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A NAKAI-MOISHEZON CRITERION FOR NON-KÄHLER SURFACES

by Nicholas BUCHDAHL

0. Introduction.

In Corollary 15 of [B], the classical Nakai-Moishezon criterion for a compact complex surface X was generalised to yield a characterization of the set of classes in $H_{\mathbb{R}}^{1,1}(X)$ which can be represented by a Kähler form, a result obtained independently by Lamari [L]. Under the assumption that $b_1(X)$ is even, this result was further generalised in Theorem 16 of [B] to the case of $\bar{\partial}\partial$ -closed modulo $\bar{\partial}\partial$ -exact $(1, 1)$ -forms. The purpose of this paper is to demonstrate that the assumption on $b_1(X)$ can be dropped entirely. Namely, the following will be proved:

THEOREM. — *Let X be a compact complex surface equipped with a positive $\bar{\partial}\partial$ -closed $(1, 1)$ -form ω and let φ be a smooth real $\bar{\partial}\partial$ -closed $(1, 1)$ -form satisfying $\int_X \varphi \wedge \varphi > 0$, $\int_X \varphi \wedge \omega > 0$ and $\int_D \varphi > 0$ for every irreducible effective divisor $D \subset X$ with $D \cdot D < 0$. Then there is a smooth function g on X such that $\varphi + i\bar{\partial}\partial g$ is positive.*

Theorem 16 of [B] differs from this only in that it assumes $b_1(X)$ is even and that $\int_D \varphi > 0$ for every effective divisor $D \subset X$; however, this inequality must hold for any effective divisor D with $D \cdot D \geq 0$ by Proposition 5 of that paper.

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1. Proof of the theorem.

Let X be a compact complex surface. Since the theorem has already been proved in the case of even first Betti number, it will be assumed henceforth that $b_1(X)$ is odd. The same notation as in [B] is employed throughout, so $\Lambda^{p,q}$ denotes the sheaf of germs of smooth (p, q) -forms on X , with $\Lambda^{p,q}(X)$ denoting the global sections. A $\bar{\partial}\partial$ -closed positive $(1, 1)$ -form $\omega \in \Lambda_{\mathbb{R}}^{1,1}(X)$ is chosen once and for all, its existence guaranteed by Gauduchon's theorem [G].

For any $f \in \Lambda^{1,1}(X)$ there is a function $g \in \Lambda^{0,0}(X)$, unique up to the addition of a constant, such that $\omega \wedge (f + g'')$ is a constant multiple of ω^2 where $g'' := i\bar{\partial}\partial g$. Since $b_1(X)$ is odd, the proof of Lemma 8 in [B] implies that there is a unique form $\sigma_0 \in \Lambda_{\mathbb{R}}^{1,1}(X)$ with the properties that it is d -exact and satisfies $\omega \wedge \sigma_0 = \omega^2$. The harmonic representative of a closed $(1, 1)$ -form f on X satisfying $\omega \wedge f = c\omega^2$ for some constant c is then $f - c\sigma_0$. This form is anti-self-dual with respect to ω , a manifestation of the fact that the intersection form on $H^2(X, \mathbb{R})$ restricted to $H_{\mathbb{R}}^{1,1}(X)$ is negative definite ([BPV], IV 2.13).

For a holomorphic line bundle L on X , there is a unique hermitian metric on L such that the corresponding hermitian connection has curvature f_L satisfying $\omega \wedge f_L = \text{Const} \cdot \omega^2$. If $s \in \Gamma(X, \mathcal{O}(L))$ is non-zero and E is the associated effective divisor $s^{-1}(0)$, the equation of currents $2\pi E = i f - i\bar{\partial}\partial \log |s|^2$ holds by the Poincaré-Lelong theorem ([GH]). Therefore $\int_E \varphi = \frac{i}{2\pi} \int_X f_L \wedge \varphi$ for any smooth $\bar{\partial}\partial$ -closed $(1, 1)$ -form φ . When the divisor E is given without reference to L , the notation f_E will be used to denote f_L for $L = \mathcal{O}(E)$.

