



ANNALES

DE

L'INSTITUT FOURIER

Satoshi KOIKE & Adam PARUSIŃSKI

Motivic-type invariants of blow-analytic equivalence

Tome 53, n° 7 (2003), p. 2061-2104.

http://aif.cedram.org/item?id=AIF_2003__53_7_2061_0

© Association des Annales de l'institut Fourier, 2003, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

MOTIVIC-TYPE INVARIANTS OF BLOW-ANALYTIC EQUIVALENCE

by S. KOIKE and A. PARUSIŃSKI⁽¹⁾

We develop techniques that allow us to study and distinguish different blow-analytic classes of analytic function germs $f: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$. For this we adapt and apply to the real analytic set-up the ideas coming from motivic integration, in particular the concept of motivic zeta function due to Denef and Loeser.

The notion of blow-analytic equivalence was introduced by T.-C. Kuo [22] and [23]. Recall briefly that analytic function germs $f, g: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exist real modifications $\mu: (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$, $\mu': (M', \mu'^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$ and an analytic isomorphism $\Phi: (M, \mu^{-1}(0)) \rightarrow (M', \mu'^{-1}(0))$ which induces a homeomorphism $\phi: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$. In this paper we suppose additionally that μ , resp. μ' , is an isomorphism over the complement of $f^{-1}(0)$, resp. $g^{-1}(0)$. The blow-analytic equivalence is interesting because it does not allow continuous moduli for families of isolated singularities cf. [23], and it preserves a deep information on the algebraic structure of the singularity. For real singularities, unlike for the complex ones, the topological classification is too crude, e.g., $x^{2k} + y^{2m}$ and $x^{2n} + y^{2l}$ are always topologically equivalent. The blow-analytic equivalence was invented to overcome this problem. Moreover, as follows from various examples, the blow-analytic equivalence of real analytic function germs behaves in

⁽¹⁾ This research was originated during the conference New Developments in Singularity Theory, Cambridge 2000, at the Isaac Newton Institute, and was subsequently supported by the University of Angers and Grant-in-Aid for Scientific Research (No. 13640070) of Ministry of Education, Science and Culture of Japan. We would like to thank these institutions for their support and hospitality.

Keywords : Blow-analytic equivalence – Motivic integration – Zeta functions – Thom-Sebastiani formulae.

Math. classification : 14B05 – 32S15 – 57R45.

a similar way to the topological equivalence in the complex case, though there is no precise result in this direction. This observation seems to be confirmed by the main results of this paper.

There exist various criteria of blow-analytic triviality of families of analytic function germs, based mainly on toric equi-resolutions [10], [13], [1], but there were till now very few results allowing to distinguish different blow-analytic types and hence to attempt a classification even in the simplest cases. The only known up to now invariant of blow-analytic equivalence was introduced by Fukui in [11], see also Section 5 below. In this paper we introduce new invariants that allow us to start such a classification.

The main results of this paper are the following. In Section 1 we associate to each real analytic function germ $f: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ its zeta functions: $Z_f, Z_{f,+}, Z_{f,-} \in \mathbb{Z}[[T]]$. We show that they are blow-analytic invariants in Section 4. In order to compute the zeta functions we propose formulae in terms of a resolution (Denef and Loeser formulae), see Section 1, and the Thom-Sebastiani Formulae in Section 2. Sections 6 and 7 contain classification results, in particular a complete classification of blow-analytic types of Brieskorn polynomials of two variables and a partial classification in three dimensional case.

Our main idea of construction of new invariants is based on the following simple observations. Suppose that $f, g: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent via a (blow-analytic) homeomorphism $\phi: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$, $f = g \circ \phi$. Then, firstly, f and g admit isomorphic resolutions. Secondly, let $\mathcal{L}(\mathbb{R}^d, 0)$ denote the set of germs of analytic arcs at the origin in \mathbb{R}^d . Then ϕ induces a bijection $\phi_*: \mathcal{L}(\mathbb{R}^d, 0) \rightarrow \mathcal{L}(\mathbb{R}^d, 0)$ by composition $\phi_*(\gamma(t)) = (\phi \circ \gamma)(t)$. In Section 1 below, using the integration with respect to the Euler characteristic with compact supports on these sets of arcs, we associate to each real analytic function germ $f: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ its zeta functions: $Z_f, Z_{f,+}, Z_{f,-} \in \mathbb{Z}[[T]]$. Here we follow the path introduced by Denef and Loeser [5], [9], and inspired by work of Kontsevich [18]. The zeta function of Denef and Loeser, and the related topological zeta function cf. [8], provides an important information on the local topology of complex analytic function germs, see a new proof of Thom-Sebastiani theorem for the Hodge spectrum [7] or works on the monodromy conjecture, see for instance [9], [35]. We refer the reader to the survey [8] for more information on the Denef and Loeser construction and its applications.

In Section 4 we show that our zeta functions are invariants of blow-analytic equivalence in the sense of Kuo. The proof is based on formulae (1.1), (1.2), analogous to the formulae of Denef and Loeser, that express the zeta functions of f in terms of a resolution. These formulae are proven by a version of the change of variable formula, Corollary 4.4. Note that these results do not follow automatically from the analogous ones in the algebraic case, due to the necessity of working with non-compact subanalytic sets. This difficulty is overcome thanks to the Łojasiewicz's theory of relatively semi-algebraic, semi-analytic sets [27].

Thom-Sebastiani Formulae, showed in Section 2, express the zeta functions of $f(x) + g(y)$ in terms of the ones of f and g . They have interesting consequences. For instance we get a suspension property: if the zeta functions of $x^m + g_1(y)$ and $x^m + g_2(y)$, m even, coincide then so do the zeta functions of g_1 and g_2 . One may speculate that if $x^2 + g_1(y)$ and $x^2 + g_2(y)$ are blow-analytically equivalent so are g_1 and g_2 (this is for instance the case if we know that the zeta functions distinguish the blow-analytic types of g_1 and g_2 , we use this in some special cases). We do not know the answer to this question. We use the Thom-Sebastiani Formulae to compute the zeta functions for all Brieskorn polynomials $f(x_1, \dots, x_d) = \pm x_1^{p_1} \pm \dots \pm x_d^{p_d}$. In Section 6 we compute the blow-analytic equivalence classes of Brieskorn polynomials of two variables and in Section 7 most of the equivalence classes of Brieskorn polynomials of three variables. This classification differs from the analytic one. For instance, thanks to a phenomenon typical for real algebraic geometry, the functions $x^p + y^{kp}$ and $x^p - y^{kp}$, p odd, k even, are blow-analytically equivalent but not analytically equivalent (over real numbers).

As we mentioned before the blow-analytic equivalence behaves in a similar way to the topological equivalence of complex analytic function germs. Consider for instance the following example. The germs at the origin of $f(x, y, z) = x^3 + xy^5 + z^3$ and $g(x, y, z) = x^3 + y^7 + z^3$ are not topologically equivalent as complex germs. One may show that any complex analytic function germ with the 6th jet equal to f is topologically equivalent either to f or g , thus there are exactly two possible topological types. On the other hand any real analytic function germ with the 6th jet equal to f is blow-analytically equivalent either to f or g . Of course, f and g as real analytic functions germs are topologically equivalent (they are equivalent to a nonsingular germ). We show in Subsection 7.2 that f and g are not blow-analytically equivalent.

Due to the presence of some phenomena typical for the real algebraic geometry it is interesting to compare the properties of our zeta functions to the ones of Denef and Loeser. For instance our sign zeta functions, Z_+, Z_- , correspond to the monodromic zeta function of Denef and Loeser, a phenomenon similar to the one studied in [29] in a different context. Note also that our zeta functions are not really motivic and have only integer coefficients. This is due to the fact that the Euler characteristic with compact supports is the only purely topologically defined additive invariant of semi-algebraic sets, cf. [32].

Moreover the zeta functions introduced in this paper do not distinguish all classes of blow-analytic equivalence and we are far from a complete classification even in the weighted homogeneous non-degenerate case. This problem may be attacked by hunting new motivic invariants in the real algebraic, and not semi-algebraic, set-up. Even if one knows such invariants it is not clear whether one can apply them to study the equivalence that is merely blow-analytic (and not ‘blow-algebraic’). On the other hand there is a variety of work in real algebraic and analytic geometry related to the space of analytic arcs that can be probably approached by the techniques of motivic integration, cf. [24], [2], [25].

We finish the introduction with more precise open questions. Let $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be weighted homogeneous polynomials with isolated singularities. It is known after [31], [33], [36], [37], for $n = 2, 3$, that if $(\mathbb{C}^n, f^{-1}(0))$ and $(\mathbb{C}^n, g^{-1}(0))$ are homeomorphic as germs at $0 \in \mathbb{C}^n$, then their systems of weights coincide. We propose the following corresponding question.

QUESTION 1. — *Let $f, g: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be weighted homogeneous polynomials with isolated singularities. Suppose that f and g are blow-analytically equivalent. Then, do their systems of weights coincide?*

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and let $J_{\mathbb{K}}^r(n, 1)$ denote the set of r -jets of analytic function germs $(\mathbb{K}^n, 0) \rightarrow (\mathbb{K}, 0)$. We say that $w \in J_{\mathbb{K}}^r(n, 1)$ is sufficient if any two analytic function germs with the r -th jets equal to w are blow-analytically equivalent. Identify r -jets with polynomial representatives of degree not exceeding r . We say that $w \in J_{\mathbb{K}}^r(n, 1)$ satisfies *the Kuiper-Kuo condition* (see [20], [21]) if there are $C, \alpha > 0$, such that

$$|\text{grad } w(x)| \geq C|x|^{r-1} \quad \text{for } |x| < \alpha.$$

T.-C. Kuo proposed the following conjecture and has affirmatively proved it in the two variables case.

CONJECTURE 1. — Let $w \in J_{\mathbb{R}}^r(n, 1)$. Suppose that w satisfies the Kuiper-Kuo condition as a complex r -jet. Then w is blow-analytically sufficient in C^ω -functions.

Convention. — By the Brieskorn polynomials of d variables we mean $f(x_1, \dots, x_d) = a_1x_1^{p_1} + a_2x_2^{p_2} + \dots + a_dx_d^{p_d}$, $a_i \neq 0$. Since their analytic types depend only on the signs of a_i , in order to simplify the notation, we consider only the Brieskorn polynomials of the form

$$f(x_1, \dots, x_d) = \pm x_1^{p_1} \pm x_2^{p_2} \pm \dots \pm x_d^{p_d}.$$

1. Motivic zeta function of analytic function germ.

1.1. Definition of the zeta functions.

Consider the space of analytic arcs at the origin $0 \in \mathbb{R}^d$,

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) := \{ \gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^d, 0); \gamma \text{ analytic} \}$$

and the one of truncated arcs

$$\mathcal{L}_n := \{ \gamma \in \mathcal{L}; \gamma(t) = \mathbf{a}_1t + \mathbf{a}_2t^2 + \dots + \mathbf{a}_nt^n, \mathbf{a}_i \in \mathbb{R}^d \}.$$

Given an analytic function $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$. For $n \geq 1$ we denote

$$\mathcal{X}_{n,+}(f) := \{ \gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c > 0 \},$$

$$\mathcal{X}_{n,-}(f) := \{ \gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c < 0 \},$$

$$\mathcal{X}_n(f) := \{ \gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c \neq 0 \}.$$

We define the *positive, negative, and total zeta function* of f by

$$Z_{f,+}(T) := \sum_{n \geq 1} (-1)^{-nd} \chi^c(\mathcal{X}_{n,+}) T^n,$$

$$Z_{f,-}(T) := \sum_{n \geq 1} (-1)^{-nd} \chi^c(\mathcal{X}_{n,-}) T^n,$$

$$Z_f(T) := \sum_{n \geq 1} (-1)^{-nd} \chi^c(\mathcal{X}_n) T^n = Z_{f,+}(T) + Z_{f,-}(T),$$

where χ^c denotes the Euler characteristic with compact supports. If f is fixed we shall often drop f and write simply $\mathcal{X}_{n,+}$ for $\mathcal{X}_{n,+}(f)$, Z_+ for $Z_{f,+}$, and so on.

Remark 1.1. — The map $\varphi : \mathcal{X}_n \rightarrow \mathbb{R}^*$ that associates to γ the first non-zero coefficient of $f \circ \gamma$, that is $\varphi(\gamma) = c$ if $f \circ \gamma = ct^n + \dots$, is a trivial

fibration over $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ (for n odd it is trivial over \mathbb{R}^*). This can be easily shown using the following action of \mathbb{R}^* :

$$\varphi(\gamma(\alpha t)) = \alpha^n \varphi(\gamma(t)), \quad \alpha \in \mathbb{R}^*.$$

Remark 1.2. — Our zeta function is an incarnation of the motivic zeta function of Denef and Loeser [8], [7], [9]. Instead of using the algebraic motifs we use just the Euler characteristic with compact supports that is the Euler characteristic of the sheaf cohomology with compact supports, with coefficients in the constant sheaf \mathbb{Z} . By the long exact cohomology sequence of the pair it satisfies the following additivity property: $\chi^c(A) = \chi^c(A \setminus B) + \chi^c(B)$ for all locally compact semialgebraic A and B , B closed in A . One may show that χ^c is the only topological invariant of semi-algebraic sets additive in this sense, cf. [32].

1.2. Denef and Loeser’s formulae.

Let $\sigma : (M, \sigma^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$ be a modification of \mathbb{R}^d such that $f \circ \sigma$ and the jacobian determinant $\text{jac } \sigma$ of σ are normal crossings simultaneously (we may define $\text{jac } \sigma$ locally using any local system of coordinates on M). For instance if σ is a composition of blowings-up with smooth centers that are in normal crossings with the old exceptional divisors then $\text{jac } \sigma$ is normal crossings. We also assume that σ is an isomorphism over the complement of the zero set of f . The existence of such a modification is guaranteed by [16], [3]. We denote by $E_i, i \in J$, the irreducible components of $(f \circ \sigma)^{-1}(0)$ (in $\sigma^{-1}(B_\varepsilon)$, where B_ε is a small ball in \mathbb{R}^d centered at the origin). We may also suppose that $\sigma^{-1}(0)$ is the union of some of E_i . For each $i \in J$ we denote

$$N_i = \text{mult}_{E_i} f \circ \sigma \quad \text{and} \quad \nu_i = \text{mult}_{E_i} \text{jac } \sigma + 1.$$

Denote for $i \in I$ and $I \subset J$,

$$\mathring{E}_i = E_i \setminus \bigcup_{j \neq i} E_j, \quad E_I = \bigcap_{i \in I} E_i, \quad \mathring{E}_I = E_I \setminus \bigcup_{j \in J \setminus I} E_j.$$

Using the change of variables formula in the motivic integral [18], [6], [28] we shall show in Section 4 that

$$(1.1) \quad Z(T) = \sum_{I \neq \emptyset} (-2)^{|I|} \chi^c(\mathring{E}_I \cap \sigma^{-1}(0)) \prod_{i \in I} \frac{(-1)^{\nu_i} T^{N_i}}{1 - (-1)^{\nu_i} T^{N_i}}.$$

Let $\mathring{E}_{I,k}$ be a connected component of \mathring{E}_I and let $x \in \mathring{E}_{I,k}$. Then, near x , the complement of $(f \circ \sigma)^{-1}(0)$ consists of $2^{|I|}$ chambers, f being non-zero

on each of them. Denote by $\alpha_+(\mathring{E}_{I,k})$, resp. $\alpha_-(\mathring{E}_{I,k})$, the number of such chambers where $f \circ \sigma$ is positive, resp. negative. Again using the change of variables formula one gets

$$(1.2) \quad Z_{\pm}(T) = \sum_{I \neq \emptyset} (-1)^{|I|} \left(\sum_k \alpha_{\pm}(\mathring{E}_{I,k}) \chi^c(\mathring{E}_{I,k} \cap \sigma^{-1}(0)) \right) \times \prod_{i \in I} \frac{(-1)^{\nu_i} T^{N_i}}{1 - (-1)^{\nu_i} T^{N_i}}.$$

The formulae (1.1), (1.2) will be shown in Section 4 below.

