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EXTREMAL PROBLEMS FOR CONDITIONED BROWNIAN MOTION AND THE HYPERBOLIC METRIC

by R. BAÑUELOS^(*) & T. CARROLL^(†)

1. Introduction.

The expected lifetime of conditioned Brownian motion in a simply connected planar domain D is by now quite well understood. It is sufficient to consider the case when the Brownian motion is conditioned to exit the domain at a particular boundary point β . P. Griffin, T. McConnell and G. Verchota [7] have shown that the expected lifetime increases as the starting point moves away from β along a hyperbolic geodesic of D . In effect, then, the greatest expected lifetime occurs when the Brownian motion is conditioned to travel between two boundary points α and β of D . An analytic expression for the expected lifetime in this case is, in terms of the Poisson kernel $K_D(z, \zeta)$ at z for a boundary point ζ ,

$$\frac{2}{\pi} \int_D K_D(z, \alpha) K_D(z, \beta) dm(z)$$

where the above product of Poisson kernels is normalized to have the constant value 1 along the hyperbolic geodesic joining α and β : see [7].

In the case of a disk, Griffin, McConnell and Verchota were able to show that the greatest expected lifetime occurs when the Brownian

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motion is conditioned to travel between diametrically opposite points and its numerical value is $(4 \ln 2 - 2)r^2$, r being the radius of the disk. An alternative proof of this fact was found by Bañuelos and Housworth [4], p. 608. Griffin, McConnell and Verchota also found the best constant in the Cranston-McConnell theorem in the case of simply connected domains. Cranston and McConnell [6], in answer to a question of Kai Lai Chung, proved that there is an absolute constant C such that the expected lifetime of any conditioned Brownian motion in any planar domain is at most C times the area of the domain. If the domain is simply connected, Griffin, McConnell and Verchota [7] proved that the constant C can be taken to be $1/\pi$, that there are no simply connected domains for which the constant is attained but that long, thin rectangles are asymptotically extremal.

In the other direction, it is in general not possible to give a lower bound for the maximal expected lifetime of conditioned Brownian motion in simply connected domains in terms of area as there are simply connected domains of infinite area but with finite maximal lifetime. The first such examples were constructed by Xu [12]; further examples may be found in Bañuelos and Carroll [3] and Griffin, Verchota and Vogel [8]. However, for convex domains Xu [12] proved that the maximum lifetime is bounded below by a constant times the area of the domain. The following problem, raised in [7], goes far beyond this.

PROBLEM. — *Is it the case that among all convex domains of prescribed area a disk of that area has the smallest maximal expected lifetime of conditioned Brownian motion?*

In [3] we found an expression which involves only the hyperbolic geometry of a simply connected domain and is an equivalent expression for the maximal lifetime of the domain. This formula gives a deeper understanding of the behavior of conditioned Brownian motion in simply connected domains and it makes precise the statement that “conditioned Brownian motion paths tend to follow hyperbolic geodesics.” On interpreting the above problem in these terms, several more general and more geometric problems arise which lead us to several conjectures. The purpose of this paper is to discuss these more general geometric conjectures and to prove various special cases. In §2 we give the details which lead to these new conjectures; in §3 we prove a sharp upper bound estimate which arises out of these conjectures. This result is a generalization of the Griffin, McConnell and Verchota result and is in complete analogy with the classical result for the Green’s function of the unconditioned Brownian motion proved in

Bandle [2], p. 61. In §4 we verify the lower bound conjectures in certain special cases.

2. A geometric conjecture.

Suppose that $p_D(t, x, y)$ is the transition density function for Brownian motion killed on the boundary of a domain D . Analytically, this quantity is the Dirichlet heat kernel for $\frac{1}{2}\Delta$ in D . Let \mathcal{H}^+ denote the class of all positive harmonic functions in D . Doob's conditioned Brownian motion is a stochastic process in D which, given a function h in \mathcal{H}^+ , has transition density function

$$\frac{1}{h(x)} p_D(t, x, y) h(y).$$

The measure on path space induced by this transition function is denoted by P_x^h and the corresponding expectation by E_x^h . If we denote by τ_D the first time the Brownian path leaves the domain, the quantity of interest in the present setting is

$$\mathcal{L}_D = \sup \{ E_x^h \tau_D : x \in D \text{ and } h \in \mathcal{H}^+ \}.$$

It is the maximal expected lifetime of conditioned Brownian motion in D when the starting point and the conditioning positive harmonic function are allowed to vary. Since the integral in time of the heat kernel gives the Green's function for $\frac{1}{2}\Delta$ in D , we arrive at the analytic expression

$$E_x^h \tau_D = \frac{1}{h(x)} \int_D G_D(x, y) h(y) dy.$$

It then follows from the Martin representation of positive harmonic functions that one need only consider Martin kernel functions $h(\cdot) = K_D(\cdot, \beta)$ in the supremum defining \mathcal{L}_D . In this case, the paths of the conditioned Brownian motion start from x and are conditioned to exit the domain at the Martin boundary point β . In this case we write $E_x^\beta \tau_D$.

When the domain D is simply connected, it is shown in [7] that, for a fixed Martin boundary point or prime end β , the expected lifetime of Brownian motion conditioned by the Martin/Poisson kernel function $K_D(\cdot, \beta)$ increases as the starting point x moves away from β along the hyperbolic geodesic joining x and β . If the second endpoint of this geodesic is the prime end α , as x moves away from β towards α along the geodesic, $E_x^\beta \tau_D$ approaches the quantity

$$\frac{2}{\pi} \int_D K_D(z, \alpha) K_D(z, \beta) dm(z).$$

Here the product of Poisson kernels is normalized to have the constant value 1 along the hyperbolic geodesic joining α and β . We write $E_\alpha^\beta \tau_D$ for this limit and think of it as the expected lifetime when the Brownian motion is conditioned to travel between the boundary points α and β . It then follows, see [7], p. 234, that

$$(2.1) \quad \mathcal{L}_D = \sup \left\{ \frac{2}{\pi} \int_D K_D(z, \alpha) K_D(z, \beta) dm(z) : \alpha, \beta \in \partial D \right\}.$$

In [3] we arrived at the same quantity, but only as an equivalent expression for the maximal lifetime. We were, however, able to go one step further and make the connection between the product of Poisson kernels and the hyperbolic geometry of D much more precise; hence facilitating the introduction of geometric function theory as a tool. We now present a refinement of the expression given in [3].

We denote by $d(a, b; D)$ the hyperbolic distance between the points a and b in D arising from the hyperbolic metric $\sigma_D(z)$ in D . The normalization we adopt is that in which the metric $\sigma_D(z)$ has constant curvature -4 . Alternatively, taking any conformal mapping $g(z)$ of D onto the unit disk U ,

$$(2.2) \quad \sigma_D(z) = \sigma_U(g(z)) |g'(z)| = \frac{|g'(z)|}{1 - |g(z)|^2}.$$

We refer the reader to [5] and [10] for an account of the hyperbolic metric in the simply connected case.

Suppose that α and β are prime ends of D and that Γ is the hyperbolic geodesic that joins them. In [3] we were able to compare the quantities $d(z, \Gamma; D)$ and $K_D(z, \alpha)K_D(z, \beta)$ and hence obtained an expression involving hyperbolic distance which is, up to a constant, the maximal expected lifetime. In retrospect, it is clear that there has to be an exact relationship between these two quantities, in accordance with the principle outlined by Ahlfors [1], p. 6: $d(z, \Gamma; D)$ is a conformal invariant which depends on one internal point and two boundary points, namely the endpoints of Γ . Hence, any function of $d(z, \Gamma; D)$ has the same dependence and, more importantly, any other conformal invariant of this type – for example, $K_D(z, \alpha)K_D(z, \beta)$ – arises in this way. We now determine explicitly the functional dependence in this specific case, the result being an improvement on Lemma 1 in [3], both in terms of its content and its proof.

