ANNALES DE L'INSTITUT FOURIER

TAKEO NODA

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Annales de l'institut Fourier, tome 50, nº 5 (2000), p. 1617-1647

<http://www.numdam.org/item?id=AIF_2000__50_5_1617_0>

Ann. Inst. Fourier, Grenoble
50, 5 (2000), 1617–1647

PROJECTIVELY ANOSOV FLOWS WITH DIFFERENTIABLE (UN)STABLE FOLIATIONS

by Takeo NODA (1)

1. Introduction.

Recently, some relationships between flows, contact structures and foliations on 3-manifolds have been noticed by several authors (for example [4], [5], [10]). As is remarked in [10], the typical example is the Anosov flows. For an Anosov flow on a 3-manifold, there exist two invariant foliations called (weak) unstable and stable foliations. Then, by rotating the plane fields which are tangent to these foliations by $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$, we obtain a transverse pair of contact structures with different signs, which is called a *bi-contact structure* in [10].

Bi-contact structures are not always induced by Anosov flows. However, Mitsumatsu defined *projectively Anosov flows* in [10] and showed that they have associated bi-contact structures and bi-contact structures are always associated to projectively Anosov flows. For projectively Anosov flows, there are two invariant plane fields and they are integrable, although the integral submanifolds passing through an orbit may not be determined uniquely in general.

It is shown by Mitsumatsu in [11] that every closed oriented 3-manifold admits a bi-contact structure. But the invariant plane fields are of class C^0 in general and they may not be uniquely integrable, so it is difficult to study projectively Anosov flows in general case.

(1) This author's research is supported by Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists.
Keywords : Projectively Anosov flows – Stable foliations – Bi-contact structures.
Math. classification : 57R30 – 58F18 – 53C12 – 53C15 – 58F15.

Ghys showed in [5], [6] and [7] that if an Anosov flow has smooth invariant foliations then it can be represented as an algebraic Anosov flow or a quasi-Fuchsian flow after changing the parametrization. Similarly, if we assume the smoothness of the invariant plane fields, we can see the topology of the flow more precisely. Indeed, it is proved in [4] that if the invariant plane fields are of class $\mathcal{C}^1$ then they are uniquely integrable and induce invariant foliations of class $\mathcal{C}^1$. We study projectively Anosov flows by making use of informations on the invariant foliations.

In this paper, we consider projectively Anosov flows with differentiable invariant foliations. The remarkable difference between Anosov flows and projectively Anosov flows is the fact that invariant foliations of projectively Anosov flows may have compact leaves. If there are compact leaves, they are homeomorphic to the 2-torus as there are non-singular flows on them. These compact leaves play an important role in considering the topology of invariant foliations.

We first study the flows on the 2-torus which are extended in the neighbourhood of the torus to projectively Anosov flows with differentiable invariant foliations such that the torus is a compact leaf of one of these foliations. We will give the necessary and sufficient condition for such flows. This is a condition on the configuration of Reeb components and the linear holonomies of closed orbits.

We further study the projectively Anosov flows on the torus bundle over the circle. There exist models on $T^2 \times I$ and if the invariant foliations are of class $\mathcal{C}^2$ and with compact leaves then the projectively Anosov flows on the torus bundles over the circle are topologically isotopic to finite unions of these models. In this case, all the flows on the compact leaves are topologically isotopic to linear foliations. In contrast with the local study mentioned above, we will see that the global topology of the 3-manifolds plays an essential role in extending flows to projectively Anosov flows in the whole manifolds.

This paper is organized as follows. In Section 2, we review the definition of projectively Anosov flows and some equivalent conditions, which have been known in [4] and [10].

In Section 3, we introduce models on $T^2 \times I$ and by using this, we construct our main example. At the end of this section, we refer to an example with non-uniquely integrable invariant plane fields.

In Section 4, we study the necessary and sufficient condition for flows on T^2 to be extended to projectively Anosov flows in the neighbourhood of T^2.

In Section 5, we state and prove the classification theorems of the projectively Anosov flows on the torus budle over the circle.

I should like to express my gratitude to Professor Takashi Tsuboi for his useful advice. I also thank Professor Yoshihiko Mitsumatsu for his suggestions. In particular, Proposition 3.3 is due to him.

2. Definitions.

Throughout this paper, M is a closed oriented 3-manifold and $\phi^t : M \to M$ is a smooth flow.

A flow ϕ^t is called an *Anosov flow* if there exist a continuous Riemannian metric on M and a continuous splitting $TM = T\phi \oplus E^{uu} \oplus E^{ss}$ of the tangent bundle into a direct sum of line bundles, such that the splitting is invariant under the flow and that for some $A > 0$, $C > 0$ the following inequality holds:

$$\left\|d\phi^t(v^u)\right\| \geq A\,\mathrm{e}^{Ct}\|v^u\|, \quad \left\|d\phi^t(v^s)\right\| \leq A^{-1}\,\mathrm{e}^{-Ct}\|v^s\|,$$

for all $t > 0$, $v^u \in E^{uu}$, $v^s \in E^{ss}$.

We can see that the constant A can be taken to be 1 by replacing the original metric g_0 with an average $g = T^{-1}\int_0^T \phi^* g_0$ for sufficiently large T. So we will assume that $A = 1$ and that any non-zero vector in E^{uu} begins growing bigger immediately along the flow.

It is well-known that the invariant plane fields $E^u = T\phi \oplus E^{uu}$ and $E^s = T\phi \oplus E^{ss}$ are of class C^1 and induce two codimension one foliations $\mathcal{F}^u$ and $\mathcal{F}^s$, which are called the *(weak) unstable and stable foliation* respectively.

Notice that among the plane fields which are tangent to the flow, E^u and E^s are the only ones which are preserved by ϕ^t. It follows that if we take two plane fields ξ and η which are obtained by rotating E^u by the angle of $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$ respectively then they cannot be integrable anywhere and thus define a pair of contact structures with different signs. Recall that if a contact structure is defined by a 1-form α then the sign of that contact structure is determined by the sign of $\alpha \wedge d\alpha$ with respect to some fixed volume form of M

Definition 2.1. — A pair of tangent plane fields (ξ,η) is said to be a *bi-contact structure* if ξ and η are contact structures with different signs and intersect transversely.

It has been seen that if there is an Anosov flow then there is a bi-contact structure such that the intersection of the plane fields is the tangent space of the flow. But the converse does not hold in general. The *projectively Anosov flow* is defined to be the flow along intersection of the plane fields of bi-contact structure.

In the following definition, the *oriented projectified S^1-bundle* $S^1(TM/T\phi)$ is defined to be the associated $(\mathbb{R}^2 - \{0\})/\mathbb{R}_+$-bundle of the set of oriented lines in the normal bundle of the flow ϕ^t.

DEFINITION 2.2. — A flow ϕ^t is called a *projectively Anosov flow* if there are four continuous sections $\mathcal{E}^u_\pm, \mathcal{E}^s_\pm$ in $S^1(TM/T\phi)$ which are invariant under the action induced by ϕ^t, and if any orbit which is not contained in $\mathcal{E}^u_\pm$ or $\mathcal{E}^s_\pm$ is attracted to $\mathcal{E}^u_\pm$ (resp. $\mathcal{E}^s_\pm$) when $t \to \infty$ (resp. $t \to -\infty$).

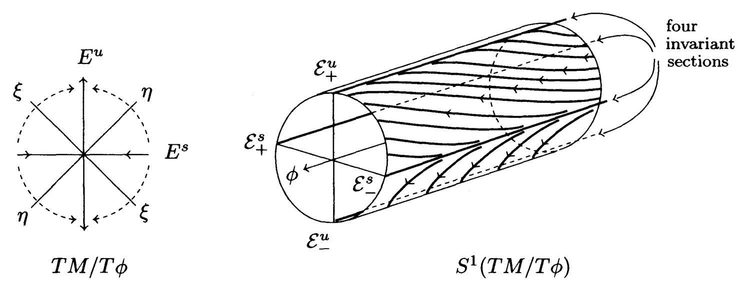


Figure 1

PROPOSITION 2.3 (see [10]). — *A flow ϕ^t is a projectively Anosov flow if and only if there exists a bi-contact structure (ξ, η) which satisfies $\xi \cap \eta = T\phi$.*

Remark. — Eliashberg and Thurston defined in [4] a *conformally Anosov flow* to be a flow ϕ^t if there exists a continuous Riemannian metric and a continuous splitting $TM/T\phi = \widehat{E}^u \oplus \widehat{E}^s$, invariant under the action $d\phi^t$, such that for some $C > 0$, the following inequality holds:

$$\frac{\|d\phi^t(v^u)\|}{\|d\phi^t(v^s)\|} \geq \mathrm{e}^{Ct} \frac{\|v^u\|}{\|v^s\|},$$

for all $t > 0$, $v^u \in \widehat{E}^u$, $v^s \in \widehat{E}^s$.

It is proved in [4] that a flow ϕ^t is a conformally Anosov flow if and only if there exists a bi-contact structure (ξ, η) which satisfies $\xi \cap \eta = T\phi$. Hence it is equivalent to the projectively Anosov flows.

The sections $\mathcal{E}^u_\pm$ and $\mathcal{E}^s_\pm$ naturally define continuous tangent plane fields E^u and E^s, which are called *unstable and stable plane fields* respectively. If a bi-contact structure (ξ,η) is given, E^u and E^s are described as follows:

$$E^u = \lim_{t\to+\infty} d\phi^t(\xi) = \lim_{t\to+\infty} d\phi^t(\eta),$$

$$E^s = \lim_{t\to-\infty} d\phi^t(\xi) = \lim_{t\to-\infty} d\phi^t(\eta).$$

It is known that E^u and E^s are integrable in the sense that for every point of M there exists some integral submanifold passing through that point, but unlike the Anosov case, these integral submanifolds may not be determined uniquely in general.

