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p-ADIC MEASURES ATTACHED TO SIEGEL MODULAR FORMS

by S. BÖCHERER and C.-G. SCHMIDT

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0. Introduction.

Let f be a holomorphic Siegel modular form of weight l for the congruence subgroup $\Gamma_0(N)$ in Sp_n which is a Hecke eigenform. The aim of this article is to study certain special values of the standard L-function $D^{(M)}(f,s)$ attached to f and of twists $D^{(M)}(f,s,\chi)$ of the L-function by

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Dirichlet characters χ, where M denotes a common multiple of N and the conductor of χ, and the Euler factors at primes dividing M are removed. The analytic properties of these L-functions have been investigated by several authors (cf. [1], [6], [14], [22]) and are more or less well-known. Also special values $D^{(M)}(f,t,\chi)$ of the L-function have been proven to become algebraic numbers after multiplication by explicitely given complex numbers under certain restricting conditions (cf. [15], [36]); in particular the behaviour of these algebraic numbers under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is known. We do remove these restrictions in the Appendix. Our main object is for a fixed rational prime number p to interpolate p-adically the essentially (i.e., up to an explicit factor) algebraic special L-values $D^{(Np)}(f,t,\chi)$ as t and χ vary. In fact we prove the existence of two p-adic (by definition bounded) measures μ and ν attached to f provided that f satisfies a certain p-ordinarity condition (Theorem 9.3). Such a condition guarantees the occurence of a p-adic unit root in a certain p-Euler polynomial and is quite familiar from other situations. The special L-values under consideration are in a sense all *critical* values, which basically only can exist for $l > n$. By a different method Panchishkin [21] got a related result assuming that n is even and $l > 2(n+1)$. For $n = 1$ our result was known by [23], where moreover a simple relationship between the two measures μ and ν was proven. However some twists had to be excluded in that paper. Since our new result holds for all twists, we get an improvement even for $n = 1$ (for the incomplete L-function with finitely many bad Euler factors removed). In general we cannot relate μ and ν since the functional equation of the complex L-function has not yet been worked out as explicitely as necessary for our purposes.

The main body of our paper (i.e., Sections 1-7) provides background material on Siegel modular forms. Some of the results are new (and perhaps of independent interest), others are more or less well known (but cannot be found in the literature in the generality needed here). At some points our results are formulated in a more general setting than actually necessary for the purposes of this paper; we think this is convenient for future generalizations and refinements of our main results. As an example we mention that our setting is general enough to make the transition from our p-adic interpolation to S-adic interpolation in the sense of [20] a routine exercise.

As a starting point we need (as always) a good integral representation for the twisted L-function in question. Here *good* means that the automorphic form appearing in the integrand should be independent of the Dirichlet character. There are basically two different integral representations of standard L-functions for automorphic forms on the symplectic group $\mathrm{Sp}(n)$: The method of Andrianov/Kalinin [1] (and its representation-theoretic version by Piatetski-Shapiro/Rallis [22]) and the method of "doubling the variables" (as presented in a classical setting in [6] and in representation-theoretic terms in [14]). The former method immediately generalizes to twists by Dirichlet characters, but it has the disadvantage that it involves Eisenstein series of integral or half-integral weight depending on the parity of n; therefore the cases n even or odd must often be treated separately (e.g. the Fourier expansions of Eisenstein series are quite different for integral and half-integral weights; this is the main reason, why Panchishkin [20], [21] only treats the case of even n). One of the main points in our paper is to show that the method of doubling the variables admits a modification which produces a good integral representation for twists of the standard L-function. This is essentially done in Section 2 (in the framework of holomorphic Siegel modular forms) and involves in a crucial way the set of variables which are put zero in the unmodified version. This method is new and of independent interest. In Section 3 we describe a Hecke algebra and Euler factors for "bad primes". We avoid holomorphic projection in the sense of Sturm [36] by using holomorphic differential operators introduced in [4], see Section 1. The Fourier expansions of certain Siegel type Eisenstein series is investigated in Sections 5 and 7; it is remarkable that we only need information about the Fourier coefficients of maximal rank. This allows us to stay essentially selfcontained and to avoid the use of the more sophisticated results of Shimura [28] and Feit [11]. Sections 4 and 6 contain straightforward computations.

Preliminaries.

For matrices A and B we denote by A^t the transposed matrix and A^tBA by $B[A]$ (if it makes sense); for an invertible matrix A we write A^{-t} instead of $(A^{-1})^t$. It will be clear from the context, whether Γ means an arithmetic group or the Gamma function. We also use $\Gamma_n(s) := \pi^{\frac{n(n-1)}{4}} \prod_{i=1}^n \Gamma(s - \frac{i-1}{2})$.

For generalities on Siegel modular forms we refer to [12] or [2]. We denote by $(M, Z) \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$ the action of the group $G^+\mathrm{Sp}(n, \mathbb{R})$ of proper symplectic similitudes on Siegel's upper half space $\mathbb{H}_n$ where as usual $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For any function f on $\mathbb{H}_n$, any

$M \in G^+\mathrm{Sp}(n,\mathbb{R})$ we write

$$f|_{\alpha,\beta}M(Z) = \det(M)^{\frac{\alpha+\beta}{2}} f(M\langle Z\rangle)\det(CZ+D)^{-\alpha}\det(C\bar{Z}+D)^{-\beta}$$

and $f|_\alpha M$ for $f|_{\alpha,0}M$; if f depends on several variables, we indicate by $f|_{\alpha,\beta}^{Z}M$ that Z is the variable relevant at this moment. Sometimes we write $j(M,Z)$ instead of $\det(CZ+D)$. The Petersson scalar product of two cusp forms f and g of weight l will be denoted by $\langle f,g\rangle$. We do not use the normalized version of this scalar product, therefore we use (sometimes) the notation $\langle f,g\rangle_\Gamma$ to indicate that we mean the integral of $f\bar{g}\cdot\det(\mathrm{Im}(Z))^l$ over a fundamental domain for the congruence group Γ. We shall mainly be concerned with the subgroups $\Gamma_0(N)$ and $\Gamma^0(N)$ of the modular group $\mathrm{Sp}(n,\mathbb{Z})$, defined by the property that the block C (respectively B) in the lower left corner (respectively in the upper right corner) is congruent to 0 modulo N. For a Dirichlet character φ mod N we denote by $M_n^l(\Gamma_0(N),\varphi)$ the space of holomorphic Siegel modular forms of degree n, weight l and nebentype φ and by $S_n^l(\Gamma_0(N),\varphi)$ the corresponding subspace of cusp forms. For two natural numbers L and R we define an operator $U_L(R)$ acting on L-periodic functions F with Fourier expansion $F(Z)=\sum_{T\in\Lambda_n} a(T,Y)e^{2\pi i\frac{1}{L}\mathrm{tr}(TX)}$ by $(F|U_L(R))(Z) = \sum_{T\in\Lambda_n} a(RT,\frac{1}{R}Y)e^{2\pi i\frac{1}{L}\mathrm{tr}(TX)}$. Here Λ_n denotes the set of all half-integral symmetric matrices of size n and Λ_n^* , Λ_n^+ are the subsets of matrices of maximal rank and of positive definite matrices respectively. We write $U(R)$ for $U_1(R)$. This operator $U(R)$ describes a "Hecke operator" acting on $M_n^l(\Gamma_0(N),\varphi)$ if $R|N^\infty$.

1. Differential operators.

We review (mainly without proofs) the basic properties of certain holomorphic differential operators introduced in [4]. Those operators act on functions defined on $\mathbb{H}_{2n}$ and have some automorphy properties for the two copies of $\mathrm{Sp}(n,\mathbb{R})$ embedded in $\mathrm{Sp}(2n,\mathbb{R})$ in the usual way:

$$(1.1)\quad \mathrm{Sp}(n,\mathbb{R})^\uparrow := \left\{ \begin{pmatrix} a & 0_n & b & 0_n \\ 0_n & 1_n & 0_n & 0_n \\ & & & \\ c & 0_n & d & 0_n \\ 0_n & 0_n & 0_n & 1_n \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \right\}$$

$$
(1.2) \quad \mathrm{Sp}(n,\mathbb{R})^{\downarrow} := \left\{ \begin{pmatrix} 1_n & 0_n & 0 & 0_n \\ 0_n & a & 0_n & b \\ & & & \\ 0_n & 0_n & 1_n & 0_n \\ 0_n & c & 0_n & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \right\}.
$$

The differential operators are built up from the operators (with $1 \leqslant i, j \leqslant 2n$)

$$
(1.3) \qquad \partial_{ij} = \begin{cases} \dfrac{\partial}{\partial z_{ii}} & i = j \\ \dfrac{1}{2}\dfrac{\partial}{\partial z_{ij}} & i \neq j \end{cases}
$$

which we put together in the symmetric $2n \times 2n$ matrix

$$
(1.4) \qquad \partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}
$$

where the ∂_i are block matrices of size n, which correspond to the decomposition

$$
\mathfrak{z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}
$$

of $\mathbb{H}_{2n}$ into block matrices (with $z_3 = z_2^t$).

In [4, p. 86] certain polynominals $\Delta(r,q)$, $r + q = n$ in the ∂_{ij} were defined, their coefficients being polynomials in the entries of z_2; we just quote from [loc.cit.] without explaining the notation

(15)

$$
\Delta(r,q) = \sum_{a+b=q} (-1)^b \binom{n}{b} z_2^{[a]}\,\partial_4^{[a]} \sqcap \left(\left(1_n^{[r]} \sqcap z_2^{[b]}\,\partial_3^{[b]} \right) \left(Ad^{[r+b]}\,\partial_1 \right) \partial_2^{[r+b]} \right)
$$

in particular

$$
(1.6) \qquad \Delta(n,0) = \det(\partial_2)
$$

$$
(1.7) \qquad \Delta(0,n) = \det(z_2)\det(\partial).
$$

Using the (standard) notation

$$
(1.8) \qquad C_q(s) = s\left(s + \frac{1}{2}\right)\dots\left(s + \frac{q-1}{2}\right) = \frac{\Gamma_q\left(s + \frac{q+1}{2}\right)}{\Gamma_q\left(s + \frac{q-1}{2}\right)}
$$

we define (with a normalization slightly different from [loc.cit.]) for any $\alpha \in \mathbb{C}$

$$(1.9) \qquad \mathfrak{D}_{n,\alpha} := \sum_{r+q=n} (-1)^r \binom{n}{r} C_r \left(\alpha - n + \frac{1}{2}\right) \Delta(r,q).$$

This operator satisfies the important relations:

$$(1.10) \qquad \mathfrak{D}_{n,\alpha}\left(F|_{\alpha,\beta} M^{\uparrow}\right) = \left(\mathfrak{D}_{n,\alpha} F\right)|_{\alpha+1,\beta} M^{\uparrow}$$

$$(1.11) \qquad \mathfrak{D}_{n,\alpha}\left(F|_{\alpha,\beta} M^{\downarrow}\right) = \left(\mathfrak{D}_{n,\alpha} F\right)|_{\alpha+1,\beta} M^{\downarrow}$$

$$(1.12) \qquad \mathfrak{D}_{n,\alpha}(F|V) = \left(\mathfrak{D}_{n,\alpha} F\right)|V$$

for all $\beta \in \mathbb{C}$, all $F \in C^{\infty}(\mathbb{H}_{2n})$ and all $M \in \mathrm{Sp}(n, \mathbb{R})$; in (1.12) V denotes the operator

$$(1.13) \qquad (F|V)(\mathfrak{Z}) = F\left(\begin{pmatrix} z_4 & z_3 \\ z_2 & z_1 \end{pmatrix}\right).$$

For $\nu \in \mathbb{N}$ we put

$$(1.14) \qquad \mathfrak{D}^{\nu}_{n,\alpha} := \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,\alpha}$$

and

$$(1.15) \qquad \overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha} := \left(\mathfrak{D}^{\nu}_{n,\alpha}\right)\Big|_{z_2=0}.$$

In particular, $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha}$ maps ($C^{\infty}-$) automorphic forms of type (α, β) on $\mathbb{H}_{2n}$ to functions on $\mathbb{H}_n \times \mathbb{H}_n$, which are automorphic of type $(\alpha+\nu, \beta)$ with respect to z_1 and z_4. If F is a holomorphic modular form on $\mathbb{H}_{2n}$, then $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha} F$ becomes a cusp form with respect to z_1 and z_2 (if $\nu > 0$).

For $\mathfrak{T} \in \mathbb{C}^{2n,2n}_{\mathrm{sym}}$ we define a polynominal $\mathfrak{P}^{\nu}_{n,\alpha}(\mathfrak{T})$ in the entries $t_{ij} (1 \leqslant i \leqslant j \leqslant 2n)$ of $\mathfrak{T}$ by

(1.16)

$$\overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha}\left(e^{\mathrm{tr}(\mathfrak{T}Z)}\right) = \mathfrak{P}^{\nu}_{n,\alpha}(\mathfrak{T}) e^{\mathrm{tr}(T_1 z_1 + T_4 z_4)}, \quad \mathfrak{T} = \begin{pmatrix} T_1 & T_2 \\ T_2^t & T_4 \end{pmatrix} \text{ with } T_i \in \mathbb{C}^{(n,n)}.$$

The $\mathfrak{P}^{\nu}_{n,\alpha}$ are homogenous polynomials of degree $n\nu$.

For $X, Y \in \mathbb{C}^{m,n}$, m even, the polynomial

$$(1.17) \qquad Q(X,Y) = \mathfrak{P}^{\nu}_{n,\frac{m}{2}} \begin{pmatrix} X^t X & X^t Y \\ Y^t X & Y^t Y \end{pmatrix}$$

is a harmonic form of degree ν in both matrix variables X and Y and it is symmetric in X and Y [loc.cit. Satz 15]. Since $O(m,\mathbb{C})$ acts irreducibly on the space of all such harmonic forms, this implies that (at least for $m = 2k \geqslant 2n$) the operator $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha}$ is (up to a constant) uniquely determined by the transformation properties (1.10), (1.11), (1.12) at $z_2 = 0$.

$\overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha}$ is a polynomial in the ∂_{ij}, homogenous of degree $n\nu$ with at most one term free of the entries of ∂_1 and ∂_4, namely the term

$$c^{\nu}_{n,\alpha} \det(\partial_2)^{\nu} \tag{1.18}$$

with a certain constant $c^{\nu}_{n,\alpha}$; this can be seen e.g. from (1.16) and (1.10) by looking at $\mathfrak{T}$ of type

$$\mathfrak{T} = \begin{pmatrix} 0_n & T \\ T' & 0_n \end{pmatrix}, \; T \in \mathbb{C}^{(n,n)}. \tag{1.19}$$

To determine the constant $c^{\nu}_{n,\alpha}$ explicitely we first observe that (for arbitrary $\alpha, s \in \mathbb{C}$)

$$\mathfrak{D}_{n,\alpha}(\det(z_2)^s) = (-1)^n C_n\left(\frac{s}{2}\right) C_n\left(\alpha - n + \frac{s}{2}\right) \det(z_2)^{s-1} \tag{1.20}$$

which implies

$$\begin{aligned} \overset{0}{\mathfrak{D}}{}^{\nu}_{n,\alpha}(\det(z_2)^{\nu}) &= \left(\prod_{\mu=1}^{\nu} C_n\left(\frac{\mu}{2}\right)\right) c^{\nu}_{n,\alpha} \\ &= (-1)^{n\nu} \prod_{\mu=1}^{\nu} \left(C_n\left(\frac{\mu}{2}\right) C_n\left(\alpha - n + \nu - \frac{\mu}{2}\right)\right). \end{aligned} \tag{1.21}$$

In particular we shall apply these differential operators to functions of type

$$f_s(\mathfrak{Z}) := \det(z_1 + z_2 + z_2' + z_4)^{-s}, \; s \in \mathbb{C}. \tag{1.22}$$

The following formulas will be used (see [loc. cit. p. 97]):

$$\Delta(r,q)\, f_s = 0 \text{ for } q > 0 \tag{1.23}$$

$$\Delta(n,0)\, f_s = C_n(-s) f_{s+1} \tag{1.24}$$

$$\mathfrak{D}^{\nu}_{n,s} f_s = \frac{\Gamma_n(s+\nu)}{\Gamma_n(s)} \cdot \frac{\Gamma_n\left(s+\nu-\frac{n}{2}\right)}{\Gamma_n\left(s-\frac{n}{2}\right)} \cdot f_{s+\nu}\,; \tag{1.25}$$

to get the last formula from (1.23) and (1.24) one has to use the elementary formula

$$C_n(-s) = (-1)^n C_n\left(s - \frac{n}{2} + \frac{1}{2}\right). \tag{1.26}$$

In the sequel we shall also use a "disturbed version" of the operator $\overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha}$, namely (for $s \in \mathbb{C}$ arbitrary)

$$(1.27) \qquad \underset{\sim}{\overset{\circ}{\mathfrak{D}}}{}^{\nu}_{n,\alpha,s} := \det(y_1)^s \det(y_4)^s \; \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha+s} \left(\det(\mathfrak{Y})^{-s} \times \underline{\quad}\right).$$

It is clear that this operator has exactly the same transformation properties as $\overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha}$. Moreover these two operators are essentially the same as far as (for $k = \alpha \in \mathbb{N}$) their behaviour with respect to the Petersson scalar product against holomorphic modular forms is concerned.

To see this, we first remark that the operator $\underset{\sim}{\overset{\circ}{\mathfrak{D}}}{}^{\nu}_{n,k,s}$ is a homogenous polynomial of degree $n\nu$ in three sets of variables, namely the entries of y_1^{-1}, the entries of y_4^{-1} and the $\partial_{ij|z_2=0}$ $(1 \leqslant i \leqslant j \leqslant 2n)$.

A structure theorem of Shimura about nearly holomorphic functions (or more precisely its proof [32, Prop. 3.4], [33, Prop. 3.3] implies the validity of an operator identity of type

$$(1.28) \qquad p_s(k) \det(y_1)^s \det(y_4)^s \; \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha+s} \left(\det(\mathfrak{Y})^{-s} \times \underline{\quad}\right) = \sum_{i,j=0}^{n\nu} \left(\mathbb{D}_1^{(i)} \otimes \mathbb{D}_4^{(j)}\right) D^{(i,j)}.$$

Here the $D^{(i,j)}$ are polynominals in the $\partial_{rt|z_2=0}(1 \leqslant r \leqslant t \leqslant 2n)$ (with coefficients depending on k and s) mapping scalar valued automorphic forms on $\mathbb{H}_{2n}$ to certain vector-valued forms on $\mathbb{H}_n \times \mathbb{H}_n$ and the $\mathbb{D}^{(i)}$ are (non-holomorphic) differential operators mapping vector valued automorphic forms to scalar-valued ones (Shimura's notation is $\Theta_{\mathbb{C}} D_*^{(i)}$). The $\mathbb{D}_1^{(i)}$ ($\mathbb{D}_4^{(j)}$ respectively) are homogenous polynomials in the entries of y_1^{-1} and ∂_1 (y_4^{-1} and ∂_4 resp.) of degree i (j resp.). The polynomial $p_s(k)$ in (1.28) assures that the identity is valid for all k (not just for k large enough as in Shimura's papers). We use the normalization $\mathbb{D}_1^{(0)} = \mathbb{D}_4^{(0)} = 1$, therefore the summand in (1.28) for $i = j = 0$ is just $D^{(0,0)}$; this operator has exactly the same properties as $\overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,k}$, therefore (by the uniqueness property mentioned earlier) we obtain (at least for $k \geqslant n$)

$$(1.29) \qquad D^{(0,0)} = d_s(k) \, \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,k}$$

with a certain complex number $d_s(k)$. We should mention that Garrett and Harris consider similar kinds of decompositions of differential operators in [13, §2].

On the other hand Shimura (see [29], [30], [34], [35]), by studying the adjoint operators of $\Theta_{\mathbb{C}} D_*^{(i)}$, showed that holomorphic modular forms are always orthogonal (w.r.t. Petersson scalar product) to elements of the image of $\Theta_{\mathbb{C}} D_*^{(i)}$, $i > 0$.

If Γ is a congruence subgroup of $\mathrm{Sp}(n, \mathbb{Z})$, g and h holomorphic cusp forms in $S_n^{k+\nu}(\Gamma)$ and $F : \mathbb{H}_{2n} \longrightarrow \mathbb{C}$ a C^∞-automorphic form of weight k for a group containing $\Gamma^\downarrow \times \Gamma^\uparrow$ and satisfying a suitable growth condition, we get (at least for $k \geqslant n$)

$$p_s(k) \left\langle \left\langle \underset{\sim}{\overset{\circ}{\mathfrak{D}}}{}^{\nu}_{n,k,s} F, g \right\rangle_\Gamma, h \right\rangle_\Gamma = d_s(k) \left\langle \left\langle \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha} F, g \right\rangle_\Gamma, h \right\rangle_\Gamma . \tag{1.30}$$

To determine $p_s(k)$ and $d_s(k)$ (or merely their quotient) it is sufficient to compare the coefficients of $\det(\partial_2)^\nu$ on both sides of (1.28); this gives

$$p_s(k) c^\nu_{n,k+s} = d_s(k) c^\nu_{n,k} \tag{1.31}$$

which implies

$$\frac{d_s(k)}{p_s(k)} = \frac{c^\nu_{n,k+s}}{c^\nu_{n,k}} = \prod_{\mu=1}^{\nu} \frac{C_n\left(k+s-n+\nu-\frac{\mu}{2}\right)}{C_n\left(k-n+\nu-\frac{\mu}{2}\right)} . \tag{1.32}$$

For later use we state here some arithmetic properties of the $\mathfrak{P}^\nu_{n,\alpha}$, which are immediate consequences of the considerations above when combined with the simple observation that the operator

$$2^{n\nu} \cdot \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,\alpha} \tag{1.33}$$

is a polynomial with integer coeffizients in α, and the $\partial_{i,j}$ $(1 \leqslant i < j \leqslant 2n)$ evaluated at $z_2 = 0$.

Remark 1.1.— For $\alpha \in \mathbb{Z}$, the $4^{n\nu} \cdot \mathfrak{P}^\nu_{n,\alpha}(\mathfrak{T})$ are polynomials in the entries $t_{ij} (1 \leqslant i < j \leqslant 2n)$ of $\mathfrak{T}$ with coefficients in $\mathbb{Z}$. They satisfy the congruence

$$4^{n\nu} \mathfrak{P}^\nu_{n,\alpha}(\mathfrak{T}) \equiv \left(2^{n\nu} C^\nu_{n,\alpha}\right) \det(2T)^\nu \bmod L \tag{1.34}$$

for any integer L and any half-integral

$$\mathfrak{T} = \begin{pmatrix} T_1 & T \\ T^t & T_4 \end{pmatrix} \text{ with } \frac{1}{L} T_1 \,,\ \frac{1}{L} T_4 \text{ both half-integral}.$$

We also mention that the integer $\left(2^{n\nu} C^\nu_{n,\alpha}\right)$ in (1.35) is certainly nonzero for $\alpha > n$.