LEMMA 1. — *Suppose that D is a simply connected domain and that Γ is a hyperbolic geodesic in D whose endpoints are the prime ends α*

and β . Then, for z in D ,

$$K_D(z, \alpha)K_D(z, \beta) = \operatorname{sech}^2[2d(z, \Gamma; D)],$$

where the above product of Poisson kernel functions is normalized to equal 1 on the hyperbolic geodesic Γ .

Proof. — We map D conformally onto the unit disk U so that the prime ends α and β correspond to -1 and 1 respectively. By composing with an appropriate automorphism $M(z) = (z - r)/(1 - rz)$, $-1 < r < 1$, which fixes -1 and 1 , we may arrange that the image of z lies on the imaginary axis, say at ir . By conformal invariance,

$$K_D(z, \alpha)K_D(z, \beta) = K_U(ir, -1)K_U(ir, 1) = \left(\frac{1 - r^2}{1 + r^2}\right)^2.$$

It also follows from conformal invariance that

$$d(z, \Gamma; D) = d(ir, (-1, 1); U) = d(ir, 0; U) = \frac{1}{2} \log \frac{1 + r}{1 - r}.$$

So solving for r we find that $r = \tanh d(z, \Gamma; D)$ and substituting this into the above expression for the product of Poisson kernels, we obtain

$$K_D(z, \alpha)K_D(z, \beta) = \left(\frac{1 - \tanh^2 d(z, \Gamma; D)}{1 + \tanh^2 d(z, \Gamma; D)}\right)^2 = \frac{1}{\cosh^2[2d(z, \Gamma; D)]},$$

which proves the lemma. \square

From Lemma 1 and the results of Griffin, McConnell and Verchota (2.1) we obtain an expression for the maximal lifetime of conditioned Brownian motion in a simply connected domain D which is an improvement of Theorem 1 in [3], namely

$$\mathcal{L}_D = \sup_{\Gamma} \left\{ \frac{2}{\pi} \int_D \operatorname{sech}^2[2d(z, \Gamma; D)] \, dm(z) \right\},$$

the supremum being over all hyperbolic geodesics in D .

We now propose to review Griffin, McConnell and Verchota's Problem in the light of this expression for \mathcal{L}_D . It is suggested that the maximal expected lifetime of conditioned Brownian motion in a convex domain is greater than that for a disk of the same area. The maximal expected lifetime of Brownian motion occurs when the Brownian motion is conditioned to travel between boundary points and, in the case of a disk, the maximum occurs when the boundary points are diametrically opposite. Thus an equivalent question is whether there corresponds to each convex domain

D of finite area, boundary points α and β such that the expected lifetime $E_{\alpha}^{\beta} \tau_D$ of Brownian motion conditioned to travel between α and β in D is greater than the lifetime of Brownian motion conditioned to travel between diametrically opposite boundary points of a disk of the same area as D . Bringing Lemma 1 and (2.1) to bear at this point, we see that the problem may be thought of as follows: is it the case that, for each convex domain D , there is a hyperbolic geodesic Γ in D such that

$$\frac{2}{\pi} \int_D \operatorname{sech}^2[2d(z, \Gamma; D)] \, dm(z) \geq \frac{2}{\pi} \int_{D^*} \operatorname{sech}^2[2d(z, \Gamma^*; D^*)] \, dm(z)$$

where D^* is a disk of the same area as D and Γ^* is a diameter of that disk?

When written in this way, the function $(2/\pi) \operatorname{sech}^2(2x)$ seems out of place and indeed it would be very surprising if the resolution of the problem was to depend on this specific function rather than on its general properties. The most obvious of these is that $(2/\pi) \operatorname{sech}^2(2x)$ is a decreasing function on $[0, \infty)$. Hence we arrive at the more general problem, which we choose to state as a conjecture.

CONJECTURE 1. — *To each convex domain D of finite area there corresponds a hyperbolic geodesic Γ such that for each non-negative, non-increasing function $\psi(x)$ on $[0, \infty)$,*

$$(2.3) \quad \int_D \psi(d(z, \Gamma; D)) \, dm(z) \geq \int_{D^*} \psi(d(z, \Gamma^*; D^*)) \, dm(z)$$

where D^* is a disk of the same area as D and Γ^* is a diameter of D^* . Equality holds for some non-constant function ψ if and only if D is a disk.

For example, if in addition D was symmetric in its longest diameter it would be natural to expect that the correct choice of Γ would be this diameter.

Given c positive, we may take $\psi(x) = 1_{[0, c)}(x)$ in (2.3), in which case (2.3) becomes

$$(2.4) \quad \operatorname{area} \{z \in D : d(z, \Gamma; D) < c\} \geq \operatorname{area} \{z \in D^* : d(z, \Gamma^*; D^*) < c\}.$$

On the other hand, suppose that (2.4) holds for a certain geodesic Γ in D and all positive c . We wish to show that (2.3) holds for the same geodesic Γ and each permissible function ψ . The validity of inequality (2.4) for each positive c is equivalent to the statement that (2.3) holds for each ψ of the form $1_{[0, c)}(x)$, with c positive. Hence, (2.3) holds for those simple functions ψ of the form

$$(2.5) \quad \sum_{i=0}^n \alpha_i 1_{[0, c_i)}(x),$$

where each α_i and each c_i is positive. Suppose that a non-negative, non-increasing function $\psi(x)$ on $[0, \infty)$ is given. By redefining ψ at a countable set of points, an operation which affects neither of the integrals in (2.3), we may assume that ψ is right continuous. The function ψ may then be written as the pointwise limit of an increasing sequence of functions $\{\psi_n(x)\}_{n=1}^\infty$, each having the form (2.5). (We omit the straightforward proof.) It then follows from monotone convergence that (2.3) holds also for the given function ψ . Thus Conjecture 1 may be reformulated as follows.

CONJECTURE 2. — *To each convex domain D of finite area there corresponds a hyperbolic geodesic Γ such that, for each positive c , the Euclidean area of the region $\{z \in D : d(z, \Gamma; D) < c\}$ is at least as great as that of the region $\{z \in D^* : d(z, \Gamma^*; D^*) < c\}$, where D^* is a disk of the same area as D and Γ^* is a diameter of D^* . Equality holds for some positive c if and only if D is a disk.*

Regions of the form $\{z \in D : d(z, \Gamma; D) < c\}$ are conformally invariant although their Euclidean areas are not. The conformal invariance of these regions allows us to refine our conjecture even further. In the case of the strip $S = \{z : |\operatorname{Im} z| < \pi/2\}$, when the geodesic Γ is the real axis, the region $\{z \in S : d(z, \Gamma; S) < c\}$ is simply a substrip $S_t = \{z : |\operatorname{Im} z| < t\}$ where $0 < t < \pi/2$. We need to know how c and t are related.