On the other hand, if E^u and E^s are C^1-smooth then they are uniquely integrable and define foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. We call them *(weak) unstable and stable foliations* respectively.

PROPOSITION 2.4 (see [4]). — *Suppose ϕ^t is a projectively Anosov flow with C^1-smooth unstable and stable foliations $\mathcal{F}^u$ and $\mathcal{F}^s$. Then $\mathcal{F}^u$ and $\mathcal{F}^s$ can be defined by 1-forms α_u and α_s such that*

$$\langle \alpha_u, \alpha_s\rangle \overset{\text{def}}{=\!=} \alpha_u \wedge \mathrm{d}\alpha_s + \alpha_s \wedge \mathrm{d}\alpha_u > 0. \tag{1}$$

Conversely, suppose forms α_u and α_s satisfy the inequality (1) *and define the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$, then any non-vanishing vector field on $\mathcal{F}^u \cap \mathcal{F}^s$ defines a projectively Anosov flow.*

In the case of Anosov flows, there exist foliations $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ determined by the line fields E^{uu} and E^{ss} respectively and they are called *strong unstable and stable foliations.* However, in the case of projectively Anosov flows, they do not always exist.

DEFINITION 2.5. — A *strong Anosov splitting* of a projectively Anosov flow ϕ^t is a continuous splitting $TM = T\phi \oplus E^{uu} \oplus E^{ss}$ of the tangent bundle into a direct sum of line bundles which is invariant under the flow.

Let L be a leaf in $\mathcal{F}^\sigma$ $(\sigma = u, s)$ and $\widetilde{L}$ be its universal covering. Then ϕ^t naturally induces a flow $\tilde{\phi}^t|_{\widetilde{L}}$ on $\widetilde{L}$. Let $\pi_L : \widetilde{L} \to \mathcal{O}_L$ be the natural projection to the orbit space $\mathcal{O}_L$ of the flow $\tilde{\phi}^t|_{\widetilde{L}}$.

PROPOSITION 2.6. — *Suppose that a projectively Anosov flow ϕ^t admits a strong Anosov splitting. Then for each leaf L in $\mathcal{F}^\sigma$ $(\sigma = u,s)$, the orbit space $\mathcal{O}_L$ of the induced flow $\tilde{\phi}^t|_{\widetilde{L}}$ on $\widetilde{L}$ is Hausdorff.*

Proof. — Since ϕ^t admits a strong Anosov splitting, there exists a continuous line field $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ which is invariant under $\tilde{\phi}^t|_{\widetilde{L}}$ on $\widetilde{L}$ and transverse to the orbits of $\tilde{\phi}^t|_{\widetilde{L}}$ on $\widetilde{L}$. This line field is integrable though the integral submanifolds may not be determined uniquely.

CLAIM 1. — *There exists a C^0-foliation $\mathcal{G}$ tangent to $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ and invariant under $\tilde{\phi}^t|_{\widetilde{L}}$.*

Proof of Claim 1. — If $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ is uniquely integrable at every point of $\widetilde{L}$, then the integral submanifolds give $\mathcal{G}$.

Suppose that the integral submanifold of $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ is not determined uniquely at a point p of $\widetilde{L}$. Since the integral manifolds are locally represented as the solutions of an ordinary differential equation of class C^0, there exist infinitely many integral submanifolds bounded by two of them, which are called *maximal and minimal solutions* (see [2], for example).

It is clear that $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ is not integrated uniquely at each point of the orbit of p. Let ℓ_1 be one of the integral submanifolds of $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ at p and let q be a point of ℓ_1 which is different from p. For another integral submanifold ℓ_2 at p sufficiently near ℓ_1, there exists some real number $s_0 \neq 0$ such that $\tilde{\phi}^{s_0}|_{\widetilde{L}}(q) \in \ell_2$. Then $\tilde{\phi}^{-s_0}|_{\widetilde{L}}(\ell_2)$ is an integral submanifold at q, but $\tilde{\phi}^{-s_0}|_{\widetilde{L}}(\ell_2) \neq \ell_1$ since $\tilde{\phi}^{s_0}|_{\widetilde{L}}(p) \neq p$. Thus we have shown that $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ is not integrated uniquely at q. By applying the procedures above repeatedly, we can see that $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$ is nowhere uniquely integrable.

For each point of $\widetilde{L}$, take the maximal solution on the right side of the flow $\tilde{\phi}^t|_{\widetilde{L}}$ and the minimal on the left. This gives a C^0-foliation $\mathcal{G}$ tangent to $\widetilde{E^{\sigma\sigma}}|_{\widetilde{L}}$. We can see that $\mathcal{G}$ is invariant under $\tilde{\phi}^t|_{\widetilde{L}}$ since the maximal and minimal solutions are invariant under $\tilde{\phi}^t|_{\widetilde{L}}$.

The image of a leaf ℓ of $\mathcal{G}$ by π_L defines an open set U_L in $\mathcal{O}_L$. For leaves ℓ, m of $\mathcal{G}$, it holds that $U_\ell \cap U_m \neq \emptyset$ if and only if $U_\ell = U_m$. In fact, if $U_\ell \cap U_m \neq \emptyset$ then there exists an orbit of $\tilde{\phi}^t|_{\widetilde{L}}$ which intersects ℓ and m. It implies that $\widetilde{\phi^\tau}|_{\widetilde{L}}(\ell) = m$ for some real number τ, so $U_\ell = U_m$.

Take two distinct orbits o_1, o_2 in $\mathcal{O}_L$ and points x_1, x_2 of the orbits o_1, o_2 respectively. Then the leaves ℓ_1, ℓ_2 of $\mathcal{G}$ passing through x_1, x_2 define two open sets U_1, U_2 in $\mathcal{O}_L$. If $U_1 \cap U_2 = \emptyset$ then o_1 and o_2 are separated by U_1 and U_2.

Suppose that $U_1 \cap U_2 \neq \emptyset$. In this case it holds that $U_1 = U_2$ and there exists a real number t_0 such that $\tilde{\phi}^{t_0}|_{\widetilde{L}}(\ell_1) = \ell_2$. Since ℓ_2 is homeomorphic to the real line, the points $\tilde{\phi}^{t_0}|_{\widetilde{L}}(x_1)$ and x_2 are separated in ℓ_2 by open neighbourhoods V_1 and V_2 of $\tilde{\phi}^{t_0}|_{\widetilde{L}}(x_1)$ and x_2 respectively. It is enough to show that $\pi_L^{-1}(V_1) \cap \pi_L^{-1}(V_2) = \emptyset$. If $\pi_L^{-1}(V_1) \cap \pi_L^{-1}(V_2) \neq \emptyset$, there exists an orbit o' of $\tilde{\phi}^t|_{\widetilde{L}}$ which intersects ℓ_2 at least two times. Then there exists a 2-disk in $\widetilde{L}$ bounded by segments in o' and ℓ_2. By considering the index of the vector field on the disk which generates $\tilde{\phi}^t|_{\widetilde{L}}$, we can see that this cannot occur. This implies that the preimages of V_1 and V_2 by π_L separate o_1 and o_2 in $\mathcal{O}_L$. Thus we have proved that $\mathcal{O}_L$ is Hausdorff.

By this proposition, we can construct examples which do not admit strong Anosov splittings. See Remark 2 after Proposition 3.3.

3. Examples.

In this section, we give the models for projectively Anosov flows on $T^2 \times I$.

By definition, Anosov flows are projectively Anosov flows. There are examples of projectively Anosov flows which are not Anosov flows. We only consider such examples.

To study projectively Anosov flows, there are two viewpoints; one is from bi-contact structures and the other is from invariant plane fields.

However, it is difficult to compute the invariant plane fields from a given bi-contact structure. Moreover, even if it is possible, these plane fields are only of class $\mathcal{C}^0$ in general, so there is another difficulty in studying the properties of integral submanifolds.

On the other hand, once invariant plane fields are given by $\mathcal{C}^1$-1-forms which satisfy the condition in Proposition 2.4, a corresponding bi-contact structure is easily obtained; indeed, given 1-forms α_u, α_s such that $\langle \alpha_u, \alpha_s \rangle > 0$, it follows that $(\alpha_u + \alpha_s) \wedge \mathrm{d}(\alpha_u + \alpha_s) > 0$ and $(\alpha_u - \alpha_s) \wedge \mathrm{d}(\alpha_u - \alpha_s) < 0$, so the plane fields defined by $(\alpha_u + \alpha_s)$ and $(\alpha_u - \alpha_s)$ form a bi-contact structure.

Our first example is of this type.

Example 3.1 (see [4]). — Let (x, y, z) be coordinates in $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Take two 1-forms $\alpha = \mathrm{d}z - \cos(2\pi z)\,\mathrm{d}x$ and $\beta = \mathrm{d}z - \sin(2\pi z)\,\mathrm{d}y$. Then $\langle \alpha, \beta \rangle > 0$ and hence the foliations $\{\alpha = 0\}$ and $\{\beta = 0\}$ serve as the invariant foliations of a projectively Anosov flow.