2. Twisted Eisenstein series.

For a Dirichlet character ψ mod M, $M > 1$, a weight $k \in \mathbb{N}$ with $\psi(-1) = (-1)^k$ and a complex parameter s with $\mathrm{Re}(s) \gg 0$ we define an Eisenstein series

$$(2.1)\qquad \hat{\mathbb{F}}_n^k(Z, M, \psi, s) \text{ and } \mathbb{F}_n^k(Z, M, \psi, s) := \det(Y)^s \hat{\mathbb{F}}_n^k(Z, M, \psi, s)$$

of degree n (with $Z = X + iY \in \mathbb{H}_n$) by

$$(2.2)\quad \hat{\mathbb{F}}_n^k(Z, M, \psi, s) = \sum_{\{C,D\}} \psi(\det C) \det(CZ+D)^{-k} \,|\det(CZ+D)|^{-2s};$$

here $\{C, D\}$ runs over all "non-associated coprime symmetric pairs" with $\det C$ coprime to M.

A more conceptual definition is as follows:

(2.3)

$$\mathbb{F}_n^k(Z, M, \psi, s) = \sum_{R \in T^n(M)_\infty \backslash T^n(M)} \tilde{\psi}(R)\, j(R, Z)^{-k} \det(\mathrm{Im}(R\langle Z\rangle)^s)$$

with

$$T^n(M) = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \Gamma_0(M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid A \equiv 0 \bmod M \right\},$$

$$T^n(M)_\infty = \left(\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \Gamma_0(M) \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}^{-1} \right)_\infty$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \,\middle|\, C = 0, B \equiv 0 \bmod M \right\}$$

and

$$(2.4)\qquad \tilde{\psi}(R) = \psi(\det C) \text{ for } R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in T^n(M).$$

It is well-known that this series converges for $k + 2\mathrm{Re}(s) > n + 1$ and (as a function of s) has a meromorphic continuation to the whole complex plane.

A key ingredient in our subsequent calculations is the following variant of Proposition 5 in [3]:

Proposition 2.1. — *A complete set of representatives for $T^{2n}(M)_\infty \backslash T^{2n}(M)$ is given by*

$$(2.5)\qquad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\uparrow} \begin{pmatrix} W^t & 0_{2n} \\ 0_{2n} & W^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\downarrow} \,\middle|\, \text{(i), (ii), (iii)} \right\}$$

with

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T^n(M)_\infty \setminus T^n(M)$, (ii) $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in T^n(M)_\infty \setminus T^n(M)$,

(iii) $W \in \left\{ \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix} \in GL(2n,\mathbb{Z}) \,\middle|\, \begin{matrix} \omega_2 \equiv 0 \bmod M \\ (\det\omega_1, M) = 1 \end{matrix} \right\} \Big/ \left\{ \begin{pmatrix} GL(n,\mathbb{Z}) & M\cdot\mathbb{Z}^{(n,n)} \\ 0_n & GL(n,\mathbb{Z}) \end{pmatrix} \right\}.$

Proof. — [loc. cit].

Remark 2.1.

a) A useful reformulation of (iii) is:

The "first column" $\begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix}$ of W runs through

$$\left\{ \begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix} \in \mathbb{Z}^{(2n,n)} \,\middle|\, \begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix} \text{ primitive}, (\det\omega_1, M) = 1,\ \omega_1 \in \mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z}) \right\}.$$

b) Suppose that $\mathfrak{M} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} \in T^{2n}(M)$ is decomposed as in the proposition above. Then

$$\mathfrak{C} = \begin{pmatrix} c\,\omega_1^t & * \\ 0_n & u_4\gamma \end{pmatrix} \quad \text{with } W^{-1} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$$

and therefore

$$\psi(\det\mathfrak{C}) = \psi(\det c)\,\psi(\det\gamma)\,\psi^2(\det\omega_1)\,\psi(\det W)^{-1}. \tag{2.6}$$

Making use of (2.6) and the (obvious) relations

$$\mathfrak{Z}[W] = \begin{pmatrix} \mathfrak{Z}\begin{bmatrix} \omega_1 \\ \omega_3 \end{bmatrix} & * \\ * & * \end{pmatrix}, \quad \mathfrak{Z} \in \mathbb{H}_{2n}, \tag{2.7}$$

$$j\left(\begin{pmatrix} W^t & 0_n \\ 0_n & W^{-1} \end{pmatrix}, \mathfrak{Z} \right) = \det(W)^{-1}, \tag{2.8}$$

$$\psi(\det W) = \det(W)^k \text{ for } W \in GL(2n,\mathbb{Z}), \tag{2.9}$$

we obtain from Proposition 2.1 (in the same way as in [3, Section 3]) an expression for the Eisenstein series of degree $2n$ (essentially the Fourier-Jacobi-expansion for the decomposition of

$$\mathfrak{Z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{H}_{2n} \tag{2.10}$$

into n-rowed block matrices):

$$(2.11)\qquad \hat{\mathbb{F}}^k_{2n}(3, M, \psi, s)$$
$$= \sum_{R \in T^n(M)_\infty \backslash T^n(M)} \sum_{\left(\begin{smallmatrix}\omega_1\\ \omega_3\end{smallmatrix}\right) \in \mathbb{Z}^{2n,n}/GL(n,\mathbb{Z})} \psi^2(\det \omega_1)\, \tilde{\psi}(R)$$
$$j(R, z_4)^{-k}\, |j(R, z_4)|^{-2s} \times \hat{\mathbb{F}}^k_n \left(R^{\downarrow} \langle 3 \rangle \begin{bmatrix} \omega_1 \\ \omega_3 \end{bmatrix}, M, \psi, s \right).$$

As mentioned above, $\left(\begin{smallmatrix}\omega_1\\ \omega_3\end{smallmatrix}\right)$ must satisfy the additional conditions that $\left(\begin{smallmatrix}\omega_1\\ \omega_3\end{smallmatrix}\right)$ is primitive and that $\det \omega_1$ is coprime to M.

We want to twist these Eisenstein series of degree $2n$ in a certain way by a Dirichlet character. To do this, we first observe that for

$$(2.12)\qquad R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}),\ X \in \mathbb{R}^{(n,n)}$$

$$(2.13)\qquad W = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix} \in GL(2n, \mathbb{R})$$

the relation

$$\begin{pmatrix} W^t & 0_{2n} \\ 0_{2n} & W^{-1} \end{pmatrix} R^{\downarrow} \begin{pmatrix} 1_{2n} & \begin{matrix} 0_n & X \\ X^t & 0_n \end{matrix} \\ 0_{2n} & 1_{2n} \end{pmatrix} = \begin{pmatrix} 1_{2n} & S \\ 0_{2n} & 1_{2n} \end{pmatrix} \begin{pmatrix} \tilde{W}^t & 0_{2n} \\ 0_{2n} & \tilde{W}^{-1} \end{pmatrix} R^{\downarrow}$$

holds with

$$(2.15)\qquad \tilde{W} = \begin{pmatrix} \omega_1 & \omega_2 \\ -\gamma X^t \omega_1 + \omega_3 & -\gamma X^t \omega_2 + \omega_4 \end{pmatrix}$$

$$(2.16)\qquad S = W^t \begin{pmatrix} -X\alpha^t \gamma X^t & X\alpha^t \\ \alpha X^t & 0_n \end{pmatrix} W.$$

In particular, the symmetric matrix S has integral entries, if $W \in GL(2n, \mathbb{Z})$, $X = \frac{\tilde{X}}{N}$ with $\tilde{X} \in \mathbb{Z}^{(n,n)}$, $N \in \mathbb{N}$ and $R \in \mathrm{Sp}(n, \mathbb{Z})$ with $\alpha \equiv 0 \bmod N^2$.

This implies that for any $\tilde{X} \in \mathbb{Z}^{(n,n)}$ and any $N \in \mathbb{N}$ with $N^2 | M$ we have (with the same additional conditions on ω_1, ω_3 as before)

$$(2.17)\qquad \hat{\mathbb{F}}^k_{2n}(3, M, \psi, s)\Big|_k \begin{pmatrix} 1_{2n} & \begin{matrix} 0_n & \frac{\tilde{X}}{N} \\ \frac{\tilde{X}^t}{N} & 0_n \end{matrix} \\ 0_{2n} & 1_{2n} \end{pmatrix}$$

$$\sum_{R=\left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right)\in T^n(M)_\infty\backslash T^n(M)} \quad \sum_{\omega_1\in \mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})} \quad \sum_{\omega_3\in \mathbb{Z}^{(n,n)}} \psi^2(\det\omega_1)\,\tilde{\psi}(R)$$

$$j(R,z_4)^{-k}\,|j(R,z_4)|^{-2s}\,\hat{\mathbb{F}}^k_n\left(R^{\downarrow}\langle 3\rangle\begin{bmatrix}\omega_1 \\ \omega_3-\gamma\frac{\tilde{X}^t}{N}\omega_1\end{bmatrix},M,\psi,s\right).$$

Here we tacitly used the fact that the matrix S from (2.16) does not contribute anything because $\hat{\mathbb{F}}^k_n(z_1,M,\psi,s)$ is a periodic function of $z_1\in\mathbb{H}_n$.

Now let χ be a Dirichlet character mod N, $N^2|M$ and consider

$$\sum_{\tilde{X}\in\mathbb{Z}^{(n,n)}\bmod N} \chi(\det\tilde{X})\,\hat{\mathbb{F}}^k_{2n}(-,M,\psi,s)\Big|_k\begin{pmatrix}1_{2n} & S\left(\frac{\tilde{X}}{N}\right)\\ 0_{2n} & 1_{2n}\end{pmatrix} \tag{2.18}$$

where $S(X)$ denotes the $2n$-rowed symmetric matrix

$$\begin{pmatrix}0_n & X\\ X^t & 0_n\end{pmatrix}. \tag{2.19}$$

We put $\tilde{\omega}_3 := N\omega_3-\gamma\tilde{X}^t\omega_1$.

If ω_1,γ are fixed and $\omega_3,\tilde{X}$ are varing as in (2.17) and (2.18) then $\tilde{\omega}_3$ runs through all elements of $\mathbb{Z}^{(n,n)}$ with the properties

$$\begin{pmatrix}\omega_1\\ \tilde{\omega}_3\end{pmatrix}\ \text{primitive},\quad \det\tilde{\omega}_3\ \text{coprime to}\ N \tag{2.20}$$

and we have

$$\chi(\det\tilde{X})=\chi(\det\tilde{\omega}_3)\,\bar{\chi}(\det\gamma)\bar{\chi}(\det(-\omega_1)). \tag{2.21}$$

Collecting all these facts (writing ω_3 again instead of $\tilde{\omega}_3$) we obtain

Proposition 2.2. — *For a Dirichlet character ψ* mod M, $M>1$, *a Dirichlet character χ* mod N, $N^2|M$ *and $k\in\mathbb{N}$ with $\psi(-1)=(-1)^k$, the twisted Eisenstein series (2.18) can be written as (for* $\mathrm{Re}(s)\gg 0$*)*

$$\sum_{R\in T^n(M)_\infty\backslash T^n(M)} \quad \sum_{\omega_1\in\mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})} \quad \sum_{\omega_3\in\mathbb{Z}^{(n,n)}} \tag{2.22}$$

$$\psi^2(\det\omega_1)\,\bar{\chi}\,(\det(-\omega_1))\,\chi\,(\det\omega_3)\,\bar{\chi}^{\sim}(R)\,\tilde{\psi}(R)\,j(R,z_4)^{-k}\,|j\,(R,z_4)|^{-2s}$$

$$\times\hat{\mathbb{F}}^k_n\left(R^{\downarrow}\langle 3\rangle\left[\begin{pmatrix}\omega_1\\ \frac{\omega_3}{N}\end{pmatrix}\right],M,\psi,s\right).$$

Here ω_1, ω_3 satisfy the additional conditions

$$(2.23)\qquad \begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix} \text{ primitive}, \quad \det \omega_1 \text{ coprime } to\, M, \quad \det \omega_3 \text{ coprime } to\, N.$$

Remark 2.2. — It is obvious from the expansion (2.22) that the twisted Eisenstein series (2.18) has the transformation properties of an automorphic form for the group $\Gamma_0^n(M)^{\downarrow}$ (and then by symmetry also for $\Gamma_0^n(M)^{\uparrow}$) with nebentype $\bar{\chi}\psi$.

This can also be seen directly from the definition (2.18) by a slightly different, but similar computation. Actually this twisting process makes sense not only for these Eisenstein series, but also for any automorphic form on $\mathbb{H}_{2n}$ with respect to $\Gamma_0^{2n}(M)$.

For future purposes we now change notation:

We put

$$(2.24)\qquad \varphi := \psi\bar{\chi} \quad (\text{ Dirichlet character mod } M)$$

and we work with φ and χ (instead of ψ and χ).

The expression

$$\psi^2(\det \omega_1)\, \bar{\chi}\,(\det(-\omega_1))\, \chi\,(\det \omega_3)\, \bar{\chi}^{\sim}(R)\, \tilde{\psi}(R)$$

becomes

$$\bar{\chi}\,(-1)^n(\chi\varphi^2)(\det \omega_1)\, \chi\,(\det \omega_3)\, \tilde{\varphi}(R)$$

and $\psi(-1) = (-1)^k$ becomes $\chi(-1) = (-1)^k\, \varphi(-1)$.

With the notations above and $l = k + \nu, \;\; \nu \geqslant 0$ we define a function on $\mathbb{H}_n \times \mathbb{H}_n$ (with $z = x + iy, \;\; w = u + iv$) by

$$(2.25)\qquad \mathfrak{E}_{2n}^{k,\nu}(w, z, M, N, \varphi, \chi, s) := \det(v)^s \det(y)^s \; \overset{0}{\mathfrak{D}}{}_{n,k+s}^{\,\nu}$$
$$\left(\sum_{\substack{X \in \mathbb{Z}^{(n,n)} \\ X \bmod N}} \chi\,(\det X) \hat{\mathbb{F}}_{2n}^{k}(-, M, \varphi\chi, s) \Big|_k \begin{pmatrix} 1_{2n} & S\left(\frac{X}{N}\right) \\ 0_{2n} & 1_{2n} \end{pmatrix} \right) \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

The remark above implies that this function (when considered as a function of z or w) defines an element of $C^\infty M_n^l(\Gamma_0(M), \varphi)$.

An elementary calculation (using basic properties of the differential operator $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,k+s}$) shows that (2.25) is equal to

$$(2.26)\quad \frac{\Gamma_n(k+\nu+s)}{\Gamma_n(k+s)}\cdot\frac{\Gamma_n\left(k+\nu+s-\frac{n}{2}\right)}{\Gamma_n\left(k+s-\frac{n}{2}\right)}\det(v)^s\det(y)^s\,\bar{\chi}(-1)^n$$

$$\times\sum_{R\in T^n(M)_\infty\backslash T^n(M)}\ \sum_{\omega_1\in\mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})}\ \sum_{\omega_3\in\mathbb{Z}^{(n,n)}}$$

$$(\chi\varphi^2)(\det\omega_1)\,\chi\,(\det\omega_3)\,\tilde{\varphi}(R)\det(\omega_1)^\nu\det(\omega_3)^\nu N^{-n\nu}$$

$$\times\hat{\mathbb{F}}^{k,\nu}_n\left(z\left[\omega_1\right]+R\langle w\rangle\left[\frac{\omega_3}{N}\right],M,\varphi\chi,s\right)$$

where we use an auxiliary function $\hat{\mathbb{F}}^{k,\nu}_n$ on $\mathbb{H}_n$, defined by

$$(2.27)\quad \hat{\mathbb{F}}^{k,\nu}_n(z,M,\varphi\chi,s)$$

$$=\sum_{\left(\begin{smallmatrix}* & *\\ c & d\end{smallmatrix}\right)\in T^n(M)_\infty\backslash T^n(M)}(\varphi\chi)(\det c)^\nu\det(cz+d)^{-k-\nu}|\det(cz+d)|^{-2s}$$

$$=\sum_{\mathfrak{R}}(\varphi\chi)\left(\nu(\mathfrak{R})\right)\nu(\mathfrak{R})^{-k-2s}\det(z+\mathfrak{R})^{-k-\nu}|\det(z+\mathfrak{R})|^{-2s}.$$

Here $\mathfrak{R}=\mathfrak{R}^t=c^{-1}d$ runs through all rational symmetric matrices and $\nu(\mathfrak{R})=|\det c|$ is the absolute value of the product of the denominators of the elementary divisors of $\mathfrak{R}$. We remark, that the condition "$\det(c)$ coprime to M" is taken care of by the factor $(\varphi\chi)\left(\nu(\mathfrak{R})\right)$.

For a cusp form $g\in S^l_n(\Gamma_0(M),\varphi)$ we want to compute (for $\mathrm{Re}(s)\gg 0$) the scalar product

$$(2.28)\quad \left\langle g,\mathfrak{E}^{k,\nu}_{2n}\left(*,-\bar{z},M,N,\varphi,\chi,\bar{s}\right)\right\rangle_{\Gamma_0(M)}.$$

It is clear that by (2.28) we define an element of $C^\infty M^l_n(\Gamma_0(M),\bar{\varphi})$.

The expansion (2.26) allows us to apply the usual "unfolding trick": The scalar product (2.28) is equal to

$$(2.29)\quad \frac{\Gamma_n(k+\nu+s)}{\Gamma_n(k+s)}\cdot\frac{\Gamma_n\left(k+\nu+s-\frac{n}{2}\right)}{\Gamma_n\left(k+s-\frac{n}{2}\right)}\det(y)^s\bar{\chi}\,(-1)^nN^{-n\nu}$$

$$\times\int_{T^n(M)_\infty\backslash\mathbb{H}_n}g\Big|_l\begin{pmatrix}0_n & -1_n\\ 1_n & 0_n\end{pmatrix}(w)\,\overline{\{\qquad\}}\det(v)^{l-n-1}du\,dv,$$

where $\{\qquad\}$ stands for

$$\sum_{\omega_1,\omega_3} \det(\omega_1)^\nu \det(\omega_3)^\nu (\chi\varphi^2)(\det\omega_1)\,\chi(\det\omega_3)$$
$$\hat{\mathbb{F}}_n^{k,\nu}\left(-\bar{z}\,[\omega_1] + w\left[\frac{\omega_3}{N}\right], M, \varphi\chi, \bar{s}\right)\det(v)^{\bar{s}}.$$

We now proceed in essentially the same way as in [6, §5]:

We apply the unfolding trick again, this time with respect to the summation over ω_3: we integrate over $\mathbb{H}_n \bmod M$ (instead of $\mathbb{H}_n$ mod $T^n(M)_\infty$) and we change the summation over $\omega_3 \in \mathbb{Z}^{(n,n)}$ into summation over $\omega_3 \in GL(n,\mathbb{Z})\backslash\mathbb{Z}^{(n,n)}$ (the additional conditions on ω_1, ω_3 remain valid). A factor 2 comes in because -1_{2n} acts trivially on $\mathbb{H}_n$.

Up to elementary factors the contribution of a fixed pair (ω_1, ω_3) to (2.29) is

$$(2.30)\qquad \int_{\mathbb{H}_n \bmod M} g\Big|_l \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}(w) \times$$
$$\overline{\sum_{\mathfrak{R}} \det\left(-\bar{z}\,[\omega_1] + w\left[\frac{\omega_3}{N}\right] + \mathfrak{R}\right)^{-1} \left|\det\left(-\bar{z}[\omega_1] + w\left[\frac{\omega_3}{N}\right] + \mathfrak{R}\right)\right|^{-2\bar{s}}}$$
$$\det(v)^{l+s-n-1} du\, dv.$$

We decompose $\mathfrak{R}$ as

$$(2.31)\qquad \mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2 \ \text{ with } \ \mathfrak{R}_2 \in \frac{M}{N^2}\mathbb{Z}_{\text{sym}}^{(n,n)}[\omega_3]\,.$$

This allows us to apply the unfolding trick again (for the summation with respect to $\mathfrak{R}_2$); the integral (2.30) becomes

$$(2.32)\qquad \sum_{\mathfrak{R}_1} \det\left(\frac{\omega_3}{N}\right)^{-2l-4s} \int_{\mathbb{H}_n} g\Big|_l \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$
$$\overline{\det\left(-\bar{z}\left[\omega_1\omega_3^{-1}\right]N^2 + \mathfrak{R}_1\left[\omega_3^{-1}\right]N^2 + w\right)^{-l}}$$
$$\times \overline{\left|\det(-\bar{z}\left[\omega_1\omega_3^{-1}\right]N^2 + \mathfrak{R}_1\left[\omega_3^{-1}\right]N^2 + w\right|^{-2\bar{s}}} \det(v)^{l+s-n-1} du\, dv.$$

The Selberg reproducing formula for holomorphic functions (in the version stated in [6]) yields for each summand of (2.32)

$$(2.33)\qquad (-1)^{\frac{nl}{2}}\, 2^{n(n+1)-2ns-nl}\, I_n(l+s-n-1)\det(\omega_3)^{-2l-4s}\,(N^n)^{2l+4s}$$

$$\times \det\left(y\left[\omega_1\omega_3^{-1}N\right]\right)^{-s} g\Big|_l \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \left(z\left[\omega_1\omega_3^{-1}\right]N^2 - \mathfrak{R}_1\left[\omega_3^{-1}\right]N^2\right)$$

with

$$I_n(s) = \pi^{\frac{n(n+1)}{2}} 2^{-\frac{n(n+1)}{2}} \frac{\Gamma_n\left(s+1+\frac{n-1}{2}\right)}{\Gamma_n(s+n+1)} \tag{2.34}$$

denoting the well-known integral of Hua [16].

We decompose the summation over $\mathfrak{R}_1$ into

$$\mathfrak{R}_1 = \mathfrak{R}_0 + \mathfrak{T}[\omega_3], \tag{2.35}$$

$$\mathfrak{R}_0 \in \mathbb{Q}_{\text{sym}}^{(n,n)} \bmod \mathbb{Z}_{\text{sym}}^{(n,n)}[\omega_3],\quad \mathfrak{T} \in \mathbb{Z}_{\text{sym}}^{(n,n)}[\omega_3] \bmod \frac{M}{N^2}\mathbb{Z}^{(n,n)}[\omega_3].$$

An easy calculation shows that

$$\sum_{\mathfrak{T}} g\Big|_l \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \left(z\left[\omega_1\omega_3^{-1}\right]N^2 - \mathfrak{R}_0\left[\omega_3^{-1}\right]N^2 - N^2\mathfrak{T}\right) \tag{2.36}$$

$$= \left(\frac{M}{N^2}\right)^{\frac{n(n+1)}{2}} M^{-\frac{nl}{2}} g\Big|_l \begin{pmatrix} 0_n & -1_n \\ M & 0_n \end{pmatrix} \Big| U\left(\frac{M}{N^2}\right)\Big|_l \begin{pmatrix} \omega_3^{-t}\omega_1^t & \omega_3^{-t}\mathfrak{R}_0\,\omega_1^t \\ 0_n & \omega_3\,\omega_1^{-1} \end{pmatrix}$$

$$\det\left(\omega_3\,\omega_1^{-1}\right)^l.$$

Summarizing all these computations (and simplifying the constants) we obtain

$$\left\langle g, \mathfrak{E}_{2n}^{k,\nu}\left(*, -\bar{z}, M, N, \varphi, \chi, \bar{s}\right)\right\rangle_{\Gamma_o(M)} \tag{2.37}$$

$$= (-1)^{\frac{nl}{2}} 2^{\frac{n(n+1)}{2}+1-2ns} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n\left(1+s-\frac{n}{2}\right)\Gamma_n\left(1+s-\frac{n+1}{2}\right)}{\Gamma_n(k+s)\Gamma_n\left(k+s-\frac{n}{2}\right)}$$

$$\times\chi(-1)^n \left(N^n\right)^{2k+\nu+2s-n-1} M^{\frac{n(n+1)}{2}-\frac{nl}{2}}$$

$$\times \sum_{\omega_1 \in \mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})} \sum_{\omega_3 \in GL(n,\mathbb{Z})\backslash\mathbb{Z}^{(n,n)}} \sum_{\mathfrak{R}_0 \in \mathbb{Q}_{\text{sym}}^{(n,n)} \bmod \mathbb{Z}_{\text{sym}}^{(n,n)}[\omega_3]} \left(\bar{\chi}\bar{\varphi}^2\right)(\det\omega_1)$$

$$\bar{\chi}(\det\omega_3)\ (\bar{\varphi}\bar{\chi})\ (\nu(\mathfrak{R}_0))\ \nu(\mathfrak{R}_0)^{-k-2s}\det(\omega_1)^{-k-2s}\det(\omega_3)^{-k-2s}$$

$$\times g\Big|_l \begin{pmatrix} 0_n & -1_n \\ M & 0_n \end{pmatrix} \Big| U\left(\frac{M}{N^2}\right)\Big|_l \begin{pmatrix} \omega_3^{-t}\omega_1^t & \omega_3^{-t}\,\mathfrak{R}_0\,\omega_1^{-1} \\ 0_n & \omega_3\,\omega_1^{-1} \end{pmatrix}.$$

We mention again, that ω_1, ω_3 have to satisfy the additional conditions

$$\begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix} \text{ primitive,} \quad \det \omega_1 \text{ coprime to } M, \quad \det \omega_3 \text{ coprime to } N.$$

Appendix: The trivial character.