Taking $F(z) = (e^z - 1)/(e^z + 1)$ in (2.2), we may verify that the hyperbolic metric for the strip S is

$$\sigma_S(z) = \frac{1}{2 \cos(\operatorname{Im} z)}.$$

Suppose that z is in S and that x_0 is real. The hyperbolic geodesic from z to \bar{z} in S is the Euclidean line segment $[z, \bar{z}]$. If γ is any curve from z to x_0 , we may reflect it in the real axis and adjoin this to γ to produce a curve joining z to \bar{z} . This last curve has hyperbolic length twice that of γ and at least that of the line segment $[z, \bar{z}]$. Since the hyperbolic length of $[z, \bar{z}]$ is twice that of $[z, \operatorname{Re} z]$, we deduce that the hyperbolic length of γ is at least that of the line segment $[z, \operatorname{Re} z]$. Thus $\operatorname{Re} z$ is the point on the real axis which is hyperbolically closest to z and

$$\begin{aligned} d(z, \mathbb{R}; S) &= d(z, \operatorname{Re} z; S) \\ &= d(i|\operatorname{Im} z|, 0; S) \\ &= \int_0^{|\operatorname{Im} z|} \sigma_S(iy) dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^{|\operatorname{Im} z|} \frac{dy}{2 \cos y} \\
&= \frac{1}{2} \ln(\sec |\operatorname{Im} z| + \tan |\operatorname{Im} z|).
\end{aligned}$$

The function $c = 2^{-1} \ln(\sec t + \tan t)$ is increasing on $[0, \pi/2)$ and its inverse function on $[0, \infty)$ is $t = \arctan[\sinh(2c)]$. Hence,

$$(2.6) \quad \{z \in S : d(z, \mathbb{R}; S) < c\} = S_t \quad \text{where } t = \arctan[\sinh(2c)].$$

Suppose now that D is a simply connected domain and that Γ is a hyperbolic geodesic in D . There is a conformal mapping $f(z)$ of the strip S onto D under which the real axis, a geodesic in S , corresponds to the geodesic Γ in D . By conformal invariance of hyperbolic distance,

$$\begin{aligned}
\{z \in D : d(z, \Gamma; D) < c\} &= \{z \in D : d(f^{-1}(z), \mathbb{R}; S) < c\} \\
&= \{z \in D : f^{-1}(z) \in S_t\} \\
&= f(S_t),
\end{aligned}$$

where, by (2.6), $t = \arctan[\sinh(2c)]$. When D is the unit disk U and Γ is the interval $(-1, 1)$, we may take the conformal mapping to be $F(z) = (e^z - 1)/(e^z + 1)$. Thus Conjecture 2, and hence Conjecture 1, may be reformulated as follows.

CONJECTURE 3. — *To each convex domain D of area π there corresponds a conformal mapping $f(z)$ of the strip S onto D such that, for each t in $(0, \pi/2)$, the area of $f(S_t)$ is at least as great as the area of $F(S_t)$ where $F(z) = (e^z - 1)/(e^z + 1)$. Equality holds for some t if and only if D is a disk of area π .*

For a simply connected domain D , a geodesic Γ in D and for t in $(0, \pi/2)$, we set

$$(2.7) \quad S(t, \Gamma; D) = \{z \in D : d(z, \Gamma; D) < 2^{-1} \ln(\sec t + \tan t)\}$$

and

$$(2.8) \quad A(t, \Gamma; D) = \operatorname{area} S(t, \Gamma; D).$$

Hence $S(t, \Gamma; D)$ is $f(S_t)$ where f maps the strip S conformally onto D and maps the real line onto Γ . Our aim is to study the Euclidean area $A(t, \Gamma; D)$ of this region with particular reference to convex domains and Conjecture 3. However, we first investigate the corresponding upper bound for which we have a complete and satisfactory solution.

3. A generalization of the Griffin, McConnell, Verchota inequality.

With the above more general geometric formulations of the original problem, it is natural to ask about a sharp upper bound for $A(t, \Gamma; D)$ and about extensions of the sharp inequality of Griffin–McConnell–Verchota. We begin here with such a sharp inequality for $A(t, \Gamma; D)$ and then relate it to the Griffin–McConnell–Verchota result in Theorems 2 and 3 below.

THEOREM 1. — *Suppose that D is a simply connected domain of finite area and that Γ is a hyperbolic geodesic in D . Then the function $A(t, \Gamma; D)$ is an infinitely differentiable function of t on $[0, \pi/2)$. Moreover, $A''(t, \Gamma; D) > 0$ on $(0, \pi/2)$ so that $A(t, \Gamma; D)$ is strictly convex. In particular,*

$$(3.1) \quad A(t, \Gamma; D) < \frac{2}{\pi} \text{area}(D)t \quad \text{for } 0 < t < \pi/2.$$

For a given positive A , we denote by R_L a rectangle of side length L and width A/L and denote by Γ_L the line of symmetry of length L of R_L . Then Γ_L is a hyperbolic geodesic in R_L and to each positive ϵ there corresponds a positive L_0 such that, for $L \geq L_0$,

$$(3.2) \quad A(t, \Gamma_L; R_L) \geq \frac{2 - \epsilon}{\pi} At \quad \text{for } 0 < t < \pi/2.$$

In terms of conformal mapping the inequality (3.1) asserts that for any conformal mapping of the strip S onto a domain of finite area A , the area of the image of the substrip S_t is strictly less than $2At/\pi$. The second part of the theorem asserts that long, thin rectangles are asymptotically extremal for the estimate (3.1). They were shown by Griffin, McConnell and Verchota to be asymptotically extremal for the Cranston–McConnell Theorem on conditioned Brownian motion for simply connected domains. We shall return to this point later.

Proof. — Let $f(z)$ be a conformal mapping of the strip S onto D under which the real axis in S and the geodesic Γ in D correspond. Since $A(t, \Gamma; D)$ is the area of $f(S_t)$,

$$\begin{aligned} A(t, \Gamma; D) &= \iint_{S_t} |f'(z)|^2 dm(z) \\ &= \int_{-t}^t \int_{-\infty}^{\infty} |f'(x + iy)|^2 dx dy \\ &= \int_0^t H(y) dy, \end{aligned}$$

where

$$H(y) = \int_{-\infty}^{\infty} (|f'(x + iy)|^2 + |f'(x - iy)|^2) dx.$$

In [7], Griffin, McConnell and Verchota show that the function $H(y)$ is bounded on $[0, a)$ for each a with $0 < a < \pi/2$. Then, by the Paley-Wiener Theorem [9], p. 174, $f'(z)$ is the inverse Fourier transform of a function $\phi(\zeta)$ for which $e^{a|\zeta|}\phi(\zeta)$ belongs to $L^2(\mathbb{R})$ for each a in $[0, \pi/2)$. Thus,

$$f'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\zeta) e^{i\zeta z} d\zeta, \quad \text{for } z \in S.$$

Plancherel's theorem then yields

$$\begin{aligned} H(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(\zeta)|^2 (e^{-2\zeta y} + e^{2\zeta y}) d\zeta \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} |\phi(\zeta)|^2 \cosh(2\zeta y) d\zeta. \end{aligned}$$

This representation for the L^2 integral means of $f'(z)$ together with the integrability property of $\phi(\zeta)$ imply that $H(y)$ is infinitely differentiable on $[0, \pi/2)$. Thus $A(t, \Gamma; D)$ is differentiable on $[0, \pi/2)$ with derivative $H(t)$ and

$$\frac{d^2}{dt^2} A(t, \Gamma; D) = H'(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} 2\zeta \sinh(2\zeta t) |\phi(\zeta)|^2 d\zeta.$$

This last integral is positive for t in $(0, \pi/2)$ and it follows that the function $A(t, \Gamma; D)$ is strictly convex on $(0, \pi/2)$. We note that $A''(0, \Gamma; D) = 0$.

We now turn to the proof of the estimate (3.2) for long, thin rectangles. First we observe that because of the convexity of the function $A(t, \Gamma_L; R_L)$, it is sufficient to prove that $A'(0, \Gamma_L; R_L)$, the derivative of the area function at 0, satisfies

$$A'(0, \Gamma_L; R_L) \geq \frac{2 - \epsilon}{\pi} A.$$

For a positive, the Schwarz-Christoffel transformation

$$f_a(z) = K_a \int_0^z \frac{dw}{\sqrt{w^2 - a^2} \sqrt{w^2 - 1}},$$

maps the upper half plane $H = \{z : \operatorname{Im} z > 0\}$ onto a rectangle R and we assume that the scaling factor K_a is chosen so that the area of R is the given number A . The imaginary axis in the upper half plane is mapped by $f_a(z)$ to a line of symmetry of the rectangle and will be our geodesic Γ .