These foliations have two compact leaves respectively and the flows on the compact leaves are linear flows. In the neighbourhood of compact leaves of the stable foliaton, the orbits flow away from the compact leaves with curving and finally wind around the compact leaves of the unstable foliation as $t \to \infty$.

In this case, the bi-contact structure is naturally given by the 1-forms $\omega = \alpha + \beta$ and $\eta = \alpha - \beta$. But there is another interpretation for this bi-contact structure, which is mentioned in [10].

Consider the two 1-forms $\omega_0 = -\cos(2\pi z)\,\mathrm{d}x - \sin(2\pi z)\,\mathrm{d}y$ and $\eta_0 = -\cos(2\pi z)\,\mathrm{d}x + \sin(2\pi z)\,\mathrm{d}y$. The plane fields defined by these forms are tangent to the lines parallel to the z-axis and induce linear foliations on $T^2 \times \{z\}$ for all $z \in S^1$, the direction of which rotates counterclockwise and clockwise respectively when z increases. Then ω_0 and η_0 define positive and negative contact structures respectively (in [4], this way of construction is called *propeller construction*). They are not transverse on $\{z = 0\}, \{z = \frac{1}{4}\}, \{z = \frac{1}{2}\}, \{z = \frac{3}{4}\}$ (Mitsumatsu in [10] calls such a pair a *pre-bi-contact structure*). Now take a function $\varepsilon(z)$ which does not vanish on these tori, then $\omega_0 + \varepsilon(z)\,\mathrm{d}z$ and η_0 become transverse and therefore they define a bi-contact structure. In particular, if we take $\varepsilon(z) \equiv 2$, then they coincide with ω and η above (see Figure 2).

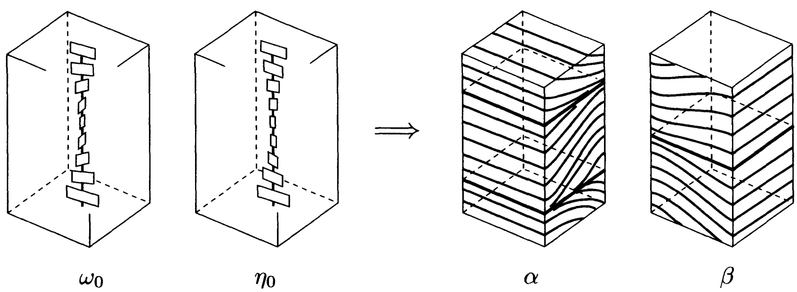


Figure 2. Projectively Anosov flow by propeller construction

Remark. — As to the construction by bi-contact structures, a similar way works for any mapping tori of the 2-torus. However, the invariant plane fields are more complicated and their smoothness is not guaranteed.

We construct our main example by generalizing the previous example.

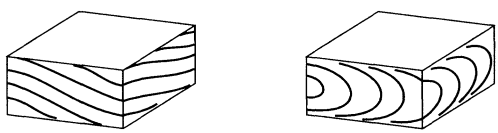
Example 3.2 ($T^2 \times I$-model). — Let us review two basic foliations $\mathcal{F}([\omega], i)$ for $i = 1, 2$ on $T^2 \times I$, which is defined by Moussu and Roussarie in [12] (see Figure 3).

These foliations are defined respectively by the 1-forms

$$\Omega_1 = \theta_1(z)\,\mathrm{d}z + \psi(z)\omega, \quad \Omega_2 = \theta_2(z)\,\mathrm{d}z + \psi(z)\omega,$$

where

- ω is a linear 1-form on T^2;
- $\theta_1 : I \to \mathbb{R}$ is a smooth function such that $\theta_1(0) = \theta_1(1) = 1$, $\theta(\frac{1}{2}) = 0$, θ_1 decreases strictly for $0 < z < \frac{1}{2}$ and increases strictly for $\frac{1}{2} < z < 1$, and that all derivatives vanish at $z = 0, \frac{1}{2}, 1$;
- $\theta_2 : I \to \mathbb{R}$ is a smooth function such that $\theta_2(0) = 1$, $\theta_2(\frac{1}{2}) = 0$, $\theta_2(1) = -1$, θ_2 decreases strictly for $0 < z < \frac{1}{2}$ and $\frac{1}{2} < z < 1$ and that all derivatives vanish at $z = 0, \frac{1}{2}, 1$;
- $\psi : I \to \mathbb{R}$ is a smooth function such that $\psi(0) = \psi(1) = 0$, $|\psi(\frac{1}{2})| = 1$ and $|\psi(z)|$ increases strictly for $0 < z < \frac{1}{2}$ and decreases for $\frac{1}{2} < z < 1$.



$i = 1$ $i = 2$

Figure 3. Basic foliations on $T^2 \times I$

Now take two linear 1-forms ω_u and ω_s on T^2 such that $\omega_u \wedge \omega_s > 0$ and define two foliations on $T^2 \times I$ as follows.

First, let $\mathcal{F}^u$ be one of the basic foliations $\mathcal{F}([\omega_u], i)$. Then it can be defined by a 1-form $\Omega_u = \theta_u(z)\,\mathrm{d}z + \psi_u(z)\omega_u$.

Secondly, let $\mathcal{F}^s$ be a smooth foliation on $T^2 \times I$ which coincides with the upper half part of some $\mathcal{F}([\omega_s], 2)$ in $T^2 \times [0, \frac{1}{2}]$ and the lower half part of the $\mathcal{F}([\omega_s], 2)$ in $T^2 \times [\frac{1}{2}, 1]$. Then $T^2 \times \{\frac{1}{2}\}$ is its unique compact leaf of $\mathcal{F}^s$. Let $\Omega_s = \theta_s(z)dz + \psi_s(z)\omega_s$ be a 1-form which defines this foliation.

Suppose further that the linear holonomies of these foliations at compact leaves are not trivial. This implies $\psi_u'(0), \psi_u'(1), \psi_s'(\frac{1}{2}) \neq 0$.

It is easy to see that ψ_u, ψ_s satisfy

$$\langle \Omega_u, \Omega_s \rangle = (\psi_u' \psi_s - \psi_u \psi_s')\, \omega_u \wedge \omega_s \wedge \mathrm{d}z \neq 0.$$

Hence Ω_u and Ω_s define the invariant foliations of a projectively Anosov flow by Proposition 2.4.

The projectively Anosov flows constructed on $T^2 \times I$ by this way are called $T^2 \times I$*-models* (see Figure 4).

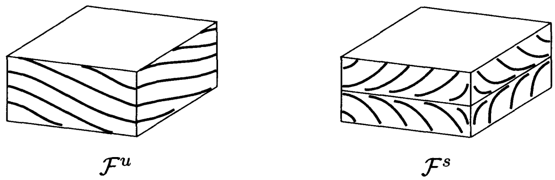


Figure 4. $T^2 \times I$*-model*

By gluing together finite number of $T^2 \times I$-models so that the resulting foliations are smooth and transversely oriented, we can obtain examples of projectively Anosov flows with smooth invariant foliations.

Remark. — In this example, the corresponding bi-contact structure is interpreted in the same way as 3.1. The perturbation $\varepsilon(z)$ is defined by θ_u and θ_s and it varies with respect to z. It is easily observed that the type $\mathcal{F}([\omega], i)$ of invariant foliations is determined by the sign of $\varepsilon(z)$ at compact leaves.

PROPOSITION 3.3. — *Let ϕ^t be a flow on T^2. Then there exists a projectively Anosov flow $\tilde{\phi}^t$ on $T^3 = T^2 \times S^1$ such that $\tilde{\phi}^t|_{T^2 \times \{0\}} = \phi^t$.*

Proof. — Let $\theta : T^2 \to S^1 = \mathbb{R}/\mathbb{Z}$ be a function such that $\theta(x, y)$ is an angle which is made by the flow at (x, y) and the x-axis.

Then the two 1-forms on T^3

$$\alpha = \sin\big(\theta(x,y) - 2\pi z\big)\,\mathrm{d}x - \cos\big(\theta(x,y) - 2\pi z\big)\,\mathrm{d}y + \mathrm{d}z,$$

$$\beta = \sin\big(\theta(x,y) + 2\pi z\big)\,\mathrm{d}x - \cos\big(\theta(x,y) + 2\pi z\big)\,\mathrm{d}y$$

satisfy $\alpha \wedge \mathrm{d}\alpha > 0$ and $\beta \wedge \mathrm{d}\beta < 0$ and the plane fields defined by $\ker\alpha$ and $\ker\beta$ intersect transversely. It follows that they define a bi-contact structure on T^3 and therefore a non-vanishing vector field on $\ker\alpha \cap \ker\beta$ is a projectively Anosov flow. Furthermore, its restriction to $T^2 \times \{0\}$ is the original flow.

Remark 1. — By this proposition we can give an example of a projectively Anosov flow on T^3 with non-uniquely integrable invariant plane fields.

Consider a 2-torus T^2 and a foliation which consists of two Reeb components in the same direction such that their linear holonomies along closed orbits are all equal to 1.

If we extend this foliation by Proposition 3.3, the original torus is invariant under the flow, therefore it is a leaf of invariant foliations. We may assume that this torus is a compact leaf of the unstable foliation. Since the linear holonomies of the stable foliation along the two closed orbits are both trivial, those of the unstable foliation must be both less than 1. However, these closed orbits have the opposite direction to each other and therefore the unstable foliation cannot be uniquely integrable.

