factors as follows:

\[
\rho_i(x) = \left( x_i^{2p A(i)} + \sum_{\alpha \in \text{Ver}(A) \setminus \{ \alpha^i \}} x^{2p \alpha} \right)^{\frac{A(i)}{2p}}, \quad i = 1, \ldots, n.
\]

It is easy to check that these functions \( \rho_i \) are compensation factors associated with \( A \) (see [1, 11] for details).
The following compensation factors are inspired by the work of Damon-Gaffney in [5]. For all integers \( l \geq 0 \), we let

\[ R_{l,i} = \{ \alpha \in \mathbb{Q}^n_+ : \langle \alpha, w^F \rangle \geq l + w_i^F, \forall F \in \mathcal{F}(A) \} \quad \text{for } i = 1, \ldots, n. \]

We may introduce the compensation factors as follows:

\[ \rho_{l,i}(x) = \left( \sum_{\alpha \in \text{Ver}(R_{l,i})} \frac{x^{2\alpha}}{\rho^2} \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n. \]

It is easy to see that for any integers \( l \geq 0 \), we have that \( \rho_{l,i}(x) \preceq \rho_{m,i}(x) \), where \( m_i = \min_{F \in \mathcal{F}(A)} \{ w_i^F \} \), which implies that \( \rho_{l,i} \) is continuous at the origin. On the other hand, by the construction of \( \rho_{l,i} \) we can deduce that \( |\rho_{l,i}\partial_x g| \preceq \rho^{\text{fil}(g)} \) for all \( g \in C_n \). Hence, we get that these functions \( \rho_{l,i} \) are compensation factors associated with \( A \).

**Observation.** We should note that in the case where \( \#\mathcal{F}(A) = 1 \) (i.e., weighted filtration associated with \( w = (w_1, \ldots, w_n) \)), the natural choice of compensation factor is that given by L. Paunescu in [10] as follows:

\[ \rho_i = \rho^{w_i} \quad \text{for } i = 1, \ldots, n. \]

Moreover, for any other compensation factors \( \xi_1, \ldots, \xi_n \) associated with the weighted filtration, we have that \( \xi_i \preceq \rho^{w_i}, \quad i = 1, \ldots, n \). Unfortunately, in the general case we have not succeeded in finding the best compensation factors \( \rho_1, \ldots, \rho_n \) such that for any other compensation factors \( \xi_1, \ldots, \xi_n \), we have that \( \xi_i \preceq \rho_i \). However, for each \( \gamma \in \mathbb{Q}^n_+ \) such that the monomial \( x^\gamma \) is \( i \)th compensation factor, we have \( |x^\gamma| \preceq \rho_{l,i} \), where \( \rho_{l,i} \) are the compensation factors defined in (iii).

Now we fix the compensation factors \( \rho_i \) for \( i = 1, \ldots, n \) relative to the Newton filtration, and consider the singular metric of \( M = \mathbb{R}^{n+m} \) defined by

\[
\langle \rho_i(x) \frac{\partial}{\partial x_i}, \rho_j(x) \frac{\partial}{\partial x_j} \rangle = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},
\]

\[
\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t_j} \rangle = 0 \quad \text{and} \quad \langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \rangle = \delta_{i,j}.
\]

Here, \( (x,t) = (x_1, \ldots, x_n, t_1, \ldots, t_p) \) denotes a system of coordinates of \( \mathbb{R}^{n+m} \). By elementary calculation we have

\[
\langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \rangle = \rho_I := \rho_{i_1} \cdots \rho_{i_k}.
\]
2.2. \((w)\)-regularity associated with \(A\).

Let \(F : (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)\) be analytic. We next assume that

\[
(2.6) \quad Y = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^m : x_1 = \cdots x_n = 0\} \quad \text{and} \quad X = F^{-1}(0) - Y.
\]