We now find a formula for $A'(0, \Gamma; R)$ in terms of the conformal mapping $f_a(z)$. The function $F_a(z) = f_a(ie^z)$ maps the strip S conformally

onto the rectangle and sends the real axis to Γ . Thus,

$$\begin{aligned}
 A'(0, \Gamma; R) &= 2 \int_{-\infty}^{\infty} |F'_a(x)|^2 dx \\
 &= 2 \int_{-\infty}^{\infty} e^{2x} |f'_a(ie^x)|^2 dx \\
 &= 2 \int_0^{\infty} y |f'_a(iy)|^2 dy \\
 &= 2K_a^2 \int_0^{\infty} \frac{y dy}{(a^2 + y^2)(1 + y^2)} \\
 &= \frac{2K_a^2}{1 - a^2} \int_0^{\infty} \left(\frac{y}{a^2 + y^2} - \frac{y}{1 + y^2} \right) dy \\
 &= \frac{2K_a^2}{1 - a^2} \ln(1/a).
 \end{aligned}$$

The lengths of the sides of R are $K_a I_1$ and $K_a I_2$ where

$$I_1 = \int_a^1 \frac{dx}{\sqrt{x^2 - a^2} \sqrt{1 - x^2}}$$

and

$$I_2 = 2 \int_0^a \frac{dx}{\sqrt{a^2 - x^2} \sqrt{1 - x^2}}.$$

Since the factor K_a was chosen so that $(K_a I_1)(K_a I_2) = A$, we find that

$$(3.3) \quad A'(0, \Gamma; R) = \frac{2A}{I_1 I_2 (1 - a^2)} \ln(1/a).$$

Thus we need upper bounds for I_1 and I_2 , for a close to 0.

First, since $\sqrt{1 - a^2} \leq \sqrt{1 - x^2}$ for $0 \leq x \leq a$, we find that

$$I_2 \leq \frac{2}{\sqrt{1 - a^2}} \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{\sqrt{1 - a^2}} = (1 + o(1))\pi$$

as $a \rightarrow 0^+$. The estimate for I_1 requires a little more work. We set $\alpha = \alpha(a) = 1/\sqrt{\ln(1/a)}$, so that $a < \alpha(a) < 1$ for small a and $\alpha(a)$ approaches 0 as a tends to 0. We also use the estimate $\cosh^{-1} x \leq \ln(2x)$

valid for $x > 1$. Then, for all sufficiently small positive a ,

$$\begin{aligned}
 I_1 &= \int_a^\alpha \frac{dx}{\sqrt{x^2 - a^2} \sqrt{1 - x^2}} + \int_\alpha^1 \frac{dx}{\sqrt{x^2 - a^2} \sqrt{1 - x^2}} \\
 &\leq \frac{1}{\sqrt{1 - \alpha^2}} \int_a^\alpha \frac{dx}{\sqrt{x^2 - a^2}} + \frac{1}{\sqrt{\alpha^2 - a^2}} \int_\alpha^1 \frac{dx}{\sqrt{1 - x^2}} \\
 &= \frac{1}{\sqrt{1 - \alpha^2}} \cosh^{-1}(\alpha/a) + \frac{1}{\sqrt{\alpha^2 - a^2}} (\pi/2 - \sin^{-1} \alpha) \\
 &\leq \frac{1}{\sqrt{1 - \alpha^2}} \ln(2\alpha/a) + \frac{\pi}{2\sqrt{\alpha^2 - a^2}} \\
 &= \frac{1}{\sqrt{1 - \alpha^2}} [\ln(1/a) - 2^{-1} \ln \ln(1/a) + \ln 2] + \frac{\pi \sqrt{\ln(1/a)}}{2\sqrt{1 - \alpha^2 \ln(1/a)}} \\
 &= (1 + o(1)) \ln(1/a) \quad \text{as } a \rightarrow 0^+.
 \end{aligned}$$

The estimates for I_1 and I_2 together with (3.3) lead to

$$\begin{aligned}
 A'(0, \Gamma; R) &= \frac{2A}{I_1 I_2 (1 - a^2)} \ln(1/a) \\
 &\geq \frac{2A}{(1 - a^2) \pi \ln(1/a) (1 + o(1))} \ln(1/a) \\
 &= \frac{2A}{\pi} (1 + o(1))
 \end{aligned}$$

as $a \rightarrow 0^+$. The line of symmetry Γ of the rectangle is parallel to the side of length $K_a I_1$ and we complete the proof by showing that this side becomes arbitrarily long as a tends to 0^+ . First, $I_1 \rightarrow \infty$ as $a \rightarrow 0^+$ since

$$I_1 \geq \int_a^{\sqrt{a}} \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(1/\sqrt{a}) \geq \ln(1/\sqrt{a}).$$

Since $A = K_a^2 I_1 I_2$ and I_2 is clearly greater than π , it must be that the scale factor K_a tends to 0 as $a \rightarrow 0^+$. Hence, the side length $K_a I_1 = A/(K_a I_2)$ becomes unbounded as $a \rightarrow 0^+$. \square

Just as Conjectures 1 and 2 are equivalent, the distributional-type inequality (3.1) has as a consequence an inequality which may be considered to be the companion upper bound to the lower bound conjectured in (2.3).

THEOREM 2. — Suppose that D is a simply connected domain of finite area A and that Γ is a hyperbolic geodesic in D . Then, for each non-negative, non-increasing function $\psi(x)$ on $[0, \infty)$

$$(3.4) \quad \int_D \psi(d(z, \Gamma; D)) \, dm(z) \leq \frac{2A}{\pi} \int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) \, dt.$$

Strict inequality holds in (3.4) unless $\psi(x)$ is constant. Moreover, for each fixed function $\psi(x)$ having the stated properties, asymptotic equality holds in (3.4) for the rectangles R_L and geodesics Γ_L of Theorem 1 in that, as $L \rightarrow \infty$,

$$(3.5) \quad \int_{R_L} \psi(d(z, \Gamma_L; R_L)) dm(z) \rightarrow \frac{2A}{\pi} \int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) dt.$$

Proof. — Given c positive, we let $t = \arctan[\sinh(2c)]$, so that $0 < t < \pi/2$ and $2^{-1} \ln(\sec t + \tan t) = c$. Since, by (2.7) and (2.8), $A(t, \Gamma; D)$ is the Euclidean area of $\{z \in D : d(z, \Gamma; D) < c\}$, we obtain from (3.1) that

$$\text{area} \{z \in D : d(z, \Gamma; D) < c\} < \frac{2A}{\pi} \arctan[\sinh(2c)].$$

We may assume that the function $\psi(x)$ occurring in (3.4) is right continuous. As stated earlier, each right continuous, non-negative, non-increasing function ψ on $[0, \infty)$ is an increasing limit of step functions $\{\psi_n\}_{n=1}^\infty$ and each ψ_n may be written in the form

$$(3.6) \quad \psi_n(x) = \sum_{i=1}^{k(n)} \alpha_{i,n} 1_{[0, c_{i,n})}(x),$$

where $\alpha_{i,n}$ is positive for each i and the $c_{i,n}$ are positive and strictly increasing. Because of monotone convergence, it suffices to prove (3.4) for each step function ψ_n , and in this case

$$\begin{aligned} \int_D \psi_n(d(z, \Gamma; D)) dm(z) &= \sum_{i=1}^{k(n)} \alpha_{i,n} \int_D 1_{[0, c_{i,n})}(d(z, \Gamma; D)) dm(z) \\ &= \sum_{i=1}^{k(n)} \alpha_{i,n} \text{area} \{z \in D : d(z, \Gamma; D) < c_{i,n}\} \\ &< \sum_{i=1}^{k(n)} \alpha_{i,n} \frac{2A}{\pi} \arctan[\sinh(2c_{i,n})]. \end{aligned}$$

We may write

$$\begin{aligned} \arctan[\sinh(2c_{i,n})] &= |\{t \in [0, \pi/2) : 2^{-1} \ln(\sec t + \tan t) < c_{i,n}\}| \\ &= \int_0^{\pi/2} 1_{[0, c_{i,n})}(2^{-1} \ln(\sec t + \tan t)) dt, \end{aligned}$$

and so it follows that

$$\begin{aligned} \int_D \psi_n(d(z, \Gamma; D)) dm(z) &< \frac{2A}{\pi} \sum_{i=1}^{k(n)} \alpha_{i,n} \int_0^{\frac{\pi}{2}} 1_{[0, c_{i,n})}(2^{-1} \ln(\sec t + \tan t)) dt \\ &= \frac{2A}{\pi} \int_0^{\frac{\pi}{2}} \sum_{i=1}^{k(n)} \alpha_{i,n} 1_{[0, c_{i,n})}(2^{-1} \ln(\sec t + \tan t)) dt \\ &= \frac{2A}{\pi} \int_0^{\pi/2} \psi_n(2^{-1} \ln(\sec t + \tan t)) dt. \end{aligned}$$

Thus, (3.4) does hold for each step function ψ_n of the form (3.6).