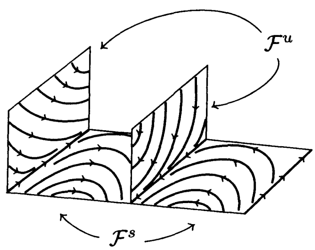


Figure 5

Remark 2. — The example constructed above is also an example of projectively Anosov flows without strong Anosov splittings. Since there exist Reeb components on the flow on the compact leaf, the orbit space of the flow induced on the universal covering of the compact leaf is non-Hausdorff. By Proposition 2.6, this flow does not admit a strong Anosov splitting.

4. Flows on the compact leaves.

One of the greatest differences between Anosov flows and projectively Anosov flows is the fact that the invariant foliations of projectively Anosov flows may have compact leaves. If compact leaves exist, they must be 2-tori since there is the restricted flow on every leaf.

We have already seen in Proposition 3.3 that every flow on T^2 can be extended to a projectively Anosov flow on T^3 but we have also remarked that the invariant foliations of the extended flow are often not smooth, or even non-uniquely integrable.

In this section, we observe equivalent conditions for a flow on T^2 to be extended locally to a projectively Anosov flow with invariant foliations of class $\mathcal{C}^1$ (this implies that these foliations have linear holonomies along closed orbits). Here a local extension means to give a pair of foliations in the neighbourhood of T^2 which satisfies the condition in Proposition 2.4.

As preliminary, let us review the foliations on T^2 with closed leaves. These foliations are topologically conjugate to unions of the foliations on $S^1 \times [0,1]$ which are defined by the following types of 1-form:

$$\omega_1 = \mathrm{d}y + \psi(y)\,\mathrm{d}x, \quad \omega_2 = (1-2y)\,\mathrm{d}y + \psi(y)\,\mathrm{d}x,$$

where $(x,y) \in S^1 \times [0,1]$, and $\psi(y)$ is a smooth function such that $\psi(0) = \psi(1) = 0$ and $|\psi(y)| > 0$. The foliations defined by ω_1 and ω_2 are called *slope components* and *Reeb components* respectively. Furthermore, the Reeb components is called *plus* (resp. *minus*) if $\psi(y) \le 0$ (resp. $\psi(y) \ge 0$) with respect to the positive direction of S^1.

The following theorem gives equivalent conditions for a flow to be extended to a projectively Anosov flow.

THEOREM 4.1. — *A C^1-flow on T^2 can be extended to a projectively Anosov flow on $T^2 \times (-\varepsilon,\varepsilon)$ so that the invariant foliations are of class C^1 and T^2 is a compact leaf of the unstable foliation if and only if the foliation defined by the flow has no Reeb component or all of the following three conditions hold*:

(a) *there exist only minus Reeb components*;

(b) *the closed orbits in the negative direction are between two Reeb components*;

(c) *all linear holonomies of the flow along closed orbits in the positive direction are greater than the inverse numbers of those in the negative direction.*

Here the direction of the closed orbits is defined as the direction of some closed orbit with non-expanding holonomy.

We may remark that if a flow on the 2-torus is extended to a projectively Anosov flow near the 2-torus, that 2-torus is a leaf of one of the invariant foliations, since it is invariant under the flow.

To prove this, we need the following lemma.

LEMMA 4.2. — *Let $\mathcal{F}$ be a transversely oriented codimension one foliation of class C^r $(r \ge 2)$ on a 2-manifold and let c be a closed leaf of $\mathcal{F}$ with a holonomy map $h(y)$, where $h(0) = 0$. Then in the neighbourhood of c, $\mathcal{F}$ is C^r-conjugate to a foliation defined by C^r-1-form $\alpha = \mathrm{d}y + \psi(x,y)\,\mathrm{d}x$ such that $\psi_y(x,0) = -\log h'(0)$, where (x,y) are coordinates in $S^1 \times (-a,a)$.*

Proof of 4.2. — It is enough to show that the holonomy map $h(y)$ is C^r-conjugate to that of such a foliation.

Suppose that $h'(0) \neq 1$. By the theorem of Sternberg [16], which is improved by Yoccoz [20], h is C^r-conjugate to the linear function $y \mapsto (h'(0))y$. Then $\mathcal{F}$ is C^r-conjugate to the foliation defined by the 1-form $\alpha = \mathrm{d}y + (\log h'(0))y\,\mathrm{d}x$, which has the holonomy map $y \mapsto (h'(0))y$.

In the case of $h'(0) = 1$, we cannot apply the theorem of Sternberg, so we have to rewrite a foliation with holonomy $h(y)$ explicitly.

To do this, let us define the following functions:

$$s(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x^2}} & (x > 0), \\ 0 & (x \leq 0), \end{cases} \qquad u(x) = \frac{s(x)}{s(x) + s(1-x)}.$$

Then $u(x)$ is a smooth increasing function such that $u(x) = 0$ for $x \leq 0$, $u(x) = 1$ for $x \geq 1$ and $0 < u(x) < 1$ for $0 < x < 1$.

Consider the foliation on $S^1 \times (-a, a) = [0,1] \times (-a,a)/(0,y) \sim (1,y)$ such that a leaf passing through $(0, y_0)$ is defined by the equation

$$y = y_0 + u(x)\big(h(y_0) - y_0\big).$$

This foliation has a compact leaf $S^1 \times \{0\}$ and the holonomy map along this leaf is $h(y)$. Put the function $g(x,y)$ to be the y-coordinate of the intersection of the leaf passing through (x,y) and $\{0\} \times (-a,a)$ (see Figure 6). Then we can generally describe the leaf passing through a point (x_1, y_1) by the equation

$$y = g(x_1, y_1) + u(x)\big(h(g(x_1,y_1)) - g(x_1,y_1)\big).$$

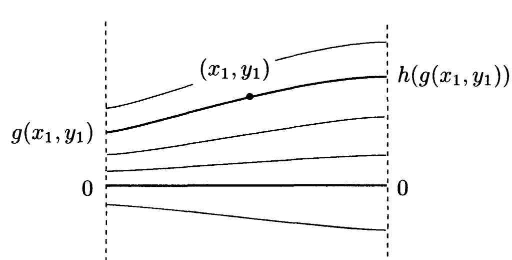


Figure 6

This foliation is defined by the 1-form

$$\alpha = dy + \psi(x,y)dx, \quad \text{where} \quad \psi(x,y) = -u'(x)\big(h(g(x,y)) - g(x,y)\big).$$

It holds that

$$\begin{aligned}\psi_y(x,0) &= -u'(x)\big(h'(g(x,0)) - 1\big)g_y(x,0)\\ &= -u'(x)\big(h'(0) - 1\big)g_y(x,0) = 0.\end{aligned}$$

Thus we have proved the lemma.

Proof of 4.1. — Suppose that a flow on T^2 with Reeb components is extended to such a projectively Anosov flow on $T^2 \times (-\varepsilon, \varepsilon)$. Let λ^u be the linear holonomy of $\mathcal{F}^u$ along the closed orbit in the positive direction of the original flow. By the definition of the direction of closed orbits, it is clear that λ^u is less than 1.

If there exists a plus Reeb component, there exists a closed orbit in the negative direction which contained in the boundary of the Reeb component. However, the holonomy of $\mathcal{F}^u$ along this orbit is expanding and that of $\mathcal{F}^s$, which coincides with that of the flow, is contracting. This is a contradiction.

Consider a closed orbit in the negative direction. Since the holonomy of $\mathcal{F}^u$ along this orbit is expanding, that of $\mathcal{F}^s$ must be also expanding. If this orbit is a component of the boundary of a slope component then the other component of the boundary is also a closed orbit in the negative direction but the holonomy of $\mathcal{F}^s$ along it is contracting. This cannot occur.

By comparing the linear holonomies along the closed orbits, we can deduce that all of the linear holonomies of $\mathcal{F}^s$ along the closed orbits in the positive direction are greater than λ^u and that those in the negative direction are greater than $(\lambda^u)^{-1}$. This implies the condition (c), and thus we have proved the necessity.

Now, let us prove the sufficiency. To do this, it is enough to find a pair of C^1-1-forms α, β which satisfies $\langle \alpha, \beta \rangle > 0$ and such that

- α defines the product foliation of the foliation by the flow on $T^2 \times (-\varepsilon, \varepsilon)$;
- β defines a foliation on $T^2 \times (-\varepsilon, \varepsilon)$ with a compact leaf $T^2 \times \{0\}$.

Remark that the property $\langle \alpha, \beta \rangle > 0$ is invariant under the action of an orientation preserving C^1-diffeomorphism H of $T^2 \times (-\varepsilon, \varepsilon)$, since $\langle H^*\alpha, H^*\beta \rangle = H^*\langle \alpha, \beta \rangle$. Furthermore, we only have to show the case where the stable foliation, which is defined by α, is of class $\mathcal{C}^2$. In fact, by the Whitney approximation theorem, the $\mathcal{C}^1$-1-form α can be approximated by a $\mathcal{C}^2$-1-form $\widetilde{\alpha}$ and if $\langle\widetilde{\alpha},\beta\rangle > 0$ for sufficiently good approximation $\widetilde{\alpha}$ then $\langle\alpha,\beta\rangle > 0$.

If the flow has no Reeb component, we can define the stable foliation by the 1-form

$$\alpha = \mathrm{d}y + \psi(x,y)\,\mathrm{d}x$$

for some coordinates. Consider the foliation such that a leaf passing through (x_0, y_0, z_0) is defined by the equation

$$z = \mathrm{e}^{-A(x-x_0)} z_0,$$

where $A > 0$. This foliation can also be defined by the 1-form

$$\beta = Az\,\mathrm{d}x + \mathrm{d}z$$

for the same A. Then it holds that

$$\langle\alpha,\beta\rangle = \big(A - \psi_y(x,y)\big)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z.$$

This is positive for sufficiently large A.

