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TRANSFORMATION OF MARKOV PROCESSES BY MULTIPLICATIVE FUNCTIONALS

by Kiyosi ITÔ and Shinzo WATANABE

1. Introduction.

Let $X_t = X_t(S, P_a)$ be a Markov process with the state space S and the probability law P_a of the path starting at a and let α_t be a multiplicative functional (m.f.) of X_t. A Markov process $X_t^* = X_t^*(S, P_a^*)$ is called the *α-subprocess* of X_t if

$$(1.1) \qquad P_a^*[X_t^* \in E] = \int_{X_t \in E} \alpha_t \, dP_a \qquad (E \subset S).$$

An important example of the α-subprocess is the Doob h-process which is the subprocess of the Brownian motion with respect to the m.f.

$$(1.2) \qquad \alpha_t = \frac{h(X_t)}{h(X_0)},$$

where h is a positive harmonic function. The h-process plays an important role in the potential theory. Another important example is a diffusion process with the generator

$$(1.3) \qquad u = \tfrac{1}{2}\Delta u - ku \qquad (k \geqslant 0)$$

This is the subprocess of the Brownian motion with respect to the m.f.

$$(1.4) \qquad \alpha_t = \exp\left[\int_0^t -k(X_s)\, ds\right].$$

The transformation to get this subprocess from the Brownian motion is *killing* with the killing rate k.

The general α-subprocess was discussed by E. B. Dynkin [1] and independently by H. Kunita and T. Watanabe [4] under the natural assumption

(1.5) $$E_a(\alpha_t) \leqslant 1.$$

In this note we shall give another general method of the transformation which seems to be more probabilistic than theirs.

Let us now describe the outline of our construction of the α-subprocess.

If there exists an increasing sequence of Markov times T_n whose limit is no less than the least zero point T_α of the given m.f. α such that $\alpha_{t \wedge T_n}$ is a martingale, we call α_t *regular*. We shall define the *subregularity* and the *superregularity* of α_t by replacing martingale with submartingale and supermartingale respectively in this definition; the superregularity of α_t is equivalent to the condition (1.5). We can prove that the *factorization theorem* that any superregular m.f. is expressed as the product of a regular m.f. $\alpha_t^{(0)}$ and a decreasing m.f. $\alpha_t^{(1)}$. $\alpha_t^{(0)}$ is called the *regular factor* of α_t and $\alpha_t^{(1)}$ the *decreasing factor*.

In order to construct the α-subprocess, we shall first distort the probability law P_a of the original process as

(1.6) $$dP_a^{(0)} = \lim_{n \to \infty} \alpha_{T_n}^{(0)} \, dP_a|\mathbf{F}_{T_n},$$

$$T_n \uparrow T_\alpha, \; P_a|\mathbf{F}_n = \text{restriction of } P_a \text{ to } \mathbf{F}_{T_n}$$

to get a Markov process $X_t^{(0)}(P_a^{(0)}, S)$ which is *semi-conservative* in the sense that the life time T_Δ can be approximated strictly from below by a sequence of Markov times almost surely and *next* kill $X_t^{(0)}$ by the rate $d\alpha_t^{(1)}/\alpha_t^{(1)}$ to get the α-subprocess X_t^*; the precise meaning of (1.6) will be given in Section 4.

In Section 2 we shall prove a factorization theorem for positive supermartingales. In Section 3 we shall use this to get a similar theorem for superregular m.f.'s which will be useful in the construction of the α-subprocess in Section 4. Interesting examples are given in Section 5. Our idea can apply to the transformation by a subregular m.f. in which case we should introduce a creation instead of killing as is seen in Section 6.

We would like to express our hearty thanks to Professors M. Brelot, G. Choquet and J. Deny who organized the Colloquium of Potential Theory at University of Paris where the idea of our construction was presented by one of us, Itô, and also to Professors S. Karlin and

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2. Factorization theorem for positive supermartingales.

Let $\Omega(\mathbf{F}, \mathrm{P})$ be the basic probability measure space where $\mathbf{F}$ is complete with respect to P and suppose that we are given an increasing and right-continuous family of σ-algebras $\mathbf{F}_t \subset \mathbf{F}$, $0 \leqslant t < \infty$, each containing all null sets. A non-negative random variable T is called a *Markov time* if $\{T < t\} \in \mathbf{F}_t$ for every t. Given a Markov time T we shall define $\mathbf{F}_T$ as the system of all sets $A \in \mathbf{F}$ for which $A \cap \{T < t\} \in \mathbf{F}_t$ for every $t \geqslant 0$. $\mathbf{F}_T$ is clearly a σ-algebra complete with respect to P and it is easy to see by the right-continuity of $\mathbf{F}_t$ that $\mathbf{F}_T = \mathbf{F}_t$ for $T \equiv t$. $\mathbf{F}_T$ is clearly *strongly right-continuous* in the sense that $\bigcap_n \mathbf{F}_{T_n} = \mathbf{F}_T$ for $T_n \downarrow T$. However, we shall here assume that it is also *strongly left-continuous* i.e. that $\bigcap_n \mathbf{F}_{T_n} = \mathbf{F}_T$ for $T_n \uparrow T$.

To avoid constant repetition of qualifying phrases, we assume that T, T_1, T_2, etc., denote Markov times and that A_t, X_t, etc., are stochastic processes measurable $(\mathbf{F}_t)$ at each time point t.

By a theorem due to P. Meyer [6] we have a decomposition of Ω for a given T

$$(2.1) \qquad \Omega = \Omega_T \vee (\Omega - \Omega_T) \qquad \Omega_T \in \mathrm{F}_T$$

such that

$$(2.2) \qquad \Omega_T \underset{\text{a.s.}}{\subset} \{\forall_n \quad T_n < T\}$$

for some $T_n \uparrow T$ and that

$$(2.3) \qquad \Omega - \Omega_T \underset{\text{a.s.}}{\subset} \{\exists n \quad T_n = T\}$$

for every $T_n \uparrow T$, where « a.s. » means the inclusion modulo null sets.