We now turn our attention to the case of equality in (3.4). If $0 < a < b < \infty$ then

$$\psi(a) - \psi(b) = \lim_{n \rightarrow \infty} (\psi_n(a) - \psi_n(b)) = \lim_{n \rightarrow \infty} \sum_{i=1}^{k(n)} \alpha_{i,n} 1_{(a,b]}(c_{i,n}).$$

This last limit does exist as a consequence and, if ψ is not constant on $[0, \infty)$, we may choose the numbers a and b so that the limit is positive. Since the area function $A(t, \Gamma; D)$ is strictly convex on $(0, \pi/2)$ and since $A(0, \Gamma; D) = 0$ and $A(\pi/2, \Gamma; D) = A$, there is a positive number ϵ such that

$$A(t, \Gamma; D) \leq \frac{2A}{\pi} t - \epsilon \quad \text{for } \arctan[\sinh(2a)] \leq t \leq \arctan[\sinh(2b)],$$

or, equivalently,

$$\text{area} \{z \in D : d(z, \Gamma; D) < c\} \leq \frac{2A}{\pi} \arctan[\sinh(2c)] - \epsilon, \quad \text{for } c \in [a, b].$$

Using this inequality we find that, for each n ,

$$\begin{aligned} &\int_D \psi_n(d(z, \Gamma; D)) dm(z) \\ &= \sum_{i=1}^{k(n)} \alpha_{i,n} \text{area} \{z \in D : d(z, \Gamma; D) < c_{i,n}\} \\ &\leq \sum_{i=1}^{k(n)} \alpha_{i,n} \frac{2A}{\pi} \arctan[\sinh(2c_{i,n})] - \epsilon \sum_{i=1}^{k(n)} \alpha_{i,n} 1_{(a,b]}(c_{i,n}) \\ &= \frac{2A}{\pi} \int_0^{\pi/2} \psi_n(2^{-1} \ln(\sec t + \tan t)) dt - \epsilon \sum_{i=1}^{k(n)} \alpha_{i,n} 1_{(a,b]}(c_{i,n}). \end{aligned}$$

On taking the limit, we obtain that strict inequality holds in (3.4).

Next we show that long, thin rectangles are asymptotically extremal for (3.4) for each fixed function $\psi(x)$ under consideration. We again use

the sequence (3.6) of step functions $\{\psi_n\}_1^\infty$. Given a positive number ϵ , the monotone convergence theorem allows n to be chosen so that

$$\int_0^{\pi/2} \psi_n(2^{-1} \ln(\sec t + \tan t)) dt \geq \int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) dt - \epsilon.$$

For this n and for sufficiently large L ,

$$\begin{aligned} & \int_{R_L} \psi(d(z, \Gamma_L; R_L)) dm(z) \\ & \geq \int_{R_L} \psi_n(d(z, \Gamma_L; R_L)) dm(z) \\ & = \sum_{i=1}^n \alpha_{i,n} \text{area}\{z \in R_L : d(z, \Gamma_L; R_L) < c_{i,n}\} \\ \text{(by (3.2)) } & \geq \sum_{i=1}^n \alpha_{i,n} \frac{2-\epsilon}{\pi} A \arctan[\sinh(2c_{i,n})] \\ & = \frac{(2-\epsilon)A}{\pi} \int_0^{\pi/2} \psi_n(2^{-1} \ln(\sec t + \tan t)) dt \\ & \geq \frac{(2-\epsilon)A}{\pi} \int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) dt - \epsilon \frac{(2-\epsilon)A}{\pi}. \end{aligned}$$

Since this holds for each positive ϵ , once the rectangle is sufficiently long and thin, asymptotic equality holds in (3.4) for such regions. \square

Strict inequality holds in (3.4) if the function ψ is non-constant. We now prove that strict inequality holds uniformly over all geodesics in the domain, at least when the function ψ is continuous. This will be important when we come to make the connection with the Cranston-McConnell Theorem.

THEOREM 3. — *Suppose that D is a simply connected domain of finite area A and that ψ is a non-negative, non-increasing, non-constant and continuous function on $[0, \infty)$. Then*

$$(3.7) \quad \sup_{\Gamma} \int_D \psi(d(z, \Gamma; D)) dm(z) < \frac{2A}{\pi} \int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) dt,$$

where the supremum is over all geodesics Γ in the domain D .

Proof. — We choose a conformal mapping $f(z)$ of the unit disk U onto the simply connected domain D . There is a one-to-one correspondence between geodesics in U and geodesics in D under the mapping f . Let us

denote the geodesic in the unit disk with endpoints e^{is} and e^{it} by $\gamma_{s,t}$ and its image under f by $\Gamma_{s,t}$. Then,

$$\begin{aligned} \int_D \psi(d(z, \Gamma_{s,t}; D)) dm(z) &= \int_U \psi(d(f(z), f(\gamma_{s,t}); D)) |f'(z)|^2 dm(z) \\ &= \int_U \psi(d(z, \gamma_{s,t}; U)) |f'(z)|^2 dm(z). \end{aligned}$$

Let us consider the function

$$g(s, t) = \int_U \psi(d(z, \gamma_{s,t}; U)) |f'(z)|^2 dm(z)$$

on the rectangle

$$R = \{(s, t) : s \in [0, 2\pi], t \in [0, 2\pi]\}.$$

By (3.4) the function g is finite at each point of R and we claim that $g(s, t)$ is continuous on R . It will follow that $g(s, t)$ assumes its supremum value on R for a certain choice of the pair (s, t) . Because ψ is non-constant, strict inequality will hold in (3.4) for this extremal geodesic $\Gamma_{s,t}$ and (3.7) will follow.

We complete the proof of the theorem by proving the continuity of the function $g(s, t)$. Suppose that (s_0, t_0) is a point in R and that ϵ is positive. Since D has finite area A , it is possible to find an r in $(0, 1)$ for which the area of $f(D(0, r))$ exceeds $A - \epsilon/(4\psi(0))$. For each z in $D(0, r)$, we denote by w_z the point on γ_{s_0, t_0} which is hyperbolically closest to z in U . We take some fixed point w on γ_{s_0, t_0} and note that the hyperbolic distance from w to z is uniformly bounded over all z in $D(0, r)$. It follows that there is a positive C_0 such that $d(z, \gamma_{s_0, t_0}; U) \leq C_0$ for z in $D(0, r)$. In addition, there is a limit to how close w_z can be to the unit circle when z is in $D(0, r)$: that is, the set of points w_z as z varies over the disk $D(0, r)$ will lie in a compact subset of the unit disk.