Suppose that the flow has Reeb components which satisfy the conditions (a), (b) and (c).

In this case, we decompose the flow along closed orbits into three types of components, which are homeomorphic to $S^1\times[0,1]$, and construct corresponding unstable foliations separately.

First of all, by condition (c), we can define the linear holonomy λ^u of the unstable foliation along closed orbits in the positive direction so that λ^u which is smaller than all linear holonomies along the closed orbit in the positive direction and greater than the inverse numbers of those in the negative direction. Put $A = -\log\lambda^u$, which is a positive number.

Type I: neighbourhoods of accumulating closed orbits. If the flow has an infinite number of closed orbits, there exist accumulating closed orbits. It is clear by condition (b) that these orbits are in the positive direction. Consider the neighbourhood of such a closed orbits. Let $h(y)$ be a holonomy map of the flow along this orbit. By the same argument as in the proof of Lemma 4.2, we can describe the stable foliation in the neighbourhood of this orbit by the 1-form

$$\alpha = \mathrm{d}y + \psi(x,y)\,\mathrm{d}x, \quad \text{where} \quad \psi(x,y) = -u'(x)\big(h(g(x,y)) - g(x,y)\big).$$

By the smoothness, we can see that if $|y|$ is small, $h(y)$ and $g(x,y)$ are C^1-close to the identity map and the projection to the second coordinate respectively. Then by taking sufficiently small neighbourhood, it holds that

$$\big|\psi_y(x,y)\big| = \big|-u'(x)\big(h'(g(x,y)) - 1\big)g_y(x,y)\big| < \delta$$

for arbitrarily small $\delta > 0$. Take the 1-form

$$\beta = Az\,\mathrm{d}x + \mathrm{d}z,$$

then for sufficiently small δ it holds that

$$\langle \alpha, \beta \rangle = \big(A - \psi_y(x,y)\big)dx \wedge dy \wedge dz > 0.$$

Type II: *pairs of Reeb components.* By condition (b), we can see that Reeb components always appear in pairs, since one of the closed orbits in a Reeb component is in the negative direction and this orbit is put between two Reeb components. Consider a pair of minus Reeb components on $S^1 \times [0,1]$ such that $S^1 \times \{0\}$, $S^1 \times \{\frac{1}{2}\}$ and $S^1 \times \{1\}$ are the closed orbits. We will rewrite this foliation into some adequate form and afterwards construct the unstable foliation, which is defined by 1-form β.

Take a small number $a > 0$. By Lemma 4.2, this foliation can be defined by the 1-form $\alpha_0 = dy + \psi_0(x,y)\,\mathrm{d}x$ in $S^1 \times [0, 2a)$. Consider the foliation on $S^1 \times (a, 3a)$ defined by the 1-form

$$\alpha_1 = u\Big(\frac{3a - y}{2a}\Big)\,\mathrm{d}y + u\Big(\frac{y - a}{2a}\Big)\,\mathrm{d}x,$$

where u is the function defined in Lemma 4.2. Using a partition of unity, we obtain a 1-form $\alpha = \theta(x,y) + \psi(x,y)$ on $S^1 \times [0, 3a)$ which coincides α_0 on $S^1 \times [0,a]$ and α_1 on $S^1 \times [2a, 3a)$. Apply the same argument to the neighbourhoods of the other closed orbits, $S^1 \times (\frac{1}{2} - 3a, \frac{1}{2} + 3a)$ and $S^1 \times (1 - 3a, 1]$. Here let $\theta(x,y) = -1$ in the neighbourhood of $S^1 \times \{\frac{1}{2}\}$ by reason of orientation. As to the remaining parts, $S^1 \times [3a, \frac{1}{2} - 3a]$ and $S^1 \times [\frac{1}{2} + 3a, 1 - 3a]$, define α by describing each leaf explicitly in the same way as the proof of Lemma 4.2. For example, if a leaf passing through $(p_0, 3a)$ is connected to $(p_1, \frac{1}{2} - 3a)$, the leaf is defined by the equation

$$x = p_0 + u\Big(\frac{y - 3a}{\frac{1}{2} - 6a}\Big)(p_1 - p_0)$$

and we can define α by taking the total differential (see Figure 7).

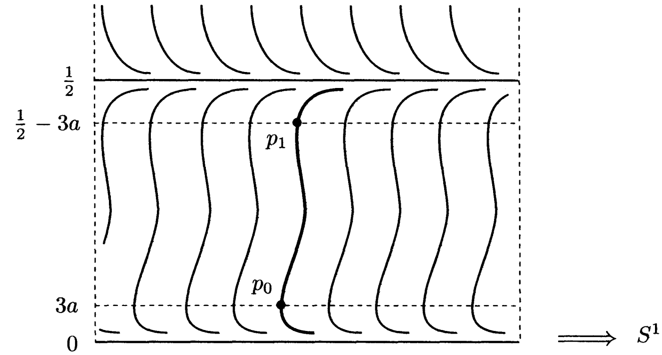


Figure 7

By construction, the functions $\theta(x,y)$ and $\psi(x,y)$ have the following properties:

- $\theta(x,y) = 1$ for $0 \le y \le a$, $1-a \le y \le 1$ and $\theta(x,y) = -1$ for $\frac{1}{2} - a \le y \le \frac{1}{2} + a$;
- $\theta_x(x,y) = 0$ for $0 \le y < 3a$, $\frac{1}{2} - 3a < y < \frac{1}{2} + 3a$, $1 - 3a < y \le 1$;
- $\psi(x,y) > 0$ for $a < y < \frac{1}{2} - a$ and $\psi(x,y) < 0$ for $\frac{1}{2} + a < y < 1 - a$;
- $\mathrm{e}^{-\psi_y(x,0)}$, $\mathrm{e}^{-\psi_y(x,\frac{1}{2})}$ and $e^{-\psi_y(x,1)}$ are the linear holonomies along the corresponding closed orbits.

Let us construct the unstable foliation. Consider a foliation such that the leaf passing through the point (x_0, y_0, z_0) is defined by the equation

$$z = \mathrm{e}^{-A(x-x_0)}\,\frac{w(y)}{w(y_0)}\,z_0,$$

where $w(y) > 0$ for $0 \le y \le 1$. This foliation can also be defined by the 1-form

$$\beta = Az\,\mathrm{d}x - \frac{w'(y)}{w(y)}\,z\,\mathrm{d}y + \mathrm{d}z.$$

For these α and β, it holds that

$$(2)\quad \langle\alpha,\beta\rangle = \Big(A\theta(x,y)+\psi(x,y)\,\frac{w'(y)}{w(y)}+\theta_x(x,y)-\psi_y(x,y)\Big)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z.$$

Let us show that (2) is positive. By condition (b), we have

$$\psi_y(x,0),\psi_y(x,1)<A,\quad \psi_y(x,\tfrac{1}{2})<-A,$$

and we can easily see that (2) is positive for $0\le y\le a$, $\frac{1}{2}-a\le y\le\frac{1}{2}+a$, $1-a\le y\le 1$ by taking smaller a according to circumstances. Since $|\psi(y)|>0$ for $a<y<\frac{1}{2}-a$ and $\frac{1}{2}+a<y<1-a$, it is possible to take a function $v(y)$ so that $\psi(x,y)v(y)$ is arbitrarily large for $a<y<\frac{1}{2}-a$ and $\frac{1}{2}+a<y<1-a$. Put $w(y)=\mathrm{e}^{\int v(y)\,\mathrm{d}y}$, then (2) is positive for an adequate function $v(y)$. In particular, we can define $w(y)$ so that all derivatives vanish at $y=0,1$.

Type III: slope components. Consider a slope component on $S^1\times[0,1]$. By the same argument as above, we can define the stable foliation by 1-form $\alpha=\theta(x,y)\,\mathrm{d}y+\psi(x,y)\,\mathrm{d}x$ such that

- $\theta(x,y)=1$ for $0\le y\le a$, $1-a\le y\le 1$;
- $\theta_x(x,y)=0$ for $0\le y<3a$, $1-3a<y\le 1$;
- $\psi(x,y)>0$ for $a<y<1-a$;
- $\mathrm{e}^{-\psi_y(x,0)}$, $\mathrm{e}^{-\psi_y(x,1)}$ are the linear holonomies along the corresponding closed orbits.

Construct the unstable foliation in the same way as above. Then the same equation as (2) holds and by the same reason, we can find β such that $\langle\alpha,\beta\rangle>0$.

Thus we have constructed local extensions for all parts of the flow and by construction, we can connect these with each other. Therefore there exist local extensions for all flows which satisfy (a), (b) and (c).

We can see by this proposition that if there are Reeb components, the orbits near the Reeb components wind around the closed orbits, which are contained in the boundaries of the Reeb components, as $t\to\pm\infty$. See Figure 8.

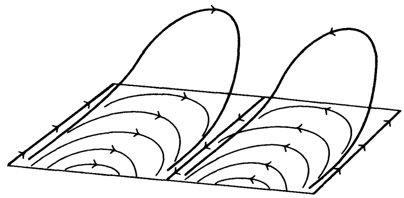


Figure 8. Orbits near Reeb component

Remark. — The extension we have constructed above is only a local one and it seems difficult to extend these flows globally into flows on some 3-manifolds. At present, no examples are known such that the invariant foliations are of class $\mathcal{C}^1$ and have a compact leaf where the restricted flow has Reeb components. On the other hand, we see in the next section that if the manifold is a T^2-bundle over S^1 then the flow restricted to a compact leaf must be isotopic to a linear flow.