Following P. Meyer we shall call a right-continuous process A_t, $t \geqslant 0$, *natural* if

$$(2.4) \qquad \{\Delta A_T > 0, T < \infty\} \underset{\text{a.s.}}{\subset} \Omega_T$$

for every T, where $\Delta A_T = A_T - A_{T-}$.

Since every supermartingale has a right-continuous version, we assume that the supermartingale in consideration is always right continuous.

Given a positive (= nonnegative) supermartingale, we set

$$(2.5) \qquad T_X = \inf(t : X_t = 0) \qquad \text{if } X_t = 0 \text{ for some } t$$
$$= \infty \qquad \text{if otherwise}$$

and denote with Ω_X the Ω_T in (2.1) for $T = T_X$. It is clear that $X_t = 0$ for $t \geqslant T_X$.

DEFINITION. — X_t, $0 \leqslant t < \infty$, is called a *local martingale* if there exists a sequence $T_n \uparrow \infty$ with $P(T_n < \infty) = 1$ such that $X_{T_n \wedge t}$, $0 \leqslant t < \infty$, is a martingale.

Remark. — It is easy to see that if X_t, $0 \leqslant t < \infty$ is a local martingale, then we can take $T'_n \uparrow \infty$ such that $X_t \wedge T'_n$ is a martingale on $[0, \infty]$.

The aim of this section is to prove the

FACTORIZATION THEOREM FOR POSITIVE SUPERMARTINGALES. — A positive (= nonnegative) supermartingale X_t with $P(T_X > 0) = 1$ is factorized as

$$(2.6) \qquad X_t = X_t^{(0)} . X_t^{(1)}$$

with a positive local martingale $X_t^{(0)}$ and a natural decreasing process $X_t^{(1)}(X_0^{(1)} = 1)$. If there are two such factorizations, then they are identical in $0 \leqslant t < T_X$.

Before proving this we shall prove some preliminary facts.

Let X_t be a positive supermartingale for which there exist a constant M and an almost surely finite Markov time T such that

$$(2.7\,a) \qquad \frac{1}{M} \leqslant X_t \leqslant M \qquad \text{for every } t < T,$$

$$(2.7\,b) \qquad X_t = X_T \qquad \text{for every } t \geqslant T.$$

Let $X_t = Y_t - A_t$ be the Meyer decomposition of X_t which is possible because X_t is a supermartingale of class D by virtue of (2.7 *a*, *b*). We shall further impose the following conditions:

$$(2.7\,c) \qquad A_t \leqslant M \qquad \text{for every } t < T.$$

Now, writing A_t^c for the continuous part of A_t and setting

$$(2.8\,a) \qquad X_t^{(0)} = X_{t \wedge T} \exp\left(\int_0^{t \wedge T} \frac{dA_u^c}{X_u}\right) . \prod_{\substack{0 \leqslant u \leqslant t \wedge T \\ \Delta A_u > 0}} \left(1 + \frac{\Delta A_u}{X_u}\right)$$

$$(2.8\,b)\quad X_t^{(1)} = \exp\left(-\int_0^{t\wedge T} \frac{dA_u^c}{X_u}\right) \prod_{\substack{0\leqslant u\leqslant t\wedge T\\ \Delta A_u>0}} \left(1+\frac{\Delta A_u}{X_u}\right)^{-1}.$$

If $t \geqslant T$ and $X_T = 0$, then the last factor $1 + \dfrac{\Delta A_T}{X_T}$ is meaningless and is to be interpreted by the following convention:

$$(2.8\,c)\qquad X_T\left(1+\frac{\Delta A_T}{X_T}\right) = X_T + \Delta A_T \qquad \text{in } (2.8\,a)$$

$$\left(1+\frac{A_T}{X_T}\right)^{-1} = 0 \qquad \text{in } (2.8\,b)$$

Then $X_t^{(i)}$, $i = 0, 1$, are well-defined by virtue of (2.7 a, b, c).

LEMMA 2.1. — $X_t = X_t^{(0)}X_t^{(1)}$ is the unique factorization of X_t into a martingale $X_t^{(0)}$ and a natural decreasing process $X_t^{(1)}$ with

$$X_t^{(i)} = X_{t\wedge T}^{(i)}\,(i = 1, 2), \qquad X_0^{(1)} = 1 \qquad \text{and} \qquad X_t^{(0)} \leqslant Z$$

for some integrable Z.

Proof. — Let $\delta = (0 = u_0 < u_1 < \cdots < u_n \cdots \to \infty)$ be a division of $[0, \infty)$ with $|\delta| \equiv \sup_i |u_i - u_{i-1}| < \infty$ and define

$$(2.9)\quad X_{t,\delta}^{(0)} = \prod_{i\leqslant m}\left(1+\frac{A_{u_i}-A_{u_{i-1}}}{X_{u_i}}\right)(X_{t\wedge T} + A_{t\wedge T} - A_{u_m})$$

where $m = m(t, T)$ is the maximum number for which $u_m < t \wedge T$. Then it is easy to see that

$$(2.10)\qquad \lim_{|\delta|\to 0} X_{t,\delta}^{(0)} = X_t^{(0)}, \qquad 0 \leqslant X_{t,\delta}^{(0)} < e^{M^2}(X_T + A_T).$$

Denoting with $\phi(x)$ a function equal to 0 or x^{-1} according as $x = 0$ or $x > 0$, and noticing

$$(2.11)\qquad X_{u_k,\delta}^{(0)} = X_{u_{k-1},\delta}^{(0)}(1 + \phi(X_{u_{k-1}})(Y_{u_k} - Y_{u_{k-1}})),$$

we can see

$$E(X_{u_k,\delta}^{(0)}|\mathbf{F}_{u_{k-1}}) = X_{u_{k-1},\delta}^{(0)}$$

and so

$$(2.12)\qquad E(X_{u_k,\delta}^{(0)}|\mathbf{F}_{u_l}) = X_{u_l,\delta}^{(0} \qquad (l < k).$$

Given any pair $s < t$, let us consider only $\delta \ni s, t$ such that $|\delta| \to 0$ and notice (2.10) to derive from (2.12)

$$\text{(2.13)} \qquad E(X_t^{(0)}|\mathbf{F}_s) = X_s^{(0)},$$

which proves that $X_t^{(0)}$ is a martingale.

