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# SEMI-INFINITE COHOMOLOGY AND SUPERCONFORMAL ALGEBRAS

by Elena POLETAEVA

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## 1. Introduction.

B. Feigin and E. Frenkel have introduced a semi-infinite analogue of the Weil complex based on the space

$$(1.1) \quad W^{\frac{\infty}{2}+*}(\mathfrak{g}) = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}).$$

In their construction  $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$  is a graded Lie algebra,  $S^{\frac{\infty}{2}+*}(\mathfrak{g})$  and  $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$  are some semi-infinite analogues of the symmetric and exterior power modules, [FF]. As in the classical case, two differentials,  $d$  and  $h$ , are defined on  $W^{\frac{\infty}{2}+*}(\mathfrak{g})$ . They are analogous to the differential in Lie algebra (co)homology and the Koszul differential, respectively. The semi-infinite Weil complex

$$(1.2) \quad \{W^{\frac{\infty}{2}+*}(\mathfrak{g}), d + h\}$$

is acyclic similarly to the classical Weil complex. The cohomology of the complex

$$(1.3) \quad \{W^{\frac{\infty}{2}+*}(\mathfrak{g}), d\}$$

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is called the *semi-infinite cohomology* of  $\mathfrak{g}$  with coefficients in its “adjoint semi-infinite symmetric powers”  $H^{\frac{\infty}{2}+*}(\mathfrak{g}, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ . One can also define the *relative semi-infinite Weil complex*  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\mathfrak{g})$  (relatively  $\mathfrak{g}_0$ ), and the *relative semi-infinite cohomology*  $H^{\frac{\infty}{2}+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$ , [FF].

E. Getzler has shown that the semi-infinite Weil complex of the Virasoro algebra admits an action of the  $N = 2$  *superconformal algebra*, [G].

Recall that a *superconformal algebra* (SCA) is a simple complex Lie superalgebra  $\mathfrak{s}$ , such that it contains the centerless Virasoro algebra (i.e. the Witt algebra)  $\text{Witt} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n$  as a subalgebra, and has growth 1. The  $\mathbb{Z}$ -graded superconformal algebras are ones for which  $\text{ad}L_0$  is diagonalizable with finite-dimensional eigenspaces, [KL]:

$$(1.4) \quad \mathfrak{s} = \bigoplus_j \mathfrak{s}_j, \mathfrak{s}_j = \{x \in \mathfrak{s} \mid [L_0, x] = jx\}.$$

In this work we consider the semi-infinite Weil complex constructed for the next natural (after the Virasoro algebra) class of graded Lie algebras: the loop algebras of the complex finite-dimensional Lie algebras. The action of the Virasoro algebra on such complex is ensured by the fact that it has a structure of a vertex operator superalgebra (see [Ak]).

Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra, and  $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  be the corresponding loop algebra. We obtain a representation of the  $N = 2$  SCA in the semi-infinite Weil complex  $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  and in the semi-infinite cohomology  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$  with central charge  $3\dim \mathfrak{g}$ . We extend the representation of the  $N = 2$  SCA in  $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  to a representation of the one-parameter family  $\hat{S}'(2, \alpha)$  of deformations of the  $N = 4$  SCA (see [Ad] and [KL]). In the case, when  $\mathfrak{g}$  is endowed with a non-degenerate invariant symmetric bilinear form, we obtain a representation of  $\hat{S}'(2, 0)$  in  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ . Finally, there exists a representation of a central extension of the Lie superalgebra of all derivations of  $S'(2, 0)$  in the relative semi-infinite cohomology  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ .

It was shown in [FGZ] that the cohomology of the relative semi-infinite complex  $C_{\infty}^*(\mathfrak{l}, \mathfrak{l}_0, V)$ , where  $\mathfrak{l}$  is a complex graded Lie algebra, and  $V$  is a graded Hermitian  $\mathfrak{l}$ -module, has (under certain conditions) a structure analogous to that of the de Rham cohomology in Kähler geometry.

Recall that given a compact Kähler manifold  $M$ , there exists a number of classical operators on the space of differential forms on  $M$ , such as the differentials  $\partial, \bar{\partial}, d, d_c$ , their corresponding adjoint operators and the associated Laplacians (see [GH]). There also exists an action of  $\mathfrak{sl}(2)$  on

$H^*(M)$  according to the Lefschetz theorem. All these operators satisfy a series of identities known as Hodge identities, [GH]. Naturally, the classical operators form a finite-dimensional Lie superalgebra.

We show that given a complex finite-dimensional Lie algebra  $\mathfrak{g}$  endowed with a non-degenerate invariant symmetric bilinear form, there exist the analogues of the classical operators on the complex  $W_{\text{rel}}^{\infty+*}(\tilde{\mathfrak{g}})$ . We prove that the exterior derivations of  $S'(2, 0)$  form an  $\mathfrak{sl}(2)$ , and observe that they define an  $\mathfrak{sl}(2)$ -module structure on  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ , which is the analogue of the  $\mathfrak{sl}(2)$ -module structure on the de Rham cohomology in Kähler geometry.

The action of  $\hat{S}'(2, 0)$  provides  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$  with eight series of quadratic operators. In particular, they include the semi-infinite Koszul differential  $h$ , and the semi-infinite analogue of the homotopy operator (cf. [Fu]). We prove that the degree zero part of the  $\mathbb{Z}$ -grading of  $S'(2, 0)$  defined by the element  $L_0 \in \text{Witt}$ , is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

It would be interesting to interpret the superconformal algebra  $S'(2, 0)$  as “affinization” of the classical operators in the case of an infinite-dimensional manifold.

This work is partly based on [P1]-[P3].

## 2. Semi-infinite Weil complex.

The semi-infinite Weil complex of a graded Lie algebra was introduced by B. Feigin and E. Frenkel in [FF]. Recall the necessary definitions. More generally, let  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  be a graded vector space over  $\mathbb{C}$ , such that  $\dim V_n < \infty$ . Let  $V' = \bigoplus_{n \in \mathbb{Z}} V'_n$  be the restricted dual of  $V$ . The linear space  $V \oplus V'$  carries non-degenerate skew-symmetric and symmetric bilinear forms:  $(\cdot, \cdot)$  and  $\{\cdot, \cdot\}$ . Let  $H(V)$  and  $C(V)$  be the quotients of the tensor algebra  $T^*(V \oplus V')$  by the ideals generated by the elements of the form  $xy - yx - (x, y)$  and  $xy + yx - \{x, y\}$ , respectively, where  $x, y \in V \oplus V'$ . We fix  $K \in \mathbb{Z}$ . Let  $V = V_+ \oplus V_-$  be the corresponding polarization of  $V$ :  $V_+ = \bigoplus_{n > K} V_n$ ,  $V_- = \bigoplus_{n \leq K} V_n$ .

The symmetric algebra  $S^*(V_+ \oplus V'_-)$  is a subalgebra of  $H(V)$  and the exterior algebra  $\Lambda^*(V_+ \oplus V'_-)$  is a subalgebra of  $C(V)$ . Let  $S^{\frac{\infty}{2}+*}(V)$ ,  $\Lambda^{\frac{\infty}{2}+*}(V)$  be the representations of  $H(V)$  and  $C(V)$  induced from the trivial representations  $\langle 1_S \rangle$  and  $\langle 1_\Lambda \rangle$  of  $S^*(V_+ \oplus V'_-)$  and of  $\Lambda^*(V_+ \oplus V'_-)$ ,

respectively. Thus we obtain some semi-infinite analogues of symmetric and exterior power modules. Denote the actions of  $H(V)$  and  $C(V)$  on these modules by  $\beta(x), \gamma(x')$  and  $\tau(x), \varepsilon(x')$ , respectively, for  $x \in V, x' \in V'$ . Notice that each element of  $S^{\frac{\infty}{2}+*}(V)$  and of  $\Lambda^{\frac{\infty}{2}+*}(V)$  is a finite linear combination of the monomials of the type  $\gamma(x'_1) \dots \gamma(x'_k)\beta(y_1) \dots \beta(y_m)\mathbf{1}_S$  and of the type  $\varepsilon(x'_1) \dots \varepsilon(x'_k)\tau(y_1) \dots \tau(y_m)\mathbf{1}_\Lambda$ , respectively, where  $x'_1, \dots, x'_k \in V'_+, y_1, \dots, y_m \in V_-$ . Let  $\text{Deg}\varepsilon(x') = \text{Deg}\gamma(x') = 1$ , and  $\text{Deg}\tau(x) = \text{Deg}\beta(x) = -1$ . Correspondingly, we obtain  $\mathbb{Z}$ -gradings on the spaces of semi-infinite power modules:  $S^{\frac{\infty}{2}+*}(V) = \bigoplus_{i \in \mathbb{Z}} S^{\frac{\infty}{2}+i}(V), \Lambda^{\frac{\infty}{2}+*}(V) = \bigoplus_{i \in \mathbb{Z}} \Lambda^{\frac{\infty}{2}+i}(V)$ .