5. Topology of the invariant foliations.

In this section, we assume that the invariant foliations are $\mathcal{C}^2$-smooth, and study the relationship between the topology of these foliations and the global properties of the manifold.

The following is one of the results from this viewpoint, which is stated in [10].

PROPOSITION 5.1. — *If the unstable and stable foliations of a projectively Anosov flow are differentiable then they cannot have Reeb components.*

To prove this, we need the following theorem of Tamura and Sato in [18].

THEOREM 5.2. — *Let $\mathcal{F}$ be a Reeb foliation on $S^1 \times D^2$ and let $\mathcal{F}'$ be a transversely oriented foliation on $S^1 \times D^2$ transverse to $\mathcal{F}$. Then there exists an annular leaf L of $\mathcal{F}'$ and a half Reeb component $\mathcal{F}'_{R/2}$ in $\mathcal{F}'$ such that $\partial L = L \cap \partial(S^1 \times D^2)$ consists of two compact leaves of $\mathcal{F}'|_{\partial(S^1 \times D^2)}$ and that L coincides the compact leaf of $\mathcal{F}'_{R/2}$.*

Proof of 5.1. — Assume that the unstable foliation $\mathcal{F}^u$ and the stable foliation $\mathcal{F}^s$ are of class $\mathcal{C}^1$ and that $\mathcal{F}^u$ has a Reeb component. Then by

Theorem 5.2, there exists an annular leaf of the restriction of $\mathcal{F}^s$ to the Reeb component. The restriction of this annular leaf to the compact leaf of the Reeb component consists of two circles and they are two closed orbits c_0 and c_1 of the flow (see Figure 9).

Let λ_i^σ be linear holonomies of $\mathcal{F}^\sigma$ along c_i for $\sigma = u, s$ and $i = 0, 1$. Since c_0 and c_1^{-1} are isotopic in the compact leaf of the Reeb component and in the annular leaf of the half Reeb component, it holds that

$$\lambda_0^\sigma = (\lambda_1^\sigma)^{-1},$$

for $\sigma = u, s$. However, since c_0 and c_1 are closed orbits of a projectively Anosov flow, both of the ratios of linear holonomies λ_0^s/λ_0^u and λ_1^s/λ_1^u must be greater than one. This is a contradiction.

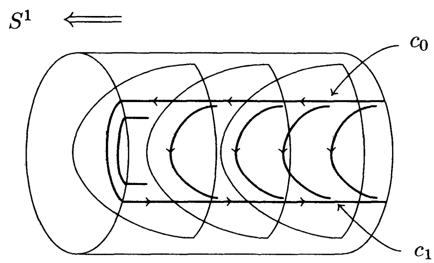


Figure 9

Applying the theorems of Novikov [14] (see also [1], [17], for example), we can obtain two corollaries from this proposition.

COROLLARY 5.3. — *Projectively Anosov flows on a 3-manifold whose universal cover is S^3 or $S^2 \times \mathbb{R}$ cannot have differentiable unstable and stable foliations.*

COROLLARY 5.4. — *If a projectively Anosov flow has differentiable invariant foliations, then every compact leaf of those foliations is incompressible.*

As the preliminaries for the proof of the main results, we must study the relationship between certain orbits of the flow and the topology of the invariant foliations.

Let p be a point such that its orbit converges to p transversely (it may be a closed orbit), and let D be a disk passing through p and transverse to the flow. Then there exists an increasing sequence t_n such that the points $\phi^{t_n}(p)$ are contained in D and converge to p as $n \to \infty$. We can take an orthogonal coordinate induced by $\mathcal{F}^u$ and $\mathcal{F}^s$ on D which has p as the origin and satisfies the inequality

$$\frac{\|d_{(x,y)}\phi^t(v_u)\|}{\|d_{(x,y)}\phi^t(v_s)\|} \geq A\,\mathrm{e}^{Ct}\frac{\|v_u\|}{\|v_s\|}, \tag{3}$$

for some $A, C > 0$, for any $(x,y) \in D, t \in \mathbb{R}$ such that $\phi^t(x) \in D$, and for all $v_u \in T_{(x,y)}(\mathcal{F}^u \cap D), v_s \in T_{(x,y)}(\mathcal{F}^s \cap D)$. Let (ξ_n, η_n) be the coordinates of $\phi^{t_n}(p)$.

For $\sigma = u, s$, we can take holonomy maps Θ_n^σ of $\mathcal{F}^\sigma$ along the orbit from p to $\phi^{t_n}(p)$. Then we can define the *return map* $r_n = (\Theta_n^u, \Theta_n^s)$ on a sufficiently small neighbourhood of the origin $N_{\varepsilon_n}(0,0)$, where

$$N_\varepsilon(\xi,\eta) = \{(x,y) \in D : |x-\xi| < \varepsilon,\ |y-\eta| < \varepsilon\}.$$

Since r_n preserves the foliations defined by $\mathcal{F}^u \cap D, \mathcal{F}^s \cap D$, the image of $N_{\varepsilon_n}(0,0)$ will be represented as the following:

$$r_n(N_{\varepsilon_n}(0,0)) = \{(x,y) \in D : \xi_n - a_n^- < x < \xi_n + a_n^+,\\ \eta_n - b_n^- < y < \eta_n + b_n^+\},$$

where $a_n^\pm, b_n^\pm > 0$. If ε_n is sufficiently small, we can uniquely define the *returning time* $\tau_n(x,y)$ for $(x,y) \in N_{\varepsilon_n}(0,0)$ which satisfies $\phi^{\tau_n(x,y)}(x,y) = r_n(x,y)$.

LEMMA 5.5. — *There exists an increasing, diverging sequence* $\{t'_n\}$ *such that for all* $n > 0$, *the following inequality holds*:

$$\frac{\min\{b_n^\pm\}}{\max\{a_n^\pm\}} \geq A\,\mathrm{e}^{Ct'_n}.$$

Proof. — First, take the $\{t'_n\}$ as follows:

$$t'_n = \inf_{(x,y)\in N_{\varepsilon_n}(0,0)} \tau_n(x,y).$$

Suppose that there exists $k > 0$ such that

$$\frac{\min\{b_k^\pm\}}{\max\{a_k^\pm\}} < A\,\mathrm{e}^{Ct'_k}.$$

If we consider the images of the points the lines which pass through the origin and have slopes ± 1, we can find from the mean value theorem that there exists a point such that the slope of the image of that point is less than $A\,\mathrm{e}^{Ct'_k}$. This contradicts (3).

Now we introduce the following important proposition, which describes a relationship between the dynamical systems and the invariant foliations of projectively Anosov flows.

PROPOSITION 5.6. — *Let the orbit of a point p converge to itself transversely. Then there exists a leaf of $\mathcal{F}^u$ or $\mathcal{F}^s$ near p whose holonomy is non-trivial.*

To prove this proposition, we need the following theorem of Imanishi.

THEOREM 5.7 (see [8]). — *Let $(M,\mathcal{F},\phi)$ be a triple consisting of a compact manifold M of dimension n, a codimension one foliation $\mathcal{F}$ on M and a flow $\phi: M \times \mathbb{R} \to M$ with the orbits transverse to leaves of $\mathcal{F}$. Let ℓ be a leaf curve and let $(-t_0', t_0)$ be the domain of the holonomy map $\Theta(\ell)$. If t_0 is finite then the leaf L passing through $\phi(\ell(0), t_0)$ is a holonomy limit leaf.*

Here the *holonomy map* $\Theta(\ell)$ is defined such that the lift of ℓ to the leaf of $\mathcal{F}$ containing $\phi(\ell(0), t)$ along orbits of ϕ passes through $\phi(\ell(1), \Theta(\ell)(t))$, and the *domain* $(-t_0', t_0)$ is the maximal connected subset containing zero where $\Theta(\ell)$ can be defined. We say that a leaf L_x is a *holonomy limit leaf* if for any $\varepsilon > 0$ there exists t, $-\varepsilon < t < \varepsilon$, such that the leaf passing through $\phi(x,t)$ has holonomy.

Proof of 5.6. — Suppose that there exists some $\delta > 0$ such that all leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ which intersect $N_\delta(0,0)$ have trivial holonomies. Then by Theorem 5.7, the two holonomy maps Θ_n^σ, $(\sigma = u, s)$ can be defined on $N_\delta(0,0)$ for all $n > 0$. So we can take $\varepsilon_n = \delta$ for all n.

Taking a sufficiently large n_1, we may assume that $(\xi_{n_1}, \eta_{n_1}) \in N_{\delta/2}(0,0)$, and that

$$\frac{\min\{b_{n_1}^\pm\}}{\max\{a_{n_1}^\pm\}} > 4,$$

by the previous lemma.

Since $\Theta_{n_1}^u$ cannot be a contraction, $\xi_{n_1} - a_{n_1}^- < -\delta$ or $\delta < \xi_{n_1} + a_{n_1}^+$ must hold. Then it implies that $b_{n_1}^\pm > 2\delta$ therefore $(\Theta_{n_1}^s)^{-1}$ is a contraction map. This contradicts the triviality of the holonomy of $\mathcal{F}^s$.

In the remainder of this paper, we consider the manifolds which are the torus bundles over the circle. For an orientation preserving diffeomorphism $f : T^2 \to T^2$, let $T_f = T^2 \times [0,1]/\sim$, where the equivalence relation $\sim$ is defined by $(x,0) \sim (f(x),1)$.