Since $X_t^{(1)}$ has the same discontinuity points as A_t by the definition, $X_t^{(1)}$ is natural.

It is now easy to see that the factorization $X_t = X_t^{(0)} X_t^{(1)}$ satisfies the conditions stated in the lemma.

Now we shall prove the uniqueness.

Let $X_t = \Phi_t^{(0)} . \Phi_t^{(1)}$ be any such factorization and consider a decomposition of X_t:

$$\text{(2.14)} \qquad X_t = Y_t' - A_t'$$

$$Y_t' = X_t - \int_0^t \Phi_u^{(0)}\, d\Phi_u^{(1)}, \qquad A_t' = -\int_0^t \Phi_u^{(0)}\, d\Phi_u^{(1)}.$$

Then it is clear that A_t' is a natural increasing process. Now we shall prove that Y_t' is a martingale. Using the division

$$\delta = (0 = u_0 < u_1 < \cdots \to \infty)$$

used above, we shall define

$$\text{(2.15)} \quad Y_t^\delta = X_t - \sum_{i \leqslant m} \Phi_{u_i}^{(0)}(\Phi_{u_i}^{(1)} - \Phi_{u_{i-1}}^{(1)}) - \Phi_t(\Phi_t^{(1)} - \Phi_{u_m}^{(11)})$$

$$(u_m < t \leqslant u_{m+1}).$$

Writing $Y_{u_k}^\delta$ as

$$\text{(2.16)} \qquad Y_{u_k}^\delta = X_0 + \sum_{i=1}^k \Phi_{u_{i-1}}^{(1)}(\Phi_{u_i}^{(0)} - \Phi_{u_{i-1}}^{(0)})$$

to see

$$\text{(2.17)} \qquad E(Y_{u_k}^\delta|\mathbf{F}_u) = Y_{u_l}^\delta \qquad (l < k),$$

and noticing

$$\text{(2.18)} \qquad \lim_{|\delta|\to 0} Y_t^\delta = Y_t', \qquad |Y_t^\delta| \leqslant X_t + Z,$$

we can see

$$\text{(2.19)} \qquad E(Y_t'|\mathbf{F}_s) = Y_s' \qquad (s < t),$$

which proves that Y_t' is a martingale.

Therefore $X_t = Y'_t - A'_t$ is the Meyer decompositions of X_t, so that $Y'_t = Y_t$ and $A'_t = A_t$. Thus we have

$$(2.15)\qquad A_t = -\int_0^t \Phi_u^{(0)}\, d\Phi_u^{(1)} \qquad \text{i.e.} \qquad dA_t = -\Phi_u^{(0)}\, d\Phi_u^{(1)}$$

where dF_t means the Lebesgue-Stieltjes measure induced by F_t.

Let $A_t = A_t^c + A_t^d$ be the decomposition of A_t into the continuous part A_t^c and the discontinuous part A_t^d and $\Phi_t^{(1)} = \Phi_{t,c}^{(1)}\Phi_{t,d}^{(1)}$ be the factorization of $\Phi_t^{(1)}$ into the continuous factor $\Phi_{t,c}^{(1)}$ and the discontinuous factor $\Phi_{t,d}^{(1)}$. Then (2.15) is written as

$$(2.16)\qquad dA_t^c + dA_t^d = -\Omega_t^c\Phi_{t,d}^{(1)}\, d\Phi_{t,c}^{(1)} - \Phi_t^{(0)}\Phi_{t,c}^{(1)}\, d\Phi_{t,d}^{(1)}$$

Equating the continuous parts and the discontinuous parts of both sides of (2.16) respectively, we get

$$(2.17)\qquad dA_t^c = -\Phi_t^{(0)}\Phi_{t,d}^{(1)}\, d\Phi_{t,c}^{(1)}, \qquad dA_t^d = -\Phi_t^{(0)}\Phi_{t,c}^{(1)}\, d\Phi_{t,d}^{(1)}.$$

Consider now the case $t < T$, so that $X_t > 0$. Recalling

$$X_t = \Phi_t^{(0)}\Phi_t^{(1)} = \Phi_t^{(0)}\Phi_{t,c}^{(1)}\Phi_{t,d}^{(1)}$$

we can derive from (2.17)

$$(2.18\ a)\qquad \frac{d\Phi_{t,c}^{(1)}}{\Phi_{t,c}} = -\frac{dA_t^c}{X_t}$$

$$(2.18\ b)\qquad \frac{d\Phi_{t,d}^{(1)}}{\Phi_{t,d}} = -\frac{dA_t^d}{X_t}.$$

Since $\Phi_{t,c}^{(1)}$ is continuous and $\Phi_{0,c}^{(1)} = 1$, we get from (2.18 *a*)

$$\Phi_{t,c}^{(1)} = \exp\left(-\int_0^t \frac{dA_u^c}{X_u}\right).$$

Since $\Phi_{t,d}^{(1)}$ is purely discontinuous and $\Phi_{0,d}^{(1)} = 1$, we get from (2.18 *b*)

$$\Phi_{t,d}^{(1)} = \prod_{\substack{\Delta A_u > 0\\ 0 \leqslant u \leqslant t}} \frac{\Phi_{u,d}^{(1)}}{\Phi_{u-,d}^{(1)}} = \prod_{\substack{\Delta A_u > 0\\ 0 \leqslant u \leqslant t}} \left(1 + \frac{\Delta A_u}{X_u}\right)^{-1}.$$

Thus we have proved $\Phi_t^{(1)} = \Phi_{t,c}^{(1)}\Phi_{t,d}^{(1)} = X_t^{(1)}$ and so $\Phi_t^{(0)} = X_t^{(0)}$, as far as $t < T$.