For the trivial character $\chi = 1$ the construction considered above does not produce the kind of result needed later on. In this appendix we therefore present a modified twisting process, which is also interesting in its own right.

We fix a prime p with $p^2 \big| M$; for $0 \leqslant i \leqslant n$ we consider the double coset

$$GL(n,\mathbb{Z}) \begin{pmatrix} 1_{n-i} & 0 \\ 0 & p \cdot 1_i \end{pmatrix} GL(n,\mathbb{Z}) = \dot{\bigcup_j} \, GL(n,\mathbb{Z}) g_{ij}. \tag{2.38}$$

Instead of (2.18) we now look at

$$\hat{\mathbb{F}}^k_{2n}(-,M,\psi,s,i) := \sum_j \sum_X \hat{\mathbb{F}}^k_{2n}(-,M,\psi,s)\Big|_k \begin{pmatrix} 1_{2n} & S(X^t) \\ 0_{2n} & 1_{2n} \end{pmatrix} \tag{2.18'}$$

where X runs over $\mathbb{Z}^{(n,n)} g_{ij}^{-t} / \mathbb{Z}^{(n,n)}$.

The same computation as in the proof of Proposition 2.2 shows that (2.18′) is equal to

$$\begin{aligned} \sum_j \sum_X \sum_R \sum_{\omega_1} \sum_{\omega_3} & \psi^2 (\det \omega_1) \, \tilde{\psi}(R) \, j(R,z_4)^{-k} \\ & \times |j(r,z_4)|^{-2s} \, \hat{\mathbb{F}}^k_{2n} \left(R^{\downarrow} \langle 3 \rangle \left[\begin{pmatrix} \omega_1 \\ \tilde{\omega}_3 \end{pmatrix} \right], M, \psi, s \right) \end{aligned} \tag{2.39}$$

with

$$\tilde{\omega}_3 = \omega_3 - \gamma X \omega_1 \, . \tag{2.40}$$

These $\tilde{\omega}_3$ can be described in a better way:

LEMMA 2.1. — *If one fixes ω_1 in its right coset modulo $GL(n,\mathbb{Z})$, then there is a permutation $j \longmapsto \hat{j}$ such that for j fixed, $p\tilde{\omega}_3$ runs precisely over those elements of $\mathbb{Z}^{(n,n)} p \cdot g_{i,\hat{j}}^{-t}$, which are coprime to ω_1.*

Proof. — Exercise on integral matrices (left to the reader).

We may combine the construction (2.18′) with the previous twisting process (2.18) in an obvious way:

Let $N = N' \cdot p$ be given with N' *coprime to* $p, N^2 | M$. Let χ be a Dirichlet character mod N'; generalizing (2.25) we define (*with* $0 \leqslant i \leqslant n$)

(2.25′) $\mathfrak{E}_{2n}^{k,\nu}(w, z, M, N', \varphi, \chi, s, i)$

$$:= \det(v)^s \det(y)^s \overset{0}{\mathfrak{D}}{}^{\nu}_{n,k+s} \sum_{\substack{X \in \mathbb{Z}^{(n,n)} \\ X \bmod N'}} \sum_j \sum_{\tilde{X} \in \mathbb{Z}^{(n,n)} g_{ij}^{-t} / \mathbb{Z}^{(n,n)}} \chi(\det X)$$

$$\hat{\mathbb{F}}^k_{2n}(-, M, \varphi\chi, s)\Big|_l \left(\begin{pmatrix} 1_{2n} & S\left(\frac{X}{N}\right) \\ 0_{2n} & 1_{2n} \end{pmatrix} \begin{pmatrix} 1_{2n} & S\left(\tilde{X}^t\right) \\ 0_{2n} & 1_{2n} \end{pmatrix} \right) \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

Then (2.37) becomes

(2.37′) $\left\langle g, \mathfrak{E}_{2n}^{k,\nu}(*, -\bar{z}, M, N', \varphi, \chi, \bar{s}, i) \right\rangle_{\Gamma_0(M)}$

$$= (-1)^{\frac{nl}{2}} 2^{\frac{n(n+1)}{2}+1-2ns} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n\left(l+s-\frac{n}{2}\right)\Gamma_n\left(s+l-\frac{n+1}{2}\right)}{\Gamma_n(k+s)\,\Gamma_n\left(k+s-\frac{n}{2}\right)}$$

$$\times \chi(-1)^n (N'p)^{n(2k+\nu+2s-n-1)} M^{\frac{n(n+1)-nl}{2}}$$

$$\times \sum_j \sum_{\omega_1} \sum_{\omega_3} \sum_{\mathfrak{N}_0} \left(\bar{\chi}\bar{\varphi}^2\right) (\det\omega_1)\, \bar{\chi}(\det\omega_3)\, (\bar{\varphi}\bar{\chi})\, (\nu(\mathfrak{N}_0))$$

$$\times \nu(\mathfrak{N}_0)^{-k-2s} \det(\omega_1)^{-2s-k} \det(\omega_3)^{-k-2s}$$

$$\times g\Big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \Big| U\left(\frac{M}{N'^2p^2}\right)\Big|_l \begin{pmatrix} \omega_3^{-t}\omega_1 & \omega_3^{-t}\mathfrak{N}_0\,\omega_1^{-1} \\ 0 & \omega_3\,\omega_1^{-1} \end{pmatrix}.$$

The summation is over

$$\omega_1 \in \mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z}), \quad \omega_3 \in GL(n,\mathbb{Z})\backslash\mathbb{Z}^{(n,n)},$$
$$\mathfrak{N}_0 \in \mathbb{Q}^{(n,n)}_{\text{sym}} \bmod \mathbb{Z}^{(n,n)}_{\text{sym}}[\omega_3]$$

and subject to the additional conditions

$$\begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix} \text{ primitive}, \quad \det(\omega_1) \text{ coprime } to\ M, \quad \nu(\mathfrak{N}_0) \text{ coprime } to\ M$$

$$\omega_3 \in \mathbb{Z}^{(n,n)} p g_{ij}^{-t}, \quad \det(\omega_3) \text{ coprime } to\ N'.$$

3. Hecke operators.

In order to define automorphic L-functions properly, we need some notations on Hecke algebras, in particular we must define a certain Hecke subalgebra for the bad primes. For standard facts about Hecke pairs, Hecke algebras etc. we refer to [2], [12]. Let $\mathfrak{H}$ be the abstract $\mathbb{C}$-Hecke algebra associated to the Hecke pair $(\mathrm{Sp}(n,\mathbb{Q}), \Gamma_o^n(M))$. As a vector space this is the set of all finite formal linear combinations of all double cosets $\Gamma_0^n(M)\, g\, \Gamma_0^n(M)$, $g \in \mathrm{Sp}(n,\mathbb{Q})$. We are interested in very special double cosets, namely those of type

$$\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W\end{pmatrix}\Gamma_0^n(M)\ ,\quad W \in M_n(\mathbb{Z})^*, \tag{3.1}$$

where $M_n(\mathbb{Z})^*$ denotes the non-singular integral matrices of size n; of course W may be chosen as an elementary divisor matrix.

We denote by $\mathfrak{H}^\circ$ the $\mathbb{C}$-linear span of all these double cosets; we shall soon see that this is indeed a subalgebra of $\mathfrak{H}$.

We first remark that the left cosets in double cosets of type (3.1) have "upper triangular" representatives

(3.2)
$$\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W\end{pmatrix}\Gamma_0^n(M) = \bigcup \Gamma_0^n(M)\, g_i,\quad g_i = \begin{pmatrix} * & * \\ 0_n & *\end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Q})$$

(by direct calculation). From this we may obtain (e.g. by counting the number of left cosets on both sides) that *for all* $V, W \in M_n(\mathbb{Z})^*$ *with coprime determinants*

$$\begin{aligned}\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W\end{pmatrix}&\Gamma_0^n(M)\cdot\Gamma_0^n(M)\begin{pmatrix} V^{-t} & 0 \\ 0 & V\end{pmatrix}\Gamma_0^n(M)\\ &= \Gamma_0^n(M)\begin{pmatrix} (WV)^{-t} & 0 \\ 0 & WV\end{pmatrix}\Gamma_0^n(M)\,.\end{aligned} \tag{3.3}$$

For $M > 1$ not all upper triangular matrices occur as a representatives of left cosets. We first look at the "bad primes":

LEMMA 3.1. — *Each double coset* $\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W\end{pmatrix}\Gamma_0^n(M)$ *with* $\det(W)\big|M^\infty$ *decomposes into left cosets according to*

$$\dot{\bigcup_{\tilde W, R}} \Gamma_0^n(M)\begin{pmatrix} \tilde W^{-t} & \tilde W^{-t}R \\ 0 & \tilde W\end{pmatrix} \tag{3.4}$$

with

$$GL(n,\mathbb{Z})\,W\,GL(n,\mathbb{Z}) = \dot{\bigcup_{\tilde{W}}} GL(n,\mathbb{Z})\,\tilde{W} \text{ and } R = R^t \in \mathbb{Z}_{\mathrm{sym}}^{(n,n)}/\mathbb{Z}_{\mathrm{sym}}^{(n,n)}[\tilde{W}].$$

Proof. — Elementary calculation.

Using (3.3) we may combine Lemma 3.1 with similar results in [7] on the case of "good primes" (i.e., $\det(W)$ coprime to M):

COROLLARY 3.1. — *The set of upper triangular matrices in* $\mathrm{Sp}(n,\mathbb{Q})$, *which occur in double cosets of type (3.1) is equal to*

(3.5)
$$\left\{ \begin{pmatrix} \omega_3^{-t}\omega_1^t & \omega_3^{-t}\,B\,\omega_1^{-1} \\ 0 & \omega_3\omega_1^{-1} \end{pmatrix} \middle| \begin{array}{l} \omega_1,\,\omega_3 \in M_n(\mathbb{Z})^*,\ \omega_1 \text{ coprime to } \omega_3 \\ B \in \mathbb{Q}_{\mathrm{sym}}^{(n,n)},\ \det(\omega_1) \text{ and } \nu(B) \text{ both coprime to } M \end{array} \right\}.$$

We omit the (easy) proof.

We remark that from an upper triangular matrix in (3.5) we cannot easily rediscover the elementary divisor matrix W describing its double coset, we only have the relation [7, Prop. 4]

$$\det(W) = \pm\det(\omega_1)\det(\omega_3)\,\nu(B).$$

The set (3.5) is a semigroup, therefore $\mathfrak{H}^\circ$ is indeed a Hecke algebra. It is commutative, because

$$g \longmapsto \begin{pmatrix} 0 & \frac{1}{M} \\ -1 & 0 \end{pmatrix} g^{-1} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \tag{3.7}$$

is an involutive anti-automorphism of $\mathfrak{H}^\circ$.

As a consequence of (3.3) we get as usual a decomposition (restricted tensor product) of our Hecke algebra into "p-components":

$$\mathfrak{H}^\circ \simeq \bigotimes_p{}' \mathfrak{H}^\circ_{M,p} \tag{3.8}$$

the p-component being defined by double cosets (3.1) with $\det(W) =$ power of the prime p.

The structure of these p-components is well known for p coprime to M:

$$\mathfrak{H}^\circ_{M,p} \simeq \mathbb{C}\left[X_1^{\pm1},\ldots,X_n^{\pm1}\right]^{W_n} \tag{3.9}$$

with W_n denoting the Weyl group generated by the permutations of the X_i and the mappings $(1 \leqslant j \leqslant n)$

$$X_j \longmapsto X_j^{-1} \qquad X_i \longmapsto X_i \quad (i \neq j). \tag{3.10}$$

From Lemma 3.1 we get

COROLLARY 3.2. — *For $p|M$ the Hecke algebra $\mathfrak{H}^\circ_{M,p}$ is isomorphic to the Hecke algebra attached to the Hecke pair*

$$\left(GL_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \cap M_n(\mathbb{Z}),\, GL_n(\mathbb{Z})\right),$$

the isomorphism being given by

$$\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W \end{pmatrix}\Gamma_0^n(M) \longmapsto GL_n(\mathbb{Z})\, W\, GL_n(\mathbb{Z}). \tag{3.11}$$

In terms of the Satake isomorphism this means

$$\mathfrak{H}^\circ_{M,p} \simeq \mathbb{C}\,[X_1, \ldots, X_n]^{S_n}. \tag{3.12}$$

For the explicit description of the Satake isomorphisms (3.9) and (3.12) in terms of left cosets we refer to [12], however we shall use a normalization of the Satake isomorphism (3.12) different from [12].

For $w \in \mathbb{Z}$ we define

$$(w)_M := \prod_{p|M} p^{\nu_p(w)}\,; \tag{3.13}$$

note that

$$\Gamma_0^n(M)\begin{pmatrix} W^{-t} & 0 \\ 0 & W \end{pmatrix}\Gamma_0^n(M) \longmapsto \det(W) \tag{3.14}$$

does not define an algebra homomorphism from $\mathfrak{H}^\circ$ to $\mathbb{C}$, but $(\det W)_M$ does.

For a double coset (3.1) we define a Hecke operator $T_M(W)$ acting on $C^\infty M_n^l\,(\Gamma_0^n(M),\, \psi)$ by

$$f\,|\, T_M(W) := \sum_i \psi\,(\det{}_M(W)\cdot \det \alpha_i)\, f\big|_l \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \tag{3.15}$$

with

$$\Gamma_0^n(M)\begin{pmatrix} W'^{-1} & 0 \\ 0 & W \end{pmatrix}\Gamma_0^n(M) = \bigcup_i \Gamma_0^n(M)\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

The reader should convince himself that $\det(\alpha_i)\det_M(W)$ is always M-integral and coprime to M and that equation (3.15) defines a homomorphism from $\mathfrak{H}^\circ$ to $\mathrm{End}(\,C^\infty M_n^l\,(\Gamma_0^n(M),\,\psi))$.

For $r|M$ and a cusp form $f \in S_n^l(\Gamma_0(M),\psi)$ let us consider (with $\mathrm{Re}(s) \gg 0$)

$$\sum_D f|T_M(D)\det(D)^{-s} \tag{3.16}$$

where $D = \mathrm{diag}\,(d_1,\ldots,d_n)$, $d_i > 0$, runs over all elementary divisor matrices of size n, $d_i|d_{i+1}$, with $(\det D,\, r) = 1$.

Using (3.5) and (3.6) we may describe the action of the operator (3.16) as follows:

$$\sum^*_{\omega_1 \in M_n(\mathbb{Z})^*/GL_n(\mathbb{Z})}\ \sum^*_{\omega_3 \in GL_n(\mathbb{Z})\backslash M_n^*(\mathbb{Z})}\ \sum_{B \in \mathbb{Q}_{\mathrm{sym}}^{(n,n)} \bmod \mathbb{Z}_{\mathrm{sym}}^{(n,n)}[\omega_3]} \tag{3.17}$$

$$\nu(B)^{-s}|\det\omega_1|^{-s}|\det\omega_3|^{-s}\psi\left(((\det(\omega_1)\det(\omega_3)\,\nu(B))_M\det(\omega_3)^{-1}\det(\omega_1)\right)$$
$$\times f\big|_l\begin{pmatrix} \omega_3'^{-1}\omega_1 & \omega_3'^{-1}B\omega_1^{-1} \\ 0 & \omega_3\omega_1^{-1} \end{pmatrix}.$$

Here $\sum^*$ indicates that ω_1, ω_3 have to satisfy the extra conditions

$$\omega_1 \text{ coprime to } \omega_3;\quad \det\omega_1 \text{ coprime to } M;\quad \det\omega_3 \text{ coprime to } r\,.$$

Assume now that f is an eigenform of all the operators $T_M(D)$, $(\det D,\, r) = 1$:

$$f|T_M(D) = \lambda_D(f)\,f. \tag{3.18}$$

The mapping

$$\Gamma_0^n(M)\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}\Gamma_0^n(M) \longmapsto \lambda_D(f) \tag{3.19}$$

induces (for all p coprime to r) homomorphisms $\chi_p : \mathfrak{H}_{M,p}^\circ \longrightarrow \mathbb{C}$, which are parameterized by "Satake parameters"

$$\alpha_{1,p}^{\pm 1},\ldots,\alpha_{n,p}^{\pm 1} \qquad \text{for } p \nmid M$$
$$\beta_{1,p},\ldots,\beta_{n,p} \qquad \text{for } p|M,\ p \nmid r.$$

Of course the operator (3.16) is also built up from its "p-components" and it follows from Tamagawa's rationality theorem [37] for local Hecke series ($p|M$) and the rationality theorem proven in [7] that (3.16) is equal to

$$(3.20)\quad \prod_{\substack{p|M\\ p\nmid r}}\left(\frac{1}{\prod_{i=1}^{n}(1-\beta_{i,p}p^{-s+n})}\right)$$
$$\prod_{p\nmid M}\left(\frac{(1-p^{-s})\prod_{i=1}^{n}(1-p^{2i-2s})}{(1-p^{-s+n})\prod_{i=1}^{n}(1-\alpha_{i,p}p^{-s+n})(1-\alpha_{i,p}^{-1}p^{-s+n})}\right)f.$$

In order to get smooth formulas, we have normalized the Satake-isomorphism for the GL_n-Hecke-algebra as follows:

$$(3.21)\qquad A=\begin{pmatrix} p^{k_1} & & * \\ & \ddots & \\ 0 & & p^{k_n}\end{pmatrix}\longmapsto \det(A)^{n+1}\prod_{\nu=1}^{n}\left(\frac{X_\nu}{p^\nu}\right)^{k_\nu},$$

which differs from [12, IV, §2] by the factor $\det(A)^{n+1}$.

For f as above, we now define the (standard-) L-function by

$$(3.22)\quad D^{(M)}(f,s)$$
$$:=\prod_{p\nmid M}\left(\frac{1}{1-\psi(p)\,p^{-s}}\prod_{i=1}^{n}\frac{1}{\left(1-\psi(p)\,\alpha_{i,p}\,p^{-s}\right)\left(1-\psi(p)\,\alpha_{i,p}^{-1}p^{-s}\right)}\right)$$

and the r'-complement L-function $\left(r':=\prod_{\substack{p|M\\ p\nmid r}}p\right)$ by

$$(3.23)\qquad D^{(M,r')}(f,s)=\prod_{p|r'}\left(\prod_{i=1}^{n}\frac{1}{(1-\beta_{i,p}p^{-s})}\right)D^{(M)}(f,s).$$

With all these notations we can now give the main result of this chapter, reformulating (2.37) in terms of Euler products:

THEOREM 3.1.— *Let* φ *be a Dirichlet character* mod $M>1$, χ *a Dirichlet character* mod N, $N^2|M$, $l=k+\nu\in\mathbb{N}$ *with* $\chi(-1)=$

$(-1)^k\varphi(-1)$ *and* $g\big|_l\left(\begin{smallmatrix}0 & -1\\ M & 0\end{smallmatrix}\right) \in S^l_n\left(\Gamma^n_0(M),\, \bar{\varphi}\right)$ *an eigenform of the Hecke algebra* $\prod\limits_{p\nmid N} \mathfrak{H}^\circ_{M,p}$. *Then we have for* $M_0' := \prod\limits_{\substack{p\mid M\\ p\nmid N}} p$ *and* $\mathrm{Re}(s) \gg 0$*:*

$$(3.24)\quad \left\langle g, \mathfrak{E}^{k,\nu}_{2n}(*, -\bar{z}, M, N, \varphi, \chi, \bar{s})\right\rangle_{\Gamma^n_0(M)}$$

$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s,\, \bar{\chi}\bar{\varphi})}\left(N^n\right)^{2k+\nu+2s-n-1} M^{\frac{n(n+1)-nl}{2}} \chi(-1)^n$$

$$\times D^{(M,M_0')}\left(g\big|_l\begin{pmatrix}0 & -1\\ M & 0\end{pmatrix}, k+2s-n, \bar{\chi}\right) g\big|_l\begin{pmatrix}0 & -1\\ M & 0\end{pmatrix}\Bigg| U\left(\frac{M}{N^2}\right)$$

with

$$\mathfrak{L}(s,\psi) = L(s,\psi)\prod_{i=1}^{n} L\left(2s-2i,\, \psi^2\right)$$

and

$$\Omega_{l,\nu} = (-1)^{\frac{nl}{2}}\, 2^{1+\frac{n(n+1)}{2}-2ns}\, \pi^{\frac{n(n+1)}{2}}\, \frac{\Gamma_n\left(l+s-\frac{n}{2}\right)\Gamma_n\left(l+s-\frac{n+1}{2}\right)}{\Gamma_n\Big(k+s\Big)\,\Gamma_n\left(k+s-\frac{n}{2}\right)}.$$

COROLLARY 3.3. — *The twisted L-function*

$$D^{(M,\,M_0')}\left(g\big|_l\begin{pmatrix}0 & -1\\ M & 0\end{pmatrix}, s, \bar{\chi}\right)$$

has a meromorphic continuation to the whole complex plane.

In fact, Theorem 3.1 produces a whole series of integral representations of our twisted L-function (depending on the choice of k and ν). Using standard properties of Eisenstein series we may deduce the corollary from any of these integral representations - if we take for granted that (for $\nu > 0$) the differential operators $\mathfrak{D}^\nu_{n,k+s}$ do not destroy the "slow growth" of the Eisenstein series. This is not completely obvious, so we give a sketch of proof here:

In [26] a differential operator D^r, $r \geqslant 1$ is introduced (generalizing the well-known operators of Maaß [18]), which maps Eisenstein series like $\mathbb{F}^k_n(Z, M, \psi, s)$ to vector-valued Eisenstein series. For each monomial $\Delta \in \mathbb{C}[\partial_{ij}]$ of degree r there is a component of that vector-valued Eisenstein series, which has $\Delta\mathbb{F}^k_n(Z, M, \psi, s)$ as its "main term", the remaining terms being sums of terms like (derivative of $\mathbb{F}_n^k(Z, M, \psi, s)$ of *degree* $< r$) $\times$ (elementary function of Y).