Let  $\{e_i\}_{i \in \mathbb{Z}}$  be a homogeneous basis of  $V$  so that if  $i \in \mathbb{Z}$ , then  $e_i \in V_n$  for some  $n \in \mathbb{Z}$ , and if  $e_i \in V_n$ , then  $e_{i+1} \in V_n$  or  $e_{i+1} \in V_{n+1}$ . Let  $\{e'_i\}_{i \in \mathbb{Z}}$  be the dual basis. Let  $i_0 \in \mathbb{Z}$  be such that  $e_{i_0} \in V_K$  and  $e_{i_0+1} \in V_{K+1}$ .

Notice that one can think of  $\Lambda^{\frac{\infty}{2}+*}(V)$  as the vector space spanned by the elements  $w = e'_{i_1} \wedge e'_{i_2} \wedge \dots$  such that there exists  $N(w) \in \mathbb{Z}$  such that  $i_{n+1} = i_n - 1$  for  $n > N(w)$ . Then  $\mathbf{1}_\Lambda = e'_{i_0} \wedge e'_{i_0-1} \wedge \dots$  is a vacuum vector in this space. The actions of  $\varepsilon(x'), \tau(x)$  are, respectively, the exterior multiplication and contraction in the space of semi-infinite exterior products.

Let  $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$  be a graded Lie algebra over  $\mathbb{C}$ , such that  $\dim \mathfrak{g}_n < \infty$ . Let  $\phi$  be a representation of  $\mathfrak{g}$  in  $V$  so that

$$(2.1) \quad \phi(\mathfrak{g}_n)V_k \subset V_{k+n}.$$

One can define the projective representations  $\rho$  and  $\pi$  of  $\mathfrak{g}$  in  $\Lambda^{\frac{\infty}{2}+*}(V)$  and  $S^{\frac{\infty}{2}+*}(V)$ , respectively

$$(2.2) \quad \rho(x) = \sum_{i \in \mathbb{Z}} : \tau(\phi(x)e_i)\varepsilon(e'_i) :,$$

$$(2.3) \quad \pi(x) = \sum_{i \in \mathbb{Z}} : \beta(\phi(x)e_i)\gamma(e'_i) :,$$

where  $x \in \mathfrak{g}$ , and where the double colons  $: \quad :$  denote a normal ordering operation:

$$(2.4) \quad \begin{aligned} : \tau(e_j)\varepsilon(e'_i) : &:= \begin{cases} \tau(e_j)\varepsilon(e'_i) & \text{if } i \leq i_0 \\ -\varepsilon(e'_i)\tau(e_j) & \text{if } i > i_0 \end{cases}, \\ : \beta(e_j)\gamma(e'_i) : &:= \begin{cases} \beta(e_j)\gamma(e'_i) & \text{if } i \leq i_0 \\ \gamma(e'_i)\beta(e_j) & \text{if } i > i_0 \end{cases}. \end{aligned}$$

Thus

$$(2.5) \quad \rho(x)\mathbf{1}_\Lambda = \pi(x)\mathbf{1}_S = 0 \text{ for } x \in \mathfrak{g}_0$$

and

$$(2.6) \quad \begin{aligned} [\rho(x), \rho(y)] &= \rho([x, y]) + c_\Lambda(x, y), \\ [\pi(x), \pi(y)] &= \pi([x, y]) + c_S(x, y), \end{aligned}$$

where  $x, y \in \mathfrak{g}$  and  $c_\Lambda, c_S$  are 2-cocycles. Notice that  $c_\Lambda = -c_S$ . Let

$$(2.7) \quad W^{\infty+*}(V) = S^{\infty+*}(V) \otimes \Lambda^{\infty+*}(V).$$

Since the cocycles corresponding to the projective representations cancel, the representation  $\theta(x) = \rho(x) + \pi(x)$  of  $\mathfrak{g}$  in  $W^{\infty+*}(V)$  is well-defined. We define a  $\mathbb{Z}$ -grading on  $W^{\infty+*}(V)$  setting

$$(2.8) \quad W^{\infty+i}(V) = \bigoplus_{2l+j=i} S^{\infty+l}(V) \otimes \Lambda^{\infty+j}(V).$$

Let  $V = \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$  and  $\phi$  be the adjoint representation of  $\mathfrak{g}$ . We define two differentials on the space  $W^{\infty+*}(\mathfrak{g})$ :

$$(2.9) \quad \begin{aligned} d &= \sum_{i < j} : \tau([e_i, e_j])\varepsilon(e'_j)\varepsilon(e'_i) : + \sum_{i, j} : \beta([e_j, e_i])\gamma(e'_i)\varepsilon(e'_j) :, \\ \mathfrak{h} &= \sum_i \gamma(e'_i)\tau(e_i). \end{aligned}$$

We obtain the *semi-infinite Weil complex*

$$(2.10) \quad \{W^{\infty+*}(\mathfrak{g}), d + \mathfrak{h}\}.$$

The differential  $d$  is the analogue of the classical differential for the Lie algebra (co)homology, and  $\mathfrak{h}$  is the analogue of the Koszul differential. Notice that

$$(2.11) \quad d^2 = 0, \mathfrak{h}^2 = 0, [d, \mathfrak{h}] = 0, (d + \mathfrak{h})^2 = 0.$$

Notice also that if  $\mathfrak{g}$  is a finite-dimensional Lie algebra, then applying the definitions given above to the polarization  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ , where  $\mathfrak{g}_+ = \mathfrak{g}$ ,  $\mathfrak{g}_- = 0$ , we obtain the classical Weil complex.

As in the case of the classical Weil complex, one can construct two filtrations,  $F_1^*$  and  $F_2^*$ , on  $W^{\infty+*}(\mathfrak{g})$ :

$$(2.12) \quad F_1^p = \bigoplus_{l+j \geq p} S^{\infty+l}(\mathfrak{g}) \otimes \Lambda^{\infty+j}(\mathfrak{g}), \quad F_2^p = \bigoplus_{2l \geq p} S^{\infty+l}(\mathfrak{g}) \otimes \Lambda^{\infty+*}(\mathfrak{g}).$$

For filtration  $F_1^*$  the complex is acyclic, the second term of the spectral sequence associated to filtration  $F_2^*$  is the *semi-infinite cohomology* of Lie algebra  $\mathfrak{g}$  with coefficients in its “adjoint semi-infinite symmetric powers”  $H^{\infty+*}(\mathfrak{g}, S^{\infty+*}(\mathfrak{g}))$  (see [FF]). Let

$$(2.13) \quad W_{\text{rel}}^{\infty+*}(V) = \{w \in W^{\infty+*}(V) \mid \tau(x)w = 0 \text{ for all } x \in V_0, \theta(x)w = 0 \text{ for all } x \in \mathfrak{g}_0\}.$$

The differential  $d$  preserves the space  $W_{\text{rel}}^{\infty+*}(\mathfrak{g})$  since

$$(2.14) \quad [d, \tau(x)] = d\tau(x) + \tau(x)d = \theta(x),$$

and

$$(2.15) \quad [d, \theta(x)] = 0,$$

for any  $x \in \mathfrak{g}$ . The complex  $\{W_{\text{rel}}^{\infty+*}(\mathfrak{g}), d\}$  is called the *relative semi-infinite Weil complex*. Its cohomology is called the *relative semi-infinite cohomology*  $H^{\infty+*}(\mathfrak{g}, \mathfrak{g}_0, S^{\infty+*}(\mathfrak{g}))$ .

We fix  $K = 0$  from this point on. Correspondingly,  $V = V_+ \oplus V_-$ , where  $V_+ = \oplus_{n>0} V_n$ ,  $V_- = \oplus_{n\leq 0} V_n$ .

### 3. The $N = 2$ superconformal algebra.

Recall that the  $N = 2$  SCA is spanned by the Virasoro generators  $\mathcal{L}_n$ , the Heisenberg generators  $H_n$ , two fermionic fields  $G_r^\pm$ , and a central element  $C$ , where  $n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2$ , and where the non-vanishing commutation relations are as follows, [FST]:

$$(3.1) \quad \begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n - m)\mathcal{L}_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n,-m}, \\ [\mathcal{L}_n, H_m] &= -mH_{n+m}, [\mathcal{L}_n, G_r^\pm] = \left(\frac{n}{2} - r\right) G_{n+r}^\pm, \\ [G_r^+, G_s^-] &= 2\mathcal{L}_{r+s} + (r - s)H_{r+s} + \frac{C}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \\ [H_n, H_m] &= \frac{C}{3}n\delta_{n,-m}, [H_n, G_r^\pm] = \pm G_{n+r}^\pm. \end{aligned}$$

Let  $\text{Witt} = \oplus_{i \in \mathbb{Z}} CL_i$  be the Witt algebra:

$$(3.2) \quad [L_i, L_j] = (i - j)L_{i+j}.$$

Let  $\lambda, \mu \in \mathbb{C}$ . Let  $\mathcal{F}_{\lambda, \mu} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}u_m$  be a module over Witt defined as follows:

$$(3.3) \quad \phi(L_n)u_m = (-m + \mu - (n - 1)\lambda)u_{n+m}.$$

*Remark 3.1.* — The module  $\mathcal{F}_{\lambda, \mu} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}u_m$  is isomorphic to the module  $\mathcal{F}_{-\lambda, \mu+1} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}f_j$  over the Witt algebra defined in [Fu]. The isomorphism is given by the correspondence  $u_m \leftrightarrow f_{-m-1}$ .