Setting \(F := (F_1, \ldots, F_p)\), assume that the Jacobi matrix of \(F\) has rank \(k\) on \(X\) near 0, where \(k \leq p\) is the codimension of \(X\) in \(\mathbb{R}^{n+m}\). We note that the normal space of \(X\) is generated by the gradient of the functions \(F_j (j = 1, \ldots, p)\) at each \(P \in X\) near 0. Following [6], we define \(\|dF\|_A\), \(\|d_xF\|_A\), \(\|d^2F\|_A\) and \(D_A(\ell)\) by the following formulae:

\[
(2.7) \quad \|dF\|_A^2 = \sum_j \|dF_j\|_A^2 \quad \text{where} \quad \|dF_j\|_A^2 = \sum_{I, S} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2,
\]

\[
\|d_xF\|_A^2 = \sum_j \|d_xF_j\|_A^2 \quad \text{where} \quad \|d_xF_j\|_A^2 = \sum_I \left( \rho_I \left| \frac{\partial F_j}{\partial x_I} \right| \right)^2,
\]

\[
\|d^2F\|_A^2 = \sum_j \|d^2F_j\|_A^2 \quad \text{where} \quad \|d^2F_j\|_A^2 = \sum_{I, S : S \neq \emptyset} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2.
\]

and

\[
(2.8) \quad D_A(\ell) = \sum_j \sum_{I, S : \#S = \ell} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2. \quad \text{Here} \quad \rho_I = \prod_{i \in I} \rho_i.
\]

We first remark that \(\langle dF, dF \rangle = \|dF\|_A^2\) and \(\langle d_xF, d_xF \rangle = \|d_xF\|_A^2\).

Now using the above construction, we state the version relative to the Newton filtration of the Fukui-Paunescu Theorem ([6], Theorem 2.1).

**Theorem 4.** — *The following conditions are equivalent*

(i) \(D_A(m) \lesssim D_A(m-1) \lesssim \cdots \lesssim D_A(1) \lesssim D_A(0)\) holds on \(X\) near 0.

(ii) \(\|d^2F\|_A \lesssim \|d_xF\|_A\) holds on \(X\) near 0.

(iii) For any \(C^1\)-functions \(\varphi_j (j = 1, \ldots, p)\) near 0, and \(s = 1, \ldots, m\),

\[
\left| \sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial t_s} \right| \lesssim \sum_{i=1}^n \rho_i \left| \sum_{j=1}^p \varphi_j \frac{\partial F_j}{\partial x_i} \right| \quad \text{holds on} \quad X \quad \text{near} \quad 0.
\]
(iv) For \( J \subset \{1, \ldots, p\} \), \( I = \{i_1, \ldots, i_{k-1}\} \subset \{1, \ldots, n\} \) with \( 1 \leq i_1 < \cdots < i_{k-1} \leq n, s = 1, \ldots, m \),

\[
\rho_I \left| \frac{\partial F_J}{\partial (x_I, t_s)} \right| \lesssim \|d_x F\|_A \quad \text{holds on } X \text{ near } 0.
\]

(v) For \( J \subset \{1, \ldots, p\} \), \( i = 1, \ldots, n, s = 1, \ldots, m \),

\[
\|\langle dF_J \wedge dx_i, dF_J \wedge dt_s \rangle\| \lesssim \rho_i \|d_x F\|^2_A \quad \text{holds on } X \text{ near } 0.
\]

(vi) For some positive C1-functions \( \phi_J \) on \( X \) with \( J \subset \{1, \ldots, p\} \), \( i = 1, \ldots, n, s = 1, \ldots, m \),

\[
\left| \sum_J \phi_J \langle dF_J \wedge dx_i, dF_J \wedge dt_s \rangle \right| \lesssim \rho_i \sum_J \phi_J \|d_x F\|^2_A \quad \text{holds on } X \text{ near } 0.
\]

**Proof.** — The proof is similar to that of Fukui-Paunescu in [6]; it is enough to replace the \( \|x\|_{w} \) (resp. \( \|x\|_{w^t} \)) in the proof of Theorem 2.1 [6] by the \( \rho_i \) (resp. \( \rho_I \)).

We say that \( X \) is \((w)-regular\) over \( Y \) at 0 with respect to \( A \) (or \( w^A\)-regular), if one of the above equivalent conditions holds. When \( \#(A) = 1 \), we find that \( \rho_i(x) = \rho^{w^A}(x) \) for \( i = 1, \ldots, n \), hence our \((w^A)\)-regularity reduces to the weighted \((w)\)-regularity (see [6]). Moreover, if \( w^F = (1, \cdots, 1) \), these coincide with the usual \((w)\)-regularity (Verdier’s regularity).

We shall prove the following theorem.

**Theorem 5.** — For \( X, Y \) as above, if \((X, Y)\) is \((w^A)\)-regular, then \((X, Y)\) is \((c)\)-regular for the control function \( \bar{p} \) (we recall that \( \bar{p}(x) = \sum_{\alpha \in \text{Ver}(A)} x^{2p_\alpha} \)).