Now we use that the vector-valued Eisenstein series are also of "slow growth", which then implies that the derivatives $\mathfrak{D}_{n,k}^0 \mathbb{F}_{2n}^k(Z, M, \psi, s)$ and also $\mathfrak{D}_{n,k+s}^{\nu} \hat{\mathbb{F}}_{2n}^k(Z, M, \psi, s)$ are still of slow growth.

Appendix: The trivial character.

There is also a version of Theorem 3.1 for (2.37′):

Under the same assumption as in Theorem 3.1 (but with χ *a Dirichlet character* mod N', $N = N'p$, *p coprime to* N') we have

$$\begin{aligned}(3.26)\quad & \left\langle g, \mathfrak{E}_{2n}^{k,\nu}(*, -\bar{z}, M, N', \varphi, \chi, \bar{s}, i) \right\rangle_{\Gamma(M)} \\ &= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\chi}\,\bar{\varphi})} (N'p)^{n(2k+\nu+2s-n-1)} M^{\frac{n(n+1)-nl}{2}} \chi(-1)^n \\ &\quad \times D^{(M,M_0')}\left(g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, k+2s-n, \bar{\chi} \right) \\ &\quad \times \sum_j \bar{\chi}\left(\det(pg_{ij}^{-t})\right) \det\left(p(g_{ij}^{-t})\right)^{-k-2s} \sum_{\mathfrak{R}_0 \in \mathbb{Z}_{\text{sym}}^{(n,n)}/\mathbb{Z}_{\text{sym}}^{(n,n)}[pg_{ij}^{-t}]} \\ &\quad g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \Big| U\left(\frac{M}{N^2}\right) \Big|_l \begin{pmatrix} \frac{1}{p} g_{ij} & \frac{1}{p} g_{ij}\mathfrak{R}_0 \\ 0_n & pg_{ij}^{-t} \end{pmatrix}.\end{aligned}$$

The sum over j and $\mathfrak{R}_0$ equals

$$\begin{aligned}(3.27)\quad & \bar{\chi}\left(p^{n-i}\right)\left(p^{n-i}\right)^{-k-2s} g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \Big| U\left(\frac{M}{N^2}\right) \Big| T_M \begin{pmatrix} 1_i & \\ & p1_{n-i} \end{pmatrix} \\ &= \bar{\chi}\left(p^{n-i}\right)\left(p^{n-i}\right)^{-k-2s} \lambda_{n-i}\, g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \Big| U\left(\frac{M}{N^2}\right),\end{aligned}$$

where λ_i is the eigenvalue for the Hecke operator $T_M\begin{pmatrix} 1_{n-i} & 0 \\ 0 & p\cdot 1_i \end{pmatrix}$.

Using the "Tamagawa-identity"

$$(3.28)\qquad \prod_{i=1}^{n}(1-\beta_i X) = \sum_{i=0}^{n}(-1)^i p^{-iM+\frac{i(i-1)}{2}} \lambda_i X^i$$

we can now deduce from (3.26) the following modified version of Theorem 3.1

$$(3.24') \quad \left\langle g, \sum_{i=0}^{n} (-1)^i p^{\frac{i(i-1)}{2}} p^{-in} \mathfrak{E}_{2n}^{k,\nu}(*, -\bar{z}, M, N', \varphi, \chi, \bar{s}, i) \right\rangle_{\Gamma_0(M)}$$

$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\chi}\bar{\varphi})} (N'p)^{n(2k+\nu+2s-n-1)} M^{\frac{n(n+1)-nl}{2}} \chi(-1)^n$$

$$\times D^{(M,M_0)} \left(g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, k+2s-n, \bar{\chi} \right) (-1)^n p^{-\frac{n^2+n}{2}}$$

$$\times \prod_{i=1}^{n} \left(1 - \beta_i \bar{\chi}(p) p^{-k+n+1} p^{-2s}\right) g\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \bigg| U\left(\frac{M}{N^2}\right).$$

In other words, the right hand sides of (3.24) and (3.24′) differ by the factor

$$(3.29) \qquad (-1)^n p^{-\frac{n^2+n}{2}} \prod_{i=1}^{n} \left(1 - \beta_i \, \bar{\chi}(p) \, p^{-k+n+1-2s}\right).$$

4. Trace and shift operators.

We now apply Theorem 3.1 to the special situation which is of interest for us:

Let S be a square-free number, $p\big|S$ and $f_0 = f_{0,S} \in S_n^l(\Gamma_0(NS), \bar{\varphi})$; we assume that $f_{0,S}$ is an eigenform for the Hecke algebras

$$(4.1) \qquad \bigotimes_{q \nmid NS}' \mathfrak{H}^{\circ}_{NS,q} \quad \text{and} \quad \bigotimes_{q|S} \mathfrak{H}^{\circ}_{NS,q}$$

and also an eigenform of $U(L)$ for all $L\big|S^\infty$:

$$(4.2) \qquad f_0\big| U(L) = \alpha(L)\, f_0.$$

Furthermore let χ be a Dirichlet character *mod* RN with $\varphi(-1) = (-1)^k \chi(-1)$ and $R_0\big|S$, where $R_0 := \prod_{q|R} q$.

We put $M = R^2 N^2 \frac{S}{R_0}$; now we may apply Theorem 3.1 to

$$(4.3) \qquad g := f_0\big|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \in S_n^l(\Gamma_0(M), \varphi)$$

(with N being replaced by RN, r' by $\frac{S}{R_0}$; we consider φ as a Dirichlet character mod M).

At the same time we move the whole situation from $\Gamma_0(M)$ to $\Gamma^0(M)$ by applying $\left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$.

Using the (obvious) relations

$$g|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} = f_0 | \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad g|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} = (-1)^{nl} f_0 ,$$

$$g|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (-1)^{nl} f_0|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$$

we obtain from (3.24)

(4.4)

$$\left\langle f_0\Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{E}_{2n}^{k,\nu}(*, -\bar{z}, M, RN, \varphi, \chi, \bar{s})\Big|_l^w \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\Big|_l^z \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\right\rangle_{\Gamma^0(M)}$$

$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\varphi}\bar{\chi})} (RN)^{n(2k+\nu+2s-n-1)} \left(R^2 N^2 \frac{S}{R_0}\right)^{\frac{n(n+1)-nl}{2}}$$

$$\times \chi(-1)^n (-1)^{nl} \alpha\left(\frac{S}{R_0}\right) D^{\left(M, \frac{S}{R_0}\right)}(f, k+2s-n, \bar{\chi})\, f_0\Big|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}.$$

Later on, we shall further modify (4.4) by taking the scalar product of both sides against a suitably chosen element h_0 of $S_n^l(\Gamma^0(N^2S), \bar{\varphi})$. The following lemmas will be useful:

LEMMA 4.1. — *For f_0, M etc. as above and any $h \in S_n^l(\Gamma^0(N^2S), \bar{\varphi})$ we have*

$$(4.5) \quad \left\langle f_0\Big|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}, h\right\rangle_{\Gamma^0(M)}$$

$$= \left(\frac{R^2}{R_0}\right)^{\frac{-nl+n(n+1)}{2}} \alpha\left(\frac{R^2}{R_0}\right) \left\langle f_0\Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h\right\rangle_{\Gamma^0(N^2S)}.$$

Proof (standard).

$$(4.6) \quad \left\langle f_0\Big|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}, h\right\rangle_{\Gamma^0(M)} = \left\langle \sum_\gamma f_0\Big|_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\Big|_l \gamma, h\right\rangle_{\Gamma^0(N^2S)},$$

where γ runs over $\Gamma^0(M)\setminus\Gamma^0(N^2S)$ (trace operator); we may choose

$$\left\{\begin{pmatrix} 1_n & N^2ST \\ 0 & 1_n\end{pmatrix} \;\middle|\; T = T^t \in \mathbb{Z}^{(n,n)} \bmod \frac{R^2}{R_0}\right\} \tag{4.7}$$

as a set of representatives for γ. The lemma follows easily from the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & M\end{pmatrix}\begin{pmatrix} 1 & N^2ST \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & T \\ 0 & \frac{R^2}{R_0}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}. \tag{4.8}$$

In the next lemma we put

$$\mathfrak{E}(w,z) := \mathfrak{E}_{2n}^{k,\nu}(w,z,M,RN,\varphi,\chi,s)\,; \tag{4.9}$$

in fact, the only property of this function, which we need at the moment, is that it can be considered as an element of $C^\infty M_n^l(\Gamma_0(M),\varphi)\otimes C^\infty M_n^l(\Gamma_0(M),\varphi)$.

We denote by $\mathfrak{K}$ the operator $\left(f\middle|\mathfrak{K}\right)(z) = f(-\bar{z})$; this operator satisfies the commutation law

$$\overline{(f|\mathfrak{K})}\Big|_l\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \overline{f\Big|_l\begin{pmatrix} a & -b \\ -c & d\end{pmatrix}\,\Big|\,\mathfrak{K}}\,; \tag{4.10}$$

in particular, $f \longmapsto \overline{f|\mathfrak{K}}$ induces an isomorphism (over $\mathbb{R}$) between $C^\infty M_n^l(\Gamma_0(M),\varphi)$ and $C^\infty M_n^l(\Gamma_0(M),\bar{\varphi})$.

LEMMA 4.2.— *For f_0, M, $\mathfrak{E}(z,w)$ etc. as above and any $h\in S_n^l(\Gamma^0(N^2S),\bar{\varphi})$ we have*

(4.11)
$$\left\langle\left\langle f_0\Big|_l\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \mathfrak{E}(*,*)|^z\mathfrak{K}\Big|_l^z\begin{pmatrix} 1 & 0 \\ 0 & M\end{pmatrix}\Big|_l^w\begin{pmatrix} 1 & 0 \\ 0 & M\end{pmatrix}\right\rangle^w_{\Gamma^0(M)}, h\right\rangle^z_{\Gamma^0(M)}$$
$$= \left(\frac{R^2}{R_0}\right)^{n(n+1)-nl}\left\langle\left\langle f_0\Big|_l\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \mathfrak{F}(*,*)|^z\mathfrak{K}\right\rangle^w_{\Gamma^0(N^2S)}, h\right\rangle^z_{\Gamma^0(N^2S)}$$

with

$$\mathfrak{F}(z,w) = \mathfrak{E}(z,w)|^z U\left(\frac{R^2}{R_0}\right)\Big|^w U\left(\frac{R^2}{R_0}\right)\Big|_l^z\begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}\Big|_l^w\begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}.$$

Proof.— We may apply the same procedure as in the proof of Lemma 4.1.

These computations imply that for any $h \in S_n^l(\Gamma^0(N^2S), \bar{\varphi})$ we obtain (using the lemmas above) a "level N^2S"-identity from (4.4):

$$\left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) \Big|^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)} \tag{4.12}$$

$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\chi}\bar{\varphi})} (RN)^{n(2k+\nu+2s-n-1)} (N^2S)^{\frac{n(n+1)-nl}{2}} (-1)^{nl} \chi(-1)^n$$

$$\alpha\left(\frac{SR^2}{R_0^2}\right) D^{\left(M, \frac{S}{R_0}\right)}(f_0, k+2s-n, \bar{\chi}) \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)}.$$

In the final part of this section we want to make the identity (4.12) somewhat more flexible. Let R_1, R_2 be natural numbers with $R_i | S^\infty$; the R_i will be specified later on.

Using standard properties of the Petersson scalar product and the fact that f_0 is an eigenfunction of $U(R_1)$ we obtain

$$\alpha(R_1) \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \right\rangle_{\Gamma^0(N^2S)} \tag{4.13}$$

$$= \left\langle f_0 | U(R_1) \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \right\rangle_{\Gamma^0(N^2S)}$$

$$= \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \Big|^w U_{N^2S}(R_1) \right\rangle_{\Gamma^0(N^2S)}.$$

On the other hand, writing just c for the factor occuring in (4.12) we get

$$\alpha(R_2)\, c \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)} \tag{4.14}$$

$$= c \left\langle f_0 | U(R_2) \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)}$$

$$= c \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \Big| U_{N^2S}(R_2) h \right\rangle_{\Gamma^0(N^2S)}$$

$$= c \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h | U_{N^2S}(R_2)^* \right\rangle_{\Gamma^0(N^2S)}$$

$$= \left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*)\big|^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \big| U_{N^2S}(R_2)^* \right\rangle^z_{\Gamma^0(N^2S)}$$

$$= \left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*)\big|^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)} \Big|^z U_{N^2S}(R_2), h \right\rangle^z_{\Gamma^0(N^2S)}$$

$$= \left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*)\big|^z U_{N^2S}(R_2)\big|^z \mathfrak{K} \right\rangle_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)}.$$

Of course $U_{N^2S}(R_2)^*$ denotes the adjoint operator of $U_{N^2S}(R_2)$.

Summarizing these results, we get for $R_1|S^\infty, R_2|S^\infty$ and any $h \in S^l_n(\Gamma^0(N^2S), \bar\varphi)$ the identity

$$\text{(4.15)} \quad \left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{g}(*,*)\big|^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)}$$

$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \overline{\chi\varphi})} (RN)^{n(2k+\nu+2s-n-1)} (N^2S)^{\frac{n(n+1)-nl}{2}}$$

$$\times \chi(-1)^n (-1)^{nl} \alpha\left(\frac{SR^2}{R_0^2}\right) \alpha(R_1)\, \alpha(R_2)$$

$$\times D^{\left(M, \frac{S}{R_0}\right)}(f_0, k+2s-n, \bar\chi) \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)}$$

with

$$\text{(4.16)} \quad \mathfrak{g}(w,z) = \mathfrak{g}^{k,\nu}\left(w, z, R^2N^2\frac{S}{R_0}, RN, \varphi, \chi, s, R_1, R_2\right)$$

$$:= \mathfrak{E}^{k,\nu}_{2n}\left(w, z, R^2N^2\frac{S}{R_0}, RN, \varphi, \chi, s\right) \Big|^z U\left(\frac{R^2}{R_0}R_1\right) \Big|^w$$

$$U\left(\frac{R^2}{R_0}R_2\right) \Big|^z_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \Big|^w_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}.$$

We finally mention that we can also make the same manipulation starting from (3.24′) instead of (3.24) and produce a result (4.15′), (4.16′), which we do not want to write down explicitely.

5. Fourier expansion and regularity.

We first go back to the general setting of section 2 and study the behavior of $\mathbb{F}^k_m(Z, M, \psi, s)$ for

$$(5.1)\quad m = 2n,\ k = n + t,\ t \geqslant 1,\quad \psi \text{ a Dirichlet character mod } M,\ M > 1 \text{ with } \psi(-1) = (-1)^k$$

at two values of s, namely

$$(5.2)\qquad s_0 := 0 \text{ and } s_1 := \frac{m+1}{2} - k = \frac{1}{2} - t\,.$$

For the applications we have in mind it is however more convenient to consider the normalized Eisenstein series

$$(5.3)\qquad \mathbf{F}^k_m(Z, M, \psi, s) := \mathfrak{L}(k + 2s, \psi) \times \mathbb{F}^k_m(Z, M, \psi, s).$$

We remark that $\mathfrak{L}(k + 2s, \psi)$ is of order zero at $s = s_0$ (at $s = s_1$ however the situation is more complicated).

We describe the Fourier expansion of $\mathbf{F}^k_m(Z, M, \psi, s)$ in some detail:

$$(5.4)\qquad \mathbf{F}^k_m(Z, M, \psi, s) = \sum_{\mathfrak{T} \in \Lambda_m} a^k_m(Y, M, \psi, \mathfrak{T}, s)\, e^{2\pi i\, tr(\mathfrak{T}X)}$$

with

$$(5.5)\qquad a^k_m(Y, M, \psi, \mathfrak{T}, s) = \frac{(-1)^{kn} 2^m \pi^{m(k+2s)}}{\Gamma_m(k+s)\,\Gamma_m(s)} \det(Y)^s\, h^{(m)}_{k+s,s}(Y, \mathfrak{T})\, \mathbf{Sing}_m(\mathfrak{T}, k + 2s, \psi).$$

Here $h^{(m)}_{\alpha,\beta}(Y, \mathfrak{T})$ is (for $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$ sufficiently large) the confluent hypergeometric function defined as in [18, §18] and

$$(5.6)\qquad \mathbf{Sing}_m(\mathfrak{T}, s, \psi) = \mathfrak{L}(s, \psi) \times \mathrm{Sing}_m(\mathfrak{T}, s, \psi)$$

is the normalized "singular series" with

$$(5.7)\qquad \mathrm{Sing}_m(\mathfrak{T}, s, \psi) = \sum_{B \in \mathbb{Q}^{(m,m)}_{\mathrm{sym}} \bmod 1} \psi(\nu(B))\, \nu(B)^{-s} e^{2\pi i\, \mathrm{tr}(\mathfrak{T}B)}\,.$$

For $\mathfrak{T} \in \Lambda_m^*$ (= set of the $\mathfrak{T} \in \Lambda_m$ of maximal rank) the main properties of the normalized singular series are summarized below (for proofs see [5], [28], [11], [17]); for $\mathfrak{T} \in \Lambda_m^*$ we denote by $\epsilon_{\mathfrak{T}}$ the quadratic character

$$(5.8) \qquad \epsilon_{\mathfrak{T}}(*) := \left(\frac{(-1)^n \det(2\mathfrak{T})}{*}\right).$$

PROPOSITION 5.1.— *For every $\mathfrak{T} \in \Lambda_m^*$ and every prime q there is a polynomial $B_q^m(x,\mathfrak{T}) \in \mathbb{Z}[x]$ of degree $\leqslant m-1$ with the following properties:*

a) *$B_q^m(x,\mathfrak{T})$ depends only on $\mathfrak{T}$ mod q.*

b) *degree $(B_q^m(x,\mathfrak{T}) \leqslant g$, where $m-g =$ rank of $\mathfrak{T}$ over $\mathbb{F}_q$.*

c) *$B_q^m(x,\mathfrak{T}) = 1$ if $q \nmid det(2\mathfrak{T})$.*

d) *$\mathrm{Sing}_m^\wedge(\mathfrak{T},s,\psi) := L(s-n,\epsilon_{\mathfrak{T}}\psi) \times \prod_q B_q^m(\psi(q)q^{-s},\mathfrak{T})$ satisfies the important relation*

$$(5.9) \qquad \mathbf{Sing}_m(\mathfrak{T},s,\psi) = \sum_{G\in GL(m,\mathbb{Z})\backslash \mathbb{D}(\mathfrak{T})} \psi^2(\det G)|\det G|^{m+1-2s}\mathrm{Sing}_m^\wedge(\mathfrak{T}[G^{-1}],s,\psi)$$

where $\mathbb{D}(\mathfrak{T})$ is the "set of divisors of $\mathfrak{T}$":

$$\mathbb{D}(\mathfrak{T}) = \left\{G \in M_m(\mathbb{Z})^* \mid \mathfrak{T}[G^{-1}] \in \Lambda_m\right\}.$$

We note that (5.9) is a finite sum.

Remark 5.1.— The properties listed above imply that for any $b \big| \det(2\mathfrak{T})$ there exists an integer $d(b,\mathfrak{T})$ such that

$$(5.10) \qquad \prod_q B_q^m(q^{-s},\mathfrak{T}) = \sum_{\substack{b|\det(2\mathfrak{T})\\ b>0}} \psi(b)b^{-s}d(b,\mathfrak{T}).$$

COROLLARY 5.1.— *For all $\mathfrak{T} \in \Lambda_m^*$ and all $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 0$, we have*

$$(5.11) \qquad \mathbf{Sing}_m(\mathfrak{T},s,\psi) = \sum_{G\in GL(m,\mathbb{Z})\backslash \mathbb{D}(\mathfrak{T})} \ \sum_{\substack{b|\det(2\mathfrak{T}[G^{-1}])\\ b>0}} \psi^2(\det G)\big|\det G\big|^{m+1-2s} L(s-n,\epsilon_{\mathfrak{T}[G^{-1}]}\psi)\, b^{-s}d(b,\mathfrak{T}).$$

In particular, $\mathbf{Sing}_m(\mathfrak{T}, s, \psi)$ *has an meromorphic continuation to the whole complex plane and* $\mathbf{Sing}_m(\mathfrak{T}, s, \psi)$ *is regular at* $s = n + t$ *unless* $t = 1$ *and* $\epsilon_{\mathfrak{T}}\,\psi$ *has trivial conductor.*

At $s = k + 2s_1 = n + 1 - t$ *it is always regular.*

A theorem of Shimura [27, Thm. 4.2] implies that for $\mathfrak{T} \in \Lambda_m^*$ with signature (p, q), $p + q = m$ we have

$$(5.12) \qquad h_{\alpha,\beta}^m(Y, \mathfrak{T}) = \Gamma_p\left(\beta - \frac{m-p}{2}\right)\Gamma_q\left(\alpha - \frac{m-q}{2}\right) \times (\text{entire function of}(\alpha, \beta) \in \mathbb{C}^2)\,.$$

Therefore the "archimedian part"

$$(*) \qquad \frac{h_{k+s,s}^{(m)}(Y, \mathfrak{T})}{\Gamma_m(k+s)\,\Gamma_m(s)}$$

of the Fourier coefficient (5.5) with $\mathfrak{T} \in \Lambda_m^*$ has a zero of order $\geqslant \frac{m}{2} - \left[\frac{p}{2}\right]$ at $s = s_0$ and has a zero of order $\geqslant \frac{m}{2} - \left[\frac{p+1}{2}\right]$ at $s = s_1$.

This statement (combined with Corollary 5.1) implies

PROPOSITION 5.2. — *Let* $\mathfrak{T} \in \Lambda_m^*$; *then*

a) *For all* $t \geqslant 1$ $a_m^k(Y, M, \psi, \mathfrak{T}, s)$ *is regular at* $s = 0$.

For $t \geqslant 2$ *it is equal to zero unless* $\mathfrak{T} > 0$.

For $t = 1$ *the same is true at least if* $(\psi^2)_0 \neq 1$.