**THEOREM 3.1.** — *The space  $W^{\infty+*}(\mathcal{F}_{\lambda, \mu})$  is a module over the  $N = 2$  SCA with central charge  $3 - 6\lambda$ .*

*Proof.* — Set

$$(3.4) \quad \mathfrak{h}_n = \frac{1}{\sqrt{2}}G_{n-\frac{1}{2}}^+, \mathfrak{p}_n = \frac{1}{\sqrt{2}}G_{n+\frac{1}{2}}^-.$$

We define a representation of Witt in  $W^{\infty+*}(\mathcal{F}_{\lambda, \mu})$  as follows:

$$(3.5) \quad \theta(L_n) = \sum_{m \in \mathbb{Z}} (-m + \mu - n\lambda + \lambda) (\tau(u_{m+n})\varepsilon(u'_m) : + : \beta(u_{m+n})\gamma(u'_m) :).$$

Let us extend  $\theta$  to a representation of the  $N = 2$  SCA in  $W^{\infty+*}(\mathcal{F}_{\lambda, \mu})$ :

$$(3.6) \quad \begin{aligned} \theta(H_n) &= \lambda \sum_{m \in \mathbb{Z}} : \tau(u_m)\varepsilon(u'_{m+n}) : \\ &\quad + (\lambda - 1) \sum_{m \in \mathbb{Z}} : \beta(u_m)\gamma(u'_{m+n}) : + \mu\delta_{n,0}, \\ \theta(\mathfrak{h}_n) &= \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n})\tau(u_m), \\ \theta(\mathfrak{p}_n) &= \sum_{m \in \mathbb{Z}} (m - \mu - (n + 1)\lambda)\beta(u_{m-n})\varepsilon(u'_m), \\ \theta(\mathcal{L}_n) &= -\theta(L_{-n}) + \frac{n + 1}{2}\theta(H_n). \end{aligned}$$

We calculate the central charge by checking the commutation relations on the vacuum vector  $\mathbf{1} = \mathbf{1}_S \otimes \mathbf{1}_\Lambda$ . Let  $n > 0$ . Then

$$(3.7) \quad \begin{aligned} \theta([H_n, H_{-n}])\mathbf{1} &= -\theta(H_{-n})\theta(H_n)\mathbf{1} \\ &= -\theta(H_{-n}) \left( \lambda \sum_{m=1-n}^0 \tau(u_m)\varepsilon(u'_{m+n}) \right) \end{aligned}$$

$$\begin{aligned}
 & + (\lambda - 1) \sum_{m=1-n}^0 \beta(u_m)\gamma(u'_{m+n}) \mathbf{1} \\
 = & -\lambda^2 \sum_{m=1-n}^0 \tau(u_{m+n})\varepsilon(u'_m)\tau(u_m)\varepsilon(u'_{m+n})\mathbf{1} \\
 & - (\lambda - 1)^2 \sum_{m=1-n}^0 \beta(u_{m+n})\gamma(u'_m)\beta(u_m)\gamma(u'_{m+n})\mathbf{1} \\
 = & (-\lambda^2 n - (\lambda - 1)^2(-n))\mathbf{1} = n(1 - 2\lambda)\mathbf{1},
 \end{aligned}$$

since  $\varepsilon(u'_i)\tau(u_i) + \tau(u_i)\varepsilon(u'_i) = 1$ , and  $\gamma(u'_i)\beta(u_i) - \beta(u_i)\gamma(u'_i) = 1$ . Hence,

$$(3.8) \quad \theta([H_n, H_m])\mathbf{1} = n(1 - 2\lambda)\delta_{n,-m}\mathbf{1}.$$

Thus the central charge is  $3 - 6\lambda$ . The other commutation relations on the vacuum vector  $\mathbf{1}$  are calculated in the same way. □

*Remark 3.2.* — In the case when  $\lambda = -1, \mu = 1$ , the module  $\mathcal{F}_{\lambda,\mu}$  is the adjoint representation of Witt. Thus we obtain a representation of the  $N = 2$  SCA in the semi-infinite Weil complex of the Witt algebra (cf. [G]).

**THEOREM 3.2.** — *Let  $V$  be a complex finite-dimensional vector space,  $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$ . There exists a representation of the  $N = 2$  SCA in  $W^{\otimes}_{2^+}(\tilde{V})$  with central charge  $3\dim V$ .*

*Proof.* — There is the natural  $\mathbb{Z}$ -grading  $\tilde{V} = \bigoplus_{n \in \mathbb{Z}} \tilde{V}_n$ , where  $\tilde{V}_n = V \otimes t^n$ . Let  $u$  run through a fixed basis of  $V$ ,  $u_n$  stand for  $u \otimes t^n$ , and let  $\{u'_n\}$  be the dual basis of  $\tilde{V}'$ . Define the following quadratic expansions by analogy with (3.5) and (3.6), where  $\lambda = 0, \mu = 0$ :

$$\begin{aligned}
 L_n &= - \sum_u \sum_{m \in \mathbb{Z}} (m : \tau(u_{m+n})\varepsilon(u'_m) : + m : \beta(u_{m+n})\gamma(u'_m) :) \\
 H_n &= - \sum_u \sum_{m \in \mathbb{Z}} : \gamma(u'_{m+n})\beta(u_m) : \\
 (3.9) \quad h_n &= \sum_u \sum_{m \in \mathbb{Z}} \gamma(u'_{m+n})\tau(u_m), \\
 p_n &= \sum_u \sum_{m \in \mathbb{Z}} m\beta(u_{m-n})\varepsilon(u'_m).
 \end{aligned}$$

Set

$$(3.10) \quad \mathfrak{L}_n = -L_{-n} + \frac{n+1}{2}H_n.$$

Then  $\mathfrak{L}_n, H_n, \mathfrak{h}_n$ , and  $\mathfrak{p}_n$  span the centerless  $N = 2$  SCA.

Let  $n > 0$ . Then  $H_{-n}\mathbf{1} = 0$ . Hence

$$\begin{aligned}
 (3.11) \quad [H_n, H_{-n}]\mathbf{1} &= -H_{-n} \left( -\sum_u \sum_{m=1-n}^0 \gamma(u'_{m+n})\beta(u_m) \right) \mathbf{1} \\
 &= \left( -\sum_u \sum_{m=1}^n \gamma(u'_{m-n})\beta(u_m) \right) \left( \sum_u \sum_{m=1-n}^0 \gamma(u'_{m+n})\beta(u_m) \right) \mathbf{1} \\
 &= -\sum_u \sum_{m=1-n}^0 \gamma(u'_m)\beta(u_{m+n})\gamma(u'_{m+n})\beta(u_m)\mathbf{1} \\
 &= -\dim V(-n)\mathbf{1},
 \end{aligned}$$

since  $\gamma(u'_i)\beta(u_i) - \beta(u_i)\gamma(u'_i) = 1$ . Notice that

$$(3.12) \quad [H_n, H_m]\mathbf{1} = 0, \text{ if } m \neq -n.$$

Hence

$$(3.13) \quad [H_n, H_m]\mathbf{1} = n\dim V\delta_{n,-m}\mathbf{1}.$$

Thus the central charge is  $3\dim V$ . □

**COROLLARY 3.1.** — *Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra, let  $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ . There exists a representation of the  $N = 2$  SCA in  $H^{\otimes 2+*}(\tilde{\mathfrak{g}}, S^{\otimes 2+*}(\tilde{\mathfrak{g}}))$  with central charge  $3\dim \mathfrak{g}$ .*

*Proof.* — We will show that the expansions (3.9) commute with the differential  $d$ . Recall that

$$(3.14) \quad d = d^{(1)} + d^{(2)},$$

where

$$\begin{aligned}
 (3.15) \quad d^{(1)} &= (1/2) \sum_{u,v,i,j} : \tau([u_i, v_j])\varepsilon(v'_j)\varepsilon(u'_i) :, \\
 d^{(2)} &= \sum_{u,v,i,j} : \beta([u_i, v_j])\gamma(v'_j)\varepsilon(u'_i) :,
 \end{aligned}$$