A *real divisor* on X is by definition a finite formal sum of the form $D = \sum_i \nu_i D_i$ where $D_i \subset X$ is an irreducible effective divisor on X and ν_i is a real number; D is *effective* if $\nu_i \geq 0$ for all i , in which case the usual notation $D \geq 0$ is employed; similarly, $D \geq E$ iff $D - E \geq 0$. As for integral divisors, the notation f_D is used to denote $\sum_i \nu_i f_{D_i}$.

The intersection form on $H^2(X, \mathbb{R})$ is denoted by the dot product symbol. Thus $E \cdot E$ is the self-intersection number of an effective divisor

E in X , realised by the integral $-\frac{1}{4\pi^2} \int_X f_E \wedge f_E$. The notation extends by \mathbb{R} -linearity to all real divisors, and is further extended to denote the pairing between $\bar{\partial}\partial$ -closed $(1, 1)$ -forms: $\varphi \cdot \psi := \int_X \varphi \wedge \psi$ for $\bar{\partial}\partial$ -closed $\varphi, \psi \in \Lambda_{\mathbb{R}}^{1,1}(X)$. If $\psi = if_D$ for some real divisor D , the notation $\varphi \cdot D$ may also be used in place of $\frac{1}{2\pi} \varphi \cdot if_D$.

LEMMA 1. — *Let $E \subset X$ be an effective integral divisor such that $E \cdot E = 0$. Then for any $\varepsilon > 0$ there is a smooth function g such that $if_E + g'' \geq -\varepsilon\omega$.*

Proof. — If there is no smooth function g on X such that $if_E + g'' + \varepsilon\omega$ is positive in a neighbourhood of E , the Hahn-Banach Theorem implies the existence of a current T and a constant c such that $T(if_E + \varepsilon\omega + g'') \leq c$ for every smooth function g and $T(\psi) > c$ for every smooth 2-form ψ whose $(1, 1)$ -component is positive in a neighbourhood of E .

It follows immediately that T is a $(1, 1)$ -current, that $\bar{\partial}\partial T = 0$, that c must be non-positive, that $T(if_E + \varepsilon\omega) \leq c$, that $T(\psi) \geq 0$ for any smooth $(1, 1)$ -form ψ which is positive in a neighbourhood of E and finally that the support of T must be contained in E . By Lemma 32 of [HL], it follows that $T = \sum_i h_i E_i$ where h_i is a non-negative constant and E_1, E_2, \dots are the irreducible components of E . Since $E \cdot E = 0$ and $b_1(X)$ is odd, $[E] = 0$ in $H^2(X, \mathbb{R})$. Hence $E_i \cdot E = 0$ for all i , and this gives a contradiction since then $c \geq T(if_E + \varepsilon\omega) = T(\varepsilon\omega) > c$.

It can therefore be supposed that E is the zero set of a section s of a holomorphic line bundle L which has a hermitian connection whose curvature form f satisfies $if > -\varepsilon\omega$ in an open neighbourhood U of E . After rescaling s if necessary, it can be assumed that $\{x \in X \mid |s(x)| \leq 1\} \subset U$.

Let χ be a smooth convex increasing function on \mathbb{R} such that $0 \leq \chi'(t) \leq 1$ for all t , with $\chi(t) = t$ for $t \geq 0$ and with $\chi(t) = -1$ for $t \leq -1$. Then $i\bar{\partial}\partial(\chi(\log |s|^2)) = \chi'(\log |s|^2) i\bar{\partial}\partial \log |s|^2 + \chi''(\log |s|^2) i\bar{\partial}(\log |s|^2) \wedge \partial(\log |s|^2) \leq \chi'(\log |s|^2) if$, so $if - i\bar{\partial}\partial(\chi(\log |s|^2)) \geq (1 - \chi'(\log |s|^2)) if \geq -\varepsilon\omega$, as required. □

Remark. — The above proof also works in some cases when $b_1(X)$ is even. For example, if E is irreducible (with $E \cdot E = 0$), or if every effective divisor on X has non-negative self-intersection.

LEMMA 2. — *Suppose $\psi \in \Lambda_{\mathbb{R}}^{1,1}(X)$ satisfies $\bar{\partial}\partial\psi = 0$, $\psi \cdot \psi = 0$, $\psi \cdot \omega \geq 0$ and $\psi \cdot D \geq 0$ for every effective divisor $D \subset X$. Then for any*

$\varepsilon > 0$ there is a smooth function g such that $\psi + g'' \geq -\varepsilon\omega$.