Since $\Phi_{T-}^{(i)} = X_{T-}^{(i)}$ is now evident, it is sufficient for the proof of $\Phi_T^{(i)} = X_T^{(i)}$ to recall the special care we took when we defined $X_T^{(i)}$. If $t \geqslant T$, then $\Phi_t^{(i)} = \Phi_T^{(i)} = X_T^{(i)} = X_T^{(i)}$, which completes the proof of Lemma 1.

LEMMA 2.—A positive supermartingale X_t is decomposed uniquely as $X_t = Y_t - A_t$, where Y_t is a local martingale and A_t is a natural increasing process with $A_0 = 0$.

Proof.—Define T_n by

$$T_n = \min[\inf(t : X_t \geqslant n), n].$$

Then $X_{t \wedge T_n}$ is a supermartingale of the class (D) and has a Meyer decomposition

$$X_{t \wedge T_n} = Y_t^{(n)} - A_t^{(n)}.$$

By the uniqueness of the Meyer decomposition we have

$$Y_t^{(m)} = Y_t^{(n)} \qquad \text{and} \qquad A_t^{(m)} = A_t^{(n)}$$

for $m < n$ and $t \leqslant T_m$, so that we have a decomposition of X_t: $X_t = Y_t - A_t$ such that $Y_t = Y_t^{(n)}$ and $A_t = A_t^{(n)}$ for $t < T_n$. This decomposition of X_t satisfies the conditions in Lemma 2. The uniqueness is easy to see.

LEMMA 3.—Let X_t be a supermartingale and T a Markov time. Suppose that $X_t = X_T$ for $t \geqslant T$ and that

$$\Omega_T \cap \{T < \infty\} \underset{\text{a.s.}}{\subset} \{\Delta X_T = 0, T < \infty\}. \tag{2.19}$$

If there exists an increasing sequence T_n with $\lim T_n \geqslant T$ such that $X_{t \wedge T_n}$ is a martingale, then X_t is a local martingale.

Proof.—Let $X_t = Y_t - A_t$ be the decomposition of Lemma 2. It is clear that $A_t = 0$ for $t \leqslant T_n$ and so for $t < T$. Since $A_t = A_T$ ($t \geqslant T$) follows from $X_t = X_T$ ($t \geqslant T$), A_t is a step function with a single jump at $t = T$. Since Y_t is a local martingale, we have a sequence of Markov times $S_n \uparrow \infty$ such that $Y_{t \wedge S_n}$ is a martingale. By a theorem due to Meyer [6], we have

$$\Omega_T \cap \{T < \infty\} \underset{\text{a.s.}}{\subset} \bigcup_n \{\Delta Y_T = 0, T < S_n\} \underset{\text{a.s.}}{\subset} \{\Delta Y_{T=0}, T < \infty\},$$

and so

$$\Omega_T \cap \{T < \infty\} \underset{\text{a.s.}}{\subset} \{\Delta Y_T = 0, \Delta X_T = 0, T < \infty\} \underset{\text{a.s.}}{\subset} \{\Delta A_T = 0, T < \infty\}$$

by (2.19), while

$$\{\Delta A_T > 0, T < \infty\} \underset{\text{a.s.}}{\subset} \Omega_T$$

because A_t is natural. Thus

$$\{\Delta A_T > 0, T < \infty\} \underset{\text{a.s.}}{\subset} \{\Delta A_T = 0, T < \infty\},$$

which implies $\Delta A_T = 0$ almost surely and so $X_t \equiv Y_t$.

Now we shall come back to the

Proof of the theorem. — Let X_t be a positive supermartingale and let $X_t = Y_t - A_t$ be the decomposition of Lemma 2 and define T'_n by

$$T'_n = \inf\left(t \; : \; X_t \geqslant n \quad \text{or} \quad X_t \leqslant \frac{1}{n} \quad \text{or} \quad A_t \leqslant n \right)$$
$$= \infty$$

if there exists no such t and set $T_n = \min(T'_n, n)$. Then it is clear $T_n \uparrow T_\infty \equiv \lim T_n \geqslant T_X$.

Define $X_t^{(0)}$ by

$$X_t^{(0)} = X_t \exp\left[\int_0^t \frac{dA_u^c}{X_u}\right] \prod_{\substack{\Delta A_u > 0 \\ 0 \leqslant u \leqslant t}} \left(1 + \frac{\Delta A_u}{X_u}\right)$$

$$(A_t^c = \text{the continuous part of } A_t)$$

for $t < T_X$ and

$$X_t^{(0)} = X_{T_X}^{(0)} = \begin{cases} X_{T_X -}^{(0)} & \text{on} \quad \Omega_X \\ 0 & \text{on} \quad \Omega - \Omega_X \end{cases}$$

for $t \geqslant T_X$. Then $X_{t \wedge T_n}^{(0)}$ is a martingale by Lemma 1 and therefore a local martingale by Lemma 3.