The diffeomorphism f is isotopic to a linear map A_f defined by an element of $\mathrm{SL}(2,\mathbb{Z})$ and the resulting manifold can be classified with respect to the eigenvalues of this linear map as follows:

- A_f has no real eigenvalues.

In this case, f is called *elliptic* and f^n is isotopic to identity for some n. Then T_f is finitely covered by the 3-torus T^3, so we consider first the case of T^3 and then the elliptic case as its corollary.

- A_f has only one real eigenvalue.

In this case, f is called *parabolic* and there exists only one isotopy class of a closed curve on T^2 which is invariant under f. By taking the double-covering according to circumstances, T_f can be represented as a non-trivial S^1-bundle over T^2.

- A_f has two different real eigenvalues.

In this case, f is called *hyperbolic* and there exist two foliations on T^2 corresponding to the eigenvectors, which are invariant under f. In this case, T_f admits an Anosov flow defined as the suspension of f.

Then we have the following theorems.

THEOREM 5.8. — *Suppose that a projectively Anosov flow on T^3 has invariant foliations of class $\mathcal{C}^2$. Then it is topologically isotopic to a finite union of $T^2 \times I$-models. In particular, both invariant foliations have finite number of compact leaves.*

THEOREM 5.9. — *For a non-trivial S^1-bundle M over T^2, there exists no projectively Anosov flow on M whose unstable and stable foliations are of class $\mathcal{C}^2$. (In this case, M is represented as T_f for a parabolic diffeomorphism f.)*

THEOREM 5.10. — *Let M be a T^2-bundle over S^1 with hyperbolic monodromy. ($M = T_f$ for hyperbolic f.) Suppose that a projectively Anosov flow on M has invariant foliations of class $\mathcal{C}^2$ and that at least one of these foliations has compact leaves. Then it is topologically isotopic to a finite union of $T^2 \times I$-models. Moreover, the flows on the compact leaves are isotopic to the linear flows on T^2 in the directions of the eigenvectors of f.*

As to the eliptic case, we can deduce the following corollary from Theorem 5.8.

COROLLARY 5.11. — *For a T^2-bundle M over S^1 with elliptic monodromy ($M = T_f$ for elliptic f), there exists no projectively Anosov flow on M with differentiable (un)stable foliations.*

Proof. — Suppose that there exists such a projectively Anosov flow ϕ^t on T_f for elliptic f. We may assume that $f = A_f \in \mathrm{SL}(2,\mathbb{Z})$ such that $f^n = \mathrm{Id}_{T^2}$ for some n. Remark that there exists no invariant direction under f since f has no real eigenvalues. We can realize an n times covering of T_f as $T_{f^n} = T^2 \times [0,n]/(x,0) \sim (f^n(x),n)$ and the covering transformations are generated by $\hat{f}(x,y) = (f(x), y+1)$. Then T_{f^n} is homeomorphic to the 3-torus and the induced projectively Anosov flow $\widetilde{\phi}^t$ is invariant under the covering transformations. By Theorem 5.8, this flow is isotopic to a finite union of $T^2 \times I$-models.

Take a compact leaf L of the unstable foliation. If L is isotopic to $T^2 \times \{y\}$ then the map $\hat{f}_{|L}: L \to \hat{f}(L)$, regarded as an automorphism of the 2-torus by the identification of L with $\hat{f}(L)$ in T_{f^n}, is isotopic to f. Since L and $\hat{f}(L)$ are compact leaves of the unstable foliation, the restricted flows on them are isotopic to the same linear flow. However, there exists no linear flow preserved by f. It is a contradiction.

If L is not isotopic to $T^2 \times \{y\}$, we can take y_0 such that $T = T^2 \times \{y_0\}$ intersects L transversely. The diffeomorphism $\hat{f}_{|T}: T^2 \times \{y_0\} \to T^2 \times \{y_0 + 1\}$, which is naturally identified with f, maps $T \cap L$ to $\hat{f}(T) \cap \hat{f}(L)$. The homology classes of these intersections in $H_1(T_{f^n})$ coincide and are not trivial, since they are determined by the intersection classes $[T] \cdot [L]$ and $[\hat{f}(T)] \cdot [\hat{f}(L)]$ and T and L are isotopic to $\hat{f}(T)$ and $\hat{f}(L)$ respectively. Thus we can see that $T \cap L$ contains a non-trivial loop in T whose homology class is invariant under $\hat{f}_{|T}$. It is a contradiction.

Before proving the theorems, let us review classifications of foliations on T^3 and $T^2 \times I$, which are given by Moussu and Roussarie in [12].

THEOREM 5.12. — *Let $\mathcal{F}$ be a transversely orientable foliation of class C^2 without a Reeb component, defined on T^3 or $T^2 \times I$ (then $\mathcal{F}$ is supposed to be tangent to the boundary). Then*

(i) *If $\mathcal{F}$ has no compact leaves, it is topologically isotopic to a linear foliation $\mathcal{F}'$ on T^3.*

(ii) *If $\mathcal{F}$ has at least one compact leaf, it is topologically conjugate to a foliation $\mathcal{F}'$ on $T^2 \times$ (S^1 resp. I) which is isotopic to a finite union of foliations of type $\mathcal{F}([\omega], i)$, where ω is a linear form of T^2 and $i = 1$ or 2.*

The linear form ω and the index i can vary from one component to another; however, by reason of orientability, $i = 1$ except for a finite number of components and components such that $i = 2$ appear in pairs in the case of T^3.

Here $\mathcal{F}([\omega], i)$ $(i = 1, 2)$ are the foliations defined in 3.2.

DEFINITION 5.13. — Two C^r-foliations F_0 and F_1 on T^2 are *R-C^r-cobordant* if there exists an orientable foliation $\mathcal{F}$ of class C^r on $T^2 \times [0,1]$ which gives a cobordism between F_0 and F_1 and have neither Reeb components nor half Reeb components.

THEOREM 5.14. — *Let F_0 and F_1 be C^2-foliations on T^2 and let $\mathcal{F}$ be a C^2-foliation on $T^2 \times [0,1]$ which gives an R-C^2-cobordism between F_0 and F_1. Then*

(i) *If $\mathcal{F}$ has compact leaves in the interior of $T^2 \times [0,1]$, both F_0 and F_1 are isotopic to linear foliations on T^2.*

(ii) *If $\mathcal{F}$ has no compact leaves in the interior of $T^2 \times [0,1]$, it gives an isotopy between F_0 and F_1.*

By using these results we can show the following proposition, which tells that projectively Anosov flows on $T^2 \times I$ are represented as $T^2 \times I$-models.

PROPOSITION 5.15. — *Suppose that a projectively Anosov flow on $T^2 \times I$ has unstable foliation $\mathcal{F}^u$ and stable foliation $\mathcal{F}^s$ of class C^2, and $T^2 \times \{0\}, T^2 \times \{1\}$ are the only compact leaves of $\mathcal{F}^u$. Then*

(i) *$\mathcal{F}^s$ has a unique compact leaf in* $\mathrm{int}(T^2 \times I)$.

(ii) *The linear holonomies along the compact leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are non-trivial.*

(ii) *The flows restricted to $T^2 \times \{0\}, T^2 \times \{1\}$ are C^0-isotopic to linear flows.*

Proof. — (i) Since $\mathcal{F}^s$ is transverse to the boundaries, it gives two codimension one foliations F_0 and F_1 on the tori $T^2 \times \{0\}$ and $T^2 \times \{1\}$ respectively. By Proposition 5.1, we can apply Theorem 5.12 for $\mathcal{F}^u$ and Theorem 5.14 for $\mathcal{F}^s$ and thus we see that $\mathcal{F}^u$ is topologically equivalent to a foliation of type $\mathcal{F}([\omega], 1)$ or $\mathcal{F}([\omega], 2)$. We also see that if $\mathcal{F}^s$ has no compact leaves in the interior, $\mathcal{F}^s$ defines an isotopy between F_0 and F_1. In this case, by the theorem of Denjoy [3] (see also [1], [17], for example), F_0 and F_1 either have compact leaves or are topologically isotopic to a linear foliation.

We consider each of the cases in the following. (Actually, we will see that all of the cases in *Case I* will be eliminated.)

Case I: $\mathcal{F}^s$ does not have compact leaves in the interior.

In this case $\mathcal{F}^s$ defines an isotopy between F_0 and F_1. Then there are two subcases according to the topology of F_0 and F_1.

Case I-a: F_0 and F_1 have compact leaves.

In this case, F_0 and F_1 have slope components or Reeb components.

First, suppose that there exists a slope component in F_0. Then there exists a closed orbit c_0 which is a component of the boundary of this slope component, and the holonomy of F_0 (or $\mathcal{F}^s$) along this orbit is not expanding. Let c_1 be the closed orbit which corresponds to c_0 by the isotopy of $\mathcal{F}^s$.

If $\mathcal{F}^u$ is of type $\mathcal{F}([\omega], 1)$ then the foliation on the annular leaf bounded by c_0 and c_1 is a slope component. Hence the directions of the orbits c_0 and c_1 are the same, and one of the holonomies of $\mathcal{F}^u$ along c_0 and c_1 is not contracting. This is a contradiction (Figure 10 (1)).