Proof. — By Lemma 7 of [B], ψ can be approximated arbitrarily closely in L^2 norm by forms of the kind $p - g''$ where p is smooth and positive and g is smooth. Following exactly the same argument as used in the proof of Theorem 11 of [B], a sequence of smooth functions g_n and smooth positive $(1, 1)$ -forms p_n can be found such that $\|\psi + g_n'' - p_n\|_{L^2(\omega)}$ is converging to 0 and g_n is converging in L^1 to define an almost-positive closed $(1, 1)$ -current $P = g_\infty'' \geq -\psi$. Applying the same arguments as in the proofs of Theorems 11 and 16 in [B] shows that for any given $\varepsilon > 0$ there is a real effective divisor D_ε and a smooth function g_ε such that $-if_{D_\varepsilon} + g_\varepsilon'' \geq -\psi - \varepsilon\omega$. The construction of D_ε is such that it can be assumed that $D_{\varepsilon'} \geq D_\varepsilon$ for $\varepsilon' < \varepsilon$ and the coefficient of an irreducible component common to both D_ε and $D_{\varepsilon'}$ is the same in both.

Now take a sequence of positive numbers ε converging monotonically to 0. Since $\chi_\varepsilon := \varepsilon\omega + \psi - if_{D_\varepsilon} + g_\varepsilon''$ is positive, $0 \leq \chi_\varepsilon \cdot \chi_\varepsilon = \varepsilon^2 \omega \cdot \omega + 4\pi^2 D_\varepsilon \cdot D_\varepsilon + 2\varepsilon \omega \cdot \psi - 4\pi\varepsilon \omega \cdot D_\varepsilon - 2\pi \psi \cdot D_\varepsilon$. The hypotheses on ψ and negativity of the intersection form restricted to $H_{\mathbb{R}}^{1,1}(X)$ therefore imply that the cohomology classes $[D_\varepsilon] \in H^2(X, \mathbb{R})$ are uniformly bounded. After passing to a subsequence if necessary, the corresponding sequence of harmonic representatives can be assumed to converge smoothly. Moreover, the inequality $0 \leq \omega \cdot \chi_\varepsilon = \varepsilon \omega \cdot \omega + \omega \cdot \psi - 2\pi \omega \cdot D_\varepsilon$ implies that the increasing sequence of non-negative numbers $\{\omega \cdot D_\varepsilon\}$ is bounded above and hence converges. Therefore the sequence of forms $\{f_{D_\varepsilon}\}$ converges smoothly to a closed $(1, 1)$ -form $f_{\mathcal{D}}$ satisfying $f_{\mathcal{D}} \cdot f_{\mathcal{D}} = 0 = \psi \cdot f_{\mathcal{D}}$ and $\omega \wedge if_{\mathcal{D}} = c\omega^2$ for some constant $c \geq 0$. Since $[if_{\mathcal{D}}] = 0$ in $H^2(X, \mathbb{R})$ it follows $if_{\mathcal{D}} = c\sigma_0$.

If $c = 0$, it follows from the fact that $\{\omega \cdot D_\varepsilon\}$ is non-negative and increasing that $\omega \cdot D_\varepsilon = 0$ for all ε ; in this case $D_\varepsilon = 0$ for all ε and therefore $\psi + g_\varepsilon'' \geq -\varepsilon\omega$ as required.

If $c > 0$, the identity $\psi \cdot \sigma_0 = 0$ and Proposition 5 of [B] imply that $\psi + g''$ is a non-negative multiple of σ_0 for some smooth function g . If there is a non-zero integral effective divisor E on X such that $E \cdot E = 0$, since $[\sigma_0] = 0$ in $H^2(X, \mathbb{R})$ it follows that $\sigma_0 \cdot E = 0$ and by Proposition 5 of [B] again, that σ_0 is a positive multiple of if_E ; in this case, the desired result follows from Lemma 1. If X has algebraic dimension 1, it is well-known that X is an elliptic surface ([BPV], VI 4.1) and therefore such a divisor E exists.