Since Euclidean and hyperbolic distances are comparable in any given compact subset of the unit disk, if we are given ϵ' positive and less than 1, say, there will therefore be a positive δ such that whenever $|s - s_0| < \delta$, $|t - t_0| < \delta$ and $z \in D(0, r)$,

$$|d(z, \gamma_{s_0, t_0}; U) - d(z, \gamma_{s, t}; U)| < \epsilon'.$$

Since ψ is uniformly continuous on $[0, C_0 + 1]$, we can choose ϵ' such that

$$|\psi(d(z, \gamma_{s_0, t_0}; U)) - \psi(d(z, \gamma_{s, t}; U))| < \frac{\epsilon}{2A}$$

for all z in $D(0, r)$. Then,

$$\int_{D(0, r)} |\psi(d(z, \gamma_{s_0, t_0}; U)) - \psi(d(z, \gamma_{s, t}; U))| |f'(z)|^2 dm(z) < \frac{\epsilon}{2A} A = \frac{\epsilon}{2}.$$

Using the triangle inequality in the usual way, the remaining term in the estimate for $|g(s_0, t_0) - g(s, t)|$ is at most the integral

$$\int_{U \setminus D(0, r)} [\psi(d(z, \gamma_{s_0, t_0}; U)) + \psi(d(z, \gamma_{s, t}; U))] |f'(z)|^2 dm(z).$$

Since the function ψ is decreasing, this quantity is bounded above by $2\psi(0)$ times the area of the image of $U \setminus D(0, r)$, which we have arranged to be less than $\epsilon/(4\psi(0))$. Thus $|g(s_0, t_0) - g(s, t)| < \epsilon$ whenever $|s - s_0| \leq \delta$ and $|t - t_0| \leq \delta$, proving the desired continuity of the function g . \square

When stating the above results we chose the hyperbolic distance $d(z, \Gamma; D)$ as the representative of those conformal invariants which depend on two boundary points and one internal point, this in order to emphasize the connection with (2.3). This was simply one possible choice, however, and the choice of the product of Poisson kernels is one which brings out more strongly the connection with the maximal expected lifetime of conditioned Brownian motion. We state this as a separate theorem, though it is but a reincarnation of Theorem 2 above.

THEOREM 4. — Suppose that D is a simply connected domain of finite area A , that α and β are prime ends of D and that $K_D(z, \alpha)$ and $K_D(z, \beta)$ are Poisson kernels for D with poles at α and β respectively and normalized so that their product is 1 on the hyperbolic geodesic joining α and β . Then, for each non-negative, non-decreasing function $\phi(x)$ on $(0, 1]$

$$(3.8) \quad \int_D \phi(K_D(z, \alpha) K_D(z, \beta)) dm(z) \leq \frac{2A}{\pi} \int_0^{\pi/2} \phi(\cos^2 t) dt.$$

Strict inequality holds in (3.8) unless ϕ is constant. Moreover, for each fixed such function ϕ , asymptotic equality holds in (3.8) for the rectangles R_L and geodesics Γ_L of Theorem 1 in that, as $L \rightarrow \infty$,

$$(3.9) \quad \int_{R_L} \phi(K_{R_L}(z, \alpha) K_{R_L}(z, \beta)) dm(z) \rightarrow \frac{2A}{\pi} \int_0^{\pi/2} \phi(\cos^2 t) dt,$$

where α and β are the endpoints of the longer line of symmetry Γ_L of R_L .

Proof. — Since $\operatorname{sech}^2(2x)$ is a decreasing function on $[0, \infty)$ and takes values in $(0, 1]$, the composed function

$$\psi(x) = \phi(\operatorname{sech}^2(2x)), \quad x \in [0, \infty)$$

is a non-negative, non-increasing function on $[0, \infty)$. Since by Lemma 1, $\operatorname{sech}^2[2d(z, \Gamma; D)] = K_D(z, \alpha)K_D(z, \beta)$, where Γ is the geodesic determined by the prime ends α and β ,

$$\int_D \psi(d(z, \Gamma; D)) dm(z) = \int_D \phi(K_D(z, \alpha) K_D(z, \beta)) dm(z).$$

Since $\operatorname{sech}^2 [\ln(\sec t + \tan t)] = \cos^2 t$,

$$\int_0^{\pi/2} \psi(2^{-1} \ln(\sec t + \tan t)) dt = \int_0^{\pi/2} \phi(\cos^2 t) dt.$$

The inequality (3.8) now follows directly from (3.4), together with the case of equality, while (3.9) follows from (3.5). \square

This proof also shows that Theorem 3 may be reformulated as stating that for a non-negative, non-decreasing, non-constant and continuous function $\phi(x)$ on $(0, 1]$,

$$(3.10) \quad \sup_{\alpha, \beta} \int_D \phi(K_D(z, \alpha) K_D(z, \beta)) dm(z) < \frac{2A}{\pi} \int_0^{\pi/2} \phi(\cos^2 t) dt,$$

where the supremum is over all pairs of prime ends α and β of D . By choosing $\phi(x) = (2/\pi)x$, we deduce that for a simply connected domain D of finite area

$$\sup_{\alpha, \beta} \frac{2}{\pi} \int_D K_D(z, \alpha) K_D(z, \beta) dm(z) < \frac{1}{\pi} \operatorname{area}(D).$$

Hence, by virtue of (2.1), the maximal expected lifetime of conditioned Brownian motion in a simply connected domain satisfies

$$\mathcal{L}_D < \frac{1}{\pi} \operatorname{area}(D),$$

a result first proved by Griffin, McConnell and Verchota. In fact, their determination of the best constant, $1/\pi$, in the Cranston-McConnell Theorem in the simply connected case is the main result in [7]. There are no simply connected domains for which equality holds but they show that, for instance, long thin rectangles are asymptotically extremal.

Griffin, McConnell and Verchota also consider the situation where the Brownian motion is conditioned to start and to end in the interior of the simply connected domain and show that $1/\pi$ is still the best constant even in this case. It would therefore be interesting to know if the product of kernel functions in (3.10) can be replaced by the quantity $G_D(z, \alpha) G_D(z, \beta) / G_D(\alpha, \beta)$.

Thinking of $K_D(z, \alpha) K_D(z, \beta)$ and $K_{R_L}(z, \alpha) K_{R_L}(z, \beta)$ as the Green's function of the conditioned Brownian motion in the domain D and the rectangle R_L , respectively, we see that Theorem 3 is in complete analogy with the classical results for the Green's function of the unconditioned "ordinary" Brownian motion in D , namely

$$(3.11) \quad \sup_{w \in D} \int_D \phi(G_D(w, z)) dm(z) \leq \int_{D^*} \phi(G_{D^*}(0, z)) dt,$$

for all non-negative, non-decreasing functions ϕ . Here D^* denotes the disk of the same area as D centered at the origin. Furthermore, equality holds if and only if D is a disk. This result, which holds for general domains in any dimension, can be found in C. Bandle [2].