1.3. Examples.

1.3.1. — Let $f(x) = x^m, x \in \mathbb{R}, m > 0$. Then

$$(1.3) \quad \mathcal{X}_n = \begin{cases} \{ \gamma(t) = a_k t + \dots + a_n t^n; a_k \neq 0 \} \simeq \mathbb{R}^* \times \mathbb{R}^{n-k} & \text{if } n = km, \\ \emptyset & \text{otherwise.} \end{cases}$$

That is $\chi^c(\mathcal{X}_n) = (-2)\chi^c(\mathbb{R}^{n-k}) = (-2)(-1)^{k(m-1)}$ if $n = km$, and

$$Z(T) = \sum_{n=km>0} (-1)^{km} (-2)(-1)^{k(m-1)} T^{km} = 2(T^m - T^{2m} + T^{3m} - \dots).$$

Of course, the same formula can be obtained by (1.1) by taking σ equal to the identity

$$Z(T) = (-2) \frac{-T^m}{1 + T^m}.$$

If m is odd then $Z_+(T) = Z_-(T) = \frac{1}{2} Z(T)$. If m is even then $Z_+(T) = Z(T), Z_-(T) = 0$.

1.3.2. — Let $f(x, y) = x^{2k} + y^{2k}, (x, y) \in \mathbb{R}^2$. We may desingularize f by blowing-up the origin with the exceptional divisor \mathbb{P}^1 . Since $\chi^c(\mathbb{P}^1) = 0$ we get by (1.1) and (1.2)

$$Z_+(T) = Z_-(T) = Z(T) = 0.$$

1.3.3. — Let $f(x, y) = x^2 - y^2, (x, y) \in \mathbb{R}^2$. Since f is already normal crossing we apply (1.1) to $\sigma = \text{id}$. Then

$$Z(T) = (-2)^2 \chi^c(\text{point}) \frac{-T}{1+T} \frac{-T}{1+T} = 4 \frac{T^2}{(1+T)^2} = 4T^2(1-2T+3T^2-\dots).$$

1.3.4. — Let $f(x, y) = x^m + y^m, (x, y) \in \mathbb{R}^2, m$ odd. Then f can be desingularized by one blowing-up with the exceptional divisor \mathbb{P}^1 . Now,

$(f \circ \sigma)^{-1}(0)$ contains as well the strict transform of $f^{-1}(0)$ that is a smooth curve meeting the exceptional divisor transversally at a point. Hence

$$\begin{aligned} Z(T) &= (-2)(-1) \frac{T^m}{1 - T^m} + (-2)^2 \frac{T^m}{1 - T^m} \frac{-T}{1 + T} \\ &= 2T^m (1 - 2T + 2T^2 - \dots + 2T^{m-1} - T^m \\ &\quad + 0 + \dots + 0 + T^{2m} - 2T^{2m+1} + \dots). \end{aligned}$$

Clearly $Z_+(T) = Z_-(T) = \frac{1}{2} Z(T)$.

1.3.5. *Zeta functions of a product.* — Let $f(x, y) : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$, $f(x, y) = f_1(x)f_2(y)$ where $f_i : (\mathbb{R}^{d_i}, 0) \rightarrow (\mathbb{R}, 0)$, $i = 1, 2$. Then it is easy to check the following formulae:

$$\begin{aligned} Z_f &= Z_{f_1} Z_{f_2}, & Z_{f,+} &= Z_{f_1,+} Z_{f_2,+} + Z_{f_1,-} Z_{f_2,-}, \\ & & Z_{f,-} &= Z_{f_1,+} Z_{f_2,-} + Z_{f_1,-} Z_{f_2,+}. \end{aligned}$$

If $f(x, y) = f_1(x)f_2(y)$, $f_1(0) = 0$ but $f_2(0) > 0$, then $Z_{f,\pm} = Z_{f_1,\pm}$ and the signs are swapped if $f_2(0) < 0$.

Let $f(x) = u(x) \prod_{i=1}^k x_i^{N_i}$, $N_i \geq 1$, $k \geq 1$, with $u(0) \neq 0$. By the above and Example 1.3.1

$$(1.4) \quad Z_f(T) = (-2)^k \prod_{i=1}^k \frac{-T^{N_i}}{1 + T^{N_i}}.$$

If one of N_i is odd then $Z_{f,+}(T) = Z_{f,-}(T)$. If they are all even and $u(0) < 0$, resp. $u(0) > 0$, then $Z_{f,+}(T) \equiv 0$, resp. $Z_{f,-}(T) \equiv 0$.

2. Thom-Sebastiani Formulae.

The Thom-Sebastiani Formulae express the zeta functions of $f(x) + g(y)$ in terms of the zeta functions of $f(x)$ and $g(y)$, $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$. We denote

$$(f * g)(x, y) := f(x) + g(y).$$

For motivic zeta functions similar formulae were proposed in [7].

In what follows we denote

$$Z_{f,\pm}(T) = \sum a_i^\pm T^i, \quad Z_{g,\pm}(T) = \sum b_i^\pm T^i, \quad Z_{f * g,\pm}(T) = \sum c_i^\pm T^i.$$

Then

$$Z_f(T) = \sum a_i T^i, \quad Z_g(T) = \sum b_i T^i, \quad Z_{f * g}(T) = \sum c_i T^i,$$

where $a_i = a_i^+ + a_i^-$ and so on. Let

$$A_n = 1 - \sum_1^n a_i, n \geq 1, \quad A_0 = 1.$$

Then

$$\sum_{i \geq 0} A_i T^i = \frac{1 - Z_f(T)}{1 - T}.$$

Similarly we define $B_n, n \geq 0$.

THEOREM 2.1.

$$(2.1) \quad c_n^+ = a_n^+ b_n^+ + a_n^+ B_n + A_n b_n^+ + \sum_1^n (-1)^{n-i} (a_i^+ b_i^- + a_i^- b_i^+),$$

$$(2.2) \quad c_n^- = a_n^- b_n^- + a_n^- B_n + A_n b_n^- + \sum_1^n (-1)^{n-i} (a_i^+ b_i^- + a_i^- b_i^+),$$

$$(2.3) \quad c_n = a_n^+ b_n^+ + a_n^- b_n^- + a_n B_n + A_n b_n + 2 \sum_1^n (-1)^{n-i} (a_i^+ b_i^- + a_i^- b_i^+).$$

Note that, in general, the total zeta function $Z_{f * g}(T)$ depends on all, that is also on the positive and negative zeta functions of f and g and not only on $Z_f(T)$ and $Z_g(T)$ as the following example shows.

Example 2.2. — Let $f(x) = x^2, g(y) = y^2$. The zeta functions of f and g are computed in Subsection 1.2. The coefficients A_i are given by

$$\sum A_i T^i = \frac{1 + T}{1 + T^2} = 1 + T - T^2 - T^3 + T^4 + T^5 - \dots$$

One may compute easily the zeta functions of $f * g$ using Theorem 2.1. They, of course, coincide with the ones given by 1.3.2. The total zeta function of $h(y) = -y^2$ equals that of g . But the total zeta functions $f * g$ and $f * h$ are different, see 1.3.3.

In general formulae (2.1)–(2.3) are not easy to use. Moreover they are term by term formulae. If the zeta functions of f and g are given by rational functions of T , then Theorem 2.1 does not give a similar form for the zeta functions of $f * g$. The Thom-Sebastiani Formulae can be simplified considerably by introducing *the modified zeta functions* given by

$$\tilde{Z}_{f,+}(T) = \sum_{n \geq 1} \tilde{A}_n^+ T^n, \quad \tilde{Z}_{f,-}(T) = \sum_{n \geq 1} \tilde{A}_n^- T^n,$$

where $\tilde{A}_n^+ = A_n + a_n^+$, $\tilde{A}_n^- = A_n + a_n^-$. Then

$$(2.4) \quad \tilde{Z}_\pm(T) = \frac{1 - Z(T)}{1 - T} - 1 + Z_\pm(T)$$

and if we introduce *the total modified zeta function* $\tilde{Z}(T) := \tilde{Z}_-(T) + \tilde{Z}_+(T)$ then

$$\frac{1 - Z(T)}{1 - T} = \frac{1 + \tilde{Z}(T)}{1 + T}.$$

We can compute the zeta functions from the modified ones by the inverse formula

$$(2.5) \quad Z_\pm(T) = \frac{1 + \tilde{Z}(T)}{1 + T} + 1 + \tilde{Z}_\pm(T).$$

Let

$$(2.6) \quad \tilde{Z}_{f,\pm}(T) = \sum_{i \geq 1} \tilde{A}_i^\pm T^i, \quad \tilde{Z}_{g,\pm}(T) = \sum_{i \geq 1} \tilde{B}_i^\pm T^i, \quad \tilde{Z}_{f * g, \pm}(T) = \sum_{i \geq 1} \tilde{C}_i^\pm T^i$$

(same signs). The following formulae are equivalent to those of Theorem 2.1.

THEOREM 2.3. — *One has*

$$(2.7) \quad \tilde{C}_n^+ = \tilde{A}_n^+ \tilde{B}_n^+, \quad \tilde{C}_n^- = \tilde{A}_n^- \tilde{B}_n^-.$$

Example 2.4. (a) Let $f(x) = x^m$, m odd. Then,

$$\begin{aligned} \tilde{Z}_{f,+}(T) = \tilde{Z}_{f,-}(T) = T + T^2 + \dots + T^{m-1} \\ - T^{m+1} - \dots - T^{2m-1} + T^{2m+1} + \dots \end{aligned}$$

In particular, $\tilde{A}_n^+ = \tilde{A}_n^- = 0$ for $n \in m\mathbb{N}$.

(b) Let $f(x) = x^m$, m even. Then,

$$\begin{aligned} \tilde{Z}_{f,+}(T) = T + T^2 + \dots + T^m - T^{m+1} - \dots - T^{2m} + T^{2m+1} + \dots, \\ \tilde{Z}_{f,-}(T) = T + T^2 + \dots + T^{m-1} - T^m - \dots - T^{2m-1} + T^{2m} + \dots \end{aligned}$$

COROLLARY 2.5. — *Let $f(x) = x^m$ or $-x^m$, m even. Then $Z_{g,\pm}(T)$ can be computed from $Z_{f * g, \pm}(T)$.*

Proof. — As follows from Theorem 2.3 this suspension property holds for any function $f(x)$ for which all \tilde{A}_n^\pm are non-zero. This holds for $f(x) = \pm x^m$, m even, by Example 2.4 (b). □

If $f(x) = x^m$, m odd, then, in general, $Z_g(T)$ cannot be computed from $Z_{f * g}(T)$. Nevertheless we have the following result.

PROPOSITION 2.6. — Let $f(x) = x^m$, $m > 1$ odd, and let $g(y) = \pm y^k$. Then k is determined by the zeta functions of $f * g$. If, moreover, k is even and not divisible by m then the sign at y^k is determined by the zeta functions of $f * g$.

Proof. — We use notation (2.6) for the modified zeta functions of f, g , and $f * g$. If $\tilde{C}_n^+ = 0$ for $n \notin m\mathbb{N}$ then $\tilde{B}_n^+ = 0$, by Theorem 2.3. Then k is odd and equals the minimum of such n . Similarly, if there is $n \notin m\mathbb{N}$ such that $\tilde{C}_n^+ \neq \tilde{C}_n^-$ then, k is even and equals the minimum of such n . Thus suppose that

$$\tilde{B}_n^+ = \tilde{B}_n^- \neq 0 \quad \text{for all } n \notin m\mathbb{N}.$$

Then k is a multiple of m and equals the minimal $n = pm$ that produce a sign change $\tilde{B}_{n-1}^+ = -\tilde{B}_{n+1}^+$. Thus k is determined by the coefficients \tilde{C}_n^\pm . If k is even and not a multiple of m then $\tilde{B}_k^+ = -\tilde{B}_k^+ \neq 0$ and is minimal for this property. □

Example 2.7. — Let $f(x) = x^m$, m odd, and let $g_1(y) = y^{km}$, $g_2(y) = -y^{km}$, k even. The total zeta functions of g_1 and g_2 are equal but the positive and the negative ones are different. By Thom-Sebastiani Formulae (2.7) and Example 2.4 all zeta functions of $f * g_1$ and $f * g_2$ coincide. The functions $f * g_1$ and $f * g_2$ are not analytically equivalent but we shall show in the proof of Theorem 6.1 below that they are blow-analytically equivalent.

For the proof of Theorem 2.3 we need the following lemmas.

LEMMA 2.8. — One has

$$\begin{aligned} \tilde{A}_n^+ &= (-1)^{nd_1} \chi^c(\{\gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c \geq 0\}), \\ \tilde{A}_n^- &= (-1)^{nd_1} \chi^c(\{\gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c \leq 0\}). \end{aligned}$$

Proof. — Denote by $\pi_{n,i} : \mathcal{L}_n \rightarrow \mathcal{L}_i$ the truncation map. It is a trivial fibration with fiber isomorphic to $\mathbb{R}^{(n-i)d_1}$. Then

$$\mathcal{L}_n = \pi_{n,1}^{-1}(\mathcal{X}_1) \cup \pi_{n,2}^{-1}(\mathcal{X}_2) \cup \dots \cup \pi_{n,n-1}^{-1}(\mathcal{X}_{n-1}) \cup \mathcal{X}_{n,-} \cup \bar{\mathcal{X}}_{n,+}$$

where by $\bar{\mathcal{X}}_{n,+}$ we denote $\{\gamma \in \mathcal{L}_n; f \circ \gamma = ct^n + \dots, c \geq 0\}$. Then

$$\begin{aligned} \chi^c(\bar{\mathcal{X}}_{n,+}) &= \chi^c(\mathcal{L}_n) - \sum_1^{n-1} (-1)^{(n-i)d_1} \chi^c(\mathcal{X}_i) - \chi^c(\mathcal{X}_{n,-}) \\ &= (-1)^{nd_1} - \sum_1^{n-1} (-1)^{nd_1} a_i - (-1)^{nd_1} a_n^- = (-1)^{nd_1} (A_n + a_n^+). \quad \square \end{aligned}$$

LEMMA 2.9. — *Let the map*

$$\varphi_i : \{ \gamma \in \mathcal{L}_n; \text{ord}_t f \circ \gamma = i \} \longrightarrow \mathbb{R}$$

associate to γ , such that $(f \circ \gamma)(t) = v_i t^i + v_{i+1} t^{i+1} + \dots$, the coefficient v_n . Then φ_i is a trivial fibration for $i < n$.