$u, v$  run through a fixed basis of  $\mathfrak{g}$ , and  $i, j \in \mathbb{Z}$ . Then

$$\begin{aligned}
 (3.16) \quad [L_n, d^{(1)}] &= (1/2) \sum_{u,v,i,j} : -(i+j)\tau([u, v]_{i+j+n})\varepsilon(v'_j)\varepsilon(u'_i) : \\
 &\quad + : \tau([u_i, v_j])(j-n)\varepsilon(v'_{j-n})\varepsilon(u'_i) : \\
 &\quad + : \tau([u_i, v_j])\varepsilon(v'_j)(i-n)\varepsilon(u'_{i-n}) := 0
 \end{aligned}$$

and

$$(3.17) \quad [L_n, d^{(2)}] = \sum_{u,v,i,j} : -(i+j)\beta([u, v]_{i+j+n})\gamma(v'_j)\varepsilon(u'_i) : \\ + : \beta([u_i, v_j])(j-n)\gamma(v'_{j-n})\varepsilon(u'_i) : \\ + : \beta([u_i, v_j])\gamma(v'_j)(i-n)\varepsilon(u'_{i-n}) := 0.$$

Clearly,

$$(3.18) \quad [H_n, d^{(1)}] = 0,$$

and

$$(3.19) \quad [H_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u, v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : + \beta([u, v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : \\ = 0.$$

Next,

$$(3.20) \quad [\mathfrak{h}_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : -\tau([u, v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) : \\ + : \tau([u, v]_{i+j})\varepsilon(v'_j)\gamma(u'_{i+n}) : \\ = - \sum_{u,v,i,j} : \tau([u, v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :,$$

$$(3.21) \quad [\mathfrak{h}_n, d^{(2)}] = \sum_{u,v,i,j} : \tau([u, v]_{i+j-n})\gamma(v'_j)\varepsilon(u'_i) : \\ + : \beta([u, v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) : \\ = \sum_{u,v,i,j} : \tau([u, v]_{i+j})\gamma(v'_{j+n})\varepsilon(u'_i) :,$$

since  $\sum_{u,v,i,j} : \beta([u, v]_{i+j})\gamma(v'_j)\gamma(u'_{i+n}) := 0$ . Hence

$$(3.22) \quad [\mathfrak{h}_n, d^{(2)}] = -[\mathfrak{h}_n, d^{(1)}].$$

Finally,

$$(3.23) \quad [\mathfrak{p}_n, d^{(1)}] = (1/2) \sum_{u,v,i,j} : (i+j)\beta([u, v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) :,$$

$$(3.24) \quad [\mathfrak{p}_n, d^{(2)}] = \sum_{u,v,i,j} : -\beta([u, v]_{i+j})(j+n)\varepsilon(v'_{j+n})\varepsilon(u'_i) :$$

$$\begin{aligned}
 &= \sum_{u,v,i,j} : -\beta([u, v]_{i+j-n})j\varepsilon(v'_j)\varepsilon(u'_i) : \\
 &= -(1/2) \sum_{u,v,i,j} : (j + i)\beta([u, v]_{i+j-n})\varepsilon(v'_j)\varepsilon(u'_i) : .
 \end{aligned}$$

Hence

$$(3.25) \quad [p_n, d^{(2)}] = -[p_n, d^{(1)}].$$

□

#### 4. The superconformal algebras $S'(2, \alpha)$ .

Recall the necessary definitions, [KL]. Let  $W(N)$  be the superalgebra of all derivations of  $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$ , where  $\Lambda(N)$  is the Grassmann algebra in  $N$  variables  $\theta_1, \dots, \theta_N$ , and  $p(t) = \bar{0}$ ,  $p(\theta_i) = \bar{1}$  for  $i = 1, \dots, N$ . Let  $\partial_i$  stand for  $\partial/\partial\theta_i$ , and  $\partial_t$  stand for  $\partial/\partial t$ . Let

$$(4.1) \quad S(N, \alpha) = \{D \in W(N) \mid \text{Div}(t^\alpha D) = 0\} \text{ for } \alpha \in \mathbb{C}.$$

Recall that

$$(4.2) \quad \text{Div}\left(f\partial_t + \sum_{i=1}^N f_i\partial_i\right) = \partial_t f + \sum_{i=1}^N (-1)^{p(f_i)}\partial_i f_i$$

where  $f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$ , and

$$(4.3) \quad \text{Div}(fD) = Df + f\text{Div}D,$$

where  $f$  is an even function. Let  $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$  be the derived superalgebra. Assume that  $N > 1$ . If  $\alpha \notin \mathbb{Z}$ , then  $S(N, \alpha)$  is simple, and if  $\alpha \in \mathbb{Z}$ , then  $S'(N, \alpha)$  is a simple ideal of  $S(N, \alpha)$  of codimension 1:

$$(4.4) \quad 0 \rightarrow S'(N, \alpha) \rightarrow S(N, \alpha) \rightarrow \mathbb{C}t^{-\alpha}\theta_1 \cdots \theta_N\partial_t \rightarrow 0.$$

Notice that

$$(4.5) \quad S(N, \alpha) \cong S(N, \alpha + n) \text{ for } n \in \mathbb{Z}.$$

The superalgebra  $S'(N, \alpha)$  has, up to equivalence, only one non-trivial 2-cocycle if and only if  $N = 2$ , which is important for our task. Let

$$(4.6) \quad \{\mathcal{L}_n^\alpha, E_n, H_n, F_n, h_n^\alpha, p_n, x_n, y_n^\alpha\}_{n \in \mathbb{Z}}$$

be the basis of  $S'(2, \alpha)$  defined as follows:

$$(4.7) \quad \begin{aligned} \mathcal{L}_n^\alpha &= -t^n(t\partial_t + \frac{1}{2}(n + \alpha + 1)(\theta_1\partial_1 + \theta_2\partial_2)), \\ E_n &= t^n\theta_2\partial_1, \\ H_n &= t^n(\theta_2\partial_2 - \theta_1\partial_1), \\ F_n &= t^n\theta_1\partial_2, \\ \mathfrak{h}_n^\alpha &= t^n\theta_2\partial_t - (n + \alpha)t^{n-1}\theta_1\theta_2\partial_1, \\ \mathfrak{p}_n &= -t^{n+1}\partial_2, \\ \mathfrak{x}_n &= t^{n+1}\partial_1, \\ \mathfrak{y}_n^\alpha &= t^n\theta_1\partial_t + (n + \alpha)t^{n-1}\theta_1\theta_2\partial_2. \end{aligned}$$

The non-vanishing commutation relations between these elements are

$$(4.8) \quad \begin{aligned} [\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha] &= (n - k)\mathcal{L}_{n+k}^\alpha, \\ [E_n, F_k] &= H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, \\ [\mathcal{L}_n^\alpha, E_k] &= -kE_{n+k}, [\mathcal{L}_n^\alpha, H_k] = -kH_{n+k}, [\mathcal{L}_n^\alpha, F_k] = -kF_{n+k}, \\ [\mathcal{L}_n^\alpha, \mathfrak{h}_k^\alpha] &= \frac{1}{2}(n - 2k + 1 - \alpha)\mathfrak{h}_{n+k}^\alpha, [\mathcal{L}_n^\alpha, \mathfrak{p}_k] \\ &= \frac{1}{2}(n - 2k - 1 + \alpha)\mathfrak{p}_{n+k}, \\ [\mathcal{L}_n^\alpha, \mathfrak{x}_k] &= \frac{1}{2}(n - 2k - 1 + \alpha)\mathfrak{x}_{n+k}, [\mathcal{L}_n^\alpha, \mathfrak{y}_k^\alpha] \\ &= \frac{1}{2}(n - 2k + 1 - \alpha)\mathfrak{y}_{n+k}^\alpha, \\ [E_n, \mathfrak{y}_k^\alpha] &= \mathfrak{h}_{n+k}^\alpha, [F_n, \mathfrak{h}_k^\alpha] = \mathfrak{y}_{n+k}^\alpha, [E_n, \mathfrak{p}_k] \\ &= \mathfrak{x}_{n+k}, [F_n, \mathfrak{x}_k] = \mathfrak{p}_{n+k}, \\ [H_n, \mathfrak{h}_k^\alpha] &= \mathfrak{h}_{n+k}^\alpha, [H_n, \mathfrak{y}_k^\alpha] = -\mathfrak{y}_{n+k}^\alpha, [H_n, \mathfrak{x}_k] \\ &= \mathfrak{x}_{n+k}, [H_n, \mathfrak{p}_k] = -\mathfrak{p}_{n+k}, \\ [\mathfrak{h}_n^\alpha, \mathfrak{x}_k] &= (k + 1 - n - \alpha)E_{n+k}, [\mathfrak{p}_n, \mathfrak{y}_k^\alpha] \\ &= (k - n - 1 + \alpha)F_{n+k}, \\ [\mathfrak{h}_n^\alpha, \mathfrak{p}_k] &= \mathcal{L}_{n+k}^\alpha - \frac{1}{2}(k - n + 1 - \alpha)H_{n+k}, \\ [\mathfrak{x}_n, \mathfrak{y}_k^\alpha] &= -\mathcal{L}_{n+k}^\alpha + \frac{1}{2}(k - n - 1 + \alpha)H_{n+k}. \end{aligned}$$