If X has algebraic dimension 0, then by [BPV], IV 6.2, there are only

finitely many irreducible curves on X so that for ε sufficiently small, the real divisors D_ε are independent of ε . Hence $f_{\mathcal{D}} = f_D$ for some genuine real effective divisor D on X satisfying $D \cdot D = 0$. By Lemma 4 in §3.5 of Ch. V of [Bou], the symmetric negative semi-definite intersection matrix M associated with the irreducible components of a connected component of D has a 1-dimensional kernel, and the entries in a generating vector \mathbf{v} all have the same sign. Since \mathbf{v} must be a multiple of a column of the cofactor matrix of M , after multiplying by a real constant it has positive integer entries. This implies that there is an effective non-zero integral divisor E on X with $E \cdot E = 0$, so the desired result follows from the previous paragraph. \square

The proof of the main theorem can now be completed. Let $\varphi \in \Lambda_{\mathbb{R}}^{1,1}(X)$ be a $\bar{\partial}\partial$ -closed form satisfying the hypotheses of the theorem. By the proof of Theorem 14 of [B], there is a form $u \in \Lambda^{0,1}(X)$ such that $\tilde{\varphi} := \varphi + \partial u + \bar{\partial}\bar{u}$ is positive; (the hypothesis that $b_1(X)$ be even in that theorem is used only in the final sentence of the proof).

By Proposition 5 of [B], $\tilde{\varphi} \cdot \varphi$ is strictly positive. Let t_0 be the smaller solution of the equation $(\varphi - t_0\tilde{\varphi}) \cdot (\varphi - t_0\tilde{\varphi}) = 0$, and set $\psi := \varphi - t_0\tilde{\varphi}$. Since $(\varphi - t\tilde{\varphi}) \cdot (\varphi - t\tilde{\varphi}) > 0$ for t satisfying $0 \leq t < t_0$, the sign of $\omega \cdot (\varphi - t\tilde{\varphi})$ cannot change for such t so $\omega \cdot \psi \geq 0$. Since $(\varphi - \tilde{\varphi}) \cdot (\varphi - \tilde{\varphi}) = -2\|\partial u\|^2 \leq 0$, it follows that $t_0 \leq 1$ and therefore for any effective divisor $E \subset X$, $\psi \cdot E = (1 - t_0)\varphi \cdot E \geq 0$.

The form ψ therefore satisfies the hypotheses of Lemma 2. Applying that lemma, given $\varepsilon > 0$ there is a smooth function g_ε such that $\psi + g_\varepsilon'' \geq -\varepsilon\omega$, so if ε is chosen so small that $t_0\tilde{\varphi} - \varepsilon\omega > 0$, it follows that $\varphi + g_\varepsilon'' > 0$, as required. \square

Remark. — The methods of this paper show that if $\varphi \in \Lambda_{\mathbb{R}}^{1,1}(X)$ satisfies the hypotheses of the theorem except for the condition that $\int_E \varphi$ be positive for every effective $E \subset X$ with negative self-intersection, there is an effective real divisor D on X such that $\varphi - if_D$ is $i\bar{\partial}\partial$ -homologous to a positive form.

BIBLIOGRAPHY

- [BPV] W. BARTH, C. PETERS and A. VAN DE VEN, *Compact Complex Surfaces*, Berlin-Heidelberg-New York, Springer, 1984.

- [Bou] N. BOURBAKI, Groupes et Algèbres de Lie, Ch. 4,5,6, "Éléments de mathématiques" Fasc. XXXIV, Paris, Hermann, 1968.
- [B] N. P. BUCHDAHL, On compact Kähler surfaces, *Ann. Inst. Fourier*, 49-1 (1999), 287–302.
- [G] P. GAUDUCHON, Le théorème de l'excentricité nulle, *C. R. Acad. Sci. Paris*, 285 (1977), 387–390.
- [GH] P. A. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry*, New York, Wiley, 1978.
- [HL] R. HARVEY and H. B. LAWSON, An intrinsic characterisation of Kähler manifolds, *Invent. Math.* 74 (1983), 169–198.
- [L] A. LAMARI, Le cône kählérien d'une surface, *J. Math. Pures Appl.*, 78 (1999), 249–263.

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