**Remark 6.** — The converse of the theorem is false in general: (Kuo’s example [8])

\[ F(x, y, t) = y^2 - tx^2 - x^5, \quad X = \{y^2 = tx^2 + x^5\} - \{0\} \times \mathbb{R} \text{ and } Y = \{0\} \times \mathbb{R}. \]

We consider the usual filtration \( A = \{(1,0); (0,1)\} \). It is easy to see that \((X, Y)\) is \((c)\)-regular at 0 for the control function \( \bar{p}(x, y) = x^2 + y^2 \), but that \((X, Y)\) is not Verdier \((w)\)-regular at 0 (see [14] for details).
As an immediate corollary we have

**COROLLARY 7.** Let \( f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \), \( t \in \mathbb{R}^m \) be a family of weighted homogeneous polynomials defining an isolated singularity at the origin. We set \( F(x, t) = f_t(x) \), then the stratification \( \Sigma(V_F) \) is \((c)\)-regular. (we again recall that \( \Sigma(V_F) = \{ F^{-1}(0) - \{0\} \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m \} \))

**Proof.** Let us put \( X = F^{-1}(0) - \{0\} \times \mathbb{R}^m \) and \( Y = \{0\} \times \mathbb{R}^m \). Consider the weighted filtration associated with \( A = \{ (\frac{1}{w_1}, 0, \cdots, 0), \ldots, (0, \cdots, 0, \frac{1}{w_n}) \} \) such that \( f_t \) is a weighted homogeneous polynomial with the weight \( w = (w_1, \cdots, w_n) \in \mathbb{Z}_+^n \). Now from the Theorem 5, it is enough to show that \((X, Y)\) is \((w^-)\)-regular, that is,

\[
|\partial_t F| \lesssim \|d_x F\|_A \text{ holds on } X \text{ near } Y.
\]

Since \( f_t \) defines an isolated singularity at the origin, we can see that \( \|d_x F\|_A^2 = \sum_{i=1}^n (\rho^{w_i} \frac{\partial F}{\partial x_i})^2 \) is not zero outside the origin, and this implies our inequality. \( \square \)

**COROLLARY 8.** Let \( f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \), \( t \in \mathbb{R}^m \) be a real analytic family non-degenerate (in the sense of Kouchnirenko [7]) and \( \Gamma(f_t) = \Gamma(f_0) \), then the stratification \( \Sigma(V_F) \) is \((c)\)-regular.

**Proof.** By standard argument, based on the curve selection lemma, we can see that

\[
|\partial_t F| \lesssim \sum_{\alpha \in \text{Ver}(\Gamma(f_0))} |x^\alpha| \lesssim \sum_{i=1}^n |x_i \frac{\partial F}{\partial x_i}|.
\]

Therefore, \((X, Y)\) is \((w^-)\)-regular for any Newton filtration. In particular, \((X, Y)\) is usual \((w)\)-regular (Verdier’s regular). \( \square \)

Before starting the proofs of the above results, we will first illustrate these results with several examples.

**EXAMPLE 9 (Briançon-Speder family [4]).** Let \( f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0) \), \( t \in J = [-1, 1] \), be a family of weighted homogeneous polynomials defined by \( f_t(x, y, z) = z^5 + t y^6 z + x y^7 + x^{15} \).

We set \( F(x, t) = f_t(x) \), \( Y = \{0\} \times J \) and \( X = F^{-1}(0) - Y \). It is easy to check that \( |\partial_t F| \lesssim \|d_x F\|_A \) holds on \( X \) near 0, where \( A = \)
Thus, by Theorem 5, we have that \((X, Y)\) is \((c)\)-regular for the function \(p(x, y, z) = x^{12} + y^6 + z^4\). (It is well known that \(f_t\) is not Whitney regular and not usual \((w)\)-regular).

**Example 10 (Oka family [9]).** — Let \(f_t: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0), t \in J = [-1,1]\), be a family of polynomial functions defined by

\[ f_t(x, y, z) = x^8 + y^{16} + z^{16} + t x^5 z^2 + x^3 y z^3. \]

We set \(F(x, t) = f_t(x), Y = \{0\} \times J, X = F^{-1}(0) - Y\) and

\[ \mathcal{A} = \left\{ \left( \frac{1}{2}, 0, 0 \right), (0, 1, 0), (0, 0, 1), \left( \frac{5}{16}, 0, \frac{1}{8} \right) \right\}. \]

It is not hard to see that the inequality \(|\partial_t F|^2 \leq \|d_x F\|^2_\mathcal{A} = \sum_{i=1}^n (\rho_i \frac{\partial F}{\partial x_i})^2\) holds on \(X\) near \(Y\), where \(\rho_i\) denotes the \(i\)th compensation factor of type (ii) as defined in 2.1. It follows from Theorem 5 that \((X, Y)\) is \((c)\)-regular for the control function \(\bar{\rho}(x, y, z) = x^{16} + y^{32} + z^{32} + x^{10} z^4\).