A non-trivial 2-cocycle on  $S'(2, \alpha)$  is

$$(4.9) \quad c(\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha) = \frac{C}{12}n(n^2 - 1)\delta_{n,-k},$$

$$\begin{aligned}
 c(E_n, F_k) &= \frac{C}{6}n\delta_{n,-k}, \quad c(H_n, H_k) = \frac{C}{3}n\delta_{n,-k}, \\
 c(\mathfrak{h}_n^\alpha, \mathfrak{p}_k) &= \frac{C}{6} \left( \left( n - 1 + \frac{\alpha + 1}{2} \right)^2 - \frac{1}{4} \right) \delta_{n,-k}, \\
 c(\mathfrak{x}_n, \mathfrak{y}_k^\alpha) &= -\frac{C}{6} \left( \left( -n - 1 + \frac{\alpha + 1}{2} \right)^2 - \frac{1}{4} \right) \delta_{n,-k};
 \end{aligned}$$

see [KL]. Let  $\hat{S}'(2, \alpha)$  be the corresponding central extension of  $S'(2, \alpha)$ . In particular,  $\hat{S}'(2, 0)$  is isomorphic to the  $N = 4$  SCA (see [Ad]).

*Remark 4.1* — Notice that

$$\begin{aligned}
 (4.10) \quad S'(2, \alpha)_{\bar{0}} &= \text{Witt} \ltimes \tilde{\mathfrak{sl}}(2), \quad \text{where} \\
 \text{Witt} &= \langle \mathfrak{L}_n^\alpha \rangle_{n \in \mathbb{Z}}, \quad \tilde{\mathfrak{sl}}(2) = \langle E_n, H_n, F_n \rangle_{n \in \mathbb{Z}},
 \end{aligned}$$

and

$$(4.11) \quad S'(2, \alpha)_{\bar{1}} = \langle \mathfrak{h}_n^\alpha, \mathfrak{y}_n^\alpha \rangle_{n \in \mathbb{Z}} \oplus \langle \mathfrak{p}_n, \mathfrak{x}_n \rangle_{n \in \mathbb{Z}}$$

is a direct sum of two standard (odd)  $\tilde{\mathfrak{sl}}(2)$ -modules.

*Remark 4.2* — For any  $\alpha \in \mathbb{C}$  one can consider the subalgebra of  $\hat{S}'(2, \alpha)$ , spanned by  $\mathfrak{L}_n^\alpha, H_n, \mathfrak{h}_n^\alpha, \mathfrak{p}_n$ , and  $\mathbb{C}$ . Thus we obtain a one-parameter family of superalgebras, which are isomorphic to the  $N = 2$  SCA. The isomorphism

$$(4.12) \quad \varphi : \langle \mathfrak{L}_n^\alpha, H_n, \mathfrak{h}_n^\alpha, \mathfrak{p}_n, \mathbb{C} \rangle \longrightarrow \langle \mathfrak{L}_n, H_n, \mathfrak{h}_n, \mathfrak{p}_n, \mathbb{C} \rangle$$

is given as follows:

$$\begin{aligned}
 (4.13) \quad \varphi(\mathfrak{L}_n^\alpha) &= \mathfrak{L}_n - \frac{\alpha}{2}H_n + \frac{\alpha^2}{24}\delta_{n,0}\mathbb{C}, \\
 \varphi(H_n) &= H_n - \frac{\alpha}{6}\delta_{n,0}\mathbb{C}, \\
 \varphi(\mathfrak{h}_n^\alpha) &= \mathfrak{h}_n, \quad \varphi(\mathfrak{p}_n) = \mathfrak{p}_n, \quad \varphi(\mathbb{C}) = \mathbb{C}.
 \end{aligned}$$

Notice that formulae (4.13) correspond to the spectral flow transformation for the  $N = 2$  SCA (cf. [FST]).

Let  $\text{Der}S'(2, \alpha)$  be the Lie superalgebra of all derivations of  $S'(2, \alpha)$ , and  $\text{Der}_{\text{ext}}S'(2, \alpha)$  be the exterior derivations of  $S'(2, \alpha)$  (see [Fu]).

THEOREM 4.1.

1) If  $\alpha \in \mathbb{Z}$ , then  $\text{Der}_{\text{ext}}S'(2, \alpha) \cong \mathfrak{SL}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle$ , where

$$(4.14) \quad [\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, [\mathcal{E}, \mathcal{F}] = \mathcal{H}.$$

The action of  $\mathfrak{SL}(2)$  is given as follows:

$$(4.15) \quad \begin{aligned} [\mathcal{E}, \mathfrak{h}_k^\alpha] &= \mathfrak{x}_{k-1+\alpha}, [\mathcal{E}, \mathfrak{y}_k^\alpha] = \mathfrak{p}_{k-1+\alpha}; \\ [\mathcal{F}, \mathfrak{x}_k] &= \mathfrak{h}_{k+1-\alpha}^\alpha, [\mathcal{F}, \mathfrak{p}_k] = \mathfrak{y}_{k+1-\alpha}^\alpha; \\ [\mathcal{H}, \mathfrak{x}_k] &= \mathfrak{x}_k, [\mathcal{H}, \mathfrak{h}_k^\alpha] = -\mathfrak{h}_k^\alpha, \\ [\mathcal{H}, \mathfrak{p}_k] &= \mathfrak{p}_k, [\mathcal{H}, \mathfrak{y}_k^\alpha] = -\mathfrak{y}_k^\alpha. \end{aligned}$$

2) If  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ , then  $\text{Der}_{\text{ext}}S'(2, \alpha) = \langle \mathcal{H} \rangle$ .

*Proof.* — Recall that the exterior derivations of a Lie (super) algebra can be identified with its first cohomology with coefficients in the adjoint representation (see [Fu]). Thus

$$(4.16) \quad \text{Der}_{\text{ext}}S'(2, \alpha) \cong H^1(S'(2, \alpha), S'(2, \alpha)).$$

The superalgebra  $S'(2, \alpha)$  has the following  $\mathbb{Z} \pm \alpha$ -grading deg:

$$(4.17) \quad \begin{aligned} \deg \mathfrak{L}_n^\alpha &= n, \deg E_n = n + 1 - \alpha, \deg F_n = n - 1 + \alpha, \\ \deg H_n &= n, \deg \mathfrak{h}_n^\alpha = n, \deg \mathfrak{p}_n = n, \deg \mathfrak{x}_n = n + 1 - \alpha, \\ \deg \mathfrak{y}_n^\alpha &= n - 1 + \alpha. \end{aligned}$$

Let

$$(4.18) \quad L_0 = -\mathfrak{L}_0^\alpha + \frac{1}{2}(1 - \alpha)H_0.$$

Then

$$(4.19) \quad [L_0, s] = (\text{deg } s)s$$

for a homogeneous  $s \in S'(2, \alpha)$ . Accordingly,

$$(4.20) \quad [L_0, D] = (\text{deg } D)D$$

for a homogeneous  $D \in \text{Der}_{\text{ext}}S'(2, \alpha)$ . On the other hand, since the action of a Lie superalgebra on its cohomology is trivial (see [Fu]), then one must have

$$(4.21) \quad [L_0, D] = 0.$$

Hence the non-zero elements of  $\text{Der}_{\text{ext}}S'(2, \alpha)$  have  $\text{deg} = 0$ , and they preserve the superalgebra  $S'(2, \alpha)_{\text{deg}=0}$ . Let  $\alpha \in \mathbb{Z}$ . Then one can check that the exterior derivations of  $S'(2, \alpha)_{\text{deg}=0}$  form an  $\mathfrak{sl}(2)$ , and extend them to the exterior derivations of  $S'(2, \alpha)$  as in (4.15). One should also note that if the restriction of a derivation of  $S'(2, \alpha)$  to  $S'(2, \alpha)_{\text{deg}=0}$  is zero, then this derivation is inner.

Finally, notice that the exterior derivations  $\mathcal{E}$  and  $\mathcal{F}$  interchange  $\{h_k^\alpha\}$  with  $\{x_k\}$ . If  $\alpha \notin \mathbb{Z}$ , then  $\text{deg } h_k^\alpha - \text{deg } x_n \notin \mathbb{Z}$  for any  $k, n \in \mathbb{Z}$ . Hence  $\mathcal{E}$  and  $\mathcal{F}$  cannot have  $\text{deg} = 0$ . By this reason,  $\text{Der}_{\text{ext}}S'(2, \alpha) = \langle \mathcal{H} \rangle$  for  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ .

□

*Remark 4.3.* — If  $\alpha \in \mathbb{Z}$ , then one can identify  $\mathcal{F}$  with  $-t^{-\alpha}\theta_1\theta_2\partial_t$  (see (4.4)).

### 5. An action of $\hat{S}'(2, \alpha)$ on the semi-infinite Weil complex of a loop algebra.

We will consider a more general case, i.e. when  $V$  is a complex finite-dimensional vector space, and  $\tilde{V} = V \otimes \mathbb{C}[t, t^{-1}]$ . Let  $\hat{\text{D}}\text{er}S'(2, \alpha)$  be a non-trivial central extension of  $\text{Der}S'(2, \alpha)$ .

THEOREM 5.1.

1) The space  $W^{\frac{\infty}{2}+*}(\tilde{V})$ , where  $\alpha \in \mathbb{C}$ , is a module over  $\hat{S}'(2, \alpha)$  with central charge  $3\text{dim}V$ ;

2) if  $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ , then  $W^{\frac{\infty}{2}+*}(\tilde{V})$  is a module over  $\hat{\text{D}}\text{er}S'(2, \alpha)$ .

*Proof.* — Let  $u$  run through a fixed basis of  $V$ ,  $u_n$  stand for  $u \otimes t^n$ , and  $\{u'_n\}$  be the dual basis of  $\tilde{V}'$ . One can define a representation of Witt in  $W^{\frac{\infty}{2}+*}(\tilde{V})$  by analogy with (3.5), where  $\lambda = 0, \mu = \alpha/2$ :

$$(5.1) \quad \theta(L_n) = - \sum_u \sum_m \left( m - \frac{\alpha}{2} \right) ( : \tau(u_{m+n})\varepsilon(u'_m) : + : \beta(u_{m+n})\gamma(u'_m) : ),$$

then extend it to a representation of the  $N = 2$  SCA, and apply (4.13). We obtain the following representation of  $\hat{S}'(2, \alpha)$ :

$$(5.2) \quad \theta(H_n) = - \sum_u \sum_m : \beta(u_m)\gamma(u'_{m+n}),$$

$$\begin{aligned} \theta(\mathfrak{L}_n^\alpha) &= -\theta(L_{-n}) + \frac{n+1-\alpha}{2}\theta(H_n) + \left(\frac{\alpha}{4} - \frac{\alpha^2}{8}\right)\dim V\delta_{n,0}, \\ \theta(\mathfrak{h}_n^\alpha) &= \sum_u \sum_m \gamma(u'_{m+n})\tau(u_m), \\ \theta(\mathfrak{p}_n) &= \sum_u \sum_m \left(m - \frac{\alpha}{2}\right)\beta(u_{m-n})\varepsilon(u'_m), \\ \theta(E_n) &= -(1/2)i \sum_u \sum_m \gamma(u'_m)\gamma(u'_{1-m+n}), \\ \theta(F_n) &= -(1/2)i \sum_u \sum_m \beta(u_m)\beta(u_{1-m-n}), \\ \theta(\mathfrak{y}_n^\alpha) &= i \sum_u \sum_m \beta(u_m)\tau(u_{1-m-n}), \\ \theta(\mathfrak{x}_n) &= -i \sum_u \sum_m \left(m - \frac{\alpha}{2}\right)\gamma(u'_{1-m+n})\varepsilon(u'_m), \\ \theta(\mathcal{H}) &= -\sum_u \sum_m : \tau(u_m)\varepsilon(u'_m) : . \end{aligned}$$

One can check that the central charge is  $3\dim V$  in the same way as in Theorem 3.2. □

**THEOREM 5.2.** — *Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra endowed with a non-degenerate invariant symmetric bilinear form. Then  $H^{\infty+*}(\mathfrak{g}, S^{\infty+*}(\mathfrak{g}))$  is a module over  $\hat{S}'(2, 0)$  with central charge  $3\dim \mathfrak{g}$ .*

*Proof.* — Let  $\{v_i\}$  be a basis of  $\mathfrak{g}$  so that with respect to the given form  $\langle v_i, v_j \rangle = \delta_{i,j}$ . Let  $u$  run through this basis. Then by Theorem 5.1, there is a representation of  $\hat{S}'(2, 0)$  in  $W^{\infty+*}(\mathfrak{g})$ . Notice that we can identify the elements of  $S'(2, 0)$  with the quadratic expansions obtained by putting  $\alpha = 0$  in the equations (5.2). One can check that the commutation relations (4.8) (where  $\alpha = 0$ ) are fulfilled. One can notice that

$$(5.3) \quad [S'(2, 0), d] = 0.$$

In fact, since  $\langle \cdot, \cdot \rangle$  is an invariant symmetric bilinear form on  $\mathfrak{g}$ , then the elements  $E_n, H_n$ , and  $F_n$  commute with  $\pi(g)$  for any  $g \in \mathfrak{g}$ . Hence they commute with  $d$ . According to Corollary 3.1,

$$(5.4) \quad [\mathfrak{h}_n^0, d] = [\mathfrak{p}_n, d] = 0.$$

Recall that

$$(5.5) \quad S'(2, 0)_{\bar{1}} = \langle \mathfrak{h}_n^0, \mathfrak{y}_n^0, \mathfrak{p}_n, \mathfrak{x}_n \rangle_{n \in \mathbb{Z}}.$$

Since

$$(5.6) \quad [E_n, \mathfrak{p}_k] = \mathfrak{x}_{n+k}, [F_n, \mathfrak{h}_k^0] = \mathfrak{y}_{n+k}^0,$$

then

$$(5.7) \quad [S'(2, 0)_{\bar{1}}, d] = 0.$$

Since

$$(5.8) \quad S'(2, 0)_{\bar{0}} = [S'(2, 0)_{\bar{1}}, S'(2, 0)_{\bar{1}}],$$

then (5.3) follows. □

To define an action of  $\hat{\text{Der}}S'(2, 0)$ , one should consider a *relative* semi-infinite Weil complex.

Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra,  $\phi$  be a representation of  $\mathfrak{g}$  in  $V$ ,  $\langle \cdot, \cdot \rangle$  be a non-degenerate  $\mathfrak{g}$ -invariant symmetric bilinear form on  $V$ . One can naturally extend  $\phi$  to a representation of  $\tilde{\mathfrak{g}}$  in  $\tilde{V}$ :

$$(5.9) \quad \phi(g \otimes t^n)(v \otimes t^k) = (\phi(g)v) \otimes t^{n+k}, \text{ for } g \in \mathfrak{g}, v \in V.$$

**THEOREM 5.3.** — *The space  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{V})$  is a module over  $\hat{\text{Der}}S'(2, 0)$  with central charge  $3\dim V$ .*

*Proof.* — Let  $\{v_i\}$  be a basis of  $V$  so that  $\langle v_i, v_j \rangle = \delta_{i,j}$ . Let  $u$  run through this basis. Then by Theorem 5.1, there is a representation of  $\hat{S}'(2, 0)$  in  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{V})$ . We can identify the elements of  $S'(2, 0)$  with the expansions (5.2) where  $\alpha = 0$ .

Since the form  $\langle \cdot, \cdot \rangle$  is  $\mathfrak{g}$ -invariant, then there is an action of  $\hat{S}'(2, 0)$  on  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{V})$ . To extend this representation to  $\hat{\text{Der}}S'(2, 0)$ , we have to define it on  $\mathfrak{S}\mathfrak{L}(2) = \langle \mathcal{F}, \mathcal{H}, \mathcal{E} \rangle$ . Let

$$(5.10) \quad \begin{aligned} \mathcal{E} &= i \sum_u \sum_{m>0} m \varepsilon(u'_{-m}) \varepsilon(u'_m), \\ \mathcal{H} &= - \sum_u \sum_{m \neq 0} : \tau(u_m) \varepsilon(u'_m) :, \\ \mathcal{F} &= -i \sum_u \sum_{m>0} (1/m) \tau(u_m) \tau(u_{-m}). \end{aligned}$$

Notice that  $\mathfrak{S}\mathfrak{L}(2)$  acts on  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{V})$ . The commutation relations between  $\mathcal{E}, \mathcal{H}, \mathcal{F}$  and the elements of  $S'(2, 0)$  coincide with the relations (4.15),

where  $\alpha = 0$ , up to some terms which contain elements  $\tau(u_0)$ . Since the action of  $\tau(u_0)$  on  $W_{\text{rel}}^{\infty+*}(\tilde{V})$  is trivial, then a representation of  $\hat{\text{Der}}S'(2, 0)$  in  $W_{\text{rel}}^{\infty+*}(\tilde{V})$  is well-defined.  $\square$

COROLLARY 5.1. —  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$  is a module over  $\hat{S}'(2, 0)$  with central charge  $3\text{dim}_{\mathbb{g}}$ .