Define $X_t^{(1)}$ by

$$X_t^{(1)} = \exp\left[-\int_0^t \frac{dA_u^{(c)}}{X_u}\right] \prod_{\substack{\Delta A_u > 0 \\ 0 \leqslant u \leqslant t}} \left(1 + \frac{\Delta B_u}{X_u}\right)^{-1}$$

for $t < T_X$ and

$$X_t^{(1)} = X_{T_X}^{(1)} = \begin{cases} 0 & \text{on} \quad \Omega_X \\ X_{T_X -}^{(1)} & \text{on} \quad \Omega - \Omega_X \end{cases}$$

if $t \geqslant T_X$. It is then clear that $X_t^{(1)}$ is natural.

Thus $X_t = X_t^{(0)} . X_t^{(1)}$ is a decomposition which satisfies all conditions of our theorem. The uniqueness follows at once from the uniqueness part of Lemma 1.

3. Factorization theorem for multiplicative functionals.

Let us recall the definition and notations on Markov processes. Let S be a locally compact Hausdorff space with a countable open base and $\bar{S} = S \vee \{\Delta\}$ the one-point compactification of S in case S is not compact. If S is already compact, Δ is adjointed to S as an isolated point. The topological σ-algebra $\mathbf{B}(S)$ on S, i.e., the least σ-algebra containing all open subsets of S is denoted by $\mathbf{B}(S)$. A function $w:[0, \infty) \to S$ is called a *path* if it satisfies the following conditions:

(3.1 *a*) $w(t) = \Delta \Rightarrow w(s) = \Delta \quad \text{for} \quad s \geqslant t,$

(3.1 *b*) $w(t)$ is right-continuous,

(3.1 *c*) *either* $\lim_{s \uparrow t-} w(s)$ exists *or* $w(t) = \Delta.$

The space of all paths is denoted by W. To emphasize the fact that $w(t)$ is a function of $w \in W$ for each t, we shall write $X_t(w)$ for $w(t)$. The *terminal time* $T_\Delta(w)$ of the path w is defined as

$$(3.2) \qquad T_\Delta(w) = \begin{cases} \inf(t : X_t(w) = \Delta) & \text{if } X_t(w) = \Delta \text{ for some } t \\ \infty & \text{if otherwise} \end{cases}$$

The shift *transformation* θ_t in W is defined by

$$(3.3) \quad \theta_t w(s) = w(s + t) \qquad \text{i.e.} \qquad X_s(\theta_t w) = X_{t+s}(w).$$

$\mathbf{B}_t$ is the least σ-algebra on W for which $X_s(w)$ is measurable for every $s \leqslant t$ and $\mathbf{B}$ denotes the lattice sum of all $\mathbf{B}_t$, $t \geqslant 0$. P_x, $x \in S$, is a system of probability measures on $W(\mathbf{B})$ such that

(3.4 *a*) $P_x(B)$ is $\mathbf{B}(S)$-measurable in x for every $B \in \mathbf{B}$

(3.4 *b*) $P_x(X_0(w) = x) = 1$

(3.4 *c*) (Markov property)

$$E_x[f(\theta_t w), B \wedge (T_\Delta > t)] = E_x[E_{X_t}(f), B \wedge (T_\Delta > t)]$$

for every $B \in \mathbf{B}_t$, for every $\mathbf{B}$-measurable f, every $t \geqslant 0$ and every $B \in \mathbf{B}_t$.

The triple (X_t, P_a, S) is called a Markov process and it is also denoted by $X_t(P_a, S)$ or simply by X_t. Given a probability measure μ

on S(**B**(S)), let P_μ be the probability measure defined by

$$P_\mu(B) = \int_S P_a(B)\, d\mu(a),$$

$B \in \mathbf{B}$. Let **N** be the class of all sets N such that $P_\mu(N) = 0$ for every μ, and $\mathbf{F}_t$ be the least σ-algebra containing $\mathbf{B}_t$ and **N**. **F** is defined similarly as the least σ-algebra containing **B** and **N**. A function $T: W \to [0, \infty)$ is called a *Markov time* if $(T < t) \in \mathbf{F}_t$ for every $t \geqslant 0$. Given a Markov time T, $\mathbf{F}_T$ is defined as

$$\text{(3.5)} \qquad \mathbf{F}_T = (B \in \mathbf{F} : (T < t) \wedge B \in \mathbf{F}_t \text{ for every } t).$$

A Markov process $X_t(P_a, S)$ is called a *standard process* if the following two conditions are satisfied:

(3.6) (strict Markov property). For any Markov time

$$E_x[f(\theta_T w), B \wedge (T < T_\Delta)] = E_x[E_{X_T}(f), B \wedge (T < T_\Delta)]$$

for every $B \in \mathbf{F}_T$.

(3.7) (quasi-left continuity before T_Δ). Given any Markov time T if T_n is a sequence of Markov times such that $T_n \uparrow T$, then

$$P_x(X_{T_n} \to X_T / T < T_\Delta) = 1$$

A standard process is called a *Hunt Process* if the following two conditions are satisfied

(3.8) (existence of left-limit).

$$P_x(\lim_{t \uparrow T-} X_t \text{ exists } | T < \infty) = 1$$

for every Markov time T,

(3.9) (quasi-left continuity). Given any Markov time T, if T_n is a sequence of Markov times such that $T_n \uparrow T$, then

$$P_x(X_{T_n} \to X_T | T < \infty) = 1.$$

For a Hunt process X_t we have

(3.10) Given any Markov time T, if T_n is a sequence of Markov time such that $T_n \uparrow T$, then $\bigvee_n \mathbf{F}_{T_n} = \mathbf{F}_T$.

Let X_t be a standard process.