Proof. — Define on $\{ \gamma \in \mathcal{L}_n; \text{ord}_t f \circ \gamma = i \}$ an action of \mathbb{R} by

$$(\alpha, \gamma(t)) \longmapsto \gamma(t + \alpha t^{n-i+1}).$$

Then $(f \circ \gamma)(t + \alpha t^{n-i+1}) = v_i t^i + \dots + v_{n-1} t^{n-1} + (v_n + i v_i \alpha) t^n + \dots$ that gives $\varphi_i(\gamma(t + \alpha t^{n-i+1})) = \varphi_i(\gamma) + i v_i \alpha$. Thus this action of \mathbb{R} trivializes φ_i . □

Proof of Theorem 2.3. — We show the formula for \tilde{C}_n^+ . By Lemma 2.8

$$\tilde{C}_n^+ = (-1)^{nd} \chi^c(\bar{\mathcal{X}}_{n,+}(f * g)),$$

where $\bar{\mathcal{X}}_{n,+}(f * g) = \{ (\gamma_1, \gamma_2) \in (\mathcal{L}_n(f) \times \mathcal{L}_n(g)); f(\gamma_1(t)) + g(\gamma_2(t)) = ct^n + \dots, c \geq 0 \}$. Then either $\text{ord}_t f(\gamma_1(t)) \geq n$ and $\text{ord}_t g(\gamma_2(t)) \geq n$ or $\text{ord}_t f(\gamma_1(t)) = \text{ord}_t g(\gamma_2(t)) < n$. This gives the following decomposition:

$$(2.10) \quad \bar{\mathcal{X}}_{n,+}(f * g) = (Z \cap \bar{\mathcal{X}}_{n,+}(f * g)) \cup \bigcup_{i=1}^{n-1} (Z_i \cap \bar{\mathcal{X}}_{n,+}(f * g)),$$

where

$$Z = \{ (\gamma_1, \gamma_2); \text{ord}_t f(\gamma_1(t)) \geq n, \text{ord}_t g(\gamma_2(t)) \geq n \},$$

$$Z_i = \{ (\gamma_1, \gamma_2); \text{ord}_t f(\gamma_1(t)) = \text{ord}_t g(\gamma_2(t)) = i \}.$$

First we shall compute $\chi^c(Z \cap \bar{\mathcal{X}}_{n,+}(f * g))$. Consider the map $\Phi : Z \rightarrow \mathbb{R}_{(\varphi, \psi)}^2$ that associates to the pair of arcs (γ_1, γ_2) the coefficients at t^n of $f(\gamma_1(t))$ and of $g(\gamma_2(t))$. The map Φ is trivial over the following strata of $\Phi(Z \cap \bar{\mathcal{X}}_{n,+}(f * g)) \subset \mathbb{R}^2$:

$$\begin{aligned} & \{ \varphi > 0, \psi > 0 \}, \quad \{ \varphi + \psi > 0, \psi < 0 \}, \quad \{ \varphi + \psi > 0, \varphi < 0 \}, \\ & \{ \varphi > 0, \psi = 0 \}, \quad \{ \psi > 0, \varphi = 0 \}, \quad \{ \varphi + \psi = 0, \psi < 0 \}, \\ & \{ \varphi + \psi = 0, \varphi < 0 \}, \quad \{ \varphi = \psi = 0 \}. \end{aligned}$$

Note that Φ is trivial over $\{ \varphi + \psi \geq 0, \psi < 0 \}$ and $\{ \varphi + \psi \geq 0, \varphi < 0 \}$. By this triviality

$$\chi^c(\Phi^{-1}(\{ \varphi + \psi \geq 0, \psi < 0 \})) = 0$$

since $\chi^c(\{(\varphi, \psi) \in \mathbb{R}^2; \varphi + \psi \geq 0, \psi < 0\}) = 0$. Similarly

$$\chi^c(\Phi^{-1}(\{\varphi + \psi \geq 0, \varphi < 0\})) = 0.$$

Hence, since $Z \cap \bar{\mathcal{X}}_{n,+}(f * g) = \Phi^{-1}(\{\varphi + \psi \geq 0\})$,

$$\begin{aligned} (2.11) \quad \chi^c(Z \cap \bar{\mathcal{X}}_{n,+}(f * g)) &= \chi^c(\Phi^{-1}(\{\varphi + \psi \geq 0\})) \\ &= \chi^c(\Phi^{-1}(\{\varphi \geq 0, \psi \geq 0\})) \\ &= (-1)^{nd_1} \tilde{A}_n^+ (-1)^{nd_2} \tilde{B}_n^+ = (-1)^{nd} \tilde{A}_n^+ \tilde{B}_n^+. \end{aligned}$$

Now we compute $\chi^c(Z_i \cap \bar{\mathcal{X}}_{n,+}(f * g))$ for $0 < i < n$ fixed. Let $(\gamma_1, \gamma_2) \in Z_i \cap \bar{\mathcal{X}}_{n,+}(f * g)$. Write

$$(2.12) \quad f(\gamma_1(t)) = v_i t^i + \dots + v_n t^n + \dots, \quad g(\gamma_2(t)) = w_i t^i + \dots + w_n t^n + \dots$$

Then $v_i \neq 0, w_i \neq 0$, and $v_i + w_i = 0$. Thus v_i and w_i are of opposite signs and hence $Z_i \cap \bar{\mathcal{X}}_{n,+}(f * g)$ is the disjoint union of two sets

$$Z_i^\pm = (\pi_{n,i}^{-1}(\mathcal{X}_{i,\pm}(f)) \times \pi_{n,i}^{-1}(\mathcal{X}_{i,\mp}(g))) \cap \bar{\mathcal{X}}_{n,+}(f * g),$$

where \pm equals the sign of v_i . Consider the following map:

$$\Psi : \pi_{n,i}^{-1}(\mathcal{X}_{i,+}(f)) \times \pi_{n,i}^{-1}(\mathcal{X}_{i,-}(g)) \longrightarrow \mathbb{R}_{(\varphi,\psi)}^2$$

that associates to (γ_1, γ_2) the coefficients (v_n, w_n) of (2.12). By Lemma (2.9), Ψ is a trivial fibration and, since $\chi^c(\{(\varphi, \psi) \in \mathbb{R}^2; \varphi + \psi \geq 0\}) = 0$,

$$\chi^c(Z_i^+) = \chi^c(\Psi^{-1}(\{\varphi + \psi \geq 0\})) = 0.$$

Similarly we show that $\chi^c(Z_i^-) = 0$ and hence

$$(2.13) \quad \chi^c(Z_i \cap \bar{\mathcal{X}}_{n,+}(f * g)) = 0.$$

The required formula for \tilde{C}_n^+ now follows from (2.10), (2.11), (2.13). □

The formulae of Theorem 2.1 and the ones of Theorem 2.3 are equivalent that one may check easily by a long but elementary computation. Alternatively, Theorem 2.1 can be proved by a topological argument similar to that of the proof of Theorem 2.3. We sketch just the main steps below. The details are left to the reader.

Proof of Theorem 2.1. — First note that the proof of Lemma 2.8 gives also

$$A_n = (-1)^{nd_1} \chi^c(\{\gamma \in \mathcal{L}_n; \text{ord}_t f \circ \gamma > n\}).$$

Then, with Z and Z_i as before,

$$(\mathcal{L}_n(f) \times \mathcal{L}_n(g)) \cap \mathcal{X}_{n,+}(f * g) = (Z \cap \mathcal{X}_{n,+}(f * g)) \cup \bigcup_{i=1}^{n-1} (Z_i \cap \mathcal{X}_{n,+}(f * g)).$$

By the triviality of $\Phi: Z \rightarrow \mathbb{R}_{(\varphi,\psi)}^2$ over the strata we get

$$(-1)^{nd} \chi^c(Z \cap \{\varphi + \psi > 0\}) = a_n^+ b_n^+ + a_n^+ b_n^- + a_n^- b_n^+ + A_n b_n^+ + a_n^+ B_n.$$

Another argument based on Lemma 2.9 gives

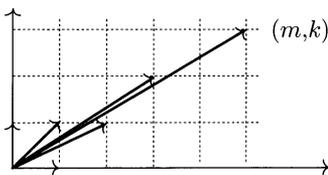
$$(-1)^{nd} \chi^c(Z_i \cap \mathcal{X}_{n,+}(f * g)) = (-1)^{n-i} (a_i^+ b_i^- + a_i^- b_i^+).$$

Formula (2.1) now follows from the additivity of Euler characteristic with compact supports. □

3. Computations in two variables case.

In this section we compute two dimensional examples using toric resolution. First we recall briefly the construction of toric resolution associated to a system of weights.

Given a weight vector $(m, k) \in \mathbb{N}^2$, m and k coprime. There is a canonical decomposition of the closed first quadrant $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ in \mathbb{R}^2 into a finite union of regular rational convex polyhedral cones that is compatible with the weight vector.



This decomposition induces a toric modification $\sigma: M_\Delta \rightarrow \mathbb{R}^2$, where Δ is the fan associated to this decomposition and M_Δ is the associated toric variety. The exceptional divisors of σ are in one-to-one correspondence with the one dimensional subcones (called rays or edges) of Δ that are not the coordinate half-axis. The integral vectors that generate these rays

can be computed out of m, k by the following procedure. Consider the Hirzebruch-Jung continued fraction

$$\frac{m}{k} = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_r}}},$$

where $a_i \geq 2$ for $i > 1$ and $a_1 \geq 1$. The coefficients a_i define the vectors $(m_i, k_i) \in \mathbb{R}^2, i = 1, \dots, r + 1$, such that

$$\begin{aligned} m_1 &= 1, & m_2 &= a_1, & m_{i+1} &= a_i m_i - m_{i-1} & \text{for } 2 \leq i \leq r, \\ k_1 &= 0, & k_2 &= 1, & k_{i+1} &= a_i k_i - k_{i-1} & \text{for } 2 \leq i \leq r, \end{aligned}$$

and then $m_{r+1} = m, k_{r+1} = k$. Similarly the coefficients b_1, \dots, b_s of the Hirzebruch-Jung continued fraction of k/m define the vectors $(m'_i, k'_i) \in \mathbb{R}^2, i = 1, \dots, s + 1$, such that

$$\begin{aligned} m'_1 &= 0, & m'_2 &= 1, & m'_{i+1} &= b_i m'_i - m'_{i-1} & \text{for } 2 \leq i \leq s, \\ k'_1 &= 1, & k'_2 &= b_1, & k'_{i+1} &= b_i k'_i - k'_{i-1} & \text{for } 2 \leq i \leq s. \end{aligned}$$

Then the vectors

$$(3.1) \quad (1, 0) = (m_1, k_1), \dots, (m_r, k_r), (m, k), (m'_s, k'_s), \dots, (m'_1, k'_1) = (0, 1)$$

are the primitive vectors of the rays of Δ . Choose a pair of subsequent vectors $\mathbf{v} = (a, b), \mathbf{w} = (c, d)$ of (3.1). They generate a two dimensional cone τ of Δ and give rise to an affine chart of $\sigma, \sigma_\tau : M_\tau \simeq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$\sigma_\tau(X, Y) = (X^a Y^c, X^b Y^d).$$

The divisor corresponding to \mathbf{v} , resp. \mathbf{w} , is given in M_τ by $X = 0$, resp. $Y = 0$. The jacobian

$$\text{jac } \sigma_\tau = \begin{vmatrix} a & c \\ b & d \end{vmatrix} X^{a+b-1} Y^{c+d-1} = X^{a+b-1} Y^{c+d-1}$$

and hence it is normal crossings. Denote by $E_{\mathbf{v}}$ the divisor corresponding to the vector \mathbf{v} . Then the multiplicity of $\text{jac } \sigma_\tau$ along $E_{\mathbf{v}}$ equals

$$\text{mult}_{E_{\mathbf{v}}} \text{jac } \sigma_\tau = a + b - 1.$$

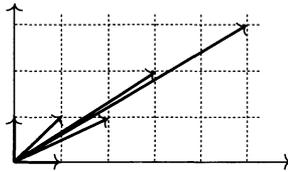
Let

$$f(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus \{(0,0)\}} a_{i,j} x^i y^j$$

and denote $\text{supp}(f) = \{(i, j); a_{i,j} \neq 0\}$. Then

$$\text{mult}_{E_\nu} f \circ \sigma = \min_{(i,j) \in \text{supp}(f)} \{ai + bj\}.$$

Example 3.1. — We compute the toric resolution and the zeta functions of $f(x, y) = x^3 + xy^5$. f is nondegenerate weighted homogeneous with weights $(5, 2)$. The toric modification associated to this system of weights is given by the vectors $(1, 0)$, $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (5, 2)$, $\mathbf{v}_3 = (2, 1)$, $\mathbf{v}_4 = (1, 1)$, $(0, 1)$.



Denote by E_i the component of the exceptional divisor corresponding to \mathbf{v}_i . Let $N_i = \text{mult}_{E_i} f \circ \sigma$, $\nu_i = \text{mult}_{E_i} \text{jac } \sigma + 1$. Then, by above, $N_1 = 8$, $\nu_1 = 4$, $N_2 = 15$, $\nu_2 = 7$, $N_3 = 6$, $\nu_3 = 3$, $N_4 = 3$, $\nu_4 = 2$. The strict transform of the zero set of f has two components: the strict transform S_1 of $x = 0$ and the strict transform S_2 of $x^2 + y^5 = 0$. The first one intersects E_1 and the second one E_2 as indicated on the resolution tree of Figure 1. Thus

$$\begin{aligned} (3.2) \quad Z(T) = & 4 \frac{T^8}{1 - T^8} - 6 \frac{T^{15}}{1 + T^{15}} - 4 \frac{T^6}{1 + T^6} + 2 \frac{T^3}{1 - T^3} \\ & - 4 \frac{T^8}{1 - T^8} \frac{T^{15}}{1 + T^{15}} + 4 \frac{T^{15}}{1 + T^{15}} \frac{T^6}{1 + T^6} \\ & - 4 \frac{T^6}{1 + T^6} \frac{T^3}{1 - T^3} - 4 \frac{T^8}{1 - T^8} \frac{T}{1 + T} + 4 \frac{T^{15}}{1 + T^{15}} \frac{T}{1 + T} \end{aligned}$$

and $Z_+(T) = Z_-(T) = \frac{1}{2} Z(T)$.

4. Zeta functions are blow-analytic invariants.

Blow-analytic equivalence is a notion introduced by T.-C. Kuo as a natural equivalence relation for real analytic function germs. He established several fundamental results on blow-analyticity. For a general review on the blow-analytic theory (until 1997), see [12]. The notion of blow-analytic equivalence is defined as follows:

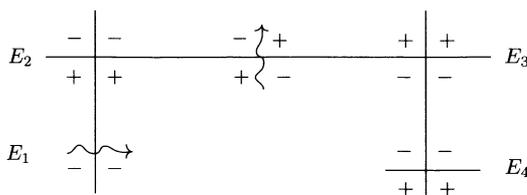


Figure 1

We say that analytic function germs $f, g: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there are real modifications

$$\mu: (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^d, 0), \quad \mu': (M', \mu'^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$$

and an analytic isomorphism $\Phi: (M, \mu^{-1}(0)) \rightarrow (M', \mu'^{-1}(0))$ which induces a homeomorphism $\phi: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$.

By a *real modification*, we mean the following. Let $\mu: M \rightarrow N$ be a proper surjective analytic map of real manifolds. It has a unique extension to a holomorphic map $\mu^*: U(M) \rightarrow U(N)$ where $U(M), U(N)$ are respectively open neighborhoods of M, N in their complexifications M^*, N^* . We say that μ is a *real modification* if μ^* is an isomorphism except on some thin subset of $U(M)$.

Let $\mu: (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$ be a real modification. Take any analytic arc at $0 \in \mathbb{R}^d, \lambda: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^d, \lambda(0) = 0$. Then λ has an analytic lifting. Namely, there is an analytic arc $\lambda': (-\epsilon, \epsilon) \rightarrow M, \lambda'(0) = P \in \mu^{-1}(0)$ such that $\lambda' \circ \mu = \lambda$. Remark that if λ is not contained in the critical value set of μ (a thin subset of \mathbb{R}^d) as set-germs at $0 \in \mathbb{R}^d$, then the lifting is unique.

We also assume the following condition for the real modifications μ and μ' in the definition of blow-analytic equivalence: the critical value sets of μ and μ' are contained in the zero-sets of f and g respectively as set-germs at $0 \in \mathbb{R}^d$. This assumption is reasonable. In fact, for any analytic function germ $f: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$, there is a real modification $\mu: (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$ with this property such that $f \circ \mu$ is a normal crossing (see [16], [3]). Any triviality theorem (see [22], [10], [13], [1] and so on) and a locally finite classification theorem (see [23]) have been established on blow-analytic equivalence with the property. A blow-analytic invariant (e.g., [11]) in the original sense is, of course, a blow-analytic invariant in our sense.

Suppose that real analytic function germs $f, g: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent in the sense of this paper. Then we can say

that the uniqueness of the arc lifting property holds for μ (resp. μ') if the arc is not contained in a subset of the zero-set $f^{-1}(0)$ (resp. $g^{-1}(0)$).

In this section we show that if two analytic function germs $f, g: (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent then their zeta functions coincide that is

$$Z_f = Z_g, \quad Z_{f,+} = Z_{g,+}, \quad Z_{f,-} = Z_{g,-}.$$

This will follow from Denef and Loeser's formulae (1.1), (1.2) that we show first. The proof will be an adaptation to the real analytic geometry, the ideas of [18], [6]. The main difficulty is that we have to use the sets that are not necessarily semi-algebraic but only subanalytic and not relatively compact, so we have to show that they have a well defined Euler characteristic with compact supports that is additive.