**2.3. Proof of Theorem 5.**

In order to show this theorem we need the following lemma.

**Lemma 11.**

1. \(||d\bar{\rho}||_\mathcal{A} \lesssim \bar{\rho}(x), x \text{ near } 0,\)
2. \(\bar{\rho} \ll \frac{||dF \wedge d\bar{\rho}||}{||dF||}\) when \((x, t) \rightarrow 0\) on \(X\).

**Proof.** — We first recall that:

\[ ||d\bar{\rho}||^2_\mathcal{A} = \sum_{i=1}^n \left( \rho_i \frac{\partial \bar{\rho}}{\partial x_i}(x) \right)^2. \]

Therefore, (1) is a simple consequence of the construction of the compensation factors and the control functions.

Let us observe that, by (1.2) we have \(|\partial \bar{\rho}|_X| = \frac{||dF \wedge d\bar{\rho}||}{||dF||}. \) On the other hand, \(\partial \bar{\rho} = \partial \bar{\rho}|_X + \partial \bar{\rho}|_N\) (where \(N\) denotes the normal space to \(X\)). Since \(N\) is generated by the gradients of \(F_j (j = 1, \ldots, p)\), we have that \(\partial \bar{\rho}|_X = \partial \bar{\rho} + \eta_1 F_1 \cdots + \eta_p F_p\). After this, (2) in the lemma, follows from the following more general proposition.
PROPOSITION 12. — Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^r, 0), \ g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be two germs of analytic maps, setting \( f := (f_1, \ldots, f_r) \). Then there exists a real constant \( C \) such that for \( p \in f^{-1}(0) \), and sufficiently close to the origin,

\[
|g(p)| \leq C \left| p \right| \inf_{(\eta_1, \ldots, \eta_n) \in \mathbb{R}^r} |\eta_1 \partial f_1(p) + \cdots + \eta_r \partial f_r(p) + \partial g(p)|.
\]

We note that if \( r = 1 \), one finds Theorem 1.1 of Adam Parusinski [12]. Moreover, the proof of this proposition is similar to that of Theorem 1.1 in [12] (we omit the details).

Now we are ready to prove Theorem 5. We assume that \((X, Y)\) is \((\omega^A)\)-regular at 0. By inequality (iii) in Theorem 4, we have

\[
\left| \frac{\partial F_J}{\partial (x_I, t_S)} \right| \lesssim \sum_{i=1}^n \rho_i \left| \frac{\partial F_J}{\partial (x_I, t_S, x_i)} \right| \quad \text{on } X \text{ near } 0,
\]

where \( \hat{S} \subset S \) such that \( \#\hat{S} = \#S - 1 \). Thus we obtain \( \|d^x F\| \ll \|dF\| \) when \((x, t) \to 0\) on \(X\) (i.e., \((X, Y)\) is \((\omega^a)\)-regular at 0), and so by Theorem 2, we only have to prove that:

\[
|\partial \bar{t}|_x | \ll \frac{\|dF \wedge d\bar{p}\|}{\|dF\|} \quad \text{as } (x, t) \in X, (x, t) \to 0.
\]

We first remark, by (1.1) the following equality:

\[
|\partial \eta \bar{p}|_x | = \left| \sum_J \sum_{I, S} \frac{\partial (F_I, t_{\eta})}{\partial (x_I, t_S, t_{\eta})} \frac{\partial (F_I, \bar{p})}{\partial (x_I, t_S, \bar{t}_{\eta})} \left\| dF \right\|^2 \right|,
\]

and hence

\[
|\partial \eta \bar{p}|_x | \lesssim \left| \sum_J \sum_{I, S} \frac{\partial (F_I, \bar{p})}{\partial (x_I, t_S, \bar{t}_{\eta})} \left\| dF \right\| \right|.
\]