4. Evidence for Conjecture 1.

Conjecture 1 is not true for general simply connected domains since, as discussed earlier, there are simply connected domains of infinite area for which the maximal expected lifetime of conditioned Brownian motion \mathcal{L}_D is finite. Here is another point of view that shows this more directly and that brings out the connection with our conjectures. We set

$$D_n = U \setminus \bigcup_{k=1}^n \left[\frac{1}{n} e^{2k\pi i/n}, e^{2k\pi i/n} \right],$$

so that D_n is the unit disk with n symmetrically arranged radial slits removed, each extending to within a distance $1/n$ of the origin. No point in D_n is more than a distance π/n from the boundary of D_n . Hence, by the Koebe 1/4-Theorem, at any point z in D_n the hyperbolic metric $\sigma_{D_n}(z)$ satisfies

$$\sigma_{D_n}(z) \geq \frac{1}{4\delta_{D_n}(z)} \geq \frac{n}{4\pi}.$$

The part of the domain D_n lying outside the disk $D(0, 1/\sqrt{n})$ consists of n truncated sectors, S_k , $k = 1, 2, \dots, n$,

$$S_k = \left\{ re^{it} : \frac{1}{\sqrt{n}} < r < 1 \text{ and } \frac{2(k-1)\pi}{n} < t < \frac{2k\pi}{n} \right\}.$$

Suppose that Γ is a geodesic in D_n which enters one of the sectors S_k and that z is a point on Γ in this sector. At z we draw the two largest disks tangent to Γ and contained in D_n , one on each side of Γ . The geodesic Γ can never enter either of these two tangent disks [11], Section 10.3. Together these disks block the entrance to the sector and so once a geodesic Γ enters a sector S_k it cannot leave it. Thus we see that a geodesic can enter at most two of the sectors S_k and these exceptional sectors, together with the disk $D(0, 2/\sqrt{n})$, have area at most $6\pi/n$. In the case of a point lying elsewhere in D_n , we let γ_z be the curve of shortest hyperbolic length from z to Γ : since it begins outside the disk of radius $2/\sqrt{n}$ and must leave the sector in which z lies in order to reach Γ , it has Euclidean length at least

$1/\sqrt{n}$. Hence, on using the lower bound on the hyperbolic metric mentioned earlier,

$$d(z, \Gamma; D_n) = \int_{\gamma_z} \sigma_{D_n}(w) |dw| \geq \frac{n}{4\pi} \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{4\pi}.$$

Given a positive number c , we choose n so that $\sqrt{n}/(4\pi) > c$. Then the area of the set $\{z : d(z, \Gamma; D_n) < c\}$ is at most $6\pi/n$ for any geodesic Γ in D_n .

This shows that there are domains D for which the area of the regions $\{z : d(z, \Gamma; D) < c\}$ in Conjecture 2 (or, equivalently, $A(t, \Gamma; D)$ if one prefers the formulation in Conjecture 3) are as small as we wish, uniformly for every geodesic Γ in the domain D . Moreover, if ψ is a non-negative, non-increasing function on $[0, \infty)$ with limit 0 at infinity, it follows that there is a domain D so that the integral $\int_D \psi(d(z, \Gamma; D)) dm(z)$, which features in Conjecture 1, is as small as we wish, uniformly for every geodesic Γ in the domain. In addition, these examples show that the condition in our conjectures that the domain be convex cannot be weakened to the condition that it be starlike.

We begin the account of our positive results in the convex case by computing explicitly the area function $A(t, \Gamma; U)$ for the unit disk $U = \{z : |z| < 1\}$ when Γ is a diameter of the disk. Related formulas may be found in [5], §7.20.

LEMMA 2. — Suppose that Γ is a diameter of the unit disk U . Then,

$$(4.1) \quad A(t, \Gamma; U) = \frac{2t - \sin(2t)}{\sin^2 t}, \quad \text{for } 0 < t < \pi/2.$$

Proof. — We may assume that the diameter Γ in U is the geodesic $(-1, 1)$ so that the required area $A(t, \Gamma; U)$ is the area of the image of the substrip S_t under the conformal mapping $F(z) = (e^z - 1)/(e^z + 1)$ of the strip S onto U . For $s \in (-\pi/2, \pi/2)$, the line $\{z : \Im z = s\}$ is mapped by the exponential function to the ray $\arg z = s$ in the right half-plane and the image of this ray under the Möbius map $(z - 1)/(z + 1)$ is a circular arc through -1 , $F(is)$ and 1 . Since $F(is) = i \tan(s/2)$, it follows that $A(t, \Gamma; U)$ is the area of the Lens-shaped region, symmetric in the real axis and bounded by two circular arcs, each passing through -1 and 1 and in one case also through $i \tan(t/2)$ and in the other also through $-i \tan(t/2)$.

Since the center of the circle, of which the upper circular arc forms a part, lies on the negative imaginary axis at a point equidistant from 1

and from $i \tan(t/2)$, we find that it lies at $-i \cot t$. Thus the radius of this circle is $\cot t + \tan(t/2) = \csc t$ and the arc subtends an angle $2t$ at the center of the circle. The area of the sector of the disk of radius $\csc t$ which is determined by the arc is therefore $t \csc^2 t$. We require only the area of the region lying above the real axis and so we subtract the area of the triangle with vertices $-1, 1$ and $-i \cot t$ which is $\cot t$. Twice the result is the area $A(t, \Gamma; U)$ of the Lens-shaped region and is $2t \csc^2 t - 2 \cot t$ as required. \square

4.1. A Differential equation for $A(t, (-1, 1); U)$.

It turns out that the function $A(t, (-1, 1); U)$, which we believe to be extremal in certain circumstances, satisfies a simple differential equation. Significantly, sub-solutions of this differential equation majorize the function $A(t, (-1, 1); U)$, which gives a possible method of verifying our conjectures.

LEMMA 3. — *The function*

$$U(t) = \frac{2t - \sin(2t)}{\sin^2 t}$$

satisfies the differential equation

$$(4.2) \quad \sin^2 t y''(t) - 6y(t) + 8t = 0 \quad \text{on } (0, \pi/2).$$

Furthermore, if $V(t)$ is a C^2 function on $[0, \pi/2]$ for which $V(0) = 0$, $V(\pi/2) = \pi$ and $V''(t) > 0$ on $(0, \pi/2)$ and that satisfies the differential inequality

$$(4.3) \quad \sin^2 t y''(t) - 6y(t) + 8t \leq 0 \quad \text{on } (0, \pi/2),$$

then

$$V(t) \geq U(t) \quad \text{for } 0 < t < \pi/2.$$

Proof. — It is a straightforward calculation to show that the function $U(t)$ satisfies the differential equation (4.2). In fact, after a simplification one obtains

$$U'(t) = \frac{4}{\sin^2 t} - \frac{4t \cos t}{\sin^3 t}.$$

Then

$$\begin{aligned}
 U''(t) &= \frac{12t \cos^2 t}{\sin^4 t} - \frac{12 \cos t}{\sin^3 t} + \frac{4t}{\sin^2 t} \\
 &= \frac{12t}{\sin^4 t} - \frac{12 \cos t}{\sin^3 t} - \frac{8t}{\sin^2 t} \\
 &= \frac{6}{\sin^2 t} \left[\frac{2t}{\sin^2 t} - \frac{2 \cos t}{\sin t} - \frac{4}{3}t \right] \\
 &= \frac{6}{\sin^2 t} \left[U(t) - \frac{4}{3}t \right].
 \end{aligned}$$

We now suppose that $V(t)$ has the properties stated in the second part of the lemma. We divide (4.3) by t and take the limit as t tends to 0 from the right. Since $V''(0)$ is finite and $V(0) = 0$, we find that $V'(0) \geq 4/3$. The strict convexity of $V(t)$ then implies that $V(t) > 4t/3$ on $(0, \pi/2)$.

We now consider the auxiliary functions $U_1(t) = U(t) - 4t/3$ and $V_1(t) = V(t) - 4t/3$: on $(0, \pi/2)$ both are positive and they satisfy

$$(4.4) \quad \sin^2 t U_1''(t) - 6U_1(t) = 0$$

and

$$(4.5) \quad \sin^2 t V_1''(t) - 6V_1(t) \leq 0.$$

We multiply (4.4) by $V_1(t)$ and (4.5) by $U_1(t)$ and subtract to obtain

$$V_1(t)U_1''(t) - U_1(t)V_1''(t) \geq 0 \quad \text{on } (0, \pi/2).$$

Integrating by parts and using $V_1(0) = U_1(0) = 0$, we find that

$$\begin{aligned}
 &\int_0^t (V_1(s)U_1''(s) - U_1(s)V_1''(s)) ds \\
 &= \left(V_1(s)U_1'(s) \Big|_0^t - \int_0^t U_1'(s)V_1'(s) ds \right) - \left(U_1(s)V_1'(s) \Big|_0^t - \int_0^t U_1'(s)V_1'(s) ds \right) \\
 &= V_1(t)U_1'(t) - U_1(t)V_1'(t),
 \end{aligned}$$

so that this last quantity is non-negative on $(0, \pi/2)$. Dividing across by $U_1(t)V_1(t)$ (which is positive) yields

$$\frac{U_1'(t)}{U_1(t)} \geq \frac{V_1'(t)}{V_1(t)}$$

and integrating from t to $\pi/2$ gives

$$\log U_1(t) \Big|_t^{\pi/2} \geq \log V_1(t) \Big|_t^{\pi/2},$$

for any t in $(0, \pi/2)$. Since $U(\pi/2) = V(\pi/2)$, it follows that $V_1(t) \geq U_1(t)$, as required. \square

We are now ready to formulate our final extremal problem for convex domains.