Let $\sigma: (M, E_0) \rightarrow (\mathbb{R}^d, 0)$, $E_0 = \sigma^{-1}(0)$, be a real modification. Consider the space of analytic arcs

$$\mathcal{L}(M, E_0) := \{ \gamma: (\mathbb{R}, 0) \rightarrow (M, E_0); \gamma \text{ analytic} \}.$$

The set of truncated arcs can be described as follows:

$$\mathcal{L}_n(M, E_0) := \mathcal{L}(M, E_0) / \sim,$$

where $\gamma_1(t) \sim \gamma_2(t)$ if $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(t) - \gamma_2(t) = O(t^{n+1})$ in a (or any) local system of coordinates at $\gamma_1(0) = \gamma_2(0)$. The set $\mathcal{L}_n(M, E_0)$ is an analytic variety, a subvariety of a similarly defined set $\mathcal{L}_n(M)$ that is an analytic manifold. The projection $\mathcal{L}_n(M, E_0) \rightarrow E_0$ is a locally trivial fibration with fiber \mathbb{R}^{nd} . Indeed, in a local system of coordinates on an open neighborhood U of $p_0 \in E_0$ we may write simply

$$(4.1) \quad \mathcal{L}_n(U, U \cap E_0) = \{ \gamma; \gamma(t) = p + \mathbf{y}_1 t + \dots + \mathbf{y}_n t^n, p \in U \cap E_0, \mathbf{y}_i \in \mathbb{R}^d \}.$$

Denote $\mathbf{y} := (\mathbf{y}_1, \dots, \mathbf{y}_n)$. Using the coordinates (p, \mathbf{y}) we identify

$$\mathcal{L}_n(U, U \cap E_0) \simeq (U \cap E_0) \times \mathbb{R}^{nd}.$$

Following Lojasiewicz [27] we call a semi-analytic subset of $\mathcal{L}_n(U, U \cap E_0)$ *relatively semi-algebraic with respect to \mathbf{y}* if it is defined by a finite number of equations and inequalities in functions that are analytic in p and polynomial in \mathbf{y} . Let us compare two such trivializations. This amounts to consider the following situation. Let U, U' be two open subsets of \mathbb{R}^d and let $h: U \rightarrow U'$

be an analytic isomorphism. Let $\gamma(t) = p + \mathbf{y}_1 t + \dots + \mathbf{y}_n t^n$ be as above. Then

$$h(\gamma(t)) = h(p) + \mathbf{a}_1(p, \mathbf{y})t + \dots + \mathbf{a}_n(p, \mathbf{y})t^n + O(t^{n+1}).$$

The coefficients $\mathbf{a}_i(p, \mathbf{y})$ are analytic in p and polynomial in \mathbf{y} . Thus two such local trivializations of $\mathcal{L}_n(M, E_0) \rightarrow E_0$ differ by an analytic isomorphism that is polynomial on the fibers. A semi-analytic subset A of $\mathcal{L}_n(M, E_0)$ will be called *relatively semi-algebraic* if for each $p \in E_0$ there is an open neighborhood U of p in M such that $A \cap \mathcal{L}_n(U, U \cap E_0)$ is relatively semi-algebraic.

Let X be an analytic manifold. If $A \subset X$ is subanalytic and relatively compact then its (co)homology groups are finitely generated. The Euler characteristic (standard or with compact supports) of such sets is well defined and the Euler characteristic with compact supports is additive. This is not, in general, true if A is no longer relatively compact. This observation justifies the following definition.

DEFINITION 4.1. — *Let X be an analytic manifold and $A \subset X$. We say that A is globally subanalytic if there is an analytic manifold \tilde{X} and an analytic embedding $i : X \rightarrow \tilde{X}$ such that $i(A)$ is relatively compact and subanalytic in \tilde{X} .*

A trivial example of globally subanalytic sets are semi-algebraic subsets of \mathbb{R}^N . The example we really have in mind are the semi-analytic and relatively semi-algebraic subsets of $\mathcal{L}_n(M, E_0) \rightarrow E_0$. Indeed, we may suppose that M is a submanifold of \mathbb{R}^N and hence $\mathcal{L}_n(M, E_0) \xrightarrow{i} \mathbb{R}^N \times \mathbb{R}^{nN}$. Choose any algebraic compactification of \mathbb{R}^{nN} , the one point compactification S^{nN} for instance. If $A \subset \mathcal{L}_n(M, E_0)$ is semi-analytic and relatively semi-algebraic then $i(A)$ is relatively compact and subanalytic (even semi-analytic) in $\mathbb{R}^N \times S^{nN}$.

Let $\pi_n : \mathcal{L}(M, E_0) \rightarrow \mathcal{L}_n(M, E_0)$ and $\pi_n : \mathcal{L}(\mathbb{R}^d, 0) \rightarrow \mathcal{L}_n(\mathbb{R}^d, 0)$ denote the truncation maps. The real modification $\sigma : (M, E_0) \rightarrow (\mathbb{R}^d, 0)$ induces a map

$$\sigma_* : \mathcal{L}(M, E_0) \longrightarrow \mathcal{L}(\mathbb{R}^d, 0),$$

defined by composition $\sigma_*(\gamma)(t) = \sigma(\gamma(t))$ that gives an analytic map on truncations

$$\sigma_{*n} : \mathcal{L}_n(M, E_0) \longrightarrow \mathcal{L}_n(\mathbb{R}^d, 0).$$

Clearly $\pi_n \circ \sigma_* = \sigma_{*n} \circ \pi_n$.

Let $\gamma \in \mathcal{L}(M, E_0)$. The jacobian determinant $\text{jac } \sigma$ of σ may be defined using any local coordinate system on M . Its order in t along $\gamma(t)$, $\text{ord}_t \text{jac } \sigma(\gamma(t))$, is independent of this choice. Given a positive integer e . Define

$$\Delta_e = \{ \gamma \in \mathcal{L}(M, E_0); \text{ord}_t \text{jac } \sigma(\gamma(t)) = e \}, \quad \Delta_{e,n} = \pi_n(\Delta_e).$$

The following lemma is an analogue of Lemma 3.4 of [6]

LEMMA 4.2. — *Let $e \geq 1$ and $n \geq 2e$.*

(a) *Let $\gamma_1, \gamma_2 \in \mathcal{L}(M, E_0)$. If $\gamma_1 \in \Delta_e$ and $\sigma(\gamma_1) \equiv \sigma(\gamma_2) \pmod{t^{n+1}}$ then $\gamma_1 \equiv \gamma_2 \pmod{t^{n+1-e}}$ and $\gamma_2 \in \Delta_e$.*

(b) *$\sigma_{*n}(\Delta_{e,n})$ is a globally subanalytic subset of $\mathcal{L}_n(\mathbb{R}^d, 0)$. There exists a subanalytic stratification of $\sigma_{*n}(\Delta_{e,n})$ such that over each stratum σ_{*n} is a trivial fibration with fiber \mathbb{R}^e .*

Proof. — Let $p \in E_0$. Choosing a local coordinate system at p we may suppose that $p = 0 \in \mathbb{R}^d$. Let $\gamma(t) \in \Delta_e$, $\gamma(0) = 0$. Denote by $J_\sigma(x)$ the jacobian matrix of σ at x . Then

$$\mathcal{M}(t) := t^e (J_\sigma(\gamma(t)))^{-1}$$

is a matrix with entries analytic functions in t . By Taylor formula

$$(4.2) \quad \sigma(\gamma(t) + t^{n+1-e}u) = \sigma(\gamma(t)) + t^{n+1-e} J_\sigma(\gamma)u + R(\gamma(t), u),$$

where $R(\gamma(t), u)$ is analytic in t and $u \in \mathbb{R}^d$. Moreover, $R(\gamma(t), u)$ is divisible by $t^{2(n+1-e)}$ and hence by t^{n+2} . Let

$$R(\gamma(t), u) = t^{n+2} \tilde{R}(\gamma(t), u), \quad \tilde{R}(\gamma(t), u) \text{ analytic.}$$

We solve the following equation with respect to $u \in \mathbb{R}^d$:

$$(4.3) \quad \sigma(\gamma(t) + t^{n+1-e}u) = \sigma(\gamma(t)) + t^{n+1}v.$$

By (4.2), (4.3) is equivalent to $t^{n+1}v = t^{n+1-e} J_\sigma(\gamma)u + R(\gamma(t), u)$, and hence to

$$u = \mathcal{M}(t)v - t\mathcal{M}(t)\tilde{R}(\gamma(t), u).$$

By the Implicit Function Theorem, for any $v_0 \in \mathbb{R}^d$, this equation has a unique analytic solution $u = u(t, v)$ defined in a neighborhood of $(v_0, 0)$.

In particular, if $v(t)$ is an analytic arc then (4.3) admits a solution being an analytic arc $u(t) = u(t, v(t))$. This solution is unique since σ is a modification (and $\text{jac } \sigma(\gamma(t) + t^{n+1-e}u(t))$ is not identically equal to 0, see (4.4) below).

Now we show (a). Let $\sigma(\gamma_1) \equiv \sigma(\gamma_2) \pmod{t^{n+1}}$ and consider a local coordinate system at $p = \gamma_1(0) = \gamma_2(0)$. By the above $\gamma_2(t)$ as the solution of (4.3) with $\gamma = \gamma_1$ and $v(t) = t^{-(n+1)}(\sigma(\gamma_2)(t) - \sigma(\gamma_1)(t))$ is of the form

$$\gamma_2(t) = \gamma_1(t) + t^{n+1-e}u(t).$$

This shows the first claim of (a). By Taylor formula

$$\begin{aligned} (4.4) \quad \text{jac } \sigma(\gamma_1(t) + t^{n+1-e}u(t)) &= \text{jac } \sigma(\gamma_1(t)) + t^{n+1-e} J_{\text{jac } \sigma}(\gamma_1)u(t) \\ &\quad + O(t^{2(n+1-e)}) \\ &\equiv \text{jac } \sigma(\gamma_1(t)) \pmod{t^{e+1}}, \end{aligned}$$

since $n + 1 - e \geq e + 1$. This completes the proof of (a).

We show (b). By (a) the set $\Delta_{e,n}$ is the union of fibers of σ_{*n} . To compute these fibers we fix $\gamma(t) \in \Delta_e$. We keep the notation of the first part of proof of lemma. By (4.3), the fiber of σ_{*n} over $\pi_n(\sigma_*(\gamma))$ equals

$$\begin{aligned} \sigma_{*n}^{-1}(\pi_n(\sigma_*(\gamma))) &= \{ \gamma(t) + t^{n+1-e}u \pmod{t^{n+1}}; \\ &\quad u = \mathbf{u}_0 + \mathbf{u}_1 t + \dots + \mathbf{u}_{e-1} t^{e-1}, \\ &\quad J_\sigma(\gamma(t))u(t) \equiv 0 \pmod{t^e} \} \end{aligned}$$

and hence is isomorphic to a linear subspace of

$$\{ u = \mathbf{u}_0 + \mathbf{u}_1 t + \dots + \mathbf{u}_{e-1} t^{e-1}; \mathbf{u}_i \in \mathbb{R}^d \} \simeq \mathbb{R}^{de}.$$

There are invertible matrices A and B with entries in $\mathbb{R}\{t\}$ such that $A J_\sigma(\gamma(t))B$ is equivalent over $\mathbb{R}\{t\}$ to a diagonal matrix with diagonal elements t^{e_1}, \dots, t^{e_d} . (For this it suffices to apply to $J_\sigma(\gamma(t))$ Gauss' elimination method.) Necessarily $e = e_1 + \dots + e_d$ and hence the fiber is isomorphic to \mathbb{R}^e .

The map $\sigma_{*n} : \mathcal{L}_n(M, E_0) \rightarrow \mathcal{L}_n(\mathbb{R}^d, 0)$ is analytic but not proper. Therefore, even if $\Delta_{e,n}$ is a semi-analytic set, it is not immediate that its image $\sigma_{*n}(\Delta_{e,n})$ is subanalytic. This follows from the relative semi-algebraicity of σ_{*n} and $\Delta_{e,n}$. By this we mean the following. Let $\Gamma \subset \mathcal{L}_n(M, E_0) \times \mathcal{L}_n(\mathbb{R}^d, 0)$ be the graph of σ_{*n} . Using a local system of coordinates at $p_0 \in E_0$ we identify an open neighborhood of p_0 in M with

an open neighborhood U of the origin in \mathbb{R}^d so that p_0 corresponds to the origin. Then σ_{*n} restricted to $\mathcal{L}_n(U, U \cap E_0)$ can be computed as follows. Write $y(t) \in \mathcal{L}_n(U, U \cap E_0)$ as in (4.1) and $x(t) \in \mathcal{L}_n(\mathbb{R}^d, 0)$ as

$$x(t) = \mathbf{x}_1 t + \cdots + \mathbf{x}_n t^n, \quad \mathbf{x}_i \in \mathbb{R}^d.$$

Denote $\mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_n)$. Then each coefficient of $x(t) = \sigma_{*n}(y(t))$, $\mathbf{x}_j(p, \mathbf{y})$ is analytic in p and polynomial in \mathbf{y} . That is in these coordinates Γ is given by an analytic equation $\mathbf{x} - \sigma_{*n}(p, \mathbf{y}) = 0$ that is polynomial in (\mathbf{y}, \mathbf{x}) . We say for short that Γ is relatively semi-algebraic with respect to the projection onto E_0 . Let $\Gamma_\Delta \subset \Gamma$ be the graph of σ_{*n} restricted to $\Delta_{e,n}$. A similar argument shows that Γ_Δ is relatively semi-algebraic with respect to the projection onto E_0 . Therefore, by Lojasiewicz's version of Tarski-Seidenberg Theorem [27], the projection $\text{pr}(\Gamma_\Delta)$ of Γ_Δ into $E_0 \times \mathcal{L}_n(\mathbb{R}^d, 0)$ is semi-analytic and relatively semi-algebraic. Finally, since E_0 is compact, the projection of $\text{pr}(\Gamma_\Delta)$ in $\mathcal{L}_n(\mathbb{R}^d, 0)$, that equals $\sigma_{*n}(\Delta_{e,n})$, is subanalytic. Moreover, it is easy to see that it is globally subanalytic.

Our original identification of $\mathcal{L}_n(\mathbb{R}^d, 0)$ with the space of truncated arcs $x(t) = \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \cdots + \mathbf{a}_n t^n$ gives an inclusion $\mathcal{L}_n(\mathbb{R}^d, 0) \hookrightarrow \mathcal{L}(\mathbb{R}^d, 0)$ that is a section of π_n . This allows us to define a section s of σ_{*n} by

$$(4.5) \quad s: \mathcal{L}_n(\mathbb{R}^d, 0) \hookrightarrow \mathcal{L}(\mathbb{R}^d, 0) \xrightarrow{\sigma_*^{-1}} \mathcal{L}(M, E_0) \xrightarrow{\pi_n} \mathcal{L}_n(M, E_0)$$

that is defined on those curves that are not entirely contained in the critical locus of σ , in particular on $\sigma_{*n}(\Delta_{e,n})$. Let s_Δ be the restriction of s onto $\sigma_{*n}(\Delta_{e,n})$ and let $\Gamma_{s,\Delta}$ be the graph of s_Δ . We shall show that $\Gamma_{s,\Delta}$ is globally subanalytic in $\mathcal{L}_n(\mathbb{R}^d, 0) \times \mathcal{L}_n(M, E_0)$. Considering $\sigma_{*n}(\Delta_{e,n})$ as a subset of $\mathcal{L}_{n+e}(\mathbb{R}^d, 0)$ by sequence of inclusions $\sigma_{*n}(\Delta_{e,n}) \subset \mathcal{L}_n(\mathbb{R}^d, 0) \subset \mathcal{L}_{n+e}(\mathbb{R}^d, 0)$ define

$$\tilde{\Gamma}_{s,\Delta} = \{(x(t), y(t)) \in \sigma_{*n}(\Delta_{e,n}) \times \mathcal{L}_{n+e}(M, E_0); x(t) = \sigma_{*(n+e)}(y(t))\}.$$

Then the graph $\Gamma_{s,\Delta}$ of s_Δ is the projection of $\tilde{\Gamma}_{s,\Delta}$ to $\mathcal{L}_n(\mathbb{R}^d, 0) \times \mathcal{L}_n(M, E_0)$. Indeed, it is clear that $\Gamma_{s,\Delta}$ is contained in this projection. On the other hand, if $x(t) = \sigma_{*(n+e)}(y(t))$ then $\sigma_{*(n+e)}(y(t)) = \sigma_{*(n+e)}(s(x(t)))$ and by (a), $y(t) \equiv s(x(t)) \pmod{t^{n+1}}$. Note that $\tilde{\Gamma}_{s,\Delta}$ is a semi-analytic set relatively semi-algebraic with respect to the projection to E_0 , and hence so is $\Gamma_{s,\Delta}$. Note also that s_Δ need not to be continuous and usually it is not, see Example 4.3 below.