According to the inequality in (iii) of Theorem 4, we have

\[
\left| \frac{\partial (F_I, \bar{p})}{\partial (x_I, t_S, t_{\eta})} \right| \lesssim \sum_{i=1}^n \rho_i \left( \left| \frac{\partial (F_I, \bar{p})}{\partial (x_I, t_S, x_i)} \right| + \left| \frac{\partial \bar{p}}{\partial x_i} \right| \left| \frac{\partial F_I}{\partial (x_I, t_S, \bar{t}_{\eta})} \right| \right).
\]
Thus, we obtain
\[ \left| \frac{\partial (F, \bar{\rho})}{\partial (x_1, t_S, t_\eta)} \right| \lesssim \| d\bar{\rho} \|_A \| dF \| + \sum_{i=1}^{n} \rho_i \| dF \land d\bar{\rho} \| \]
and, using (2.13), we obtain
\[ |\partial_i \bar{\rho}|_X \lesssim \| d\bar{\rho} \|_A + \sum_{i=1}^{n} \rho_i \frac{\| dF \land d\bar{\rho} \|}{\| dF \|} \quad \text{on } X \text{ near 0}. \]

It follows from Lemma 11 that (2.12) holds. This completes the proof of Theorem 5.

3. The Damon-Gaffney condition and (c)-regularity.

In this section we describe some definitions and notations used by Damon-Gaffney in [5].

Given a Newton filtration $A$ as above. We extend this filtration on the ring $C_{x,t}$ of formal power series in the variables $x_1, \ldots, x_n; t_1, \ldots, t_m$ around the origin by defining

\[ \text{fil}(\sum_{\nu} c_{\nu}(t)x^{\nu}) = \min\{\phi(\nu) : c_{\nu}(t) \neq 0\}. \]

Let $g = \sum_{\nu} c_{\nu}(t)x^{\nu}$ be a series in $C_{x,t}$, the support of $g$, denoted by $\text{supp}(g)$, is the set of points $\nu \in \mathbb{Z}_+^n$ such that $c_{\nu}(t) \neq 0$. We denote the set of $g$ with $\text{fil}(g) \geq l$ in $C_{x,t}$ by $A_{l,x,t}$. It is not difficult to see the following equality:

\[ A_{l,x,t} = \{ g \in C_{x,t} : \text{supp}(g) \subset \Gamma_+ (lA) \} = \{ g \in C_{x,t} : |g| \lesssim \rho^l \}. \]

We say that level $A_l$ of the Newton filtration is fit if all the vertices of $\phi^{-1}(l)$ are lattice points of $\mathbb{R}_+^n$. This says that $l \text{Ver}(A) = \text{Ver}(lA) \in \mathbb{Z}_+^n$ (because of the linearity of the Newton filtration on cones). For $A_l$ which is fit, we let

\[ \text{ver}(A_l) = \{ x^{\beta} : \beta \text{ is a vertex of } \phi^{-1}(l) \} = \{ x^{l\alpha} : \alpha \in \text{Ver}(A) \}. \]

We also let

\[ \mathcal{V}_{l,x,t} = \{ \zeta \in A_{l+1,x,t} : \zeta/\partial x_i \} : \zeta(A_{k,x,t}) \subset A_{l+k,x,t} \}. \]
with $\mathcal{A}_{t+1,x,t}\{\partial/\partial x_i\}$ denoting the $\mathcal{A}_{t+1,x,t}$-module generated by the $\partial/\partial x_i$, $i = 1, \ldots, n$. Finally, for an element $g \in \mathcal{C}_{x,t}$, we let $\mathcal{V}_{i,x,t}(g) = \{\zeta(g) : \zeta \in \mathcal{V}_{i,x,t}\}.$

Now we can announce the Damon-Gaffney Theorem.

**Theorem 13 (Damon-Gaffney [5]).** Let $f : (\mathbb{R}^{n+m}, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic deformation of a germ $f_0 : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ (i.e., $f \in \mathcal{C}_{x,t}$). Then a sufficient condition that $f$ be a topologically trivial deformation is that there exists a fit $\mathcal{A}_t$ so that

$$\text{ver}(\mathcal{A}_t) \cdot \frac{\partial f}{\partial t_j} \subset \mathcal{V}_{i,x,t}(f), \ j = 1, \ldots, m. \tag{3.5}$$

We will call condition (3.5) the Damon-Gaffney condition. Next, our principal goal will be to show that this condition implies a $(w)$-regularity condition relative to the Newton filtration, hence, these deformations will, in fact, satisfy the Bekka condition.