Moreover we have for all $\mathfrak{T} > 0$ *and all* $t \geqslant 1$ *the explicit formula*

$$(5.13) \quad a_m^k(Y, M, \psi, \mathfrak{T}, 0) = A_m^k(\det 2\mathfrak{T})^{k-\frac{m+1}{2}}\,\mathbf{Sing}_m(\mathfrak{T}, k, \psi)\, e^{-2\pi\,\mathrm{tr}(\mathfrak{T}Y)}$$

with

$$(5.16) \qquad A_m^k = (-1)^{nk}\frac{2^m}{\Gamma_m(k)}\,\pi^{mk}.$$

b) *At* $s = s_1$ *the function* $a_m^k(Y, M, \psi, \mathfrak{T}, s)$ *is regular and equal to zero unless* $\mathfrak{T} > 0$. *For* $\mathfrak{T} > 0$ *we have the explicit formula*

$$(5.15) \qquad a_m^k(Y, M, \psi, \mathfrak{T}, s_1) = B_m^k\,\mathbf{Sing}_m(\mathfrak{T}, n+1-t, \psi)\, e^{2\pi\,\mathrm{tr}(\mathfrak{T}, Y)}$$

with

$$(5.16) \qquad B_m^k = (-1)^{nk}\frac{2^{n+mt}}{\Gamma_m(n+\frac{1}{2})}\,\pi^{n+2n^2}.$$

The explicit formulas (5.13)-(5.16) follow from [27, 4.35.K]; for t large enough (5.13) and (5.14) are also in [18, §18]. The proof of Proposition 5.2 follows in a straightforward way from (*) and standard properties of Dirichlet L-series. We mention two crucial points:

i) To prove the regularity of $a_m^k(Y, M, \psi, s)$ at s_0 for the case $t = 1$, one has to observe that for $\mathfrak{T} > 0$ the character $\epsilon_{\mathfrak{T}}\, \psi$ is odd.

ii) If $\mathfrak{T}$ is of signature $(m-1, 1)$, the archimedian part (*) does not necessarily vanish at $s = s_1$. In that case however $\mathbf{Sing}_m(\mathfrak{T}, s, \psi)$ is zero at $s = n + 1 - t$ because of $L(1-t, \epsilon_{\mathfrak{T}}\, \psi) = 0$.

To see this one has to observe that for this $\mathfrak{T}$ the signature of $\epsilon_{\mathfrak{T}}\, \psi$ is $t + 1$ mod 2. For $t = 1$ one has to take into acount that $\epsilon_{\mathfrak{T}}\, \psi$ is not the trivial character.

For $k = n + t$ there are no singular modular forms of degree $2n$ and weight k, therefore the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on such modular forms can be read off from the action of σ on the non-singular Fourier coefficients.

COROLLARY 5.2. — *Let $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

a) *If $\mathbb{F}_m^k(Z, M, \psi, 0)$ is a holomorphic Siegel modular form, then*

$$\left(\frac{1}{A_m^k (\pi i)^{k-n}\, i^n}\, \mathbb{F}_m^k(Z, M, \psi, 0)\right)^{\sigma} = \frac{1}{\psi^{\sigma}(\eta)}\, \frac{1}{A_m^k (\pi i)^{k-n}\, i^n}\, \mathbb{F}_m^k(Z, M, \psi^{\sigma}, 0) \tag{5.17}$$

where $\eta \in (\mathbb{Z}/M\mathbb{Z})^*$ is defined by

$$\psi(e^{2\pi i/M}) = e^{2\pi i \eta/M}. \tag{5.18}$$

b) *If $\mathbb{F}_m^k(Z, M, \psi, s_1)$ is a holomorphic Siegel modular form, then*

$$\left(\frac{1}{B_m^k}\mathbb{F}_m^k(Z, M, \psi, s_0)\right)^{\sigma} = \frac{1}{B_m^k}\mathbb{F}_m^k(Z, M, \psi^{\sigma}, s_0). \tag{5.19}$$

This follows easily from Proposition 5.2; to prove a.) one must use the Galois properties of quotients of type

$$\frac{L(k-n, \epsilon_{\mathfrak{T}}\, \psi)}{(\pi i)^{k-n} G((\epsilon_{\mathfrak{T}}\, \psi)_0)} \tag{5.20}$$

and the fact that for $\mathfrak{T} \in \Lambda_m^{\times}$ we have $G((\epsilon_{\mathfrak{T}})_0) \in i^n \det(2\mathfrak{T})^{\frac{1}{2}} \times \mathbb{Q}^{\times}$.

Remark 5.2.— The assumption that the Eisenstein series defines a holomorphic Siegel modular form is not really necessary in the statement above: Essentially the corollary is a statement about the Galois behaviour of the $\mathfrak{T}$-Fourier coefficients of $\mathbb{F}_m^k(Z, M, \psi, s)$ at s_0 and s_1 for $\mathfrak{T} \in \Lambda_m^+$.

6. Gauß sums and the twisting process.

To study the effect of the twisting process introduced in Section 2 on Fourier expansions we first explain the properties of certain Gauß sums.

For any $T \in \mathbb{Z}^{(n,n)}$ and any Dirichlet character $\chi \bmod L$ we define

$$G_n(T, L, \chi) := \sum_{X \in \mathbb{Z} \bmod L} \chi(\det X) e^{2\pi i \operatorname{tr}(\frac{1}{L} T' X)}. \tag{6.1}$$

Special cases of such Gauß sums were formerly studied by Christian [8], [9] (using analytic tools).

PROPOSITION 6.1.— *If χ is primitive mod L we get*

$$G_n(T, L, \chi) = L^{\frac{1}{2}n(n-1)} \bar{\chi}(\det T)\, G(\chi)^n. \tag{6.2}$$

Proof (Sketch).— We may assume that T is of type $T = \begin{pmatrix} \tilde{T} & 0 \\ 0 & t \end{pmatrix}$, $\tilde{T} \in \mathbb{Z}^{(n-1,n-1)}$.

We expand $\det(X)$ as

$$\det(X) = \sum_{j=1}^{n} (-1)^{j+n} x_{jn} \det(X_{jn}) \tag{6.3}$$

where X_{jn} is the $(n-1)$-rowed matrix obtained from X by omitting the j-th row and the n-th column.

Because of

$$\operatorname{tr}\left(\frac{1}{L} T' X\right) = \frac{1}{L}\, t\, x_{nn} + \frac{1}{L} \operatorname{tr}(\tilde{T} X_{nn}) \tag{6.4}$$

we may first consider the following subsum (where x_{nn} and the first $n-1$ columns of X are fixed)

$$\sum_{(x_{1n}, \ldots, x_{n-1,n}) \in \mathbb{Z}^{n-1} \bmod L} \chi\left(\sum_{j=1}^{n} (-1)^{j+n} x_{jn} \det(X_{in}) \right). \tag{6.5}$$

Standard properties of Gauß sums attached to primitive characters imply that (6.5) is zero unless

$$(6.6) \qquad \det(X_{1n}) \equiv \ldots \equiv \det(X_{n-1,n}) \equiv 0 \bmod L\,.$$

If (6.6) is satisfied, (6.5) equals

$$(6.7) \qquad L^{n-1}\chi(x_{nn})\,\chi(\det X_{nn}).$$

Moreover, if $\det(X_{nn})$ *is coprime to* L, (6.6) implies that the first (n-1) entries of the last row of X must be zero mod L.

Therefore we get

$$(6.8) \qquad G_n(T,L,\chi) = L^{n-1}G_1(t_n,L,\chi)\,G_{n-1}(\tilde{T},L,\chi).$$

Using $G_1(t_n,L,\chi) = \bar{\chi}(t_n)\,G(\chi)$ we obtain Proposition 6.1 from (6.8) by induction.

Gauß sums like (6.1) arise naturally by the twisting process of Section 2: Starting from any periodic C^∞-function F on $\mathbb{H}_{2n}$ ($\mathfrak{Z} = \mathfrak{X} + i\mathfrak{Y}$)

$$(6.9) \qquad F(\mathfrak{Z}) = \sum_{\mathfrak{T}\,\in\,\Lambda_{2n}} a(\mathfrak{T},\mathfrak{Y})e^{2\pi i\,\mathrm{tr}(\mathfrak{T}\,\mathfrak{X})}$$

we define (with χ as above)

$$(6.10) \qquad F^{(\chi)}(\mathfrak{Z}) := \sum_{X\,\in\,\mathbb{Z}^{(n,n)} \bmod L} \chi(\det X)\,F\big|_l \begin{pmatrix} 1_{2n} & S(X) \\ 0_{2n} & 1_{2n} \end{pmatrix}$$

$$\text{where } S(X) = \frac{1}{L}\begin{pmatrix} 0_n & X \\ X^t & 0_n \end{pmatrix}.$$

It is obvious that

$$(6.11) \quad F^{(\chi)}(\mathfrak{Z}) = \sum_{\mathfrak{T}\,\in\,\Lambda_{2n}} G_n(2T_2,L,\chi)\,a(\mathfrak{T},\mathfrak{Y})\,e^{2\pi i\,\mathrm{tr}(\mathfrak{T}\,\mathfrak{X})}$$

$$\text{where } \mathfrak{T} = \begin{pmatrix} * & T_2 \\ T_2^t & * \end{pmatrix}.$$

Suppose now that in addition F is a holomorphic function of $\mathfrak{Z}$, so we can define F^σ, $\sigma \in \mathrm{Aut}(\mathbb{C})$ by acting on the Fourier coefficients of F.

If χ is a Dirichlet character *mod* L, then

$$(6.12) \qquad G_n(2T_2,L,\chi)^\sigma = \frac{1}{\chi^\sigma(\eta^n)}G_n(2T_2,L,\chi^\sigma)$$

where η is an integer *mod* L with $\sigma\left(e^{2\pi\, i/L}\right) = e^{2\pi i\eta/L}$.

This implies

$$(F^{(\chi)})^{\sigma} = \frac{1}{\chi^{\sigma}(\eta^{n})} F^{(\chi^{\sigma})}. \tag{6.13}$$

We should also mention that the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ is also compatible with the differential operators $\overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,k}$:

$$\left((2\pi i)^{-n\nu}\, \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,k} F\right)^{\sigma} = (2\pi i)^{-n\nu}\, \overset{\circ}{\mathfrak{D}}{}^{\nu}_{n,k}(F^{\sigma}). \tag{6.14}$$

Appendix: The trivial character.

To compute the exponential sums, which occur in the modified twisting process, we start with a general remark (p *a fixed prime*).

Let φ and ψ denote the characteristic functions on $GL(n,\mathbb{Q})$ of the subsets of those A with $\det A \in \mathbb{Z}_p$ and $\det A \in \mathbb{Z}_p^{\times}$ (respectively).

Obviously, for $A \in GL(n,\mathbb{Q})$

$$\varphi(A) = \sum_{X} \psi(X^{-t}A), \tag{6.15}$$

where X runs over $GL(n,\mathbb{Z})\backslash\{X \in \mathbb{Z}^{(n,n)} \mid \det X = p - \mathit{power}\}$.

By considering φ and ψ as *$GL(n,\mathbb{Z})$-left-invariant functions on* $GL(n,\mathbb{Q})$, we may describe the relation (6.15) in terms of *GL_n-Hecke operators*.

Using Tamagawa's rationality theorem for the standard *GL_n -Hecke series* we also have the relation

$$\psi(A) = \sum_{i=0}^{n} (-1)^{i} p^{\frac{i(i-1)}{2}} \sum_{j} \varphi(g_{ij}^{-t}A) \tag{6.16}$$

(with using the same notations as in Section 2).

Now we consider, for non-singular $T \in \mathbb{Z}^{(n,n)}$ the exponential sum

$$\sum_{i=0}^{n} (-1)^{i} p^{\frac{i(i-1)}{2}} p^{-in} \sum_{j} \sum_{X \in \mathbb{Z}^{(n,n)} g_{ij}^{-t}/\mathbb{Z}^{(n,n)}} e^{2\pi i\, \mathrm{tr}(TX)}. \tag{6.17}$$

The exponential sum over X is zero unless $g_{ij}^{-t}T \in \mathbb{Z}^{(n,n)}$, in which case its value is p^{in}, hence (6.17) equals

$$\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}} \sum_j \varphi(g_{ij}^{-t}T) = \psi(T). \tag{6.18}$$

If T is singular, then (6.17) is zero, so in any case (6.17) picks out those T with $\det(T)$ *coprime* to p.

7. Fourier expansions II.

We now return to the notations of Section 4 and make two additional assumptions; the first one is

(7.1) $\quad N$ *is coprime to* S *(and therefore coprime to* R*);*

this implies that the Dirichlet character $\chi \bmod RN$ may be written as a product

$$\chi = \chi^\circ \cdot \chi_1, \tag{7.2}$$

where

χ° *is a Dirichlet character mod* N *and*

χ_1 *is a Dirichlet character mod* R.

Therefore

$$G_n(T, RN, \chi) = \chi^\circ(R)^n \chi_1(N)^n \cdot G_n(T, N, \chi^\circ) \cdot G_n(T, R, \chi_1). \tag{7.3}$$

The second assumption is

(7.4) $\quad \chi_1$ *is primitive mod* R.

This allows us to compute the Gauß sum $G_n(T, R, \chi_1)$ explicitely as in Proposition 6.1. As a new ingredient we introduce a natural number L with $L|S^\infty$. Starting now from a primitive Dirichlet character χ_1 with conductor $c(\chi_1) = R|L$, we want to compute the Fourier expansion of $\mathfrak{g}(w, z)$ in (4.16) (and of some variant) as explicitely as possible with

$$R_1 = R_2 = \left(\frac{L}{R}\right)^2 \cdot R_0\,. \tag{7.5}$$

Of course we are mainly interested in $s = s_0$ and $s = s_1$.

We first observe that for an arbitrary function on $\mathbb{H}_{2n}$ of the form

$$\mathfrak{F}(\mathfrak{Z}) = \sum_{\mathfrak{T} \in \Lambda_{2n}} a(\mathfrak{T}, \mathfrak{Y})\, e^{2\pi i \operatorname{tr}(\mathfrak{T}\mathfrak{X})} \tag{7.6}$$

with

$$\mathfrak{F}^{(\chi)} = \text{twist of } \mathfrak{F} \text{ in the sense of (2.18)} \tag{7.7}$$

the Fourier expansion of

$$\mathfrak{F}^{(\chi)}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} |^{z}U(L^2)\, |^{w}U(L^2) \,\Big|_l^z \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \Big|_l^w \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \tag{7.8}$$

equals

$$R^{\frac{n(n-1)}{2}} G(\chi_1)^n (N^2S)^{-ln} \chi^\circ(R)^n\, \chi_1(N)^n \tag{7.9}$$

$$\sum_{T_1 \in \Lambda_n} \sum_{T_4 \in \Lambda_n} \sum_{2T_2 \in \mathbb{Z}^{(n,n)}} G_n(2T_2, N, \chi^\circ) \times \bar{\chi}_1(\det(2T_2)) a\left(\begin{pmatrix} L^2T_1 & T_2 \\ T_2 & L^2T_4 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} (L^2N^2S)^{-1}y_1 & 0 \\ 0 & (L^2N^2S)^{-1}y_4 \end{pmatrix}\right) e^{\frac{2\pi i}{N^2S} \operatorname{tr}(T_1x + T_4u)}.$$

In particular, if $c(\chi_1) = R \neq 1$, only those $\mathfrak{T} = \begin{pmatrix} L^2T_1 & T_2 \\ T_2' & L^2T_4 \end{pmatrix}$ contribute, which are of maximal rank, i. e. $\mathfrak{T} \in \Lambda_{2n}^*$.

The Fourier expansion of

$$\mathfrak{L}(k+2s, \varphi\chi)\, \mathfrak{E}_{2n}^{k,\nu}(w, z, R^2N^2\frac{S}{R_0}, RN, \varphi, \chi, s) \tag{7.10}$$

$$|^{z}U(L^2)\, |^{w}U(L^2) \,\Big|_l^z \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \Big|_l^w \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}$$

at $s = 0$ in the case $c(\chi_1) \neq 1$ can now easily be computed to be

$$A_{2n}^k (2\pi i)^{n\nu} R^{\frac{n(n-1)}{2}} (N^2S)^{-ln}\, G(\chi_1)^n\, \chi^\circ(R)^n\, \chi_1(N)^n \sum_{T_1 \in \Lambda_n^+} \sum_{T_4 \in \Lambda_n^+} \tag{7.11}$$

$$\Bigg(\sum_{2T_2 \in \mathbb{Z}^{(n,n)}:\ \mathfrak{T} = \begin{pmatrix} L^2T_1 & T_2 \\ T_2' & L^2T_4 \end{pmatrix} \in \Lambda_{2n}^+} \mathfrak{P}_{n,k}^{\nu}(\mathfrak{T})\, G_n(2T_2, N, \chi^\circ)\, \bar{\chi}_1(\det(2T_2)) \times$$

$$\sum_{G \in GL(2n,\mathbb{Z})\backslash \mathbb{D}(\mathfrak{T})} (\varphi\chi)^2(\det G)\det(2\,\mathfrak{T}[G^{-1}])^{k-\frac{2n+1}{2}}\, L(k-n, \epsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi)$$

$$\times \sum_{\substack{b \mid \det \mathfrak{T}[G^{-1}] \\ b>0}} (\varphi\chi)(b)\, b^{-k}\, d(b,\, \mathfrak{T}[G^{-1}])\Bigg) e^{\frac{2\pi i}{N^2 S}\operatorname{tr}(T_1 z + T_4 w)}.$$

This is true at least for $k > n$, where for $k = n+1$ we impose the extra condition

$$(7.12) \qquad (\varphi\chi)_0^2 \neq 1.$$

At $s = s_1$ we look at the Fourier expansion of

$$(7.13) \qquad \mathfrak{L}(k+2s,\varphi\chi)\ \overset{0}{\mathfrak{D}}{}^{\nu}_{n,k}\left(\mathbb{F}^k_{2n}\left(-, R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi)}\right)$$

$$\left|^z U(L^2)\right.\left|^w U(L^2)\right.\left|^z_l \begin{pmatrix} 1 & 0 \\ 0 & N^2 S\end{pmatrix}\right.\left|^w_l \begin{pmatrix} 1 & 0 \\ 0 & N^2 S\end{pmatrix}\right..$$

It is important to note that we apply the differential operator $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,k}$ to an Eisenstein series of type $\mathbb{F}^k_{2n}$ (and not $\overset{0}{\mathfrak{D}}{}^{\nu}_{n,k+s}$ to $\mathbb{F}^{\wedge}_{2n}$).

The Fourier expansion of (7.13) at $s = s_1$ is

$$(7.14) \quad B^k_{2n}(2\pi i)^{n\nu} R^{\frac{n(n-1)}{2}}(N^2S)^{-ln}\, G(\chi_1)^n\, \chi^\circ(R)^n\, \chi_1(N)^n \sum_{T_1 \in \Lambda_n^+}\ \sum_{T_4 \in \Lambda_n^+}$$

$$\Bigg(\sum_{2T_2 \in \mathbb{Z}^{(n,n)}:\ \mathfrak{T} = \begin{pmatrix} L^2T_1 & T_2 \\ T_2' & L^2T_4\end{pmatrix} \in \Lambda_{2n}^+} \mathfrak{P}^{\nu}_{n,k}(\mathfrak{T})\, G_n(2T_2, N, \chi^\circ)\, \bar{\chi}_1(\det 2T_2)$$

$$\times \sum_{G \in GL(2n,\mathbb{Z})\backslash \mathbb{D}(\mathfrak{T})} (\varphi\chi)^2(\det G)\, |\det G|^{2t-1} L(1-t, \epsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi)$$

$$\times \sum_{\substack{b \mid \det(2\mathfrak{T}[G^{-1}]) \\ b>0}} (\varphi\chi)(b)\, b^{-(n+1)+t} d(b,\, \mathfrak{T}[G^{-1}])\Bigg) e^{\frac{2\pi i}{N^2 S}\operatorname{tr}(T_1 z + T_4 w)}.$$

We describe now the analogues of these expansions for the modified twisting process under the assumptions N coprime to S, $R = R'\cdot p$, *R' coprime to*

p, $R|S^\infty$

$$\begin{aligned} &\chi' \text{ a Dirichlet character mod } R'\cdot N\\ \chi' = \chi^\circ\cdot\chi_1' &\text{ where } \chi^\circ \text{ is a Dirichlet character mod } N\\ &\chi_1' \text{ a primitive Dirichlet character mod } R'.\end{aligned}$$

With some number L such that $R|L$ we look at the Fourier expansions of

$$(7.10')\quad \mathfrak{L}(k+2s,\varphi\chi')\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}\,p^{-in}\,\mathfrak{E}^{k,\nu}(w,z,(R'p)^2N^2\frac{S}{R_0'},$$
$$R'N,\varphi,\chi',s,i)\,|^zU(L^2)\,|^wU(L^2)\,\Big|_l^z\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\Big|_l^w\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}$$

at $s=0$, and

$$(7.13')\quad \mathfrak{L}(k+2s,\varphi\chi')\,\overset{0}{\mathfrak{D}}_{k,\nu}\Bigg(\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}p^{-in}\sum_j$$
$$\mathbb{F}_{2n}^k\left(--,(R'p)^2N^2\frac{s}{R_0'},\varphi\right)^{(\chi')}\Bigg|\begin{pmatrix}1_{2n}&S(g_{ij}^*)\\0_{2n}&1_{2n}\end{pmatrix}\Bigg)_{z_2=0}$$
$$|^zU(L^2)\,|^wU(L^2)\,\Big|_l^z\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\Big|_l^w\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}$$

at $s=s_1$.

The Fourier expansion of (7.10′) is given by

(7.11′)
$$A_{2n}^k(2\pi i)^{n\nu}(R')^{\frac{n(n-1)}{2}}(N^2S)^{-ln}G(\chi_1')^n\chi^\circ(R')^n\chi_1'(N)^n\sum_{T_1\in\Lambda_n^+}\sum_{T_4\in\Lambda_n^+}$$
$$\sum_{\substack{2T_2\in\mathbb{Z}^{(n,n)}:\ \mathfrak{T}=\begin{pmatrix}LT_1&T_2\\T_2&LT_4\end{pmatrix}\in\Lambda_{2n}^+\\ p\nmid\det(2\mathfrak{T})}}\mathfrak{P}_{n,k}^{\nu}(\mathfrak{T})\,G_n(2T_2,N,\chi^\circ)\bar{\chi}_1'(\det 2T_2)$$
$$\times\sum_{G\in GL(2n,\mathbb{Z})\backslash\mathbb{D}(\mathfrak{T})}(\varphi\chi')^2(\det G)\det(2\mathfrak{T}[G^{-1}])^{k-\frac{2n+1}{2}}L(k-n,\epsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi')$$
$$\times\sum_{\substack{b\,|\,\det\mathfrak{T}[G^{-1}]\\ b>0}}(\varphi\chi')(b)\,b^{-k}\,d(g,\mathfrak{T}[G^{-1}])e^{\frac{2\pi i}{N^2S}\operatorname{tr}(T_1z+T_4w)}$$

where for $k = n+1$ we impose the extra condition

$$(\varphi\chi')_0 \neq 1\,. \tag{7.15}$$

The corresponding expansion for (7.13′) is

(7.14′)

$$B_{2n}^{k}(2\pi i)^{n\nu}(R')^{\frac{n(n-1)}{2}}(N^2S)^{-ln}\,G(\chi_1')^n\,\chi^{\circ}(R')^n\,\chi_1'(N)^n \sum_{T_1\in\Lambda_n^+}\ \sum_{T_4\in\Lambda_n^+}$$

$$\sum_{\substack{2T_2\in\mathbb{Z}^{(n,n)}:\ \mathfrak{T}=\begin{pmatrix} LT_1 & T_2\\ T_2 & LT_4\end{pmatrix}\in\Lambda_{2n}^+\\ p\nmid\det(2\mathfrak{T})}} \mathfrak{P}_{n,k}^{\nu}(\mathfrak{T})\,G_n(2T_2,N,\chi^{\circ})\,\bar{\chi}_1'(\det 2T_2)$$

$$\times \sum_{G\in GL(2n,\mathbb{Z})\backslash\mathbb{D}(\mathfrak{T})} (\varphi\chi')^2\,(\det G)\,|\det G|^{2t-1}\,L(1-t,\epsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi')$$

$$\times \sum_{\substack{b\,|\,\det\mathfrak{T}[G^{-1}]\\ b>0}} (\varphi\chi')(b)\,b^{-(n+1)+t}\,d(b,\mathfrak{T}[G^{-1}])\,e^{\frac{2\pi i}{N^2S}\operatorname{tr}(T_1z+T_4w)}\,.$$

We summarize the results of this section (using all the results obtained so far in this paper):

PROPOSITION 7.1. — *Using the basic notations*

$$l = k+\nu,\ k = n+t,\ t\geqslant 1,\ N\in\mathbb{N}$$

$$S\in\mathbb{N}\ \textit{squarefree, coprime to } N$$

we assume that $f_0\in S_n^l(\Gamma_0(NS),\bar{\varphi})$ is an eigenform of the operators $U(q)$ for $q|S$ and of the Hecke algebra $\bigotimes'_{q\nmid NS}\mathfrak{H}^{\circ}_{NS,q}\ \bigotimes_{q|S}\mathfrak{H}^{\circ}_{NS,q}$.