Now we are ready to finish the proof of (b). Fix a subanalytic stratification of $\sigma_{*n}(\Delta_{e,n})$ so that s_Δ is analytic on each stratum. Such

a stratification exists since the graph of s_Δ is globally subanalytic. Fix a stratum S . Subdividing S if necessary, we may suppose that $s(S)$ is contained in an open subset of $\mathcal{L}_n(M, E_0)$ corresponding to a local chart on M . Thus we may use local coordinates on M .

$$\sigma_{*n}^{-1}(S) = \{s(x)(t) + t^{n+1-e}u \bmod t^{n+1}; x(t) \in S, \\ J_\sigma(s(x)(t))u(t) \equiv 0 \bmod t^e\}.$$

Since the kernel of $J_\sigma(s(x)(t)) \bmod t^e$ is isomorphic to \mathbb{R}^{nd} for each $x \in S$, σ_{*n} is a locally trivial analytic fibration over S . Thus subdividing again S , if necessary, we may ensure that the fibration becomes trivial over each stratum. This ends the proof. □

Let $A \subset \mathcal{L}(M, E_0)$ (or $A \subset \mathcal{L}(\mathbb{R}^d, 0)$). We say that A is *subanalytic* if $A = \pi_n^{-1}(C)$ where C is a globally subanalytic subset of $\mathcal{L}_n(M, E_0)$ (resp. of $\mathcal{L}_n(\mathbb{R}^d, 0)$). We say that A is *n-stable* if A is subanalytic and $A = \pi_n^{-1}(\pi_n(A))$. For instance by Lemma 4.2, $\sigma_*(\Delta_e)$ is $2e$ stable. It follows from (4.4) that Δ_e is always e stable.

Example 4.3. — Let $\sigma(X, Y) = (X^2Y, XY)$, $e = 2$. Let $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $(x, y) = \sigma(x, y) = (X^2Y, XY)$. Consider the curve

$$\gamma(t) = (X(t), Y(t)) = (X_0 + X_1t + X_2t^2 + \dots, Y_0 + Y_1t + Y_2t^2 + \dots), \\ \sigma(\gamma(t)) = (x(t), y(t)) = (x_0 + x_1t + x_2t^2 + \dots, y_0 + y_1t + y_2t^2 + \dots).$$

The jacobian determinant $\text{jac } \sigma = X^2Y$ and

$$\Delta_2 = \{\gamma(t); X_0Y_0 = X_0Y_1 = 0, X_0^2Y_2 + X_1^2Y_0 \neq 0\}, \\ \sigma_*(\Delta_2) = \{\sigma(\gamma(t)); x_0 = x_1 = y_0 = 0, x_2 \neq 0\}.$$

The conditions on Δ_2 do not involve $X_i, Y_i, i > 2$, so Δ_2 is 2-stable (as always). In this example also $\sigma_*(\Delta_2)$ is 2-stable. The truncations $\Delta_{2,2}$ and $\sigma_{*2}(\Delta_{2,2})$, that are subsets of $\mathcal{L}_2(\mathbb{R}^2) \simeq \mathbb{R}^6$, are given by the same conditions. Note that $\sigma_{*2}(\Delta_{2,2})$ is irreducible but $\Delta_{2,2}$ has two irreducible components. They are 2-truncations of

$$\Delta'_2 := \Delta_2 \cap \{X_0 = 0\}, \quad \Delta''_2 := \Delta_2 \cap \{Y_0 = 0\},$$

and their images are respectively

$$\sigma_*(\Delta'_2) = \sigma_*(\Delta_2) \cap \{y_1 \neq 0\}, \quad \sigma_*(\Delta''_2) = \sigma_*(\Delta_2) \cap \{y_1 = 0\}.$$

Thus by modification σ the geometry of the set of curves in $\sigma_*(\Delta_2)$ changes dramatically and Δ_2 contains the curves of two different kinds: the ones hitting $\{X = 0, Y \neq 0\}$ transversally and the ones touching $\{Y = 0, X \neq 0\}$ with intersection number 2.

Both restrictions of $\sigma_{*2}: \Delta'_2 \rightarrow \sigma_*(\Delta'_2)$ and $\Delta''_2 \rightarrow \sigma_*(\Delta''_2)$, are trivial fibrations with fiber \mathbb{R}^2 . For instance, the first one is given by

$$x_0 = x_1 = y_0 = 0, \quad x_2 = X_1^2 Y_0, \quad y_1 = X_1 Y_0, \quad y_2 = X_1 Y_1 + X_2 Y_0.$$

The section s of σ_{*2} is defined in (4.5). We compute the restriction of s to $\sigma_{*2}(\Delta'_{2,2})$. Fix a curve in $\sigma_{*2}(\Delta'_{2,2})$,

$$(x(t), y(t)) = (x_2 t^2, y_1 t + y_2 t^2)$$

that we consider as a curve in $\sigma_*(\Delta'_2)$. It lifts to

$$\begin{aligned} X(t) &= x(t)(y(t))^{-1} = x_2 y_1^{-1} t (1 - (y_2/y_1)t + \dots), \\ Y(t) &= (x(t))^{-1} (y(t))^2 = x_2^{-1} (y_1^2 + 2y_1 y_2 t + y_2^2 t^2). \end{aligned}$$

That is s on $\sigma_{*2}(\Delta'_{2,2})$ is given by

$$s(0, 0, x_2, 0, y_1, y_2) = (0, x_2/y_1, x_2 y_1^{-2} y_2, x_2^{-1} y_1^2, 2x_2^{-1} y_1 y_2, x_2^{-1} y_2^2).$$

Recall that $x_2 \neq 0$ everywhere on $\sigma_{*2}(\Delta_{2,2})$ but y_1 vanishes on $\sigma_{*2}(\Delta''_{2,2})$. Thus s cannot be extended continuously from $\sigma_{*2}(\Delta'_{2,2})$ to $\sigma_{*2}(\Delta_{2,2})$. A similar computation shows that s on $\sigma_{*2}(\Delta''_{2,2})$ is given by $s(0, 0, x_2, 0, 0, y_2) = (x_2/y_2, 0, 0, 0, 0, y_2^2/x_2)$.

By definition each subanalytic $A \subset \mathcal{L}(M, E_0)$ is n stable for n sufficiently large. Following [18], [7], [8], we may associate to each n -stable A its motivic measure that will be in our case simply

$$\chi^c(A) := (-1)^{-(n+1)d} \chi^c(\pi_n(A)).$$

This expression is independent of n (if A is n -stable). We say that $\varphi : A \rightarrow \mathbb{Z}$ is *constructible* if the image of φ is finite and $\varphi^{-1}(m)$ is subanalytic for each $m \in \mathbb{Z}$. Then we define

$$\int_A \varphi \, d\chi^c := \sum_{m \in \mathbb{Z}} m \chi^c(\varphi^{-1}(m)).$$

The following corollary of Lemma 4.2 is a real analytic version of the change of variables formula [18], [7], [8].

COROLLARY 4.4. — *Let $\sigma : (M, E_0) \rightarrow (\mathbb{R}^d, 0)$ be a real modification. Let $A \subset \mathcal{L}(\mathbb{R}^d, 0)$ be subanalytic and suppose that $\text{ord}_t \text{jac}(\sigma)$ is bounded on $\sigma_*^{-1}(A)$. Then*

$$\chi^c(A) = \int_{\sigma_*^{-1}(A)} (-1)^{-\text{ord}_t \text{jac}(\sigma)} d\chi^c.$$

Proof. — The function $\varphi = \text{ord}_t \text{jac}(\sigma)$ is constructible on A . Thus, by additivity of χ^c , it suffices to show the formula only on

$$A_e := A \cap \sigma_*(\Delta_e),$$

for e fixed. By Lemma 4.2, for n sufficiently large, σ_{*n} is a locally trivial fibration over $\pi_n(A_e)$ with fiber \mathbb{R}^e . Hence $\chi^c(\sigma_{*n}^{-1}(\pi_n(A_e))) = \chi^c(\mathbb{R})^{-e} \chi^c(\pi_n(A_e))$. This ends the proof. \square

Proof of (1.1), (1.2). — We show only (1.1). The proof of (1.2) is similar. The set $\mathcal{Z}_n(f) = \pi_n^{-1}(\mathcal{X}_n(f))$ is subanalytic and n -stable. The zeta function of f can be equivalently written as

$$Z_f(T) = (-1)^d \sum_{n \geq 1} \chi^c(\mathcal{Z}_n(f)) T^n.$$

Let $\mathcal{Z}_n(f \circ \sigma) = \sigma_*^{-1}(\mathcal{Z}_n(f))$ and $\mathcal{Z}_{n,e}(f \circ \sigma) = \mathcal{Z}_n(f \circ \sigma) \cap \Delta_e$. Then $\mathcal{Z}_n(f \circ \sigma)$ is the disjoint union of a finite number of $\mathcal{Z}_{n,e}(f \circ \sigma)$. Indeed, by comparing the multiplicities of $f \circ \sigma$ and $\text{jac} \sigma$ along the components of the exceptional divisor we see that $\text{ord}_t \text{jac} \sigma \leq n \max_i (\nu_i - 1) / N_i$ on $\mathcal{Z}_n(f \circ \sigma)$. (Here we use the assumption that the critical locus of σ is contained in the zero set of f . Otherwise the union may be infinite.) By the change of variables formula

$$(4.6) \quad Z_f(T) = (-1)^d \sum_{n \geq 1} \sum_{e \leq nq} (-1)^{-e} \chi^c(\mathcal{Z}_{n,e}(f \circ \sigma)) T^n,$$

where $q = \max_i (\nu_i - 1) / N_i$.

Fix $p \in \mathring{E}_I$. In a local system of coordinates at p the germ of $f \circ \sigma$ at p , that we denote by $g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$, is a normal crossings $g(y) = \text{unit} \cdot \prod_{i=1}^s y_i^{N_i}$, $s = |I|$. Let $\text{jac} \sigma(y) = \text{unit} \cdot \prod_{i=1}^s y_i^{\nu_i - 1}$. We shall compute the weighted zeta function of g that is

$$\widehat{Z}_g(T) = (-1)^d \sum_{n \geq 1} \sum_{e \leq nq} (-1)^{-e} \chi^c(\mathcal{Z}_{n,e}(g)) T^n,$$

where $\mathcal{Z}_{n,e}(g) = \mathcal{Z}_n(g) \cap \Delta_e$. Note that $\mathcal{Z}_{n,e}(g)$ is non-empty iff there are k_1, \dots, k_s such that $n = \sum k_i N_i$ and $e = \sum k_i (\nu_i - 1)$. We denote the set of such $k = (k_1, \dots, k_s)$ by $A(n, e)$. Thus $\mathcal{Z}_{n,e}(g)$ is the disjoint union

$$\mathcal{Z}_{n,e}(g) = \bigsqcup_{k \in A(n,e)} \left(\prod_{i=1}^s \mathcal{Z}_{k_i}(y_i^{N_i}) \times \mathcal{L}(\mathbb{R}, 0) \right)^{d-s},$$

and the last factor comes from the remaining $d - s$ variables y_i that do not contribute to the zero of g . Hence

$$\chi^c(\mathcal{Z}_{n,e}(g)) = (-1)^{d-s} \sum_{k \in A(n,e)} \left(\prod_{i=1}^s \chi^c(\mathcal{Z}_{k_i}(y_i^{N_i})) \right).$$

Thus, by (1.4),

$$\begin{aligned} (4.7) \quad \widehat{Z}_g(T) &= (-1)^s \sum_{(k_1, \dots, k_s) \in \mathbb{N}^s} \prod_{i=1}^s \chi^c(\mathcal{Z}_{k_i}(y_i^{N_i})) ((-1)^{\nu_i-1} T^{N_i})^{k_i} \\ &= \prod_{i=1}^s \left(\sum_k (-1) \chi^c(\mathcal{Z}_k(y_i^{N_i})) ((-1)^{\nu_i-1} T^{N_i})^k \right) \\ &= (-2)^s \prod_{i=1}^s \frac{(-1)^{\nu_i} T^{N_i}}{1 - (-1)^{\nu_i} T^{N_i}}. \end{aligned}$$

Formula (1.1) follows now from (4.6) by integration (with respect to χ^c) of (4.7) along the fibers of the projection $\mathcal{L}(M, E_0) \rightarrow E_0 = \sigma^{-1}(0)$. More precisely, to establish the equality of coefficients of T^n , we integrate along the fibers of the projection $\mathcal{L}_n(M, E_0) \rightarrow E_0$ restricted to $\mathcal{X}_n(f \circ \sigma) := \sigma_{*n}^{-1} \mathcal{X}_n(f)$. Then, by (4.7), the Euler characteristic with compact support of the fiber over $p \in \overset{\circ}{E}_I$ is independent of the choice of p in $\overset{\circ}{E}_I$. If we denote this Euler characteristic by $\chi^c(\mathcal{X}_n(f \circ \sigma)_I)$ then

$$\chi^c(\mathcal{X}_n(f \circ \sigma)) = \sum_{I \neq \emptyset} \chi^c(\overset{\circ}{E}_I) \chi^c(\mathcal{X}_n(f \circ \sigma)_I),$$

and the formula follows. □

THEOREM 4.5. — *Let $f, g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ be blow-analytically equivalent function germs. Then*

$$Z_f = Z_g, \quad Z_{f,+} = Z_{g,+}, \quad Z_{f,-} = Z_{g,-}.$$

Proof. — Since $f, g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent there are real modifications $\mu : (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$, $\mu' : (M', \mu'^{-1}(0)) \rightarrow (\mathbb{R}^d, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \rightarrow (M', \mu'^{-1}(0))$ such that $f \circ \mu = g \circ \mu' \circ \Phi$.

First we show that we may assume that both μ and μ' satisfy the properties required by Denef and Loeser's formula. Let $\text{jac } \mu, \text{jac}(\mu' \circ \Phi)$ denote the jacobian determinant of μ , resp. of $\mu' \circ \Phi$. By [16], [3] there is a modification $\mu_1 : M_1 \rightarrow M$ so that $f \circ \mu \circ \mu_1, \text{jac } \mu \circ \mu_1$, and $\text{jac}(\mu' \circ \Phi) \circ \mu_1$ are normal crossings simultaneously. Moreover, we may assume that μ_1 is a composition of blowings-up with smooth centers that are in normal crossings with the old exceptional divisors and hence that $\text{jac } \mu_1$ is normal crossings. Let $\sigma := \mu \circ \mu_1$. Then $\text{jac } \sigma(x) = \text{jac } \mu_1(x) \text{jac} \mu(\mu_1(x))$ is normal crossings with respect to the same set of divisors. Set $\sigma' = \mu' \circ \Phi \circ \mu_1$. Then $g \circ \sigma' = f \circ \sigma$ is normal crossings and so is $\text{jac } \sigma'(x) = \text{jac } \mu_1(x) \text{jac}(\mu' \circ \Phi)(\mu_1(x))$. Thus both σ and σ' satisfy the required properties.

Let E_i be an irreducible component of $(f \circ \sigma)^{-1}(0)$ (in $\sigma^{-1}(B_\epsilon)$). Since $g \circ \sigma' = f \circ \sigma$ the multiplicities of these two functions coincide on E_i . Thus, by formulae (1.1), (1.2), in order to show that the zeta functions of f and g coincide it suffices to show that $\text{mult}_{E_i} \text{jac } \sigma$ and $\text{mult}_{E_i} \text{jac } \sigma'$ are of the same parity for any irreducible component E_i of the exceptional divisor of σ since $\text{mult } \text{jac } \sigma = 0$ outside the exceptional set E of σ . Recall that Φ induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$. Then $\sigma(E)$ is of dimension $\leq d - 2$ and ϕ is analytic on the complement of $\sigma(E)$. In particular the jacobian $\text{jac } \phi$ has constant sign on the complement of $\sigma(E)$. Fix $p \in \overset{\circ}{E}_i$ and a local system of coordinates on M_1 at p . Then, since $\phi \circ \sigma = \sigma'$, $\text{jac } \phi(\sigma(x)) \text{jac } \sigma(x) = \text{jac } \sigma'(x)$. In particular, $\text{jac } \sigma(x)$ changes sign across $\overset{\circ}{E}_i$ iff so does $\text{jac } \sigma'(x)$. This shows that the multiplicities $\text{mult}_{E_i} \text{jac } \sigma$ and $\text{mult}_{E_i} \text{jac } \sigma'$ are of the same parity, as claimed. This ends the proof. □

It follows that the modified zeta functions f and g are also equal if f and g are blow-analytically equivalent.