*Proof.* — Follows from Theorem 5.2.  $\square$

## 6. Relative semi-infinite cohomology and Kähler geometry.

Let  $M$  be a compact Kähler manifold with associated  $(1, 1)$ -form  $\omega$ , let  $\text{dim}_{\mathbb{C}} M = n$ . There exists a number of operators on the space  $A^*(M)$  of differential forms on  $M$  such as  $\partial, \bar{\partial}, d, d_c$ , their corresponding adjoint operators and the associated Laplacians (see [GH]). Recall that

$$(6.1) \quad \begin{aligned} \partial &: A^{p,q}(M) \rightarrow A^{p+1,q}(M), \\ \bar{\partial} &: A^{p,q}(M) \rightarrow A^{p,q+1}(M), \\ d &= \partial + \bar{\partial}, \\ d_c &= i(\partial - \bar{\partial}), \\ \Delta &= dd^* + d^*d = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}. \end{aligned}$$

The Hodge  $\star$ -operator maps

$$(6.2) \quad \star : A^{p,q}(M) \longrightarrow A^{n-q, n-p}(M),$$

so that  $\star^2 = (-1)^{p+q}$  on  $A^{p,q}(M)$ . Correspondingly, the Hodge inner product is defined on each of  $A^{p,q}(M)$ :

$$(6.3) \quad (\varphi, \psi) = \int_M \varphi \wedge \star \bar{\psi}.$$

In addition,  $A^*(M)$  admits an  $\mathfrak{sl}(2)$ -module structure. Namely,  $\mathfrak{sl}(2) = \langle L, H, \Lambda \rangle$ , where

$$(6.4) \quad [L, \Lambda] = H, [H, L] = 2L, [H, \Lambda] = -2\Lambda.$$

The operator

$$(6.5) \quad L : A^{p,q}(M) \rightarrow A^{p+1, q+1}(M),$$

is defined by

$$(6.6) \quad L(\varphi) = \varphi \wedge \omega.$$

Let  $\Lambda = L^*$  be its adjoint operator:

$$(6.7) \quad \Lambda : A^{p,q}(M) \rightarrow A^{p-1,q-1}(M),$$

and

$$(6.8) \quad H|_{A^{p,q}(M)} = p + q - n.$$

According to the Lefschetz theorem, there exists the corresponding action of  $\mathfrak{sl}(2)$  on  $H^*(M)$ . These operators satisfy a series of identities, known as the Hodge identities (see [GH]). Consider the Lie superalgebra spanned by the classical operators:

$$(6.9) \quad \mathfrak{S} := \langle \Delta, L, H, \Lambda, d, d^*, d_c, d_c^* \rangle.$$

The non-vanishing commutation relations in  $\mathfrak{S}$  are as follows:

$$(6.10) \quad \begin{aligned} [L, \Lambda] &= H, [H, L] = 2L, [H, \Lambda] = -2\Lambda, \\ [d, d^*] &= dd^* + d^*d = \Delta, \\ [d_c, d_c^*] &= d_c d_c^* + d_c^* d_c = \Delta, \\ [H, d] &= d, [H, d^*] = -d^*, \\ [H, d_c] &= d_c, [H, d_c^*] = -d_c^*, \\ [L, d^*] &= -d_c, [L, d_c^*] = d, \\ [\Lambda, d] &= d_c^*, [\Lambda, d_c] = -d^*. \end{aligned}$$

**THEOREM 6.1.** — *Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then there exist operators on  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\hat{\mathfrak{g}})$ , which are analogous to the classical operators in Kähler geometry.*

*Proof.* — It was shown in [FGZ] that a relative semi-infinite complex  $C_{\infty}^*(\mathfrak{l}, \mathfrak{l}_0, V)$ , where  $\mathfrak{l} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{l}_n$  is a complex  $\mathbb{Z}$ -graded Lie algebra, and  $V$  is a graded Hermitian  $\mathfrak{l}$ -module, has a structure, which is similar to that of the de Rham complex in Kähler geometry. It is assumed that there exists a 2-cocycle  $\gamma$  on  $\mathfrak{l}$  such that  $\gamma|_{\mathfrak{l}_n \times \mathfrak{l}_{-n}}$  is non-degenerate if  $n \in \mathbb{Z} \setminus 0$  and it is zero otherwise. Then there exist operators on  $C_{\infty}^*(\mathfrak{l}, \mathfrak{l}_0, V)$  analogous to the classical ones.

We will define analogues of the classical operators on  $W_{\text{rel}}^{\infty +*}(\tilde{\mathfrak{g}})$ . Using the form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  we obtain the 2-cocycle  $\gamma$  on  $\tilde{\mathfrak{g}}$ :

$$(6.11) \quad \gamma(g_1 \otimes t^n, g_2 \otimes t^m) = n \langle g_1, g_2 \rangle \delta_{n, -m}, \text{ for } g_1, g_2 \in \mathfrak{g}.$$

Notice that  $\gamma|_{\tilde{\mathfrak{g}}_n \times \tilde{\mathfrak{g}}_{-n}}$  is non-degenerate if  $n \in \mathbb{Z} \setminus 0$  and zero otherwise. Let

$$(6.12) \quad \Lambda_{\text{rel}}^{\infty +*}(\tilde{\mathfrak{g}}) = \bigoplus_{a, b \geq 0} \Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_{\infty}(\mathfrak{n}'_-).$$

For a homogeneous element in  $\Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_{\infty}(\mathfrak{n}'_-)$ ,  $a$  is the number of added elements, and  $b$  is the number of missing elements with respect to the vacuum vector  $\mathbf{1}_{\text{rel}}$ . Let

$$(6.13) \quad C^{a,b}(\tilde{\mathfrak{g}}) = [S^{\infty +*}(\tilde{\mathfrak{g}}) \otimes \Lambda^a(\mathfrak{n}'_+) \wedge \Lambda^b_{\infty}(\mathfrak{n}'_-)]^{\tilde{\mathfrak{g}}_0}.$$

We obtain a bigrading on the relative semi-infinite Weil complex, such that

$$(6.14) \quad W_{\text{rel}}^{\infty +i}(\tilde{\mathfrak{g}}) = \bigoplus_{a-b=i} C^{a,b}(\tilde{\mathfrak{g}}).$$

Let  $d$  be the restriction of the differential to the relative subcomplex. Notice that

$$(6.15) \quad d : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}) \oplus C^{a,b-1}(\tilde{\mathfrak{g}}).$$

Define  $d_1$  and  $d_2$  such that

$$(6.16) \quad \begin{aligned} d &= d_1 + d_2, \\ d_1 : C^{a,b}(\tilde{\mathfrak{g}}) &\longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}), \\ d_2 : C^{a,b}(\tilde{\mathfrak{g}}) &\longrightarrow C^{a,b-1}(\tilde{\mathfrak{g}}). \end{aligned}$$

Let

$$(6.17) \quad d_c = i(d_1 - d_2).$$

To define the adjoint operators, we have to introduce a Hermitian form on  $W_{\text{rel}}^{\infty +*}(\tilde{\mathfrak{g}})$ .

It was shown in [FGZ] that if a  $\mathbb{Z}$ -graded Lie algebra  $\mathfrak{l}$  admits an antilinear automorphism  $\sigma$  of order 2 such that  $\sigma(\mathfrak{l}_n) = \mathfrak{l}_{-n}$ , then there exists a Hermitian form on  $\Lambda^{\infty +*}(\mathfrak{l})$  such that

$$(6.18) \quad \varepsilon(x')^* = -\varepsilon(\sigma(x')), \quad \tau(x)^* = -\tau(\sigma(x)),$$

where  $x \in \mathfrak{l}, x' \in \mathfrak{l}'$ .