We fix numbers R and L with $R|L|S^{\infty}$ (and in the modified case a prime p such that $R = R'\cdot p$, R' coprime to p) and a Dirichlet character χ with

$$\chi(-1) = (-1)^k\varphi(-1)$$

$$\chi = \chi^{\circ}\cdot\chi_1,\quad \chi^{\circ}\ \textit{a Dirichlet character mod } N$$

$$\chi_1\ \textit{a primitive Dirichlet character mod } R$$

(in the modified case, $\chi' = \chi^{\circ}\cdot\chi_1'$, χ_1' primitive mod R').

Then (7.10), (7.13) and (7.10′), (7.13′) with Fourier expansions given in (7.11), (7.14) and (7.11′), (7.14′) define functions

$$\mathfrak{g}(z,w) \in M_n^l(\Gamma^0\,(N^2S), \bar\varphi) \otimes M_n^l(\Gamma^0\,(N^2S), \bar\varphi) \tag{7.16}$$

(for $\nu > 0$ they are actually cusp forms, for $k = n+1$ in (7.10), (7.10′) we exclude $(\varphi\chi)_0^2 = 1$, $(\varphi\chi')_0^2 = 1$ respectively)

with the property, that for any $h \in S_n^l(\Gamma^0(N^2S), \bar\varphi)$ the double scalar product

$$\left\langle \left\langle f_0 \Big|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{g}(*,*)\,|^z\mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)} \tag{7.17}$$

is equal to

<u>***case (7.10)***</u>

$$\Omega_{l,\nu}(0)\,(RN)^{n(2k+\nu-n-1)}\,(N^2S)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\,\chi(-1)^n\,(-1)^{nl} \tag{7.18}$$
$$\alpha\left(\frac{SL^4}{R^2}\right) D^{(NS,\frac{S}{R_0})}\,(f_0, k-n, \bar\chi) \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)},$$

<u>***case (7.13)***</u>

$$\left(\Omega_{l,\nu}(s)\cdot\frac{p_s(k)}{d_s(k)}\right)_{s=s_1}\cdot(RN)^{n(\nu+n)}\,(N^2S)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\,(-1)^{nl}\,\chi(-1)^n \tag{7.19}$$
$$\times\alpha\left(\frac{SL^4}{R^2}\right) D^{(NS,\frac{S}{R_0})}\,(f_0, n+1-k, \bar\chi) \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)},$$

<u>***case (7.10′)***</u>

$$\Omega_{l,\nu}(0)\,(R'pN)^{n(2k+\nu-n-1)}\,(N^2S)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\,\chi(-1)^n(-1)^{nl} \tag{7.20}$$
$$\times\alpha\left(\frac{SL^4}{(R'p)^2}\right) D^{\left(NS,\frac{S}{R_0'}\right)}\,(f_0, k-n, \bar\chi)\,(-1)^n\,p^{-\frac{n^2+n}{2}}$$
$$\times\prod_{i=1}^n\left(1-\beta_i\bar\chi(p)\,p^{-k+n+1}\right) \left\langle f_0 \Big|_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)},$$

case (7.13′)

(7.21)

$$\left(\Omega_{l,\nu}(s)\cdot\frac{p_s(k)}{d_s(k)}\right)_{s=s_1}(R'pN)^{n(\nu+n)}(N^2S)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl}$$
$$\times\alpha\left(\frac{SL^4}{(R'p)^2}\right)D^{\left(NS,\frac{S}{R'_0}\right)}(f_0,n+1-k,\bar{\chi})(-1)^n p^{-\frac{n^2+n}{2}}$$
$$\times\prod_{i=1}^{n}\left(1-\beta_i\,\bar{\chi}(p)\,p^{-n+k}\right)\left\langle f_0\Big|_l\begin{pmatrix}1&0\\0&N^2S\end{pmatrix},h\right\rangle_{\Gamma^0(N^2S)}.$$

We should remark here, that $\Omega_{l,\nu}(s)\,\frac{p_s(k)}{d_s(k)}$ is regular in $s=s_1$ and $\Omega_{l,\nu}(s)$ is regular in $s=0$.

8. Twists and congruences for Fourier coefficients.

To begin with we recall some facts about p-adic L-functions and p-adic measures. We fix a prime p and an embedding of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ into $\overline{\mathbb{Q}_p}$. Thus we can for instance interprete any Dirichlet character ψ as a $\overline{\mathbb{Q}_p}$-valued function. Kubota and Leopoldt have proven the existence of a p-adic function $L_p(s,\psi)$, continuous for $s\in\mathbb{Z}_p\setminus\{1\}$ and even at $s=1$ if ψ is not of the second kind (i.e., ψ is not a character of p-power conductor c_ψ and p-power order), satisfying the following interpolation property:

$$L_p(1-n,\psi)=L(1-n,\psi\omega^{-n})\cdot(1-\psi\omega^{-n}(p)p^{n-1}) \tag{8.1}$$

for integers $n\geqslant 1$ in terms of the classical L-function involving the Teichmüller character at p. Recall that these special L-values are linked with Bernoulli numbers by the identity

$$L(1-n,\psi)=-\frac{1}{n}B_{n,\psi}, \tag{8.2}$$

and that this value is zero in case n and ψ have different parity, i.e., $n\equiv\delta+1 \bmod 2$ where $\delta=\delta_\psi$ equals 0 or 1 according as $\psi(-1)=(-1)^\delta$. Note also that the functional equation of the L-function tells us:

$$L(1-n,\psi)=(2\pi i)^{-n}(n-1)!c_\psi^{n-1}2G(\psi)L(n,\overline{\psi}) \tag{8.3}$$

for integers $n \geqslant 1$ with $n \equiv \delta \bmod 2$. Here the Gauss sum $G(\psi)$ is given by

$$G(\psi) := \sum_{x \bmod c_\psi} \psi(x) \cdot \exp\left(\frac{2\pi i x}{c_\psi}\right). \tag{8.4}$$

Let $u \in \mathbb{Z}_p^\times$ denote a topological generator (for instance $u = 1 + p$) and put $\mathcal{F}_\psi(T) := \psi(u)(1+T) - 1$ if ψ is of the second kind and $\mathcal{F}_\psi(T) := 1$ otherwise. There exists a unique power series $\mathcal{G}_\psi(T) \in \mathbb{Z}_p[\psi][[T]]$ such that we have

$$\mathcal{G}_{\psi\chi}(T) = \mathcal{G}_\psi(\mathcal{F}_\chi(T)). \tag{8.5}$$

By the well-known correspondence between measures and power series (see for instance [38, 12.2]) any $(p-1)$-tuple of power series $\mathcal{G}_{(i)}(T) \in \mathcal{O}[[T]]$ for $i = 1, \ldots, p-1$ with coefficients in the ring of integers of some finite extension of $\mathbb{Q}_p$ uniquely determines an $\mathcal{O}$-valued measure $d\mu$ on $\mathbb{Z}_p^\times$ such that

$$\int_{\mathbb{Z}_p^\times} \psi(a)\langle a\rangle^s d\mu = \mathcal{G}_{(i)}(\psi(u)u^s - 1) \tag{8.6}$$

for all characters ψ whose restriction to $\Delta := \mu_{p-1} \subset \mathbb{Z}_p^\times$ coincides with ω^i, and where $\langle a\rangle = a \cdot \omega^{-1}(a)$ denotes the projection from $\mathbb{Z}_p^\times$ to $1 + p\,\mathbb{Z}_p$. For the remainder of this section we make the following

Hypothesis 1. — *ψ is a primitive Dirichlet character whose conductor is not a power of p.*

Thus the character $\omega^i\psi$ is never of the second kind for any i, and by (8.1), (8.4), (8.5), (8.6) there is a unique $\mathbb{Z}_p[\psi]$-valued measure $d\mu_{(\psi)}$ such that for all characters χ of p-power conductor we have

$$\int_{\mathbb{Z}_p^\times} \chi(a)a^n d\mu_{(\psi)} = (1 - \psi\chi(p)p^{n-1}) \cdot L(1-n, \psi\chi). \tag{8.7}$$

Moreover fixing n we can form the measures

$$d\mu_{(\psi,n)}(a) := a^n \cdot d\mu_{(\psi)}(a) \tag{8.8}$$

and express the p-adic integral in (8.7) as a finite sum using the corresponding set functions $\mu_{(\psi,n)}$ on compact open subsets of $\mathbb{Z}_p^\times$. By a Fourier transformation we get for arbitrary p-powers $L > 1$:

$$\mu_{(\psi,n)}(a + L\mathbb{Z}_p) = \frac{1}{\varphi(L)} \sum_{\chi, c_\chi | L} \overline{\chi}(a) \cdot (1 - \psi\chi(p)p^{n-1}) \cdot L(1-n, \psi\chi), \tag{8.9}$$

hence for all integers $t \leqslant 0$ we have the functions $M_L^{(t)} : (\mathbb{Z}/L)^\times \to \mathbb{Q}(\psi)$ given by

$$M_L^{(t)}(a) := \sum_{\chi, c_\chi | L} \overline{\chi}(a)(1 - \psi\chi(p)p^{-t}) \cdot L(t, \psi\chi)$$

with the congruence property

$$M_L^{(t)}(a) \equiv 0 \bmod \frac{L}{p}. \tag{8.10}$$

Moreover by (8.8) these functions are related by

$$M_L^{(t)}(a) \equiv a^{-t} \cdot M_L^{(0)}(a) \bmod \frac{L^2}{p}. \tag{8.11}$$

Note that we can as well define functions $M_L^{(t)}$ for *positive* integers t simply by applying the functional equation (8.3) and restricting the summation to characters of the right parity with respect to t. (For $t \leqslant 0$ we could have made the same restriction, since the remaining summands vanish anyway.) So for $t > 0$ set

$$M_L^{(t)}(a) := \frac{(t-1)!}{(2\pi i)^t} \sum_\chi \overline{\chi}(a) c_{\psi\chi}^{t-1} 2G((\psi\chi)_0)(1 - (\psi\chi)_0(p)p^{t-1}) \cdot L(t, (\overline{\psi\chi})_0)$$

where we sum over all χ with $c_\chi \mid L$ and of parity $\delta_{\psi\chi} \equiv t \bmod 2$. This is equal to $M_L^{(1-t)}(a)$, and therefore enjoys the analogous congruence properties as in (8.10) and (8.11):

$$M_L^{(t)}(a) \equiv 0 \bmod \frac{L}{p}, \tag{8.10$'$}$$

$$M_L^{(t)}(a) \equiv a^{t-1} \cdot M_L^{(1)}(a) \bmod \frac{L^2}{p}. \tag{8.11$'$}$$

We now want to apply these congruences to the Fourier coefficients which occur in (7.10) resp. (7.10 0$'$) in the previous section. So let $t := k - n \geqslant 1$ and let us assume (in the notation of Section 7) from now on:

HYPOTHESIS 2. — *The character $(\varphi\chi^\circ)^2$ has non-trivial conductor, i.e. the associated primitive character $((\varphi\chi^\circ)^2)_0$ is non-trivial.*

Also we specialize to $S = p$ and therefore L is always a power of p in the sequel. Remember that φ resp. χ° is a character defined modulo Np resp. modulo N, and that p does not divide the conductor of φ. Hence the character $\varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ$ which occurs in the L-function in (7.10) never has a p-power conductor. Therefore the associated primitive character

$$\psi := (\varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ)_0$$

satisfies Hypothesis 1 and we can apply the machinery of p-adic integration as previously described. We now write χ for χ_1. Note that for non-trivial χ in (7.10) we could only encounter a non-trivial Fourier coefficient for those $\mathfrak{T}$ which satisfy the congruence condition

$$\det(2\mathfrak{T}) \not\equiv 0 \bmod p, \tag{8.12}$$

since otherwise $\det(T_2) \equiv 0 \bmod p$ implies that the factor $\overline{\chi}(\det(2T_2))$ vanishes. Moreover for the trivial character $\chi = 1$ by (7.10′) the same holds true. Similarly the summation over G in (7.10) resp. (7.10′) may be restricted to those G satisfying

$$\det(G) \not\equiv 0 \bmod p, \tag{8.13}$$

since the character φ is defined modulo Np, hence the factor $(\varphi\chi)^2(\det(G))$ vanishes if $\det(G)$ is divisible by p.

Remark 8.1. — For $\mathfrak{T}$ and G subject to (8.12) and (8.13) the quadratic character $\varepsilon_{\mathfrak{T}[G^{-1}]}$ is defined modulo some integer prime to p and moreover we have

$$\varepsilon_{\mathfrak{T}[G^{-1}]}(p) = 1.$$

Proof. — Since L is divisible by p we easily get that $(-1)^n \cdot \det(2\mathfrak{T}[G^{-1}])$ is a square modulo p, hence the assertion.

We further make the obvious

Remark 8.2. — For $\mathfrak{T}$ subject to (8.12) the conductor of $\psi = (\varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ)_0$ is not divisible by p hence we have the following decomposition of Gauss sums:

$$G(\psi\chi) = \psi(c_\chi)\cdot\chi(c_\psi)\cdot G(\psi)\cdot G(\chi).$$

Under the same assumption our function $M_L^{(t)}$ can now be written in the form

$$M_L^{(t)}(a) = \frac{(t-1)!}{(2\pi i)^t}\, c_\psi^{t-1} 2G(\psi) \tag{8.14}$$

$$\times \sum_\chi \chi(a^{-1}c_\psi)\cdot c_\chi^{t-1}\cdot\psi(c_\chi)\cdot G(\chi)\cdot(1-\psi\chi(p)p^{t-1})\cdot L(t,\overline{\psi\chi}).$$

Remark 8.3. — For $\mathfrak{T}$ and G subject to (8.12) and (8.13) the factor $\psi(c_\chi)\cdot(1-\psi\chi(p)p^{t-1})$ is independent of $\mathfrak{T}$ and G by Remark 8.1.

Since $c_\psi^{t-1}\cdot 2\cdot G(\psi)$ is a p-adic unit (8.10′) becomes equivalent to the congruences

$$M_{L,\psi}^{(t)}(a)\equiv 0 \bmod \frac{L}{p},\, M_{L,\psi}^{(t)}(a)\equiv a^{t-1}M_{L,\psi}^{(1)}(a) \bmod \frac{L^2}{p}, \tag{8.15}$$

for all integers a not divisible by p, where

$$M_{L,\psi}^{(t)}(a):=\frac{(t-1)!}{(2\pi i)^t}$$

$$\sum_\chi \overline{\chi}(a)c_\chi^{t-1}(\varphi\chi^\circ)_0(c_\chi)\cdot G(\chi)\cdot(1-(\varphi\chi^\circ\chi)_0(p)p^{t-1})\cdot L(t,\overline{\psi\chi}).$$

Note that this implies the same congruences but where we have replaced the L-function for the *primitive* characters $\psi\chi$ by an incomplete L-function with the Euler factors at a fixed set of primes removed. This type of argument occured already in [[23] (3.18), (3.19)]. Moreover (8.11) and (8.11′) hold true also for these incomplete L-functions.

For fixed χ°,φ let now $\mathcal{H}_{L,\chi}(z,w):=\mathcal{H}_{L,\chi}^{(t)}(z,w)$ denote the function in (7.10) for non-trivial χ resp. the function in (7.10′) for $\chi=1$. Remember that the index χ here is the χ_1 in (7.10). We want to consider the Fourier expansion of the following linear combinations:

$$\mathcal{H}_{(a,L)}(z,w):=\frac{p}{L}\sum_\chi \chi(a)c_\chi^{t-1-\frac{n(n-1)}{2}}\cdot(\overline{\varphi\chi^\circ})_0(c_\chi)\chi^\circ(c_\chi)^{-n}$$
$$\frac{G(\overline{\chi})}{G(\chi)^n}(1-(\overline{\varphi\chi^\circ\chi})_0(p)p^{t-1})\mathcal{H}_{L,\chi}(z,w)$$

where χ runs over all characters of conductor $c_\chi\mid L$ which fulfill the additional parity condition

$$\varphi\chi^\circ\chi(-1)=(-1)^{t+n}. \tag{8.16}$$

When we want to emphasize the dependence on t we write $\mathcal{H}_{(a,L)}^{(t)}$ for $\mathcal{H}_{(a,L)}$.

THEOREM 8.4. — *The function*

$$\tilde{\mathcal{H}}_{(a,L)}(z,w):=(A_{2n}^k)^{-1}\cdot(2\pi i)^{-n\nu-t}\cdot p^{ln}\cdot(t-1)!\cdot\mathcal{H}_{(a,L)}(w,z)$$

has a Fourier expansion of the form

$$\tilde{\mathcal{H}}_{(a,L)}(z,w) = \sum_{T_1,T_4 \in \Lambda_n^+} \alpha_{a,L}(T_1,T_4) \cdot \exp\left(\frac{2\pi i}{N^2 p}\operatorname{tr}(T_1 z + T_4 w)\right)$$

whose coefficients $\alpha_{a,L}(T_1,T_4)$ belong to the number field

$$K := \mathbb{Q}(\varphi, \chi^\circ, i^n, \zeta_N)$$

and have bounded p-adic absolute value independent of a and L (and of T_1, T_4). For $p \neq 2$ these coefficients are even p-integral.

Proof. — We just plug in the Fourier expansion (7.10) resp. (7.10′) of the functions $\mathcal{H}_{L,\chi}(z,w)$ into the definition of $\tilde{\mathcal{H}}_{(a,L)}$. With the same summation conventions as in (7.10) we get

$$\alpha_{a,L}(T_1,T_4) = (2\pi i)^{-t} \cdot N^{-2ln} \cdot (t-1)!$$

$$\sum_{\mathfrak{T}(T_2),G,b} \mathfrak{P}_{n,k}^{\nu}(\mathfrak{T}) \cdot G_n(2T_2, N, \chi^\circ) \cdot (\varphi\chi^\circ)^2(\det G) \cdot \det(2\mathfrak{T}[G^{-1}])^{k-\frac{2n+1}{2}}$$

$$\cdot(\varphi\chi^\circ)(b)\cdot b^{-k}\cdot d(b,\mathfrak{T}[G^{-1}])\cdot \frac{p}{L}\sum_{\chi}\chi(aN^n)c_\chi^{t-1}(\overline{\varphi\chi^\circ})_0(c_\chi)\cdot G(\overline{\chi})\cdot\overline{\chi}(\det(2T_2))$$

$$\cdot\chi(\det(G^2)\cdot b)\cdot(1-(\overline{\varphi\chi^\circ\chi})_0(p)p^{t-1})\cdot L(t, \varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ\chi).$$

Note that we only sum over $\mathfrak{T} = \mathfrak{T}(T_2), G, b$ such that p does not divide $\det(2T_2)\cdot\det(G)\cdot b$ in view of (8.12) and (8.13). Thus for any fixed tripel $(\mathfrak{T}, G, b)$ in this expression by (8.15) the term

$$\beta(\mathfrak{T},G,b) := \det(2\mathfrak{T}[G^{-1}])^{k-\frac{2n+1}{2}}\frac{(t-1)!}{(2\pi i)^t}\frac{p}{L}\sum_{\chi}\chi(abN^n\det(G)^2\cdot\det(2T_2)^{-1})$$

$$\cdot c_\chi^{t-1}\cdot(\overline{\varphi\chi^\circ})_0(c_\chi)\cdot G(\overline{\chi})\cdot(1-(\overline{\varphi\chi^\circ\chi})_0(p)p^{t-1})\cdot L(t,\varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ\chi)$$

is in fact p-integral and moreover belongs to K which easily can be verified. Now we can write

$$\text{(8.17)} \qquad \alpha_{a,L}(T_1,T_4) = N^{-2ln}\cdot\sum_{\mathfrak{T},g,b}\mathfrak{P}_{n,k}^{\nu}(\mathfrak{T})\cdot\kappa(\mathfrak{T},g,b)\cdot\beta(\mathfrak{T},g,b)$$

with p-integral numbers $\kappa(\mathfrak{T},g,b) \in K$. Recall that the $\mathfrak{P}_{n,k}^{\nu}$ are polynomials with coefficients in $\mathbb{Z}[\frac{1}{2}]$, hence the expression in (8.17) is p-integral for $p \neq 2$ and has at least bounded 2-adic absolute value for $p = 2$. This completes the proof of Theorem 8.4.

By the same method we treat now the functions in (7.13) resp. (7.13′), which we denote by $\mathcal{H}'_{L,\chi}(z,w) := \mathcal{H}'^{(t)}_{L,\chi}(z,w)$. Note that again we only need to sum over $\mathfrak{T}, G$ subject to (8.12) and (8.13). We define

$$\mathcal{H}'_{(a,L)}(w,z) := \frac{p}{L} \cdot \sum_{\chi} \chi(a) \cdot c_\chi^{-\frac{n(n-1)}{2}} \cdot G(\chi)^{-n} \cdot \chi^\circ(c_\chi)^{-n} \cdot \mathcal{H}'_{L,\chi}(z,w)$$

where χ runs over all characters of conductor $c_\chi \mid L$ satisfying the parity condition

$$\varphi\chi^\circ\chi(-1) = (-1)^{t+n+1}. \tag{8.18}$$

Again we sometimes write $\mathcal{H}'^{(t)}_{(a,L)}$ for $\mathcal{H}'_{(a,L)}$.

THEOREM 8.5. — *The normalized function*

$$\tilde{\mathcal{H}}'_{(a,L)}(z,w) := (B^k_{2n})^{-1} \cdot \frac{p^{ln}}{(2\pi i)^{n\nu}} \cdot \mathcal{H}'_{(a,L)}(z,w)$$

has a Fourier expansion whose coefficients $\alpha'_{a,L}(T_1,T_4)$ belong to the number field $K' := \mathbb{Q}(\varphi,\chi^\circ,\zeta_N)$ and have bounded p-adic absolute value independent of (a,L). For $p \neq 2$ these coefficients are even p-integral.