5. Various formulae to compute the Fukui invariant.

5.1. Formulae in terms of the resolution.

Let $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function germ. Take any analytic arc $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^d, 0)$. Then $f(\gamma(t))$ is a convergent power series in t . We denote by $\text{ord}_t(f(\gamma(t)))$ its order in t . Set

$$A(f) = \{ \text{ord}_t(f(\gamma(t))) \in \mathbb{N} \cup \{ \infty \}; \gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^d, 0) C^\omega \}.$$

In [11], T. Fukui proved that $A(f)$ is a blow-analytic invariant. Namely, if analytic functions $f, g : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$. We call $A(f)$ *the Fukui invariant*. Note that the smallest number in $A(f)$ is the multiplicity of f . For a positive integer $a \in \mathbb{N}$, set

$$\mathbb{N}_{\geq a} = \{ n \in \mathbb{N}; n \geq a \}.$$

Example 5.1. — Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be a polynomial function defined by $f(x, y) = x^3 - y^5$. Then

$$A(f) = 3\mathbb{N} \cup 5\mathbb{N} \cup \mathbb{N}_{\geq 16} \cup \{ \infty \} = \{ 3, 5, 6, 9, 10, 12, 15, 16, 17, \dots \} \cup \{ \infty \}.$$

Any integer $15 + s \in A(f)$, $s \in \mathbb{N}$, is attained along $\gamma(t) = (t^5 + t^{5+s}, t^3)$.

For an analytic function germ $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$, let $\sigma : M \rightarrow \mathbb{R}^d$ be a simplification of $f^{-1}(0)$, namely, σ is a composition of a finite number of blowings-up, M is smooth and $f \circ \sigma$ is normal crossing. As in Subsection 1.2, we denote by E_i , $i \in J$, the irreducible components of $(f \circ \sigma)^{-1}(0)$ (in $\sigma^{-1}(B_\varepsilon)$, where B_ε is a small ball in \mathbb{R}^d centered at the origin). For each $i \in J$, let $N_i = \text{mult}_{E_i} f \circ \sigma$. Denote for $I \subset J$, $E_I = \bigcap_{i \in I} E_i$ and $\mathring{E}_I = E_I \setminus \bigcup_{j \in J \setminus I} E_j$. We put

$$\mathcal{C} = \{ I; \mathring{E}_I \cap \sigma^{-1}(0) \neq \emptyset \}.$$

Remark 5.2. — As stated in Section 1, we can assume that $\sigma^{-1}(0)$ is the union of some of E_i . Then $\mathcal{C} = \{ I; E_I \subset \sigma^{-1}(0) \}$.

For $A, B \subset \mathbb{N} \cup \{ \infty \}$, define

$$A + B = \{ a + b \in \mathbb{N} \cup \{ \infty \}; a \in A, b \in B \},$$

where we set $a + b = \infty$ if $a = \infty$ or $b = \infty$. Let us put

$$\Omega_I(f) = (N_{i_1}\mathbb{N} + \dots + N_{i_p}\mathbb{N}) \cup \{ \infty \},$$

for $I = (i_1, \dots, i_p) \in \mathcal{C}$.

THEOREM 5.3 (see [17]). — *Let $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function germ and let σ be a simplification of $f^{-1}(0)$. Then we have*

$$A(f) = \bigcup_{I \in \mathcal{C}} \Omega_I(f).$$

Let us put

$$\begin{aligned} \mathcal{C}^+ &:= \{I \in \mathcal{C}; \overset{\circ}{E}_I \cap \sigma^{-1}(0) \cap \overline{P(f)} \neq \emptyset\}, & P(f) &:= \{x \in M; f \circ \sigma(x) > 0\}, \\ \mathcal{C}^- &:= \{I \in \mathcal{C}; \overset{\circ}{E}_I \cap \sigma^{-1}(0) \cap \overline{N(f)} \neq \emptyset\}, & N(f) &:= \{x \in M; f \circ \sigma(x) < 0\}, \end{aligned}$$

where the overlines denote the closures in M .

Let $\lambda : U \rightarrow \mathbb{R}^d$ be an analytic arc with $\lambda(0) = 0$, where U denotes a neighborhood of $0 \in \mathbb{R}$. We call λ *nonnegative* (resp. *nonpositive*) for f if $(f \circ \lambda)(t) \geq 0$ (resp. ≤ 0) in a positive half neighborhood $[0, \delta) \subset U$. Then we define the Fukui invariants with sign by

$$\begin{aligned} A_+(f) &:= \{\text{ord}_t(f \circ \lambda); \lambda \text{ is a nonnegative arc through } 0 \text{ for } f\}, \\ A_-(f) &:= \{\text{ord}_t(f \circ \lambda); \lambda \text{ is a nonpositive arc through } 0 \text{ for } f\}, \end{aligned}$$

respectively. It is easy to see that these $A_+(f)$ and $A_-(f)$ are also blow-analytic invariants. Remark that $A(f) = A_+(f) \cup A_-(f)$. Then we have the following formulae to compute the Fukui invariants with sign:

THEOREM 5.4 (see [17]). — *Let $f : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function germ. Then we have*

$$A_+(f) = \bigcup_{I \in \mathcal{C}^+} \Omega_I(f), \quad A_-(f) = \bigcup_{I \in \mathcal{C}^-} \Omega_I(f).$$

5.2. List of the Fukui invariants for $\pm x^p \pm y^q$.

Let $p, q \in \mathbb{N}$, and let $(p, q) = d$. Here, (p, q) denotes $\text{gcd}(p, q)$. Then there are $p_1, q_1 \in \mathbb{N}$ such that $p = p_1d, q = q_1d$ and $(p_1, q_1) = 1$. Set

$$[p, q] = \text{LCM}(p, q) = p_1q_1d = pq_1 = p_1q.$$

Using the argument of Example 5.1, we compute the Fukui invariants for Brieskorn polynomials $f(x, y) = \pm x^p \pm y^q, (x, y) \in \mathbb{R}^2, p \leq q$, listed in Table 1.

Remark 5.5. — Let $f_1(x, y) = \pm x^p + y^q$ and $f_2(x, y) = \pm x^p - y^q, p$ odd, q even. If q is divisible by p , then $[p, q] = q = q_1p$. Thus $A(f_1) = A_{\pm}(f_1) = A(f_2) = A_{\pm}(f_2)$.

If q is not divisible by p , then $[p, q] > q$. Thus $A_+(f_1) \neq A_+(f_2)$ and $A_-(f_1) \neq A_-(f_2)$.

$f(x, y)$	Fukui invariants
p, q odd $\pm x^p \pm y^q,$	$A(f) = A_+(f) = A_-(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
p odd, q even $\pm x^p + y^q$	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$ $A_+(f) = A(f), A_-(f) = p\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
$\pm x^p - y^q$	$A_-(f) = A(f), A_+(f) = p\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
p even, q odd $x^p \pm y^q$	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$ $A_+(f) = A(f), A_-(f) = q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
$-x^p \pm y^q$	$A_-(f) = A(f), A_+(f) = q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
p, q even $\pm(x^p - y^q),$	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$ $A_+(f) = p\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}, A_-(f) = q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
$x^p - y^q$	$A_+(f) = p\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}, A_-(f) = q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
$-x^p + y^q$	$A_+(f) = q\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}, A_-(f) = p\mathbb{N} \cup \mathbb{N}_{\geq[p,q]} \cup \{\infty\}$
p, q even $\pm(x^p + y^q),$	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \{\infty\}$ $A_+(f) = A(f), A_-(f) = \{\infty\}$
$x^p + y^q$	$A_+(f) = A(f), A_-(f) = \{\infty\}$
$-x^p - y^q$	$A_-(f) = A(f), A_+(f) = \{\infty\}$

Table 1. Fukui invariants for $\pm x^p \pm y^q$

5.3. Thom-Sebastiani formulae for the Fukui invariant.

Let $f : (\mathbb{R}^{d_1}, 0) \rightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^{d_2}, 0) \rightarrow (\mathbb{R}, 0)$ be analytic function germs. Define $f * g : (\mathbb{R}^{d_1+d_2}, 0) \rightarrow (\mathbb{R}, 0)$ by $(f * g)(x, y) := f(x) + g(y)$ as in Section 2, and define also

$$f \cdot g : (\mathbb{R}^{d_1+d_2}, 0) \longrightarrow (\mathbb{R}, 0), \quad (f \cdot g)(x, y) := f(x) \times g(y).$$

In this subsection, we give the Thom-Sebastiani formulae expressing the Fukui invariants of $f(x) + g(y)$ and $f(x) \times g(y)$ in terms of the Fukui invariants of $f(x)$ and $g(y)$.

THEOREM 5.6. — *Let $M_1 = \min(A_+(f) \cap A_-(g))$ and $M_2 = \min(A_-(f) \cap A_+(g))$.*

$$(5.1) \quad A(f * g) = A(f) \cup A(g) \cup (M_1 + \mathbb{N}) \cup (M_2 + \mathbb{N}),$$

$$(5.2) \quad A_+(f * g) = A_+(f) \cup A_+(g) \cup (M_1 + \mathbb{N}) \cup (M_2 + \mathbb{N}),$$

$$(5.3) \quad A_-(f * g) = A_-(f) \cup A_-(g) \cup (M_1 + \mathbb{N}) \cup (M_2 + \mathbb{N}).$$

Proof. — We show only (5.1).

• *Inclusion* \subset . Take any $k \in A(f * g)$. We may assume that $k < \infty$ since $\infty \in A(f)$ or $A(g)$. Then there is an analytic arc $\nu = (\lambda, \mu) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, (0, 0))$ such that $\text{ord}_t((f * g) \circ \nu) = k$. Let

$$(f \circ \lambda)(t) = a_u t^u + a_{u+1} t^{u+1} + \dots, \quad a_u \neq 0,$$

$$(g \circ \mu)(t) = b_v t^v + b_{v+1} t^{v+1} + \dots, \quad b_v \neq 0.$$

Then $u = \text{ord}_t(f \circ \lambda) \in A(f)$ and $v = \text{ord}_t(g \circ \mu) \in A(g)$. Since $k \in A(f * g)$ and, $u \leq k$ or $v \leq k$, it suffices to consider the following three cases:

- (i) $u = k$ and $v \geq k$. In this case, $k \in A(f)$.
- (ii) $u \geq k$ and $v = k$. In this case, $k \in A(g)$.
- (iii) $u = v < k$. In this case, $u \in A_+(f)$ and $v \in A_-(g)$, or $u \in A_-(f)$ and $v \in A_+(g)$. This means $u = v \in (A_+(f) \cap A_-(g)) \cup (A_-(f) \cap A_+(g))$. It follows that $k > u = v \geq \min(M_1, M_2)$.

If $k \leq \min(M_1, M_2)$, then case (i) or case (ii) holds. Thus

$$k \in A(f) \cup A(g) \cup (M_1 + \mathbb{N}) \cup (M_2 + \mathbb{N})$$

because $k > \min(M_1, M_2)$ implies $k \in (M_1 + \mathbb{N}) \cup (M_2 + \mathbb{N})$.

• *Inclusion* \supset . It is obvious that $A(f), A(g) \subset A(f * g)$. Let us show $M_1 + \mathbb{N} \subset A(f * g)$. First we recall the reparametrization formulae of Remark 1.1 and Lemma 2.9. Let $h : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function defined by

$$h(t) = a_k t^k + a_{k+1} t^{k+1} + \dots, \quad a_k \neq 0.$$

Then, if we replace t by αt , $\alpha \neq 0$,

$$h(\alpha t) = a_k \alpha^k t^k + a'_{k+1} t^{k+1} + \dots.$$

Let $A = \{a_k \alpha^k; \alpha \in \mathbb{R}^*\}$. Then $A = \mathbb{R}^*$ for k odd, $A = \mathbb{R}_{>0}$ for k even and $a_k > 0$, and $A = \mathbb{R}_{<0}$ for k even and $a_k < 0$. Similarly, if we replace t by $t + \alpha t^{i+1}$, $i \geq 1$,

$$h(t + \alpha t^{i+1}) = a_k t^k + \dots + a_{k+i-1} t^{k+i-1} + (a_k k \alpha + a_{k+i}) t^{k+i} + \dots,$$

and in this case $\{a_k k \alpha + a_{k+i}; \alpha \in \mathbb{R}\} = \mathbb{R}$.

Take $k + j \in M_1 + \mathbb{N}$ such that $k = M_1$ and $j \in \mathbb{N}$. Then there are a nonnegative arc for $f, \lambda: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_1}, 0)$, and a nonpositive arc for $g, \mu: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_2}, 0)$, such that $\text{ord}_t(f \circ \lambda) = \text{ord}_t(g \circ \mu) = k$. Then

$$\begin{aligned} (f \circ \lambda)(t) &= a_k t^k + a_{k+1} t^{k+1} + \dots, \quad a_k > 0, \\ (g \circ \mu)(t) &= b_k t^k + b_{k+1} t^{k+1} + \dots, \quad b_k < 0. \end{aligned}$$

By the above there is a reparametrization $\mu^{(1)}: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_2}, 0)$ of μ such that

$$(g \circ \mu^{(1)})(t) = -a_k t^k + b_{k+1}^{(1)} t^{k+1} + \dots.$$

Using the second type of reparametrizations we can construct by induction on i an analytic arc $\mu^{(i)}: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_2}, 0)$ such that

$$(g \circ \mu^{(i)})(t) = -a_k t^k - a_{k+1} t^{k+1} - \dots - a_{k+i-1} t^{k+i-1} + b_{k+i}^{(i)} t^{k+i} + \dots,$$

for $2 \leq i \leq j$. Using the same argument again, we show that there is an analytic arc $\check{\mu}: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_2}, 0)$ such that

$$(g \circ \check{\mu})(t) = -a_k t^k - a_{k+1} t^{k+1} - \dots - a_{k+j-1} t^{k+j-1} + \check{b}_{k+j} t^{k+j} + \dots$$

with $a_{k+j} + \check{b}_{k+j} \neq 0$. Define $\nu: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, (0, 0))$ by $\nu(t) = (\lambda(t), \check{\mu}(t))$. Then $\text{ord}_t((f * g) \circ \nu) = k + j$. Thus $k + j \in A(f * g)$, namely, $M_1 + \mathbb{N} \subset A(f * g)$.

We can similarly show $M_2 + \mathbb{N} \subset A(f * g)$. □

Example 5.7. — Let $f(x) = x^4$ and $g(y) = y^6$. Then $A(f) = A_+(f) = 4\mathbb{N} \cup \{\infty\}$, $A(g) = A_+(g) = 6\mathbb{N} \cup \{\infty\}$, $A_-(f) = A_-(g) = \{\infty\}$ and $M_1 = M_2 = \infty$.

Thus $A(f * g) = A_+(f * g) = 4\mathbb{N} \cup 6\mathbb{N} \cup \{\infty\}$ and $A_-(f * g) = \{\infty\}$.

Concerning the Fukui invariant for $f \cdot g$, we can easily show following formulae.

PROPOSITION 5.8. — *One has*

$$(5.4) \quad A(f \cdot g) = A(f) + A(g),$$

$$(5.5) \quad A_+(f \cdot g) = (A_+(f) + A_+(g)) \cup (A_-(f) + A_-(g)),$$

$$(5.6) \quad A_-(f \cdot g) = (A_+(f) + A_-(g)) \cup (A_-(f) + A_+(g)).$$

Remark 5.9. — If f is one variable function, then

$$A(f) = (\min A(f))\mathbb{N} \cup \{\infty\}, \quad A_+(f) = (\min A_+(f))\mathbb{N} \cup \{\infty\},$$

$$A_-(f) = (\min A_-(f))\mathbb{N} \cup \{\infty\}.$$

Example 5.10. — Let $f(x, y) = cx^p y^q$, $c \neq 0$, $x, y \in \mathbb{R}^2$. Then

$$A(f) = \{ap + bq; a, b \in \mathbb{N}\} \cup \{\infty\}.$$

- 1) Let p or q be odd. Then $A_+(f) = A_-(f) = A(f)$.
- 2) Let p and q be even:
 - (i) if $c > 0$, $A_+(f) = A(f)$ and $A_-(f) = \{\infty\}$;
 - (ii) if $c < 0$, $A_+(f) = \{\infty\}$ and $A_-(f) = A(f)$.