DEFINITION.—$\alpha_t(w)$, $t \in (0, \infty) w \in W$, is called a *multiplicative functional* (= m.f.), if it satisfies the following conditions

(3.11) $\alpha_t(w)$ is $\mathbf{F}_t$-measurable in w for each fixed t, (3.12) Except on a subset N of W such that $P_x(N) = 0$ for every x, we have

(i) $\alpha_0(w) = 1$

(ii) $\alpha_t(w)$ is right-continuous in t and has finite left limits in $0 \leqslant t < T_\Delta$

(iii) $\alpha_t(w) = 0,\ t \geqslant T_\Delta$

(iv) $\alpha_{t+s}(w) = \alpha_t(w)\alpha_s(\theta_t w)$, for every pair $t, s \geqslant 0$.

Given a m.f. α_t, we shall denote $\inf(t : \alpha_t = 0)$ by T_α.

DEFINITION.—A m.f. α_t is called *regular* (superregular, subregular) if there exists an increasing sequence of Markov times T_n such that $\lim_n T_n \geqslant T_\alpha$ and that $\alpha_{t \wedge T_n}$ is a martingale (supermartingale, submartingale) for each P_x.

It is easy to see by Fatou's lemma on integrals that the following conditions are equivalent to each other.

a) α_t is superregular

b) α_t is a supermartingale for each P_x

c) $E_x(\alpha_t) \leqslant 1$ for every pair (t, x).

Suppose X_t be a Hunt process and α_t a superregular m.f. Since α_t is a supermartingale by (*b*), we can apply the results of Section 1 to see that there exist a local martingale $\beta_t^{(x)}$ and a natural decreasing process $\gamma_t^{(x)}(\gamma_0^{(x)} = 1)$ such that $\alpha_t = \beta_t^{(x)} . \gamma_t^{(x)}$. Recalling how the natural increasing part was constructed in the Meyer decomposition, we can get a version β_t of $\beta_t^{(x)}$ and a version γ_t of $\gamma_t^{(x)}$, both independent of x. β_t and γ_t satisfy

$$(3.13) \qquad \begin{aligned} \beta_{t+s}(w) &= \beta_t(w)\beta_s(\theta_t w) \\ \gamma_{t+s}(w) &= \gamma_t(w)\gamma_s(\theta_t w) \end{aligned} \qquad t + s < T_\alpha.$$

We shall discuss the details in a separate paper.

Now we shall define $\alpha_t^{(0)}$ and $\alpha_t^{(1)}$ by

$$(3.14) \qquad \alpha_t^{(0)} = \beta_t e_{[T_\alpha > t]}, \qquad \alpha_t^{(1)} = \gamma_t e_{[T_\alpha > t]}$$

where $e_{[T_\alpha > t]}$ takes 1 or 0 according as $T_\alpha > t$ or $T_\alpha \leqslant t$. It is easy to see that $T_{\alpha^{(0)}} = T_{\alpha^{(1)}} = T_\alpha$, $\alpha_t^{(0)}$ is regular. X_t is continuous at every jump point of $\alpha_t^{(1)}$ in $[0, T_\alpha)$ since γ_t is natural. Thus we have

FACTORIZATION THEOREM FOR M.F.—A m.f. α_t of a Hunt process $X_t(P_a, S)$ such that $E_x[\alpha_t] \leqslant 1$ for every (t, x) can be factorized uniquely as

$$\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$$

with a regular m.f. $\alpha_t^{(0)}$ and a decreasing m.f. $\alpha_t^{(1)}$ such that

$$T_\alpha = T_{\alpha^{(0)}} = T_{\alpha^{(1)}}$$

and that $\alpha_t^{(1)}$ has no common discontinuity points with X_t on $[0, T_\alpha)$.

4. Construction of the α-subprocess of a Hunt process X_t.

Let $X_t(P_a, S)$ be a Hunt process and α_t be a multiplicative functional for X_t. We shall construct the α-subprocess of X_t.

Regular case. — Let α_t be regular. Then we have a sequence of Markov times $T_n \uparrow T_\alpha$ such that for each n, $\alpha_{T_n \wedge t}$ be a martingale on $[0, \infty]$ for every P_x. We shall prove that *there exists one and only one standard process* $X_t^{(0)}(P_a^{(0)}, S)$ that satisfies

(4.1) $$P_x^{(0)}(B) = E_x[\alpha_{T_n}, B] \qquad B \in \mathbf{F}_{T_n}, n = 1, 2, \ldots$$

and

(4.2) $$P_x^{(0)}(T_\Delta = \lim T_n) = 1.$$

This process is semi-conservative, in fact, we have

(4.3) $$P_x^{(0)}(T_n < T_\Delta) = 1, \qquad n = 1, 2, \ldots.$$

(4.1) and (4.2) are written symbolically as

(4.4) $$dP_x = \lim_{n \to \infty} \alpha_{T_n} \, dP_a | \mathbf{F}_{T_n}.$$

Let $\alpha_t = \beta_t \cdot \gamma_t$ be the factorization that we introduced in Section 3. Since α_t is regular, it is also the regular part of the factorization of the m.f. α_t, so that we have

(4.5) $$\alpha_t = \beta_t e_{[T_\alpha > t]} = \beta_t \qquad \text{for} \qquad t < T_\alpha.$$

Setting

(4.6) $$\tau = \min[\inf(t : \beta_t > 2), 1],$$

we shall obtain

(4.7) $$E_x[\beta_\tau | \mathbf{F}_t] = \beta_{\tau \wedge t};$$

in fact, since β_t is a local martingale, there exists a sequence of Markov times $S_n \uparrow \infty$ such that $\beta_{S_n \wedge t}$ is a martingale on $0 \leqslant t \leqslant \infty$ and so we get

(4.8) $$E_x[\beta_{S_n \wedge \tau} | \mathbf{F}_t] = \beta_{S_n \wedge \tau \wedge t},$$

but since $\beta_{t\wedge\tau} \leqslant 2 + \beta_\tau$ follows from (4.6), we get (4.7) by letting n tend to ∞ in (4.8).