In the case where $\mathcal{F}^u$ is of type $\mathcal{F}([\omega], 2)$, the foliation on this annular leaf is a Reeb component and the directions of the orbits c_0 and c_1 differ. Then the holonomy of F_1 (or $\mathcal{F}^s$) along c_1 is not contracting. Now take the orbit c_1' such that c_1 and c_1' bounds the slope component of F_1 and the holonomy of F_1 along c_1' is not expanding. Then one of the holonomies of $\mathcal{F}^u$ along c_0 and c_1' is not contracting and this is also a contradiction (Figure 10 (2)).

Secondly, suppose that there exists a Reeb component. If $\mathcal{F}^u$ is of type $\mathcal{F}([\omega], 2)$ then $\mathcal{F}^u$ and $\mathcal{F}^s$ cannot be transverse (Figure 10 (3)), so we may assume that $\mathcal{F}^u$ is of type $\mathcal{F}([\omega], 1)$. Then one of the orbits which are contained in the boundary of the Reeb component has non-expanding holonomy. Let d_0 be this orbit and let d_1 be the other orbit which is a component of the boundary of the annular leaf containing d_0. The directions of the orbits d_0 and d_1 are the same, so both d_0 and d_1 have non-expanding holonomies of $\mathcal{F}^s$. However, one of the holonomies of $\mathcal{F}^u$ along d_0 and d_1 is not expanding. This is a contradiction (Figure 10 (4)).

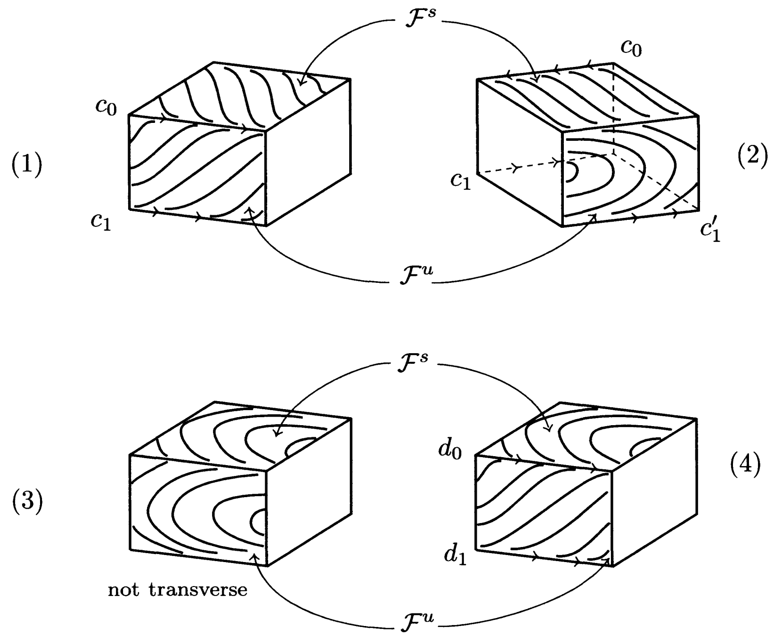


Figure 10

Case I-b: *F_0 and F_1 are isotopic to an irrational linear foliation.*

In this case, all orbits of the flows on the boundary are dense. So applying Lemma 5.5, we can see that the flows in the neighbourhood of the boundaries are absorbed into compact leaves of $\mathcal{F}^u$ as $t \to +\infty$. Then there exists $\varepsilon > 0$ such that

$$K = T^2 \times I - \Big(\bigcup_{t\in\mathbb{R}} \phi^t(T^2 \times [0,\varepsilon))\Big) \cup \Big(\bigcup_{t\in\mathbb{R}} \phi^t(T^2 \times (1-\varepsilon, 1])\Big)$$

is a non-empty closed set. Since the minimal set of the flow restricted to K has an orbit which converges to itself transversely, the unstable or the stable foliation has non-trivial holonomy by Proposition 5.6. However, both of them do not have holonomies in $\operatorname{int}(T^2 \times I)$. This is a contradiction.

Thus we have shown that *Case I* can never occur.

Case II: *$\mathcal{F}^s$ has compact leaves in the interior.*

Suppose that $\mathcal{F}^s$ has more than one compact leaf. If there exists a subset A in the interior of $T^2 \times I$ such that $\mathcal{F}^s|_A$ is topologically equivalent to the product foliation of $T^2 \times \mathrm{int}I$, then both of the holonomies of $\mathcal{F}^u$ and $\mathcal{F}^s$ are trivial in A and it contradicts 5.6. Thus we may assume that there exists a subset B in the interior of $T^2 \times I$ such that ∂B consists of two compact leaves of $\mathcal{F}^s{}_{|B}$ and that there exists no compact leaf in the interior of B. Since $\mathcal{F}^s$ can be regarded as the unstable foliation of the inverse flow ϕ^{-t}, we can apply the same discussion as above to $\mathcal{F}^u{}_{|B}$ and $\mathcal{F}^s{}_{|B}$. It follows that there exist compact leaves of $\mathcal{F}^u$ in the interior of $T^2 \times I$. This contradicts the assumption. Therefore $\mathcal{F}^s$ contains only one compact leaf and we have thus proved (i).

(ii) The flows on the compact leaves are isotopic to linear foliations. Approximating these orbits by closed curves on the leaves, we can easily see by Lemma 5.5 that linear holonomies are non-trivial.

(iii) It is obvious by (ii) and the smoothness of the foliations.

Now we can give proofs for the main theorems.

Proof of 5.8. — By Proposition 5.1, we can apply Theorem 5.12 to the unstable and stable foliations $\mathcal{F}^u, \mathcal{F}^s$. Thus we know that $\mathcal{F}^u$ and $\mathcal{F}^s$ are topologically isotopic to either linear foliations or unions of foliations of type $\mathcal{F}([\omega], i)$ where $i = 1$ or 2.

If both $\mathcal{F}^u$ and $\mathcal{F}^s$ are isotopic to linear foliations, their holonomies are trivial. However, a minimal set of the flow contains an orbit which converges to itself transversely and it contradicts Proposition 5.6. Therefore, at least one of the invariant foliations is not isotopic to a linear foliation.

Now we may assume that $\mathcal{F}^u$ is a union of foliations of type $\mathcal{F}([\omega], i)$. Then using Proposition 5.15 to each component which is homeomorphic to $T^2 \times I$, we know that $\mathcal{F}^s$ is also represented as a union of foliations on $T^2 \times I$, and that compact leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ appear alternatively in T^3.

Now it remains to prove that the number of components is finite. If there are infinitely many components, then there will be also infinitely many compact leaves, and some compact leaf must have trivial linear holonomy. This contradicts 5.15 (ii).

Proof of 5.9. — It is shown by Levitt [9] and Thurston [19] that if a C^2-foliation on a non-trivial S^1-bundle over T^2 has no compact leaves, then it is isotopic to the foliation which is a pull-back of a linear foliation on T^2 by the natural projection.

Suppose that both $\mathcal{F}^u$ and $\mathcal{F}^s$ have no compact leaves. Then they are isotopic to pull-backs of linear foliations on T^2 and the holonomies of these foliations are both trivial. This contradicts Proposition 5.6.

Next, assume that $\mathcal{F}^u$ has compact leaves. If we get rid of a compact leaf and glue two copies of it on both sides of the boundaries, we obtain a manifold with boundaries, which is homeomorphic to $T^2 \times I$, and the original manifold can be made over again by gluing together the boundaries by another parabolic diffeomorphism g. Thus we can assume that $\mathcal{F}^u$ is a foliation on $T^2 \times I$ such that the boundaries $T^2 \times \{0\}$ and $T^2 \times \{1\}$ (identified by g) correspond to a union of compact leaves.

In this case, by the theorem of Moussu and Roussarie and by a discussion similar to the one of the previous theorem, $\mathcal{F}^u$ is a finite union of $T^2 \times I$-models. Furthermore, from the gluing condition of g, the flows on the compact leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are all isotopic to the linear foliation of T^2 by closed leaves, which is g-invariant. However, the holonomies of $\mathcal{F}^u$ and $\mathcal{F}^s$ along these orbits are both trivial and it contradicts Proposition 5.6.

Proof of 5.10. — As the compact leaves are incompressible by Proposition 5.1, they are isotopic to fibers of M.

The same argument as in the previous theorem also enables us to regard the invariant foliations as a finite union of $T^2 \times I$-models. In this case, the gluing condition allows the flows on the compact leaves to be isotopic to the linear flows on T^2 in the directions of the two eigenvectors of A_f.

If the flows on compact leaves of both $\mathcal{F}^u$ and $\mathcal{F}^s$ have the same direction, the approximation of the orbits by sufficiently long closed curves on the leaves will lead to a contradiction of Lemma 5.5. Therefore they differ. We have thus proved the theorem.

Remark. — For a hyperbolic automorphism f, T_f admits an Anosov flow by the suspension of f. In the case of projectively Anosov flows on T_f which have invariant foliations without compact leaves, we have the following theorem.

THEOREM 5.16 (see [13]). — *Let M be a T^2-bundle over S^1 with hyperbolic monodromy. Suppose that a projectively Anosov flow on M has invariant foliations which are of class C^2 and do not contain compact leaves. Then the flow is actually an Anosov flow given as the suspension of the monodromy.*

The bi-contact structures for the Anosov flows by suspensions can be obtained by propeller constructions with rotations less than π. Therefore, as to bi-contact structures, we can deduce the following corollary:

COROLLARY 5.17. — *If projectively Anosov flows on T^2-bundles over S^1 have invariant foliations of class C^2, the accompanied bi-contact structures are obtained by propeller constructions.*

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Manuscrit reçu le 29 juin 1999,
accepté le 15 octobre 1999.

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