6. Two variables Brieskorn polynomials.

6.1. Classification of two variables Brieskorn polynomials.

Let $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ be a two variables Brieskorn polynomial defined by $f(x, y) = \pm x^p \pm y^q$, $p \leq q$. If $0 \in \mathbb{R}^2$ is a regular point of f , i.e., $p = 1$, then f is analytically equivalent to $g(x, y) = x$ by the Implicit Function Theorem. After this, we assume that $0 \in \mathbb{R}^2$ is a singular point of f , i.e., $2 \leq p \leq q$.

Let \mathbb{N}_e (resp. \mathbb{N}_o) denote the set of positive even integers (resp. positive odd integers). Set

$$\mathfrak{M} := \{(p, q) \in \mathbb{N}_{\geq 2} \times \mathbb{N}_{\geq 2}; p \leq q\},$$

$$\mathfrak{N} := \mathfrak{M} - \{(p, mp) \in \mathfrak{M}; p \in \mathbb{N}_o, m \in \mathbb{N}_e\}.$$

Let us consider the classification of Brieskorn polynomials by blow-analytic equivalence. We denote by $(\pm x, \pm y)$ the Klein group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ consisting of the following four transformations of \mathbb{R}^2 :

$$(x, y) \mapsto (x, y), \quad (x, y) \mapsto (-x, y), \quad (x, y) \mapsto (x, -y), \quad (x, y) \mapsto (-x, -y).$$

For a subset A of $\{f(x, y) = \pm x^p \pm y^q; (p, q) \in \mathfrak{M}\}$, let $A/b.a.e$ (resp. $A/(\pm x, \pm y)$) denote the quotient of A by blow-analytic equivalence (resp. the Klein G -equivalence). Then we have the following blow-analytic classification.

THEOREM 6.1. — *One has*

$$\begin{aligned} & \{f(x, y) = \pm x^p \pm y^q; (p, q) \in \mathfrak{M}\} / b.a.e. \\ & = \{f(x, y) = \pm x^p \pm y^q; (p, q) \in \mathfrak{N}\} \\ & \quad / (\pm x, \pm y) \cup \{x^p + y^{mp}; p \in \mathbb{N}_o \cap \mathbb{N}_{\geq 2}, m \in \mathbb{N}_e\}. \end{aligned}$$

Proof. — By our list of the Fukui invariant in Subsection 5.2, we can distinguish all real Brieskorn polynomials of two variables $f(x, y) = \pm x^p \pm y^q$, $(p, q) \in \mathfrak{M}$, up to $\{(\pm x, \pm y)\}$ by the Fukui invariant except the following two cases:

- Case (i): $x^p + y^{mp}$ for a fixed even p and $m = 1, 2, 3, \dots$, or $-x^p - y^{mp}$ for a fixed even p and $m = 1, 2, 3, \dots$

- Case (ii): $\pm x^p + y^{mp}$ and $\pm x^p - y^{mp}$ for fixed odd $p \geq 3$ and even m .

We first consider case (i). For a fixed even p , let $f_m(x, y) = x^p + y^{mp}$ and $g_m(y) = y^{mp}$, $m = 1, 2, 3, \dots$. In this case,

$$A(f_m) = A_+(f_m) = \{p, 2p, 3p, \dots\} \cup \{\infty\}, \quad A_-(f_m) = \{\infty\}.$$

Since p and mp are even, it follows from Corollary 2.5 that if $Z_{x^p * g_m}(T) = Z_{x^p * g_n}(T)$, then $Z_{g_m}(T) = Z_{g_n}(T)$. On the other hand, as seen in Example 1.3.1, $Z_{g_m}(T) \neq Z_{g_n}(T)$ if $m \neq n$. Since the zeta function is a blow-analytic invariant, f_m and f_n are not blow-analytically equivalent if $m \neq n$. The case of $-x^p - y^{mp}$ follows similarly.

We next consider case (ii). In this case, $x^p + y^{mp}$ (resp. $x^p - y^{mp}$) is equivalent to $-x^p + y^{mp}$ (resp. $-x^p - y^{mp}$) under the transformation of \mathbb{R}^2 : $(x, y) \mapsto (-x, y)$. Therefore, we treat only $f(x, y) = x^p + y^{mp}$ and $g(x, y) = x^p - y^{mp}$ for fixed odd $p \geq 3$ and even m . Remark that the Fukui invariants $A(f)$ and $A_{\pm}(f)$ and the zeta functions $Z_f(T)$ and $Z_{f, \pm}(T)$ coincide with $A(g)$, $A_{\pm}(g)$, $Z_g(T)$ and $Z_{g, \pm}(T)$, respectively.

Here we recall the Fukui-Paunescu Theorem.

LEMMA 6.2 (T. Fukui, L. Paunescu [13], T. Fukui, E. Yoshinaga [10]). *Given a system of weights $\alpha = (\alpha_1, \dots, \alpha_d)$. Let $f_s : (\mathbb{R}^d, 0) \rightarrow (\mathbb{R}, 0)$, $s \in I = [0, 1]$, be an analytic family of analytic function germs. Suppose that for each $s \in I$, the weighted initial form of f_s with respect to α is of the same weighted degree and has an isolated singularity at $0 \in \mathbb{R}^d$. Then $\{f_s\}_{s \in I}$ is blow-analytically trivial over I .*

Let $\{f_s\}$ be a family of polynomial functions defined by

$$f_s(x, y) = x^p + pxy^{m(p-1)} + sy^{mp}, \quad s \in [-1, 1].$$

Then it follows from Lemma 6.2 that $x^p + pxy^{m(p-1)} + y^{mp}$ and $x^p + pxy^{m(p-1)} - y^{mp}$ are blow-analytically equivalent.

Nextly, let $\{g_s\}$ and $\{h_s\}$ be families of polynomial functions defined by

$$g_s(x, y) = x^p + psxy^{m(p-1)} + y^{mp}, \quad s \in [0, 1],$$

$$h_s(x, y) = x^p + psxy^{m(p-1)} - y^{mp}, \quad s \in [0, 1].$$

Then, by the same reason as above, $x^p + pxy^{m(p-1)} + y^{mp}$ (resp. $x^p + pxy^{m(p-1)} - y^{mp}$) are blow-analytically equivalent to $x^p + y^{mp}$ (resp. $x^p - y^{mp}$). Since blow-analytic equivalence is an equivalence relation (see [23]), $x^p + y^{mp}$ and $x^p - y^{mp}$ are blow-analytically equivalent.

This completes the proof of the theorem. □

Concerning cases (i) and (ii) in the proof of Theorem 6.1, we have the following remarks.

Remark 6.3. — By the above proof, we see that the Fukui invariants distinguish all real Brieskorn polynomials of two variables except case (i) and are not enough to give a complete classification of Brieskorn polynomials by blow-analytic equivalence. Then it gives rise to the following natural question:

Is the blow-analytic type of Brieskorn polynomials completely determined by the zeta functions?

The answer is ‘No’. Our zeta functions distinguish the blow-analytic types of all real Brieskorn polynomials of two variables except $f_m^+(x, y) = x^{2m} + y^{2m}$ and $f_m^-(x, y) = -(x^{2m} + y^{2m})$, $m = 1, 2, 3, \dots$. We shall see this fact in the next subsection. As seen in Example 1.3.2, $Z_{f_m^\pm}(T) = Z_{f_{m,\pm}^\pm}(T) = 0$ for any m . On the other hand, $A(f_m^\pm)$ is different from $A(f_n^\pm)$ if $m \neq n$ and $A_+(f_m^+) \neq A_+(f_m^-)$.

These remarks mean that the Fukui invariants and the zeta functions are compensating each other for our blow-analytic classification.

Remark 6.4. — Consider two functions of case (ii), $f(x, y) = x^p + y^{mp}$ and $g(x, y) = x^p - y^{mp}$ for fixed odd $p \geq 3$ and even m . These functions are

exceptional in our classification since they are blow-analytically equivalent, but not Klein G -equivalent. It is easy to see that they are not analytically equivalent. In addition, it was shown recently in [15], [14] that f and g are not even bi-Lipschitz equivalent.

6.2. Distinction of Brieskorn polynomials by zeta functions.

Let $f(x, y) = \pm x^p \pm y^q$, $2 \leq p \leq q$. Considering $f(x, y)$ up to Klein G -equivalence, we assume the following:

- (i) In case p (resp. q) is odd, we consider only the positive case that is the coefficient at x^p (resp. y^q) is $+1$.
- (ii) In case $p = q$ are even, we consider $f(x, y) = x^p - y^q$ but not $f(x, y) = -x^p + y^q$.

We show that our zeta functions distinguish all real Brieskorn polynomials of two variables up to blow-analytic equivalence except

$$(6.1) \quad f(x, y) = \pm(x^p + y^p), \quad p = 2, 4, 6, \dots$$

Note that $Z_f(T) = Z_{f,\pm}(T) \equiv 0$ only for Brieskorn polynomials of form (6.1) in the two variables case.

Assume that $f(x, y) = \pm x^p \pm y^q$ is not of form (6.1). Let $Z_f(T) = \sum_{i \geq 1} c_i T^i$, $Z_{f,\pm}(T) = \sum_{i \geq 1} c_i^\pm T^i$ as above. Then, by Theorem 2.1 and Example 1.3.1, we see that $c_i = 0$, $1 \leq i \leq p - 1$, and $c_p \neq 0$. Therefore p is determined by $Z_f(T)$.

We first consider the even case that is p is even. If $f(x, y) = x^p \pm y^q$ (resp. $-x^p \pm y^q$), $p < q$, then $c_p^+ = c_p \neq 0$, $c_p^- = 0$ (resp. $c_p^- = c_p \neq 0$, $c_p^+ = 0$). Therefore the sign at x^p is determined by $Z_{f,\pm}(T)$. Let $\phi(x) = \pm x^p$. By Corollary 2.5, $Z_{y^q,\pm}(T)$ (resp. $Z_{-y^q,\pm}(T)$) or $\tilde{Z}_{y^q,\pm}(T)$ (resp. $\tilde{Z}_{-y^q,\pm}(T)$) can be computed from $Z_{\phi * y^q,\pm}(T)$ (resp. $Z_{\phi * (-y^q),\pm}(T)$). As seen in Example 2.4, $\tilde{Z}_{\pm y^q,\pm}(T)$ are different from $\tilde{Z}_{\pm y^{q'},\pm}(T)$ if $q \neq q'$, and $\tilde{Z}_{y^q,\pm}(T)$ are different from $\tilde{Z}_{-y^q,\pm}(T)$ (if q is even). Therefore q and the sign at y^q are determined by $Z_{f,\pm}(T)$.

We next consider the odd case. Then, by Proposition 2.6, q is determined by $\tilde{Z}_{f,\pm}(T)$. If q is even and not divisible by p , the sign at y^q is also determined by $\tilde{Z}_{f,\pm}(T)$. On the other hand, as shown in the preceding subsection, if q is even and divisible by p , $x^p + y^q$ and $x^p - y^q$ are blow-analytically equivalent. Therefore the zeta functions distinguish Brieskorn polynomials up to blow-analytic equivalence in this case, too.

7. Examples in three variables.

7.1. Brieskorn polynomials of three variables.

Using the zeta functions and the Fukui invariants we classify blow-analytic types of Brieskorn polynomials of three variables, except for the following families: $\{x^p + y^{kp} + z^{kp}; k \in \mathbb{N}\}$, $\{-(x^p + y^{kp} + z^{kp}); k \in \mathbb{N}\}$, p even.

The following proposition generalizes Proposition 2.6.

PROPOSITION 7.1. — *Let $f(x_1, \dots, x_d)$ be a Brieskorn polynomial, $f(x_1, \dots, x_d) = \pm x_1^{m_1} \pm \dots \pm x_d^{m_d}$, all $m_i \geq 2$, and let $g(y) = \pm y^r$. Then r is determined by the zeta functions of f and of $f * g$. If, moreover, r is even and $r \notin \bigcup_{m_i, \text{ odd}} m_i \mathbb{N}$ then the sign at y^r is determined, too.*

Proof. — We use notation (2.6) for the modified zeta functions of f, g , and $f * g$. By assumption the coefficients \tilde{A}_n^\pm , resp. \tilde{C}_n^\pm , of the modified zeta functions of f , resp. $f * g$ are given. Hence, by Thom-Sebastiani Formulae (2.7), we may determine the coefficients \tilde{B}_n^\pm of the modified zeta functions of g for all n such that $\tilde{C}_n^\pm = 0$ that is for $n \in U := \mathbb{N} \setminus \bigcup_{m_i, \text{ odd}} m_i \mathbb{N}$.

If there is $n \in U$ such that $\tilde{B}_n^+ = 0$ then r is odd and equals the minimum of such n . Similarly, if there is $n \in U$ such that $\tilde{B}_n^+ \neq \tilde{B}_n^-$ then, r is even and equals the minimum of such n . In this case we may determine the sign in $g(y) = \pm y^r$.

From now on we suppose that

$$\tilde{B}_n^+ = \tilde{B}_n^- \neq 0 \quad \text{for all } n \in U.$$

Then r is a multiple of one of odd m_i 's. We shall show that the values

$$(7.1) \quad \tilde{B}_n^\pm, \quad n \in U$$

determine r . Without loss of generality we may suppose that all odd m_i are distinct prime numbers. Otherwise, without increasing U , we replace the set of odd m_i 's by the set of all their prime divisors. Thus we assume $U = \mathbb{N} \setminus \bigcup_{p \in P} p\mathbb{N}$, where P is a finite set of odd prime numbers. Let m be the product of all $p \in P$.

First we show that $m' = (m, r)$ is determined by the coefficients (7.1).

Let $m = m'm''$. Then $(m'', r) = 1$. So there exist $a, b \in \mathbb{Z}$ such that for all $k \in \mathbb{N}$,

$$(a + km'')r = (kr - b)m'' - 1.$$

Since m'' is odd, choosing k we may suppose that $a + km''$ is even, and $a + km'', kr - b \in \mathbb{N}$ if k is sufficiently large. Then $kr - b$ is odd. Fix such natural

$$q = Ar = Bm'' - 1 \quad A \text{ even}, B \text{ odd}.$$

Each $p \in P$ divides either r or m'' and hence does not divide $q - 1$ nor $q + 2$, i.e., $q - 1, q + 2 \in U$. Thus, by Example 2.4, if $g(y) = \pm y^r$ then

$$(7.2) \quad \tilde{B}_{q-1}^+ = -1, \quad \tilde{B}_{q+2}^+ = 1.$$

Suppose now that $g(y) = \pm y^{r_1}$ gives the same coefficients (7.1) as $g(y) = \pm y^r$ and that there is $p_0 \in P$ such that p_0 divides r_1 but it does not divide r . We show that this contradicts (7.2). Note that (7.2) is possible only if either q or $q + 1$ is an even multiple of r_1 . Firstly, $q + 1 = Bm''$, as an odd number, cannot be an even multiple of r_1 . Secondly, p_0 divides $q + 1 = Bm''$ so it does not divide q . Hence r_1 cannot divide q . Thus if $g(y) = \pm y^r$ and $g(y) = \pm y^{r_1}$ give the same coefficients (7.1) they have the same factors in P . That is $(m, r) = (m, r_1)$.

Let $m' = (m, r) = (m, r_1)$, $m = m'm''$. Then $(r, r_1) = dm'$ where $(d, m'') = 1$. Suppose $r \neq r_1$. Then one of them, say r_1 , is strictly bigger than dm' . By assumptions $(r_1, rm'') = dm'$ so there is $q \in \mathbb{N}$ of the form

$$q = Arm'' = Br_1 + dm'.$$

Clearly q is a multiple of m so $q - 1, q + 1 \in U$. But then, for $g = \pm y^r$,

$$\tilde{B}_{q-1}^+ = -\tilde{B}_{q+1}^+ \neq 0.$$

But this is not possible for $g = \pm y^{r_1}$ since $Br_1 < q - 1 < q + 1 < (B + 1)r_1$. This ends the proof. □

Remark 7.2. — We recall that the smallest number in the Fukui invariant $A(f)$ is the multiplicity of f . Let

$$f(x_1, \dots, x_d) = \pm x_1^{p_1} \pm x_2^{p_2} \pm \dots \pm x_d^{p_d}, \quad 2 \leq p_1 \leq p_2 \leq \dots \leq p_d.$$

Then p_1 is determined as the smallest number in $A(f)$. Let n be the smallest number in $A(f)$ that is not divisible by p_1 . Suppose that p_1 is odd. If $kp_1 < p_2 < (k + 1)p_1$ for some positive integer k , then $n = p_2$. In case

where $p_2 = kp_1$ for some k , using the argument of Example 5.1 we see that $n = p_2 + 1$. Therefore, if $kp_1 + 1 < n < (k + 1)p_1$ then $p_2 = n$. If $n = kp_1 + 1$ then $p_2 = n - 1$ or n . This implies that if $kp_1 + 1 < n < (k + 1)p_1$ then p_2 is determined by $A(f)$.