Define $\tau_n, n = 1, 2, \ldots$ recursively by

$$(4.9) \qquad \tau_1 = \tau, \qquad \tau_2(w) = \tau_{n-1}(w) + \tau(\theta_{\alpha_{n-1}(w)}w)$$

and set

$$(4.10) \qquad \begin{aligned} \tilde{\beta}_t &= \beta_t & (0 \leqslant t \leqslant \tau_1) \\ &= \beta_{\tau_1} \cdot \beta_{t-\tau_1}(\theta_{\tau_1}w) & (\tau_1 \leqslant t \leqslant \tau_2) \\ &= \beta_{\tau_2} \cdot \beta_{t-\tau_2}(\theta_{\tau_2}w) & (\tau_2 \leqslant t \leqslant \tau_3) \\ & \textit{etc.} \end{aligned}$$

It is clear that

$$(4.11) \qquad \tilde{\beta}_t = \alpha_t \qquad \text{for} \qquad t < T_\alpha.$$

Using (4.9), we have

$$(4.12) \qquad E_x[\tilde{\beta}_{\tau_n} | \mathbf{F}_s] = \tilde{\beta}_{\tau_n \wedge s}.$$

It is easy to see that $q(x, B) = E_x[\beta_\tau, B]$ is a probability measure in $B \in \mathbf{F}_\tau$ for each x by (4.7) ($t = 0$) and measurable in x for each $B \in \mathbf{F}_\tau$, so that $p(w, B) = q(X_{\tau(w)}(w, B)$ is also a probability measure in $B \in \mathbf{F}_\tau$ for each x and $\mathbf{F}_\tau$-measurable in ω for each $B \in \mathbf{F}_\tau$. Using Ionescu Tulcea's theorem [2], we can define a probability measure on the direct product space $\Omega(\mathbf{B}_\Omega) = W(\mathbf{B}) \times \mathbf{W}(\mathbf{B}) \times \ldots$ such that

$$P(\omega \in \Omega : \omega_1 \in B_1, \omega_2 \in B_2, \ldots, \omega_n \in B_n)$$
$$= \int_{B_1} \cdots \int_{B_n} q(x, dw_1) p(w_1, dw_2) \ldots p(w_{n-1}, dw_n),$$

where ω_i denote the n-th component of $\omega \in \Omega$.

Define $\xi_t(\omega)$ by

$$(4.13) \qquad \begin{aligned} \xi_t &= X_t(\omega_1) & 0 \leqslant t \leqslant \tau(\omega_1), \\ &= X_{\tau(\omega_i)+t}(\omega_2) & 0 < t \leqslant \tau(\omega_2), \\ &= X_{\tau(\omega_1)+\tau(\omega_2)+t}(\omega_3) & 0 < t \leqslant \tau(\omega_3), \\ & \ldots \\ &= \Delta & t \geqslant \sum_i \tau(\omega_i), \end{aligned}$$

denote by $\tilde{P}_x^{(0)}$ the probability law (on $W(\mathbf{B})$) of the stochastic process ξ_t, $0 \leqslant t < \infty$, and define

$$\text{(4.14)} \qquad \begin{aligned} \xi_t(w) &= X_t(w) \qquad t < T_\alpha \\ &= \Delta \qquad t \geqslant T_\alpha \end{aligned}$$

on the probability space $W(\mathbf{B}, P_x^{(0)})$. Using the properties of α_t and β_t and noticing (4.11) and (4.12), we can prove that the probability law $P_x^{(0)}$ of the stochastic process $\xi_t, 0 \leqslant t < \infty$ satisfies (4.1) and (4.2). (4.3) follows from

$$P_x^{(0)}(T_n < T_\Delta) = E_x(\alpha_{T_n}) = 1.$$

Decreasing case. — In this case the construction of the α-process is the usual terminating procedure. Consider a probability measure space $\Omega = W \times [0, \infty]$ associated with the probability measure $\tilde{P}_x^{(1)}$ such that

$$\text{(4.15)} \qquad \tilde{P}_x^{(1)}([s, \infty] \times B) = \int_B \alpha_s(w)\, dP(w), \qquad B \in \mathbf{B},$$

and define a stochastic process $\xi_t(\omega), 0 \leqslant t < \infty$, on this probability measure space by

$$\text{(4.16)} \qquad \xi_t(\omega) = \begin{cases} X_t(w) & (t < s) \\ \Delta & (t \geqslant s) \end{cases}$$

for $\omega = (w, s)$. Let $P_x^{(1)}$ be the probability law of the process $\xi_t(\omega)$, $0 \leqslant t < \infty$. Then $X_t(P_x^{(1)}, S)$ is the α-subprocess.

General case. — Let $\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$ be the factorization of α_t into the regular part $\alpha_t^{(0)}$ and the decreasing part $\alpha_t^{(1)}$. The α-subprocess can be constructed by the superposition of the transformations by $\alpha_t^{(0)}$ and by $\alpha_t^{(1)}$ in this order, each having been explained above.

Remark. — It is to be noted that the α-subprocess is not always a Hunt process but a standard process in general, even if the original one is a Hunt process. In order to construct the α-subprocess of a standard process as Dynkin and Kunita-Watanabe did, we should overcome some technical difficulties. We would like to discuss this in a separate paper.