Proof. — By definition we have similar as for $\alpha_{a,L}(T_1,T_4)$ the formula

$$\begin{aligned}\alpha'_{a,L}(T_1,T_4) = N^{2ln} \cdot \sum_{\mathfrak{T},G,b} \mathfrak{P}^\nu_{n,k}(\mathfrak{T}) \cdot G_n(2T_2,N,\chi^\circ)\\ \times(\varphi\chi^\circ)(\det(G)^2 b) \cdot |\det G|^{2t-1} \cdot b^{t-(n+1)} \cdot d(b,\mathfrak{T}[G^{-1}])\\ \times \frac{p}{L}\sum_{\chi} \chi(abN^n \det G^2 \cdot \det(2T_2)^{-1}) \cdot L(1-t,\varepsilon_{\mathfrak{T}[G^{-1}]}\varphi\chi^\circ\chi).\end{aligned}$$

Remember that we only sum over $\mathfrak{T}, G, b$ such that p does not divide $b \cdot \det(G) \cdot \det(2T_2)$. Note also that the Euler factor at p is removed from the L-function since φ is defined modulo Np. So we can apply (8.10) to the last line of this formula, again by observing that (8.10) remains true if we replace the L-function in (8.10) by an incomplete L-function with the Euler factors at a fixed set of primes removed (see [23, (3.18), (3.19)]). This completes the proof exactly as in the proof of the previous theorem.

Eventually we describe a relationship between the functions $\tilde{\mathcal{H}}_{(a,L)}$ resp. $\tilde{\mathcal{H}}'_{(a,L)}$ for varying t. We recall from Remark 1.1 the congruence property of the polynomials $\mathfrak{P}^\nu_{n,k}(\mathfrak{T})$ saying

$$4^{n\nu} \cdot \mathfrak{P}^\nu_{n,k}(\mathfrak{T}) \equiv 2^{n\nu} \cdot c^\nu_{n,k} \cdot \det(2T_2)^\nu \bmod L$$

for any $\mathfrak{T}$ occuring as summation index in (7.10) and (7.12), where $2^{n\nu} \cdot c^{\nu}_{n,k}$ is a certain rational integer in $\mathbb{Q}^{\times}$ independent of L. If $\mathcal{O} = \mathcal{O}_K$ is the ring of p-integral elements of a number field K, which we always imbed into $\overline{\mathbb{Q}}_p$, let $\mathcal{M}(\mathcal{O})$ denote the $\mathcal{O}$-module in

$$\mathcal{M} = \mathcal{M}(\mathbb{C}) := M^l_n(\Gamma^0(N^2p), \varphi)^{\otimes 2}$$

consisting of all elements with all Fourier coefficients in $\mathcal{O}$. By Theorem 8.4 and Theorem 8.5 there is a constant 2-power 2^{κ} such that $2^{\kappa} \cdot \tilde{\mathcal{H}}_{(a,L)}, 2^{\kappa}\tilde{\mathcal{H}}'_{(a,L)} \in \mathcal{M}(\mathcal{O})$ for all a, L. We say that $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{M}(\mathcal{O})$ are congruent modulo L and write $\mathbf{g}_1 \equiv \mathbf{g}_2 \bmod L$, if this congruence simultaneously holds for the Fourier coefficients of $\mathbf{g}_1, \mathbf{g}_2$.

THEOREM 8.6. — *The functions in the previous theorems satisfy the congruence*

$$c^{l-n-1}_{n,n+1} \cdot \tilde{\mathcal{H}}^{(t)}_{(a,L)} \equiv c^{l-n-t}_{n,n+t} \cdot (-2N)^{n(t-1)} \cdot a^{t-1} \cdot \tilde{\mathcal{H}}^{(1)}_{(a,L)} \bmod L,$$

respectively

$$c^{l-n-1}_{n,n+1} \cdot \tilde{\mathcal{H}}'^{(t)}_{(a,L)} \equiv c^{l-n-t}_{n,n+t} \cdot (2^{-n}N^n a)^{1-t} \cdot \tilde{\mathcal{H}}'^{(1)}_{(a,L)} \bmod L$$

for $t = 1, \dots, l-n$ and L a p-power with $p \neq 2$. For $p = 2$ these congruences hold when we multiply both sides by the constant 2-power 2^{κ} from above.

Proof. — These congruences follow immediately from (8.11) and (8.11′) extended to L-functions with non-primitive characters, applied to the explicit formuli for the Fourier coefficients $\alpha_{a,L} = \alpha^{(t)}_{a,L}$ and $\alpha'_{a,L} = \alpha'^{(t)}_{a,L}$ as displayed in the proofs of Theorems 8.4 and 8.5. Note that $\mathfrak{P}^{\nu}_{n,k}(\mathfrak{T})$ is taken care of in both cases by the congruence quoted above, which completes the proof of the theorem.

Since by (1.11) and (1.4) we know that

$$\kappa_t := \frac{c^{l-n-1}_{n,n+1}}{c^{l-n-t}_{n,n+t}} \in \mathbb{Z}[\frac{1}{2}],$$

we can reformulate the congruences of the theorem as follows.

COROLLARY 8.7. — *Let $\hat{\mathcal{H}}^{(t)}_{(a,L)} := (-1)^{n(t+1)} \cdot \kappa_t \cdot (2N)^{-nt} \cdot \tilde{\mathcal{H}}^{(t)}_{(a,L)}$ and $\hat{\mathcal{H}}'^{(t)}_{(a,L)} := \kappa_t \cdot (N/2)^{nt} \cdot \tilde{\mathcal{H}}'^{(t)}_{(a,L)}$. Then we have*

$$\hat{\mathcal{H}}^{(t)}_{(a,L)} \equiv a^{t-1} \cdot \hat{\mathcal{H}}^{(1)}_{(a,L)} \bmod L$$

and

$$\hat{\mathcal{H}}'^{(t)}_{(a,L)} \equiv a^{1-t} \cdot \hat{\mathcal{H}}'^{(1)}_{(a,L)} \bmod L$$

for $p \neq 2$, and for $p = 2$ these congruences hold for a suitable constant 2-power multiple.

We remark that by a straightforward computation we get

$$\kappa_t = (-1)^{n(t+1)} \prod_{\mu=l-n-t+1}^{l-n-1} \frac{\Gamma(l-n+\frac{n+1-\mu}{2}) \cdot \Gamma(l-n+\frac{n-\mu}{2})}{\Gamma(l-n+\frac{1-\mu}{2}) \cdot \Gamma(l-n-\frac{\mu}{2})} \tag{8.19}$$

and

$$\kappa_t = (-2)^{-n(t-1)} \cdot \prod_{j=1}^{n} \frac{\Gamma(l+t-j)}{\Gamma(l+1-j)}.$$

9. P-adic interpolation of special L-values.

We now return to the situation in the third section. We fix a modular form $f \in S_n^l(\Gamma_0(N), \overline{\varphi})$,

$$f(z) = \sum_{T \in \Lambda_n^+} a(T) \cdot (2\pi i \mathrm{tr}(Tz)),$$

which is an eigenform of the "good" Hecke algebra $\otimes'_{q \nmid N} \mathcal{H}^\circ_{N,q}$. Also we fix a prime p which does not divide N. In order to simplify the handling of formuli we put $S := p$. The technique for dealing with special L-values as developped in previous sections demands that in a first step we must pass from our given modular form f to a form $f_0 \in S_n^l(\Gamma_0(Np), \overline{\varphi})$ as in Proposition 7.1. So we first discuss how this change of forms effects the Satake parameters at p.

Let $\mathbf{M} = \mathbf{M}_p$ denote the module generated by f under the action of the Hecke algebra $\mathcal{H}^\circ_{N,p}$ and of the operator $U(p)$. Of course $\mathbf{M}$ is a submodule of $S_n^l(\Gamma_0(Np), \overline{\varphi})$. Let $f_0 \in \mathbf{M}$ be an eigenform (i.e., in particular $f_0 \neq 0$) of all the corresponding Hecke operators. We denote by $\beta_{1,p}, \ldots, \beta_{n,p}$ the Satake parameters of f_0 for the action of $\mathcal{H}^\circ_{N,p}$ and by $\beta_{0,p}$ the eigenvalue of the operator $U(p)$. The following proposition describes the relation between the p-Satake parameters $\alpha_{0,p}, \ldots, \alpha_{n,p}$ and the parameters $\beta_{0,p}, \ldots, \beta_{n,p}$.

PROPOSITION 9.1. — *For f and f_0 as above we have*

$$\prod_{i=1}^{n}(1-\beta_{i,p}\varphi(p)Y)(1-\beta_{i,p}^{-1}\varphi^{-1}(p)Y)=\prod_{i=1}^{n}(1-\alpha_{i,p}Y)(1-\alpha_{i,p}^{-1}Y).$$

Proof. — We need some more notation. For any $g\in M_n(\mathbb{Z})$ with a p-power determinant we denote by $b(T)\mid SL_n(\mathbb{Z})gSL_n(\mathbb{Z})$ the T-Fourier coefficient of the image of a modular form

$$\sum_T b(T)\exp(2\pi i\mathrm{tr}(Tz))\in S_n^l(\Gamma_0(Np),\varphi^{-1})$$

under the Hecke operator corresponding to $SL_n(\mathbb{Z})gSL_n(\mathbb{Z})$ as in Corollary 3.3. Note that the passage between $SL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z})$ does not create problems but it is convenient in order to avoid ambiguities arising from the nebentype character φ^{-1}. More explicitely we have

$$b(T)\mid SL_n(\mathbb{Z})gSL_n(\mathbb{Z})=\det(g)^{n+1-l}\sum_j b(T[h_j^t]) \tag{9.1}$$

with $SL_n(\mathbb{Z})gSL_n(\mathbb{Z})=\bigcup_j SL_n(\mathbb{Z})h_j$. We also need a second action of this Hecke algebra on the Fourier coefficients $b(T)$ (or on any $SL_n(\mathbb{Z})$-invariant function defined on the space of symmetric matrices):

$$b(T)\|SL_n(\mathbb{Z})gSL_n(\mathbb{Z}):=\sum b(T[h_j^{-1}]). \tag{9.2}$$

With these notations we can now describe the Fourier expansion of $f_0=\sum_{T\in\Lambda_n^+}c(T)\exp(2\pi i\mathrm{tr}(Tz))$ as follows. There are finitely many $g_r\in M_n(\mathbb{Z})$ of p-power determinant, algebraic numbers λ_r and p-powers t_r such that for all $T\in\Lambda_n^+$ we have

$$c(T)=\sum_r\lambda_r a(t_rT)\mid SL_n(\mathbb{Z})g_rSL_n(\mathbb{Z}). \tag{9.3}$$

The Fourier coefficients $c(T)$ satisfy the following relations:

$$\sum_X c(T[X])\det(X)^{-1}\det(X)^{n+1}Y^{\nu_p(\det X)}=\prod_{i=1}^{n}(1-p^n\beta_iY), \tag{9.4}$$

$$c(p^2T)=\beta_0^2c(T)=p^{nl-n(n+1)/2}\beta_1\cdot\ldots\cdot\beta_n, \tag{9.5}$$

where in (9.4) X runs over $\{X\in M_n(\mathbb{Z});\det(X)=p-power\}/SL_n(\mathbb{Z})$.

To prove the proposition we try to describe the right hand side of (9.4) in terms of the $\alpha_{i,p}$. We start from the well-known "Andrianov-identity"

[2], [7] for the Fourier coefficients of f (p may be any prime not dividing N):

$$(9.6)\qquad \sum_X a(T[X])\det(X)^{n+1-l}Y^{\nu_p(\det X)}$$
$$= B_p^n(Y,T)\cdot\prod_{i=1}^n(1-\alpha_{i,p}\varphi_0^{-1}(p)p^nY)(1-\alpha_{i,p}^{-1}\varphi_0^{-1}(p)p^nY)$$
$$\times a(T)\|\left(\sum_{i=0}^n(-1)^ip^{i(i-1)/2+il}\varphi_0^{-2i}(p)\pi_iY^i\right).$$

Here again X runs as in (9.4), $B_p^n(Y,T)$ is a polynomial in Y (with coefficients depending on T; this polynomial is equal to 1 if T is p-imprimitive, i.e., $p^{-1}\cdot T\in\Lambda_n$) and π_i denotes the double coset

$$\pi_i := SL_n(\mathbb{Z})\begin{pmatrix}1_{n-i} & \\ & p\cdot 1_i\end{pmatrix}SL_n(\mathbb{Z})=\bigcup SL_n(\mathbb{Z})g_{ij}.$$

Now we fix some $T=p^2T_0$ with $T_0\in\Lambda_n^+$ and we consider

(9.7)
$$a(T)\|\pi_i=\sum_j a(p^2T_0[g_{ij}^{-1}])=\sum_j a(T_0[pg_{ij}^{-1}])=p^{(n-i)(l-n-1)}a(T_0)\|\pi_{n-i}.$$

In other words we can write the right hand side of the Andrianov-identity in terms of the $|$-action. Using the commutativity of the Hecke algebra in question and the fact that the Fourier coefficients $c(\ldots)$ arise from the $a(\ldots)$ via the $|$-action of that Hecke algebra, we get from (9.6) the following relation for the Fourier coefficients of f_0 (with $T=p^2T_0$ as above):

$$(9.8)\qquad \sum_X c(T[X])\det(X)^{n+1-l}Y^{\nu_p(\det X)}$$
$$=\prod_{i=1}^n(1-\alpha_{i,p}\varphi_0^{-1}(p)p^nY)(1-\alpha_{i,p}^{-1}\varphi_0^{-1}(p)p^nY)$$
$$\times c(T_0)\mid\sum_i(-1)^ip^{i(i-1)/2+il+(n-i)(l-n-1)}\pi_{n-i}\varphi_0^{-2i}(p)Y^i.$$

Now we use that f_0 is an eigenform with p-parameters $\beta_1,\ldots,\beta_n$, hence

$$c(T_0)\mid\pi_i=\lambda_ic(T_0),\qquad \lambda_i=p^{i(n+1)-i(i+1)/2}E_i(\beta_1,\ldots,\beta_n),$$

where $E_i(\ldots)$ is the i-th elementary symmetric polynomial. Using

$$c(p^2T) = \beta_0^2 c(T_0) = p^{nl-n(n+1)/2}\beta_1 \cdot \ldots \cdot \beta_n$$

and

$$(\beta_1 \cdot \ldots \cdot \beta_n)^{-1} E_{n-i}(\beta_1, \ldots, \beta_n) = E_i(\beta_1^{-1}, \ldots, \beta_n^{-1})$$

we obtain after some elementary calculation

$$c(T_0) \mid \sum_i (-1)^i p^{i(i-1)/2+il+(n-i)(l-n-1)} \pi_{n-i} \varphi_0^{-2i}(p) Y^i$$
$$= c(p^2T_0) \cdot \prod_{i=1}^{n} (1 - p^n \beta_i^{-1} \varphi_0^{-2}(p) Y). \tag{9.9}$$

The proposition follows by comparing (9.4) with (9.8) (and using (9.9)); we tacitly use that there exists a T_0 with $c(p^2T_0) \neq 0$, a fact which follows easily from (9.5).

Remark 9.2. — We should point out that the $\alpha_{0,p}, \ldots, \alpha_{n,p}$ are well defined only up to the action of the Weyl group W_n, generated by the permutations of the X_i $(1 \leqslant i \leqslant n)$ and by the substitutions $(1 \leqslant j \leqslant n)$

$$X_0 \mapsto X_0 X_j, X_j \mapsto X_j^{-1}, X_i \mapsto X_i$$

for $i \neq j$.

If we want to describe the relation between the α_i and the β_i more explicitely than in our proposition we may put

$$\alpha_i := \overline{\varphi}(p) \cdot \beta_i^{-1} \tag{9.10}$$

for $(1 \leqslant i \leqslant n)$. In this way the α_i are now defined up to the action of the subgroup $S_n \subset W_n$. Combining the well-known relation

$$\alpha_0^2 \alpha_1 \cdot \ldots \cdot \alpha_n = \overline{\varphi}(p)^n \cdot p^{nl-n(n+1)/2}$$

with (9.10) we obtain from (9.5) the relation

$$\alpha_0^2 = \beta_0^2. \tag{9.11}$$

Remark 9.3. — By an argument similar to the computation in [20, Chap. 2, 3] we see that β_0 must be among the reciprocal roots of the p-spinor polynomial of f. Therefore we have indeed $\alpha_0 = \beta_0$ unless $-\alpha_0$ is also among the reciprocal roots of the spinor polynomial.

DEFINITION. — *We call a Hecke eigenform $f \in S_n^l(\Gamma_0(N),\overline{\varphi})$ p-regular for a prime $p \nmid N$ if our module $\mathbf{M} = \mathbf{M}_p \subset S_n^l(\Gamma_0(Np),\overline{\varphi})$ contains an eigenform $f_0 \neq 0$ with Satake p-parameters $\beta_0, \beta_1, ..., \beta_n$ such that $|\beta_0|_p (= |p^{nl-n(n+1)/2}\beta_1 \cdot \ldots \cdot \beta_n|_p) = 1$.*

Recall that $\mathcal{M} := M_n^l(\Gamma^0(N^2p),\varphi)^{\otimes 2}$ denotes the space of modular forms $\mathbf{g}(z,w)$ which as a function of z (or w) belong to $M_n^l(\Gamma^0(N^2p),\varphi)$. Assuming that f is p-regular we have to consider the following $\mathbb{C}$-valued function $\mathcal{F} = \mathcal{F}_f$ on $\mathcal{M}$:

$$\mathcal{F}(\mathbf{g}) := \frac{\langle\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{g} \mid^z \mathfrak{K}\rangle^w_{\Gamma^0(N^2p)}, f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2p \end{pmatrix}\rangle^z_{\Gamma^0(N^2p)}}{\langle f_0, f_0\rangle^2_{\Gamma_0(N^2p)}}.$$

Note that $\mathcal{F}$ is $\mathbb{R}$-linear and for an arbitrary complex scalar λ satisfies $\mathcal{F}(\lambda\mathbf{g}) = \overline{\lambda} \cdot \mathcal{F}(\mathbf{g})$. For an arbitrary Dirichlet character ψ we introduce the modified p-Euler factor

$$E_p(s,\psi) := \prod_{j=1}^{n} \frac{(1 - \psi(p)\beta_j^{-1}p^{s-1})}{(1 - \overline{\psi}(p)\beta_j p^{-s})},$$

where the β's are the Satake parameters as in Section 3. Note that this p-Euler factor equals 1 if p divides the conductor of ψ. Recall that by Proposition 7.1 we know

(9.12)

$$\mathcal{F}(\mathcal{H}^{(t)}_{L,\chi}) = \langle f_0, f_0\rangle^{-1} \cdot \Omega_{l,\nu}(0) \cdot (N^2p)^{\frac{n(n+1)}{2} - \frac{nl}{2}} \chi\chi^\circ(-1)^n(-1)^{ln}$$
$$\cdot (Nc_\chi)^{n(l+t-1)} \alpha(pL^4c_\chi^{-2}) \cdot E_p(t,\chi^\circ\chi) \cdot \overline{\chi^\circ}(\frac{p}{(p,c_\chi)})^n \cdot D^{(Np)}(f,t,\overline{\chi^\circ\chi})$$

for any character χ whose conductor c_χ is a power of p.

Similarly by Proposition 7.1 we have

(9.13)

$$\mathcal{F}(\mathcal{H}'^{(t)}_{L,\chi}) = \langle f_0, f_0\rangle^{-1} \cdot \Omega_{l,\nu}(s_1)\frac{p_{s_1}(k)}{d_{s_1}(k)} \cdot (N^2p)^{\frac{n(n+1)}{2} - \frac{nl}{2}} \chi\chi^\circ(-1)^n(-1)^{nl}$$
$$\cdot (Nc_\chi)^{n(l-t)} \cdot \alpha(pL^4c_\chi^{-2}) \cdot E_p(1-t,\chi\chi^\circ) \cdot \overline{\chi^\circ}(\frac{p}{(p,c_\chi)})^n \cdot D^{(Np)}(f,1-t,\overline{\chi\chi^\circ})$$

for all characters χ of p-power conductor. Note that the values $\mathcal{F}(\mathcal{H}_{L,\chi})$ resp. $\mathcal{F}(\mathcal{H}'_{L,\chi})$ only depend on L by the factor $\alpha(L^4)$. Hence we get

Remark 9.4. — The values $\mathcal{F}(\overline{\alpha(L^{-4})}\cdot\mathcal{H}_{L,\chi})$ and $\mathcal{F}(\overline{\alpha(L^{-4})}\cdot\mathcal{H}'_{L,\chi})$ do not depend on L.

As a consequence we can define $\mathbb{C}$-valued distributions $\mu_{(t)}$ on $\mathbb{Z}_p^\times$ (one for each t) by prescribing their integrals on finite characters χ to be given by the formula

$$(9.14)\qquad \int_{\mathbb{Z}_p^\times}\chi d\mu_{(t)} := \alpha(L)^{-4}\times$$

$$\mathcal{F}(c_\chi^{t-1-\frac{n(n-1)}{2}}(\overline{\varphi\chi^\circ})_0(c_\chi)\cdot\chi^\circ(c_\chi)^{-n}\frac{G(\overline{\chi})}{G(\chi)^n}(1-(\overline{\varphi\chi\chi^\circ})_0(p)p^{t-1})\mathcal{H}_{L,\chi}^{(t)}(z,w))$$

where χ fulfills (8.16) and c_χ divides L. For χ not of the parity (8.16) one demands the integral to be equal to zero. In other words we have

$$(9.15)\qquad \mu_{(t)}(a+L\mathbb{Z}_p) = \alpha(L)^{-4}\cdot\frac{2}{p-1}\cdot\mathcal{F}(\mathcal{H}_{(a,L)}^{(t)}).$$

Similarly we define distributions $\mu_{(1-t)}$ by setting

$$(9.14')\qquad \int_{\mathbb{Z}_p^\times}\chi d\mu_{(1-t)} := \alpha(L)^{-4}\cdot\mathcal{F}(c_\chi^{-\frac{n(n-1)}{2}}(\chi^\circ(c_\chi)\cdot G(\chi))^{-n}\cdot\mathcal{H}_{L,\chi}^{'(t)})$$

for χ of parity (8.18) and any p-power $L\equiv 0 \bmod c_\chi$. Otherwise the integral has to be zero. For this distribution we get

$$(9.15')\qquad \mu_{(1-t)}(a+L\mathbb{Z}_p) = \alpha(L)^{-4}\cdot\frac{2}{p-1}\cdot\mathcal{F}(\mathcal{H}_{(a,L)}^{'(t)}).$$

Now let $\mathcal{K}$ denote the field generated over K by $\alpha(p)$, $\alpha(p)^\rho$ and all Fourier coefficients of f_0, f_0^ρ. We may choose f and f_0 in such a way that $\mathcal{K}$ is an algebraic number field. We normalize our distributions by replacing in (9.15) and in (9.15′) the functions $\mathcal{H}_{(a,L)}^{(t)}$ and $\mathcal{H}_{(a,L)}^{'(t)}$ by their multiples $\hat{\mathcal{H}}_{(a,L)}^{(t)}$ and $\hat{\mathcal{H}}_{(a,L)}^{'(t)}$ as defined in Corollary 8.7. Thus we create distributions $\hat{\mu}_{(t)}$ and $\hat{\mu}_{(1-t)}$. By the theorem of the appendix these distributions have algebraic values in $\mathcal{K}$ and our main result essentially says that they are even p-adically bounded, i.e., they are p-adic measures. To formulate our result let

$$\Lambda_\infty^+(s) := \frac{(2i)^{ns}\cdot\Gamma(s)}{(2\pi i)^s}\cdot\prod_{j=1}^n\Gamma_{\mathbb{C}}(s+l-j)$$

and

$$\Lambda_\infty^-(s) := (2i)^{ns}\cdot\prod_{j=1}^n\Gamma_{\mathbb{C}}(1-s+l-j)$$

where as usual $\Gamma_{\mathbb{C}}(s) := 2 \cdot (2\pi)^{-s}\Gamma(s)$. Further for any character χ of p-power conductor c_χ we let

$$A^-(\chi) := c_\chi^{nl-\frac{n(n+1)}{2}} \cdot \alpha(c_\chi)^{-2} \cdot (\chi^\circ([p, c_\chi]) \cdot \chi(-1)G(\chi))^{-n}$$

and

$$A^+(\chi) := (\overline{\chi^\circ \varphi})_0(c_\chi) \cdot \frac{A^-(\chi)}{\chi(-1)G(\chi)},$$

where $[a, b]$ denotes the least common multiple of the integers a, b. Eventually we let

$$E_p^+(s, \chi\chi^\circ) := (1 - (\overline{\varphi\chi\chi^\circ})_0(p)p^{t-1}) \cdot E_p(s, \chi\chi^\circ),$$

and $E_p^-(\ldots) := E_p(\ldots)$.