To define a Hermitian form  $\{\cdot, \cdot\}$  on  $\Lambda_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ , we set  $\{\mathbf{1}_{\text{rel}}, \mathbf{1}_{\text{rel}}\} = 1$ . We fix a basis  $\{v_i\}$  of  $\mathfrak{g}$  so that  $\langle v_i, v_j \rangle = \delta_{i,j}$ . Let  $u$  run through this basis. We define an antilinear automorphism  $\sigma$  of  $\tilde{\mathfrak{g}}$  as follows:

$$(6.19) \quad \sigma(u_n) = iu_{-n}.$$

Correspondingly,

$$(6.20) \quad \sigma(u'_n) = -iu'_{-n}.$$

We introduce a Hermitian form on  $\Lambda_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  so that the relations (6.18), where

$$(6.21) \quad x \in \tilde{\mathfrak{g}}_n, x' \in \tilde{\mathfrak{g}}'_n \text{ for } n \neq 0$$

hold. In the similar way we introduce a Hermitian form on  $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ , such that

$$(6.22) \quad \gamma(x')^* = \gamma(\sigma(x')), \quad \beta(x)^* = -\beta(\sigma(x)).$$

Then we obtain a Hermitian form  $\{\cdot, \cdot\}$  on  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  by tensoring these two forms. It gives a pairing:  $C^{a,b}(\tilde{\mathfrak{g}}) \rightarrow C^{b,a}(\tilde{\mathfrak{g}})$ . To define a Hermitian form on  $C^{a,b}(\tilde{\mathfrak{g}})$ , we use the linear map

$$(6.23) \quad * : C^{a,b}(\tilde{\mathfrak{g}}) \rightarrow C^{b,a}(\tilde{\mathfrak{g}}),$$

defined as follows:

$$(6.24) \quad \begin{aligned} & * \left( v \otimes (\varepsilon(u'_{n_1}) \cdots \varepsilon(u'_{n_a}) \tau(u_{m_1}) \cdots \tau(u_{m_b}) \mathbf{1}_{\text{rel}}) \right) \\ & = v \otimes (\varepsilon(u'_{-m_1}) \cdots \varepsilon(u'_{-m_b}) \tau(u_{-n_1}) \cdots \tau(u_{-n_a}) \mathbf{1}_{\text{rel}}), \end{aligned}$$

where  $v \in S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ ,  $\{n_i\}_{i=1}^a > 0$  and  $\{m_i\}_{i=1}^b < 0$ . Finally, the Hermitian form on  $C^{a,b}(\tilde{\mathfrak{g}})$  is defined by  $(w_1, w_2) = \{i^{a+b} * w_1, w_2\}$  (cf. [FGZ]). We introduce the adjoint operators  $d^*, d_c^*$  and the Laplace operator  $\Delta = dd^* + d^*d$ .

It was pointed out in [FGZ] that as in the classical theory (see [GH]), there exists an action of  $\mathfrak{sl}(2)$  on  $H_{\infty}^*(\mathfrak{l}, \mathfrak{l}_0, V)$ . One can identify  $\mathfrak{l}'_n$  with  $\mathfrak{l}_{-n}$  by means of the cocycle  $\gamma$ . If  $\{e_i\}$  is a homogeneous basis in  $\mathfrak{l}$ , then  $\mathfrak{sl}(2) = \langle L, H, \Lambda \rangle$  is defined as follows:

$$(6.25) \quad L = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \varepsilon(e_m) \varepsilon(e'_m),$$

$$H = - \sum_{m \in \mathbb{Z} \setminus 0} : \tau(e_m) \varepsilon(e'_m) :,$$

$$\Lambda = (i/2) \sum_{m \in \mathbb{Z} \setminus 0} \tau(e_m) \tau(e'_m).$$

We identify  $\tilde{\mathfrak{g}}'_n$  with  $\tilde{\mathfrak{g}}_{-n}$  by means of the cocycle  $\gamma$  (see (6.11)), and set

$$(6.26) \quad \mathcal{E} = L, \mathcal{H} = H, \mathcal{F} = \Lambda.$$

Then we obtain the  $\mathcal{SL}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle$  defined in (5.10). The operators

$$(6.27) \quad \{\Delta, \mathcal{E}, \mathcal{H}, \mathcal{F}, d, d^*, d_c, d_c^*\}$$

are the analogues of the classical operators (6.9).  $\square$

**THEOREM 6.2.** — *Let  $\mathfrak{g}$  be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$  is a module over  $\hat{\text{Der}}S'(2, 0)$  with central charge  $3\text{dim}_{\mathbb{g}}$ .*

*Proof.* — By Theorem 5.3,  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  is a module over  $\hat{\text{Der}}S'(2, 0)$  with central charge  $3\text{dim}_{\mathbb{g}}$ . By Corollary 5.1, there is an action of  $\hat{S}'(2, 0)$  on  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ . We have proved that

$$(6.28) \quad \text{Der}_{\text{ext}}S'(2, 0) = \mathcal{SL}(2) = \langle \mathcal{E}, \mathcal{H}, \mathcal{F} \rangle,$$

see (5.10). Notice that as in the classical case, the element  $\mathcal{F}$  and the differential  $d$  do not commute. Nevertheless, there exists an action of  $\mathcal{SL}(2)$  on the relative semi-infinite cohomology according to [FGZ].  $\square$

**THEOREM 6.3.** — *The degree zero part of the  $\mathbb{Z}$ -grading  $\text{deg}$  of  $S'(2, 0)$  is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.*

*Proof.* — Recall that the  $\mathbb{Z}$ -grading  $\text{deg}$  of  $S'(2, 0)$  is defined by the element  $L_0 \in \text{Witt}$ , see (4.17)-(4.19). One can easily check that

$$(6.29) \quad S'(2, 0)_{\text{deg}=0} = \langle L_0, E_{-1}, H_0, F_1, \mathfrak{h}_0^0, \mathfrak{p}_0, \mathfrak{x}_{-1}, \mathfrak{y}_1^0 \rangle.$$

The isomorphism of Lie superalgebras

$$(6.30) \quad \psi: \mathcal{S} \longrightarrow S'(2, 0)_{\text{deg}=0}$$

is given as follows:

$$(6.31) \quad \begin{aligned} \psi(\Delta) &= L_0, \psi(L) = E_{-1}, \psi(H) = H_0, \psi(\Lambda) = F_1, \\ \psi(d) &= \mathfrak{h}_0^0, \psi(d^*) = -\mathfrak{p}_0, \psi(d_c) = \mathfrak{x}_{-1}, \psi(d_c^*) = \mathfrak{y}_1^0. \end{aligned}$$

□

**COROLLARY 6.1.** — *The action of  $S'(2, 0)_{\text{deg}=0}$  defines a set of quadratic operators on  $W_{\text{rel}}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$  (correspondingly, on  $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ ), which are analogues of the classical ones, and include the semi-infinite Koszul differential  $\mathfrak{h} = \mathfrak{h}_0^0$  and the semi-infinite homotopy operator  $\mathfrak{p}_0$ .*

**Remark 6.1.** — In this work we have realized superconformal algebras by means of quadratic expansions on the generators of the Heisenberg and Clifford algebras related to  $\tilde{\mathfrak{g}}$ . Note that the differentials on a semi-infinite Weil complex are represented by cubic expansions. One can possibly define an additional (to the already known) action of the  $N = 2$  SCA on  $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ , considering Fourier components of the differentials  $d$  and  $d^*$ , [Fe].

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### BIBLIOGRAPHY

- [Ad] M. ADEMOLLO, L. BRINK, A. D’ADDA, R. D’AURIA, E. NAPOLITANO, S. SCIUTO, E. DEL GIUDICE, P. DI VECCHIA, S. FERRARA, F. GLIOZZI, R. MUSTO and R. RETTORINO, Dual strings with  $U(1)$  colour symmetry, Nucl. Phys., B111 (1976), 77-110.
- [Ak] F. AKMAN, Some cohomology operators in 2-D field theory, Proceedings of the conference on Quantum topology (Manhattan, KS, 1993), World Sci. Publ, River Edge, NJ (1994), 1-19.
- [Fe] B. L. FEIGIN, Private communication.
- [Fu] D. B. FUKS, Cohomology of infinite-dimensional Lie algebras, Consultants Bureau, New York and London, 1986.
- [FF] B. FEIGIN, E. FRENKEL, Semi-infinite Weil Complex and the Virasoro Algebra, Commun. Math. Phys., 137 (1991), 617-639. Erratum: Commun. Math. Phys., 147 (1992), 647-648.

- [FGZ] I. FRENKEL, H. GARLAND, G. ZUCKERMAN, Semi-infinite cohomology and string theory, *Proc. Natl. Acad. Sci. U.S.A.*, 83 (1986), 8442-8446.
- [FST] B. L. FEIGIN, A. M. SEMIKHATOV, I. Yu. TIPUNIN, Equivalence between chain categories of representations of affine  $\mathfrak{sl}(2)$  and  $N = 2$  superconformal algebras, *J. Math. Phys.*, 39, no 7 (1998), 3865-3905.
- [G] E. GETZLER, Two-dimensional topological gravity and equivariant cohomology, *Commun. Math. Phys.*, 163, no 3 (1994), 473-489.
- [GH] P. GRIFFITHS, J. HARRIS, *Principles of algebraic geometry*, Wiley-Interscience Publ., New York, 1978.
- [KL] V. G. KAC, J. W. van de LEUR, On Classification of Superconformal Algebras, in S. J. Gates et al., editors, *Strings-88*, World Scientific, 1989, 77-106.
- [P1] E. POLETAEVA, Semi-infinite Weil complex and  $N = 2$  superconformal algebra I, preprint MPI 97-78, Semi-infinite Weil complex and superconformal algebras II, preprint MPI 97-79.
- [P2] E. POLETAEVA, Superconformal algebras and Lie superalgebras of the Hodge theory, preprint MPI 99-136.
- [P3] E. POLETAEVA, Semi-infinite cohomology and superconformal algebras, *Comptes Rendus de l'Académie des Sciences*, t. 326, Série I (1998), 533-538.

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