5. Examples.

a) Let the original process be the Brownian motion in the n-dimensional domain D terminated at the boundary and let α_t be defined as

$$(5.1) \qquad \alpha_t = \begin{cases} u(X_t)/u(X_0) & t < T_\Delta \\ 0 & t \geqslant T_\Delta, \end{cases}$$

where u is a positive superharmonic function of class C^2 in D. Note that T_Δ is nothing but the exit time from D. In this case the regular and decreasing parts of α_t are expressed in $t < T_\Delta$ as

$$(5.2\,a) \qquad \alpha_t^{(0)} = \exp\left[\int_0^t \frac{\operatorname{grad} u(X_s)}{u(X_s)}\, dX_s - \frac{1}{2}\int_0^t \frac{|\operatorname{grad} u|^2(X_s)}{u^2(X_s)}\, ds\right]$$

$$(5.2\,b) \qquad \alpha_t^{(1)} = \exp\left[\frac{1}{2}\int_0^t \frac{\Delta u(X_s)}{u(X_s)}\, ds\right],$$

since a formula of stochastic integrals shows

$$(5.3) \qquad \log \alpha_t = \log u(X_t) - \log u(X_0)$$

$$= \int_0^t \frac{\operatorname{grad} u(X_s)}{u(X_s)}\, dX_s + \frac{1}{2}\int_0^t \left(- \frac{|\operatorname{grad} u|^2(X_s)}{u(X_2)^2} + \frac{u(X_s)}{u(X_s)}\right) ds.$$

In particular, α_t is regular if and only if u is harmonic.

b) Let $\tilde{X}_t(\tilde{P}_a, S)$ be a conservative Hunt process and $X_t(P_a, S)$ be the subprocess of X_t by the multiplicative functional:

$$(5.4) \qquad \tilde{\alpha}_t = \exp\left(-\int_0^t f(\tilde{X}_s)\, ds\right).$$

Then

$$(5.5) \qquad \alpha_t = e_{[T_\Delta > t]}$$

is a multiplicative functional of X_t, whose regular and decreasing parts are

$$(5.6\,a) \qquad \alpha_t^{(0)} = \alpha_t \,.\, \exp\left(\int_0^t f(X_s)\, ds\right)$$

$$\alpha_t^{(1)} = \alpha_t \exp\left(-\int_0^t f(X_s)\, ds\right). \tag{5.6 b}$$

To prove this, observe

$$\begin{aligned} E_x(\alpha_t - \alpha_0) &= \tilde{E}_x\left(\exp\left(-\int_0^t f(\tilde{X}_s)\, ds\right)\right) - 1 \\ &= -\tilde{E}_x\left(\int_0^t \exp\left(-\int_0^s f(\tilde{X}_u)\, du\right) f(\tilde{X}_s)\, ds\right) \\ &= -E_x\left(\int_0^t \alpha_s \,.\, f(X_s)\, ds\right) \end{aligned}$$

to see that the increasing process A_t appearing in the Meyer decomposition of α_t is $\int_0^t \alpha_s f(X_s)\, ds$. It is now easy to verify (5.6 *a*, *b*) by

$$\alpha_t^{(1)} = \exp\left(-\int_0^t \frac{dA_s}{\alpha_s}\right) = \exp\left(\int_0^t f(X_s)\, ds\right).$$

The $\alpha^{(0)}$-subprocess of X_t is exactly the process $\tilde{X}_t$.

c) In the construction discussed in Section 4, the $\alpha^{(0)}$-subprocess was semi-conservative. We shall prove that *it is conservative if* $E_x(\alpha_t)$ *tends to* 1 *uniformly in x as t tends to* 0.

Let $P_x^{(0)}$ be the probability law of the $\alpha^{(0)}$-subprocess. Then

$$P_x^{(0)}(T_\Delta > t) = E_x[\alpha_t^{(0)}] \geqslant E_x[\alpha_t].$$

As we saw in Section 4, we have $T_n \uparrow T$ such that

$$P_x^{(0)}(X_{T_n} \in S) = P_x^{(0)}(T_n < T_\Delta) = 1.$$

To complete the proof, we need only observe

$$\begin{aligned} P_x^{(0)}(T_\Delta < \infty) &\doteq \lim_{n\to\infty} P_x^{(0)}(T_\Delta < T_n + t) \\ &= \lim_{n\to\infty} E_x^{(0)}[P_{xT_n}^{(0)}(T_\Delta < t)] \\ &\leqslant 1 - \inf_{x\in S} E_x(\alpha_t) \to 0 \qquad (t \to 0). \end{aligned}$$

d) Using the results of (*c*) we shall prove that *a Hunt process on a compact state space* S *whose semi-group transforms* C(S) *into* C(S) *can be obtained from a conservative Hunt process by terminating procedure.*

Since $P_x(T_\Delta > t) = T_t 1(x)$ is continuous in $x \in S$ and $P_x(T_\Delta > t) \to 1$ $(t \to 0)$, it is clear by Dini's theorem that the convergence is uniform, so that the m.f. $\alpha_t = e_{(T_\Delta > t)}$ satisfies the condition of (c). Let $\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$ be the factorization of α_t. Then X_t is obtained by terminating procedure from its $\alpha^{(0)}$-subprocess which is conservative by (c).

Remark. — V. A. Volkonsky [8] discussed the same problem by a different method.

6. Subregular multiplicative functionals and creation.

Let α_t be a subregular multiplicative functional of a Hunt process $X_t(P_a, S)$. Under reasonable conditions α_t can be factorized as $\alpha_t = \alpha_t^{(0)}\alpha_t^{(1)}$ with a regular $\alpha_t^{(0)}$ and an increasing $\alpha_t^{(1)}$. Construct first the $\alpha^{(0)}$-subprocess $X_t^{(0)}$ which is semi-conservative and then the *branching process* of the particle subject to the same probability law as $X_t^{(0)}$ except the particle will create a new particle of the same probabilistic character by the rate $d\alpha_t^{(1)}/\alpha_t^{(1)}$, each performing the same random motion and creation. This branching process is called the α-subprocess of X_t, because $E(\alpha_t, X_t \in E)$ *is the expected number of the particles in* $E(\in S)$ *at time t*. Several interesting results concerning creation are found in [3] and [7], in case the original process is a Brownian motion.

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