Given an analytic function $f \in \mathcal{C}_{x,t}$, we define

$$\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m) = \{\mathbb{R}^n \times \mathbb{R}^m - f^{-1}(0), f^{-1}(0) - \{0\} \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m\},$$

which gives a stratification of $\mathbb{R}^n \times \mathbb{R}^m$ around $\{0\} \times \mathbb{R}^m$. Then, we have

**Theorem 14.** For $f \in \mathcal{C}_{x,t}$, if there is a positive integer $l$ such that

$$\text{ver}(\mathcal{A}_t) \cdot \frac{\partial f}{\partial t_j} \subset \mathcal{V}_{i,x,t}(f), \ j = 1, \ldots, m \ (\text{The Damon-Gaffney condition}),$$

then the stratification $\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m)$ is $(c)$-regular.

**Proof.** Let us put $\text{ver}(\mathcal{A}_t) = \{x^\alpha\}$ then we get the following expression:

$$x^\alpha \frac{\partial f}{\partial t_j} = \sum_{i=1}^n \xi^{(\alpha)}_{ij} \frac{\partial f}{\partial x_i} = \xi^{(\alpha)}_j(f),$$

and summing over $x^\alpha \in \text{ver}(\mathcal{A}_t)$ we obtain

$$\left(\sum_{\alpha \in \text{Ver}(\mathcal{A})} |x^\alpha|\right) \left|\frac{\partial f}{\partial t_j}\right| \lesssim \sum_{i=1}^n \left(\sum_{\alpha \in \text{Ver}(\mathcal{A})} |\xi^{(\alpha)}_{ij}|\right) \left|\frac{\partial f}{\partial x_i}\right|. \tag{3.6}$$
Since $\text{Ver}(l \mathcal{A}) = l \text{Ver}(\mathcal{A})$, which means $\rho^l \sim \sum_{\alpha \in \text{Ver}(l \mathcal{A})} |x^\alpha|$. Then we let

$$\xi'_i = \sum_{j=1}^{m} \sum_{\alpha \in \text{Ver}(l \mathcal{A})} \rho^{-l}|\xi_\alpha| \quad \text{for} \quad i = 1, \ldots, n.$$ 

It follows from (3.6) that $|\partial_l f|^2 \leq \sum_{i=1}^{n} (\xi'_i \frac{\partial f}{\partial x_i})^2$, and so by Theorem 5, it is sufficient to show that these $\xi'_i$ are compensation factors associated with $\mathcal{A}$. Indeed, for any $g \in \mathcal{C}_n$, we have from the filtration properties of the $\xi_{ij}^{(\alpha)}$ that

$$\text{fil}(\xi_{ij}^{(\alpha)}(g)) = \text{fil}(\xi_{ij}^{(\alpha)} \partial_{x_i} g) \geq \text{fil}(g) + l$$

which means

$$|\xi_{ij}^{(\alpha)} \partial_{x_i} g| \lesssim \rho^{l+\text{fil}(g)}.$$ 

Therefore, for $i = 1, \ldots, n$,

$$|\xi'_i \partial_{x_i} g| \lesssim \rho^{\text{fil}(g)}.$$ 

This completes the proof of the Theorem.

**Remark 15.** We observe that $\zeta = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \in \mathcal{V}_{l,x,l}$ if and only if $\text{supp}(\xi_i) \subset R_{l,i}$ (we recall that $R_{l,i} = \{ \alpha \in \mathbb{Q}_+^n : \langle \alpha, w^F \rangle \geq l + w_i^F, \forall F \in \mathcal{F}(\mathcal{A}) \}$) which is also equivalent to $|\xi_i| \lesssim \sum_{\alpha \in \text{Ver}(R_{l,i})} |w^\alpha|$. Hence, the Damon-Gaffney condition implies a $(w^A)$-regularity condition with $\rho_{l,i}$ as compensation factors, where $\rho_{l,i}$ denotes the $i$th compensation factor of type (iii) as defined in 2.1.

**Acknowledgement.** The author wishes to express his sincere gratitude to T. Fukui, S. Koike, T.-C. Kuo, A. Parusinski and L. Paunescu for many helpful discussion during the preparation of this paper.

**BIBLIOGRAPHY**


Manuscrit reçu le 19 août 2003,
accepté le 9 décembre 2003.

Ould M. ABDERRAHMANE,
Saitama University
Faculty of Science
Department of Mathematics
255 Shimo-Okubo
Urawa, 338-8570 (Japan).
vould@rimath.saitama-u.ac.jp