CONJECTURE 4. — *To each convex domain D of finite area A there corresponds a hyperbolic geodesic Γ such that the function $A(t, \Gamma; D)$ satisfies the differential inequality*

$$(4.6) \quad \sin^2 t y''(t) - 6y(t) + \frac{8A}{\pi} t \leq 0,$$

on $(0, \pi/2)$.

After an appropriate scaling, one sees that it follows from Lemma 3 that Conjecture 3 is correct if Conjecture 4 is so.

4.2. Verification of Conjecture 4 for Lens-shaped regions.

We saw in the course of proving Lemma 2 that each of the regions $U_t = S(t, (-1, 1); U)$, $t \in (0, \pi/2)$, is a Lens-shaped region, symmetric in its longest diameter $(-1, 1)$ and bounded by two circular arcs. Thus these regions are convex domains and one may ask if Conjecture 4 holds for these domains. We show that the conjecture does hold in this special case by showing, more generally, that from the point of view of the differential inequality (4.6), a subdomain $S(t, \Gamma; D)$ performs better than the original domain D . We first set up some notation in which the role of the geodesic is suppressed. For a simply connected domain D and a fixed hyperbolic geodesic Γ in D , we write $S(t; D)$ for $S(t, \Gamma; D)$, $t \in (0, \pi/2)$ and write $A(t; D)$ for its area, that is for $A(t, \Gamma; D)$. Now $S(t; D)$ is a simply connected domain in its own right and Γ , which is a hyperbolic geodesic in D , is also a geodesic in the hyperbolic metric for $S(t; D)$. Thus we may consider the regions $S(r; S(t; D))$, and begin by showing that each has the form $S(x; D)$ for an appropriate x depending on r and t .

LEMMA 4. — *Suppose that D is a simply connected domain, that Γ is a hyperbolic geodesic in D and that $0 < t < \pi/2$. Then $S(t; D)$ is a simply connected domain and Γ is a geodesic in the hyperbolic metric for $S(t; D)$. Moreover, for $0 < r < \pi/2$,*

$$S(r, S(t; D)) = S(2rt/\pi; D).$$

Proof. — We choose a conformal mapping $f(z)$ of the strip S onto the domain D such that the real axis in S is mapped to the geodesic Γ in D . Since $S(t; D)$ is the image under f of the substrip $S_t = \{z : |\Im z| < t\}$, we see that $f_t(z) = f(2tz/\pi)$ is a conformal map of the strip S onto $S(t; D)$. Since the real axis is a geodesic in S and f_t maps the real axis onto Γ (as did f), it follows that Γ is a geodesic in the hyperbolic metric for $S(t; D)$. For each r with $0 < r < \pi/2$, $S(r, \Gamma; S(t; D))$ is, by definition, the image of the substrip S_r under the mapping f_t . But this is also the image of the substrip S_x , with $x = 2rt/\pi$, under the mapping f , that is $S(2rt/\pi; D)$. \square

For a simply connected domain D of finite area $A(D)$ and a fixed geodesic Γ in D we write

$$L(t; D) = \sin^2 t \frac{d^2}{dt^2} A(t; D) - 6A(t; D) + \frac{8A(D)}{\pi} t,$$

for $0 < t < \pi/2$. Thus Conjecture 4 states that if D is convex then Γ may be chosen so that $L(D, t) \leq 0$ on $(0, \pi/2)$. We now prove

THEOREM 5. — *Suppose that D is a convex domain of finite area and that Γ is a geodesic in D . Then, for each r and t in $(0, \pi/2)$,*

$$(4.7) \quad L(r; S(t; D)) < L(2rt/\pi; D).$$

Proof. — Lemma 4 tells us that

$$A(r; S(t; D)) = A(2rt/\pi; D),$$

so that on differentiating with respect to r we obtain

$$A''(r; S(t; D)) = \left(\frac{2t}{\pi}\right)^2 A''(2rt/\pi; D).$$

Hence,

$$L(2rt/\pi; D) = \sin^2(2rt/\pi) A''(2rt/\pi; D) - 6A(2rt/\pi; D) + \frac{8A(D)}{\pi} \frac{2rt}{\pi}$$

and

$$L(r; S(t; D)) = \left(\frac{2t}{\pi}\right)^2 \sin^2 r A''(2rt/\pi; D) - 6A(2rt/\pi; D) + \frac{8A(t; D)}{\pi} r.$$

On subtracting, the terms involving $A(2rt/\pi; D)$ cancel and we deduce that

$$\begin{aligned} & L(2rt/\pi; D) - L(r; S(t; D)) \\ &= \left[\sin^2 \left(\frac{2rt}{\pi} \right) - \left(\frac{2t}{\pi} \right)^2 \sin^2 r \right] A'' \left(\frac{2rt}{\pi}; D \right) + \frac{8r}{\pi} \left[\frac{2tA(D)}{\pi} - A(t; D) \right]. \end{aligned}$$

The estimate (3.1) states that $A(t; D) < 2tA(D)/\pi$ and so

$$L(2rt/\pi; D) - L(r; S(t; D)) > \left[\sin^2 \left(\frac{2rt}{\pi} \right) - \left(\frac{2t}{\pi} \right)^2 \sin^2 r \right] A''(2rt/\pi; D).$$

Since $A(t; D)$ is a strictly convex function of t by Theorem 1 and since both r and t lie between 0 and $\pi/2$, it suffices to show that the function of two variables

$$f(r, t) = \sin \left(\frac{2rt}{\pi} \right) - \frac{2t}{\pi} \sin r$$

is positive on the rectangular region $R = \{(r, t) : 0 < r, t < \pi/2\}$. This is easy to show. First the function $f(r, t)$ can have no critical points inside the rectangle R since the partial derivative $f_r(r, t)$ is never 0 in R . Now, $f(0, t) = 0$ for $t \in [0, \pi/2]$. For $r \in [0, \pi/2]$, we find that $f(r, 0) = 0$ and $f(r, \pi/2) = \sin r - \sin r = 0$. Finally, for $t \in [0, \pi/2]$, $f(\pi/2, t) = \sin t - 2t/\pi$ which is non-negative. Thus $f(r, t)$ is continuous on the closed rectangle, is non-negative on the boundary and has no critical points inside the rectangle and it follows that $f(r, t)$ is positive on the rectangle. \square

COROLLARY 1. — *Conjecture 4 holds for symmetric Lens-shaped regions.*

Proof. — Suppose that D is a symmetric Lens-shaped region which may therefore be thought of as the intersection of two disks of equal radius. By scaling D if necessary, we may assume that D coincides with $S(t; U)$ for a suitable t in $(0, \pi/2)$, the implied geodesic Γ being the diameter $(-1, 1)$ of the unit disk. By Theorem 5, for each r in $(0, \pi/2)$,

$$L(r; D) = L(r; S(t; U)) < L(2rt/\pi; U).$$

But by Lemma 3, $L(\cdot; U)$ is identically zero on $(0, \pi/2)$ and we conclude that the area function $A(\cdot; D)$ satisfies (4.6). \square

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