THEOREM 7.3. — *Let $f_i(x, y, z) = \pm x^{p_i} \pm y^{q_i} \pm z^{r_i}$, $2 \leq p_i \leq q_i \leq r_i$, $i = 1, 2$, be two Brieskorn polynomials with the same Fukui invariants and the same zeta functions. Then $p_1 = p_2$ and one of the two following cases holds:*

(i) $p = p_1 = p_2$ is even and f_1 and f_2 belong to one of the following families :

$$\{x^p + y^{kp} + z^{kp}; k \in \mathbb{N}\}, \quad \{-(x^p + y^{kp} + z^{kp}); k \in \mathbb{N}\}.$$

(ii) $q_1 = q_2$, $r_1 = r_2$, and f_1 and f_2 are blow-analytically equivalent.

We make the following convention. Whenever a Brieskorn polynomial $f(x, y, z) = \pm x^p \pm y^q \pm z^r$ contains two terms with the same exponents and different signs then ‘+’ precedes ‘-’, for instance we write $x^p - y^p$ instead of $-x^p + y^p$.

Remark 7.4. — Suppose that the f_i ’s are written down according to the above convention. Then, in the second case of Theorem 7.3 the signs corresponding to the even exponents have to be the same for $i = 1, 2$, except for the case when an even exponent (q or r) is a multiple of another exponent (p or q) that is odd. In the latter case the sign cannot be determined. For instance we cannot distinguish $x^p + y^{kp}$ from $x^p - y^{kp}$, p odd, k even, cf. proof of Theorem 6.1.

Proof of Theorem 7.3. — Let $f(x, y, z) = \pm x^p \pm y^q \pm z^r$, $1 < p \leq q \leq r$. We show that except the cases considered in (i) the exponents p , q , and r are determined by the zeta functions and the Fukui invariants of f . We suppose that the signs in f satisfy the above convention.

First note that p is determined by the Fukui invariant.

If p is even then the Fukui invariants with sign determine the sign at x^p (if $p = q$ by the sign convention). Then, by Corollary 2.5, the zeta functions of $g(y, z) = \pm y^q \pm z^r$ are determined by the zeta functions of f . If $Z_{g,\pm}$ are not identically equal to zero then we may use Subsection 6.2 to determine the exponents and the blow-analytic type of g . The signs are determined as in Remark 7.4. If the zeta functions of g are identically equal

to zero then $g(y, z) = \pm(y^q + z^q)$, q even. Note that the Fukui invariants of $\pm x^p \pm (y^q + z^q)$, $p \leq q$ both even, are the same as the Fukui invariants of $\pm x^p \pm y^q$. The latter are given in Subsection 5.2. Thus these invariants determine q , and the sign, in all cases except (i) of the theorem.

Suppose that p is odd. Consider the Fukui invariant $A(f)$. Let n be the smallest number in $A(f)$ that is not divisible by p . If $kp + 1 < n < (k + 1)p$ then $q = n$. If moreover such q is even then $A_+(f), A_-(f)$ determine the sign at y^q . Note that if we determine the second exponent, for instance q but the argument works also if it is r , so that we can determine uniquely the zeta functions of $\pm x^p \pm y^q$ then the remaining third exponent is unique by Lemma 7.1. This ends the proof if $kp + 1 < n < (k + 1)p$.

Suppose $n = kp + 1$. Then $q = kp$ or $kp + 1$. Consider first the case k even. Then $kp + 1$ is odd. Let $\tilde{Z}_{f,\pm} = \sum_{n \geq 1} \tilde{A}_n^{pm}$. By Example 2.4 and by Thom-Sebastiani Formula (2.7) applied twice to f , $kp + 1$ equals q or r if and only if $\tilde{A}_{kp+1}^\pm = 0$. If this is the case then we apply Proposition 7.1 to determine the remaining exponent (and the sign as in Remark 7.4). If this is not the case then $q = kp$. The zeta functions of $\pm x^p \pm y^q$ do not depend on the signs, see Example 2.7, and we may apply again Proposition 7.1 to determine r .

Thus the only remaining case is p odd, $q = kp$ or $kp + 1$, with k odd. In this case, q can be determined by the coefficients $\tilde{A}_{kp+1}^+, \tilde{A}_{kp+1}^-, \tilde{A}_{kp+2}^+ = \tilde{A}_{kp+2}^-$ of the modified zeta function of f and the Fukui invariants, that is the knowledge whether $kp + 1 \in A_+(f)$ or $kp + 1 \in A_-(f)$. The computation is summarized in Table 2. \square

7.2. Example on blow-analytic sufficiency of jets.

The zeta functions can be used to distinguish the blow-analytic types of functions that are not necessarily Brieskorn polynomials. For such a function it may be simpler to use the standard zeta functions than the modified ones. To facilitate the computations we reduce the Thom-Sebastiani formulae of Theorem 2.1 modulo 2. Taking into account that always $Z_+ \equiv Z_- \pmod{2}$, i.e., the coefficients satisfy $a_n^+ \equiv a_n^- \pmod{2}$, we obtain easily:

$$(7.3) \quad \begin{cases} 1 + c_n^+ \equiv (1 + a_n^+)(1 + b_n^+) \pmod{2}, \\ 1 + c_n^- \equiv (1 + a_n^-)(1 + b_n^-) \pmod{2}. \end{cases}$$

Of course these both formulae are equivalent.

$g(y, z) = \pm y^q \pm z^r$	\tilde{A}_{kp+1}^+	\tilde{A}_{kp+1}^-	\tilde{A}_{kp+2}^\pm	$kp+1 \in A_+(f)$	$kp+1 \in A_-(f)$
$\pm y^{kp} \pm z^{kp}$	-1	-1	-1	yes	yes
$\pm y^{kp} + z^{kp+1}$	1	-1	-1	yes	yes
$\pm y^{kp} - z^{kp+1}$	-1	1	-1	yes	yes
$\pm y^{kp} \pm z^{kp+2}$	1	1	0	yes	yes
$\pm y^{kp} \pm z^r, r > kp + 2$	1	1	1	yes	yes
$y^{kp+1} + z^{kp+1}$	-1	-1	-1	yes	no
$y^{kp+1} - z^{kp+1}$	1	1	-1	yes	yes
$-y^{kp+1} - z^{kp+1}$	-1	-1	-1	no	yes
$y^{kp+1} \pm z^{kp+2}$	-1	1	0	yes	no
$-y^{kp+1} \pm z^{kp+2}$	1	-1	0	no	yes
$y^{kp+1} \pm z^r, r > kp + 2$	-1	1	1	yes	no
$-y^{kp+1} \pm z^r, r > kp + 2$	1	-1	1	no	yes

Table 2

Example 7.5. — Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We consider polynomial functions $f_{\mathbb{K}}, g_{\mathbb{K}} : (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}, 0)$ defined by

$$f_{\mathbb{K}}(x, y, z) = x^3 + xy^5 + z^3, \quad g_{\mathbb{K}}(x, y, z) = x^3 + y^7 + z^3.$$

Note that they are weighted homogeneous polynomials with isolated singularities at $0 \in \mathbb{K}^3$ and the Fukui invariants of $f_{\mathbb{K}}$ and $g_{\mathbb{K}}$ are the same and equal $A_+ = A_- = \{3, 4, 5, \dots\} \cup \{\infty\}$. Let $\phi : (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}, 0)$ be an analytic function germ with $j^6\phi(0) = j^6f_{\mathbb{K}}(0)$. In case $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), it follows from Theorem 6.2 (resp. [4]) that if the Taylor expansion of ϕ contains a term of the form $ay^7, a \neq 0$, then ϕ is blow-analytically equivalent (resp. topologically equivalent) to $g_{\mathbb{R}}$ (resp. $g_{\mathbb{C}}$), otherwise ϕ is blow-analytically equivalent (resp. topologically equivalent) to $f_{\mathbb{R}}$ (resp. $f_{\mathbb{C}}$). Using the formula of Milnor and Orlik [30], we have $\mu(f_{\mathbb{C}}) = 26$ and $\mu(g_{\mathbb{C}}) = 24$. Thus it follows from [26] or [34] that $f_{\mathbb{C}}$ and $g_{\mathbb{C}}$ are not topologically equivalent. The real jet $w = f_{\mathbb{R}}$ was originally given by W. Kucharz [19] as an example such that w is C^0 -sufficient in C^8 functions as a 6-jet but not C^0 -sufficient in C^7 functions as a 7-jet. Therefore $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ are topologically equivalent and w does not satisfy the Kuiper-Kuo condition even as a real 7-jet.

We show that $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ are *not* blow-analytically equivalent. As a result, $w \in J_{\mathbb{R}}^6(2, 1)$ is not blow-analytically sufficient.

Let us first compute $Z_{f_{\mathbb{R}},+}(T) \bmod 2$. By (3.2),

$$Z_{x^3+xy^5,+}(T) \equiv \frac{T^{15}}{1-T^{15}} + \frac{T^3}{1+T^3} \bmod 2.$$

Hence, by (7.3), the coefficients $a_n^+(f)$ of $Z_{f_{\mathbb{R}},+}(T) \bmod 2$ are given by

$$a_n^+(f) \equiv \begin{cases} 1 \bmod 2 & \text{if } 3 \mid n, \\ 0 \bmod 2 & \text{otherwise.} \end{cases}$$

A similar computation of $Z_{g_{\mathbb{R}},+}(T) \bmod 2$ shows that its coefficients are equal to

$$a_n^+(g) \equiv \begin{cases} 1 \bmod 2 & \text{if } 3 \mid n \text{ or } 7 \mid n, \\ 0 \bmod 2 & \text{otherwise.} \end{cases}$$

Therefore, by Theorem 4.5, $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ are not blow-analytically equivalent.

BIBLIOGRAPHY

- [1] O.M. ABDERRAHMANE Yacoub, Polyèdre de Newton et trivialité en famille, *J. Math. Soc. Japan*, 54 (2002), 513–550.
- [2] E. BIERSTONE, P.D. MILMAN, Arc-analytic functions, *Invent. Math.*, 101 (1990), 411–424.
- [3] E. BIERSTONE, P.D. MILMAN, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, *Invent. Math.*, 128 (1997), 207–302.
- [4] J. DAMON, T. GAFFNEY, Topological triviality of deformations of functions and Newton filtrations, *Invent. Math.*, 72 (1983), 335–358.
- [5] J. DENEFF, F. LOESER, Motivic Igusa zeta functions, *J. Alg. Geom.*, 7 (1998), 505–537.
- [6] J. DENEFF, F. LOESER, Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.*, 135 (1999), 201–232.
- [7] J. DENEFF, F. LOESER, Motivic exponential integrals and a motivic Thom-Sebastiani Theorem, *Duke Math. J.*, 99 (1999), 289–309.
- [8] J. DENEFF, F. LOESER, Geometry of arc spaces of algebraic varieties, *European Congress of Math. (Barcelona, July 10–14, 2000)*, 1 (2001), 325–348.
- [9] J. DENEFF, F. LOESER, Lefschetz numbers of iterates of the monodromy and truncated arcs, *Topology*, 41 (2002), 1031–1040.
- [10] T. FUKUI, E. YOSHINAGA, The modified analytic trivialization of family of real analytic functions, *Invent. Math.*, 82 (1985), 467–477.
- [11] T. FUKUI, Seeking invariants for blow-analytic equivalence, *Comp. Math.*, 105 (1997), 95–107.

- [12] T. FUKUI, S. KOIKE, T.-C. KUO, Blow-analytic equisingularities, properties, problems and progress, *Real Analytic and Algebraic Singularities* (T. Fukuda, T. Fukui, S. Izumiya and S. Koike, eds.), Pitman Research Notes in Math. Series, 381 (1998), 8–29.
- [13] T. FUKUI, L. PAUNESCU, Modified analytic trivialization for weighted homogeneous function-germs, *J. Math. Soc. Japan*, 52 (2000), 433–446.
- [14] J.-P. HENRY, A. PARUSIŃSKI, Existence of Moduli for bi-Lipschitz equivalence of analytic functions, *Comp. Math.*, 136 (2003), 217–235.
- [15] J.-P. HENRY, A. PARUSIŃSKI, Invariants of bi-Lipschitz equivalence of real analytic functions, *Banach Center Publications* (to appear).
- [16] H. HIRONAKA, Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II, *Ann. of Math.*, 79 (1964), 109–302.
- [17] S. IZUMI, S. KOIKE, T.-C. KUO, Computations and Stability of the Fukui Invariant, *Comp. Math.*, 130 (2002), 49–73.
- [18] M. KONTSEVICH, Lecture at Orsay, December 7, 1995.
- [19] W. KUCHARZ, Examples in the theory of sufficiency of jets, *Proc. Amer. Math. Soc.*, 96 (1986), 163–166.
- [20] N. KUIPER, C^1 -equivalence of functions near isolated critical points, *Symp. Infinite Dimensional Topology*, Princeton Univ. Press, Baton Rouge, 1967, R.D. Anderson ed., *Annales of Math. Studies*, 69 (1972), 199–218.
- [21] T.-C. KUO, On C^0 -sufficiency of jets of potential functions, *Topology*, 8 (1969), 167–171.
- [22] T.-C. KUO, The modified analytic trivialization of singularities, *J. Math. Soc. Japan*, 32 (1980), 605–614.
- [23] T.-C. KUO, On classification of real singularities, *Invent. Math.*, 82 (1985), 257–262.
- [24] K. KURDYKA, Ensembles semi-algébriques symétriques par arcs, *Math. Ann.*, 282 (1988), 445–462.
- [25] K. KURDYKA, Injective endomorphisms of real algebraic sets are surjective, *Math. Ann.*, 282 (1998), 1–14.
- [26] LÊ DŨNG TRÁNG, Topologie des singularités des hypersurfaces complexes, *Singularités à Cargèse, Astérisque*, 7 & 8 (1973), 171–182.
- [27] S. ŁOJASIEWICZ, Ensembles semi-analytiques, *I.H.E.S.*, 1965.
- [28] E. LOOIJENGA, Motivic Measures, *Séminaire Bourbaki*, exposé 874, mars 2000.
- [29] C. MCCRORY, A. PARUSIŃSKI, Complex monodromy and the topology of real algebraic sets, *Comp. Math.*, 106 (1997), 211–233.
- [30] J. MILNOR, P. ORLIK, Isolated singularities defined by weighted homogeneous polynomials, *Topology*, 9 (1970), 385–393.
- [31] T. NISHIMURA, Topological invariance of weights for weighted homogeneous singularities, *Kodai Math. J.*, 9 (1986), 188–190.
- [32] R. QUAREZ, Espace des germes d’arcs réels et série de Poincaré d’un ensemble semi-algébrique, *Ann. Inst. Fourier*, 51-1 (2001), 43–67.
- [33] O. SAEKI, Topological invariance of weights for weighted homogeneous isolated singularities in \mathbb{C}^3 , *Proc. Amer. Math. Soc.*, 103 (1988), 995–999.

- [34] B. TEISSIER, Cycles évanescents, sections planes, et conditions de Whitney, Singularités à Cargèse, Astérisque, 7 & 8 (1973), 285–362.
- [35] W. VEYS, The topological zeta function associated to a function on a normal surface germ, Topology, 38 (1999), 439–456.
- [36] S.-T. YAU, Topological types and multiplicity of isolated quasihomogeneous surface singularities, Bull. Amer. Math. Soc., 19 (1988), 447–454.
- [37] E. YOSHINAGA, M. SUZUKI, Topological types of quasihomogeneous singularities in \mathbb{C}^2 , Topology, 18 (1979), 113–116.

Manuscrit reçu le 4 novembre 2002,
accepté le 3 février 2003.

Satoshi KOIKE,
Hyogo University of Teacher Education
Department of Mathematics
942-1 Shimokume, Kato, Yashiro
Hyogo 673-1494 (Japan).
koike@sci.hyogo-u.ac.jp
&
Adam PARUSIŃSKI,
Université d'Angers
Département de Mathématiques
2 Bd. Lavoisier
49045 Angers cedex (France).
parus@tonton.univ-angers.fr