THEOREM 9.5. — *Suppose that f is p-regular with eigenvalue $\alpha(p)$ of f_0 under the action of $U(p)$. Then*

a) *there is a unique p-adic measure μ on $\mathbb{Z}_p^\times$ with values in $\mathcal{K}$ such that for $t = 1, \ldots, l - n$ and for each character χ of p-power conductor c_χ with parity $\varphi\chi\chi^\circ(-1) = (-1)^{t+n}$ we have*

$$\int_{\mathbb{Z}_p^\times} \chi(x)x^{-t}d\mu(x) = c_\chi^{t(n+1)}A^+(\chi) \cdot E_p^+(t, \chi\chi^\circ)\frac{\Lambda_\infty^+(t)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, t, \overline{\chi\chi^\circ}),$$

whereas the integral vanishes for the opposite parity;

b) *there is also a unique measure ν on $\mathbb{Z}_p^\times$ with values in $\mathcal{K}$ such that for $t = 1, \ldots, l - n$ and for each χ of p-power conductor c_χ with parity $\varphi\chi\chi^\circ(-1) = (-1)^{t+n+1}$ we have*

$$\int_{\mathbb{Z}_p^\times} \chi(x)x^{t-1}d\nu(x)$$
$$= c_\chi^{n(1-t)}A^-(\chi) \cdot E_p^-(1 - t, \chi\chi^\circ)\frac{\Lambda_\infty^-(t)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, 1 - t, \overline{\chi\chi^\circ}),$$

whereas the integral vanishes for the opposite parity.

Remark 9.6. — a) For even degree n and under the additional assumption for the weight $l > 2n+2$ the theorem essentially had been proven by Panchishkin [20], [21] by a different technique.

b) For degree $n = 1$, which is the elliptic modular case, the result was also known by [23]. Since in that case one knows the functional equation for the complex L-function, the two measures $d\mu$ and $d\nu$ could be easily related. For $n > 1$ the same phenomenon is expected to hold true.

c) Since in [23] the integrals of characters χ against the measures were only known to be given by the above formuli for all but two exceptional characters, our theorem is even an improvement for $n = 1$.

Proof of Theorem 9.5. — The proof relies on the following lemma whose proof will be given below.

LEMMA 9.7. — *There is a finitely generated $\mathcal{O}$-submodule in $\mathbb{C}$ containing $\mathcal{F}(\mathcal{M}(\mathcal{O}))$ for $\mathcal{O} = \mathcal{O}_K$ as in the previous section. In particular, there is a constant C, in fact a power of p such that for any $\mathcal{H} = \hat{\mathcal{H}}^{(t)}_{(a,L)}$ we have*

$$\mathcal{F}(\mathcal{H}) \in C^{-1} \cdot \mathcal{O}.$$

Hence by Corollary 8.7 we get

$$\mathcal{F}(C \cdot \hat{\mathcal{H}}^{(t)}_{(a,L)}) \equiv a^{1-t} \cdot \mathcal{F}(C \cdot \hat{\mathcal{H}}^{(1)}_{(a,L)}) \bmod L \tag{9.16}$$

and

$$\mathcal{F}(C \cdot \hat{\mathcal{H}}'^{(t)}_{(a,L)}) \equiv a^{t-1} \cdot \mathcal{F}(C \cdot \hat{\mathcal{H}}'^{(1)}_{(a,L)}) \bmod L. \tag{9.16$'$}$$

Since $\alpha(L)$ was supposed to be a unit in $\mathcal{O}$, Lemma 9.7 immediately implies that the distributions $\hat{\mu}_{(t)}$ and $\hat{\mu}_{(1-t)}$ are bounded, i.e. p-adic measures, which moreover by (9.16) and (9.16′) are related by

$$d\hat{\mu}_{(t)}(x) = x^{1-t} \cdot d\hat{\mu}_{(1)}(x) \tag{9.17}$$

and

$$d\hat{\mu}_{(1-t)}(x) = x^{t-1} \cdot d\hat{\mu}_{(0)}(x). \tag{9.17$'$}$$

Now integrating characters χ against these measures using (9.12) and (9.13) a little computation easily leads to a certain multiple of the formula in b). Thus $d\mu(x)$ is an appropriate multiple of $x d\hat{\mu}_{(1)}(x)$ and $d\nu(x)$ is an appropriate multiple of $d\hat{\mu}_{(0)}(x)$, which finishes the proof of the theorem.

Proof of Lemma 9.7. — Essentially all follows by linear algebra. There is a basis $g_1, \dots, g_d$ of $M^l_n(\Gamma^0(pN^2), \varphi)$ with Fourier coefficients in $\mathbb{Q}(\varphi)$ (see [24]). On the other hand there are matrices $T_1, \dots, T_d \in \Lambda^+_n$ such that the map

$$M^l_n(\Gamma^0(pN^2), \varphi) \to \mathbb{C}^d, g = \sum a(T) e^{(2\pi i \mathrm{tr}(Tz))} \mapsto (a(T_i))$$

is bijective. Note that singular forms do not occur here because of $l > n$. We consider now two $\mathbb{C}$-linear isomorphisms $\Phi, \tilde{\Phi}$ of $\mathcal{M}$ with $\mathbb{C}^{d^2}$:

$$\Phi : \mathbf{g}(z,w) = \sum a_{ij} \cdot g_i(z)g_j(w) \mapsto (a_{ij})$$

and

$$\tilde{\Phi} : \mathbf{g}(z,w) = \sum_{T,\tilde{T}} a(T,\tilde{T}) \cdot e^{(2\pi i \mathrm{tr}(Tz+\tilde{T}w))} \mapsto (a(T_i,T_j)).$$

Both Φ and $\tilde{\Phi}$ commute with the action of $\mathrm{Aut}(\mathbb{C})$; therefore the matrix describing the isomorphism $\Phi \circ \tilde{\Phi}^{-1} : \mathbb{C}^{d^2} \to \mathbb{C}^{d^2}$ has coefficients in $\mathbb{Q}(\varphi)$, hence for a suitable p-power C' we find

$$\Phi(\mathcal{M}(\mathcal{O})) \subseteq (\Phi \circ \tilde{\Phi}^{-1})(\tilde{\Phi}(\mathcal{M}(\mathcal{O}))) \subseteq (\Phi \circ \tilde{\Phi}^{-1})(\mathcal{O}^{d^2}) \subseteq C'^{-1} \cdot \mathcal{O}^{d^2}.$$

For any $\mathbf{g} \in \mathcal{M}(\mathcal{O})$ we therefore have $\mathbf{g} = \sum a_{ij} \cdot g_i(z)g_j(w)$ with $a_{ij} \in C'^{-1}\mathcal{O}$ and

$$\mathcal{F}(\mathbf{g}) = \sum \overline{a_{ij}} \frac{\langle f_0 \,|_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g_j \rangle^{w}_{\Gamma^0(pN^2)} \cdot \langle g_i^{\rho}, f_0 \,|_l \begin{pmatrix} 1 & 0 \\ 0 & pN^2 \end{pmatrix} \rangle^{z}_{\Gamma^0(pN^2)}}{\langle f_0, f_0 \rangle^2_{\Gamma_0(pN^2)}}.$$

This completes the proof of the lemma.

Appendix: Properties of critical values.

Let $f \in S_n^l(\Gamma_0(M), \varphi)$ be a Hecke eigenform for $\otimes_{p\nmid M} \mathfrak{H}^0_{M,p}$ and χ any Dirichlet character; we define $\delta \in \{0,1\}$ by $\varphi(-1)\chi(-1) = (-1)^{n+\delta}$. The following set of integers r will be called critical for $D^{(M)}(f,s,\chi)$:

(A1) $\{-l+n+1 \leqslant r \leqslant 0 | r-\delta \text{ odd}\} \cup \{0 < r \leqslant l-n | r-\delta \text{ even}\}$,

(A2) $\{-l+n+1 \leqslant r \leqslant 0 | r-\delta \text{ even}\} \cup \{0 < r \leqslant l-n | r-\delta \text{ odd}\}$.

We point out that there are no critical values in this sense for small weights (i.e., $l \leqslant n$). For more sophisticated notions of "critical values" we refer to [10] or [21]; we only mention here that the sets (A1) and (A2) coincide with the set of those $r \in \mathbb{Z}$ for which $\gamma(s)$ as well as $\gamma(1-s)$ are regular at $s = r$ with

$$\gamma(s) = \begin{cases} \Gamma_{\mathbb{R}}(s+\delta)\displaystyle\prod_{j=1}^{n}\Gamma_{\mathbb{C}}(s+l-j) & \text{for } n \text{ even,} \\ \Gamma_{\mathbb{R}}(s+1-\delta)\displaystyle\prod_{j=1}^{n}\Gamma_{\mathbb{C}}(s+l-j) & \text{for } n \text{ odd,} \end{cases}$$

where - as usual - $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

The aim of this appendix is to show that for critical r the standard L-function (more precisely the standard L-function suitably modified at some bad primes) is regular at $s = r$ and its value is (up to a power of π and the square of the Petersson norm) an algebraic number with good Galois properties. Results of this type can be found at many places in the literature [25], [15], [36], [19] but all with certain restrictions which we want to avoid. Our approach is a variant of [4, §6] Our proof has the remarkable property that it does not really involve the Siegel type Eisenstein series on $\mathrm{Sp}(2n)$ themselves, but only its Fourier coefficients of *maximal* rank; therefore our method may be of independant interest and (perhaps) useful for other purposes as well.

PROPOSITION. — *Let l, k, ν, t be as usual, φ (χ respectively) be a Dirichlet character mod M (N respectively) and $0 \neq f \in S_n^l(\Gamma_0(M), \overline{\varphi})$ a Hecke eigenform for $\otimes_{q\nmid M}\mathfrak{H}_{M,q}$; furthermore let p be a prime dividing M such that f is an eigenform for $U(p)$ with nonzero eigenvalue γ_p. Then $D^{(M)}(f, s, \chi)$ is regular in $s = t$ and $s = 1-t$ if $p|c(\chi)$. If p is coprime to $c(\chi)$, we assume in addition that f is an eigenform of $\mathfrak{H}_{M,p}$ with Satake parameters $\beta_1, \ldots, \beta_n$; we may assume that $N = N'p$ with N' coprime to p. Then*

$$\left(\prod_{i=1}^{n}(1-\beta_i\overline{\chi}'(p)p^{1-s})\right) D^{(M,p)}(f, s, \chi') \tag{A3}$$

is regular in $s = t$ and $s = 1-t$, where χ' is the corresponding character mod N'.

Proof. — We consider the case $p|c(\chi)$ first: By enlarging M and N if necessary we may assume that $M = N^2$. We then start from (3.24) with $f = g|_l\begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}$ and apply $U(p)$ to both sides of (3.24). Then only those $T \in \Lambda_{2n}$ which are of maximal rank can contribute to the Fourier expansion of $\mathfrak{E}_{2n}^{k,\nu}(w, z, M, N, \varphi, \chi, s)|^z U(p)$. The result follows from Proposition 5.2

and the simple observation that $\Omega_{l,\nu}(s)$ is of order zero in $s = 0$ and $s = s_1 = \frac{1}{2} - t$. If p does not divide $c(\chi)$ we may assume that $M = (N'p)^2 p^\lambda$ with some nonnegative integer λ; the assertion follows then in the same way as above by starting from (3.24′).

The statement about the Galois behaviour of the standard-L-function is (essentially) an easy consequence of the following elementary lemma:

LEMMA. — *Let φ and χ be Dirichlet characters mod M and N. Assume that for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have modular forms $G(z, w, \varphi^\sigma, \chi^\sigma) \in M_n^l(\Gamma_0(M), \varphi^\sigma) \otimes M_n^l(\Gamma_0(M), \varphi^\sigma)$, which are symmetric with respect to z and w and satisfy*

$$\text{(A4)} \qquad G(z, w, \varphi, \chi)^\sigma = \alpha(\varphi, \chi, \sigma) G(z, w, \varphi^\sigma, \chi^\sigma)$$

for a certain complex number $\alpha(\varphi, \chi, \sigma)$. Furthermore let V and W be linear endomorphisms of $S_n^l(\Gamma_1(M))$ which map subspaces of type $S_n^l(\Gamma_0(M), \psi)$ into $S_n^l(\Gamma_0(M), \overline{\psi})$and commute there with the action of $\otimes_{p\nmid M} \mathfrak{H}_{M,p}$. We assume that for any $g \in S_n^l(\Gamma_0(M), \varphi^\sigma)$, which is an eigenform of $\otimes_{p\nmid M} \mathfrak{H}_{M,p}$ an equation

$$\text{(A5)} \qquad \langle g, G(-\overline{z}, *, \varphi^\sigma, \chi^\sigma) \rangle = c(g, \chi^\sigma) g | V$$

holds with a certain complex number $c(g, \chi)^\sigma$. Then for any $\otimes_{p\nmid M} \mathfrak{H}_{M,p}$-eigenform $g \in S_n^l(\Gamma_0(M), \phi)$ such that g^σ is an eigenform of WV with eigenvalue $\lambda(\sigma)$ we have

$$\text{(A6)} \qquad \alpha(\varphi, \chi, \sigma\rho)^\rho c(g^\sigma | W, \chi^{\sigma\rho}) \lambda(\sigma) = c(g, \chi)^{\rho\sigma} \frac{\langle g^\sigma | W, g | V^{\rho\sigma\rho} \rangle}{\langle g, g \rangle^\sigma}.$$

Proof. — Let $\{g_j\}$ be an orthogonal basis of $S_n^l(\Gamma_0(M), \varphi)$ consisting of eigenforms $\otimes_{p\nmid M} \mathfrak{H}_{M,p}$; we may assume that $g = g_1$; furthermore let $\{h_r\}$ be a basis of $S_n^l(\Gamma_0(M), \varphi)^\perp$. With certain coefficients $a_{r,t}$ and $b_{r,t}$ we have

$$\text{(A7)} \qquad G(z, w, \varphi, \chi) = \sum_j c(g_j, \chi)^\rho \frac{g_j(w)(g_j | V)^\rho(z)}{\langle g_j, g_j \rangle} + \sum_{r,t} a_{r,t} h_r(w) h_t(w) + \sum_{r,j} b_{r,j} h_r(w) g_j(z).$$

The symmetry of G implies that $b_{r,j} = 0$ for all r, j. We now interchange the roles of z and w and apply $\sigma\rho$ on both sides:

$$\text{(A8)} \qquad G(z, w, \varphi, \chi)^{\sigma\rho} = \sum_j c(g_j, \chi)^{\rho\sigma\rho} \frac{(g_j | V)^{\rho\sigma\rho}(w) g_j^{\sigma\rho}(z)}{\langle g_j, g_j \rangle^{\sigma\rho}} + \sum_{r,t} a_{r,t}^{\sigma\rho} h_r^{\sigma\rho}(z) h_t^{\sigma\rho}(w).$$

We compute $\langle g^\sigma | W, G^{\sigma\rho}(-\overline{z}, *, \varphi, \chi)\rangle$ in two different ways: On one hand we get

$$\text{(A9)} \qquad \sum_j c(g_j,\chi)^{\rho\sigma} \frac{\langle g^\sigma | W, (g_j|V)^{\rho\sigma\rho}\rangle g_j^\sigma(z)}{\langle g_j, g_j\rangle^\sigma} + \sum_{r,t} a_{r,t}^\sigma \langle g^\sigma | W, h_t^{\sigma\rho}\rangle h_r^\sigma$$

and on the other hand

$$\text{(A10)} \qquad \alpha(\varphi,\chi,\sigma\rho)^\rho c(g^\sigma|W,\chi^{\sigma\rho}) g^\sigma|WV.$$

Comparing these two expressions we get the assertion of the lemma.

THEOREM. — *Under the same assertions as in the proposition above, we have for all $\sigma \in \mathrm{Aut}(\mathbb{C})$:*

Case (a):

$$\text{(A11)} \qquad \left(G(\chi)^{-n-1}G(\varphi)^{-n-1}\pi^d i^{(n+1)k}\frac{D^{(M)}(g,t,\chi)}{\langle g,g\rangle}\right)^\sigma$$
$$= G(\chi^\sigma)^{-n-1}G(\varphi^\sigma)^{-n-1}\pi^d i^{(n+1)k}\frac{D^{(M)}(g^\sigma,t,\chi^\sigma)}{\langle g^\sigma, g^{\rho\sigma\rho}\rangle}$$

with $d = \frac{3}{2}n(n+1) - 2nk - n\nu - k$; in the special case $t = 1$ we impose the additional condition $(\varphi\chi^2)_0 \neq 1$.

Case (b):

$$\text{(A12)} \qquad \left(G(\chi)^{-n}G(\varphi)^{-n}\pi^d i^{nk}\frac{D^{(M)}(g,1-t,\chi)}{\langle g,g\rangle}\right)^\sigma$$
$$= G(\chi^\sigma)^{-n}G(\varphi^\sigma)^{-n}\pi^d i^{nk}\frac{D^{(M)}(g^\sigma,1-t,\chi^\sigma)}{\langle g^\sigma, g^{\rho\sigma\rho}\rangle}$$

with $d = -\frac{1}{2}n(n+1) - n\nu$.

These assertions hold true if $p|c(\chi)$; in the case $p \nmid c(\chi)$ however we must replace $D^{(M)}(g,s,\chi)$ by

$$\text{(A13)} \qquad \left(\prod_{i=1}^n (1-\beta_i\overline{\chi}'(p)p^{1-s})\right) D^{(M,p)}(f,s,\chi').$$

Proof. — We prove this only for the case $p|c(\chi)$ (the other case will be left to the reader), so we assume that $M = N^2$.

For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we denote by $\eta = \eta_\sigma$ an integer with $\sigma(\exp(\frac{2\pi i}{M})) = \exp(\frac{2\pi i\eta}{M})$. We also mention the following properties of Gauß sums:

$$G(\varphi)^\sigma = \varphi^\sigma(\eta)^{-1} G(\varphi^\sigma) \tag{A14}$$

$$G_n(T,N,\chi)^\sigma = \chi^\sigma(\eta^n)^{-1} G_n(T,N,\chi^\sigma) \qquad \text{with} \quad T \in \mathbb{Z}^{(n,n)}. \tag{A15}$$

To prove a) we define

$$G = G(z,w,\varphi,\chi) = \left((\pi i)^{n\nu} A_{2n}^k (\pi i)^{k-n} i^n\right)^{-1} \mathcal{L}(k,\varphi\chi)\mathfrak{E}_{2n}^{k,\nu}(z,w,M,N,\varphi,\chi,s)_{|s=0}|^z U(p)|^w U(p). \tag{A16}$$

From (A15) and Corollary (5.2) we have

$$G(z,w,\varphi,\chi)^\sigma = \varphi^\sigma(\eta)^{-1}\chi^\sigma(\eta^{n+1})^{-1} G(z,w,\varphi^\sigma,\chi^\sigma) \tag{A17}$$

and therefore

$$\alpha(\varphi,\chi,\sigma) = \varphi^\sigma(\eta)^{-1}\chi^\sigma(\eta^{n+1})^{-1}. \tag{A18}$$

Using

$$A_{2n}^k = \frac{2^{2n}}{\Gamma_{2n}(k)}\pi^{2nk} \in \pi^{2nk-n^2} \times \mathbb{Q}^\times \tag{A19}$$

$$\Omega_{l,\nu} \in i^{ln}\pi^{\frac{n(n+1)}{2}} \times \mathbb{Q}^\times \tag{A20}$$

we get from (3.24) (with d as in the theorem) by transfering $|^w U(p)$ into its adjoint operator $U(p)^*$

$$\begin{aligned} &\langle g, G(-z,*,\varphi,\chi)\rangle \\ &\in i^{n\nu+k+ln}\pi^d D^{(M)}\left(g|U(p)^*|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, k-n, \chi^\rho\right) \\ &\qquad \times g|U(p)^*|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} |U(p) \times \mathbb{Q}^\times. \end{aligned} \tag{A21}$$

Hence we have

$$V = U(p)^* \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} U(p), \tag{A22}$$

$$W = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}. \tag{A23}$$

We tacitly use the commutation rules $WU(p)^* = U(p)W$ in the sequel.

We get

$$c(g,\chi) = i^{(n+1)k}\pi^d D^{(M)}(g|_l \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, k-n, \chi^\rho) \tag{A24}$$
$$= i^{(n+1)k}\pi^d D^{(M)}(g, k-n, \chi)^\rho.$$

This implies

$$\left(\frac{c(g,\chi)^\rho}{\langle g,g\rangle}\right)^\sigma = (-1)^{(n+1)k}\left(\frac{i^{n+1)k}\pi^d D^{(M)}(g,k-n,\chi)}{\langle g,g\rangle}\right)^\sigma. \tag{A25}$$

On the other hand, the lemma tells us that (A25) is equal to

$$\frac{\alpha(\varphi,\chi,\sigma\rho)^\rho c(g^\sigma|W,\chi^{\sigma\rho})\lambda(\sigma)}{\langle g^\sigma|W, g|V^{\rho\sigma\rho}\rangle}. \tag{A26}$$

We consider each of the terms on the right hand side separately:

$$\alpha(\varphi,\chi,\sigma\rho)^\rho = (\varphi\chi^{n+1})(-1)\varphi^\sigma(\eta)\chi^\sigma(\eta^{n+1}), \tag{A27}$$

$$c(g^\sigma|W,\chi^{\sigma\rho}) = i^{(n+1)k}\pi^d D^{(M)}(g^\sigma, k-n, \chi^\sigma), \tag{A28}$$

$$\lambda(\sigma) = (-1)^{nl}(\gamma(p)^2)^\sigma, \tag{A29}$$

$$\langle g^\sigma|W, g|V^{\rho\sigma\rho}\rangle = \varphi^\sigma(\eta^n)(\gamma(p)^2)^\sigma\langle g^\sigma, g^{\rho\sigma\rho}\rangle. \tag{A30}$$

We briefly point out the key steps in proving these equations:

By definition we have $\alpha(\varphi,\chi,\sigma\rho) = \varphi^{\sigma\rho}(\eta_{\sigma\rho})^{-1}\chi^{\sigma\rho}(\eta_{\sigma\rho}^{n+1})^{-1}$; we may use $-\eta_\sigma$ as $\eta_{\sigma\rho}$ to get (A27). The equation (A28) follows from the general equality (already used in (A24))

$$D^{(M)}(g|W, s^\rho, \psi)^\rho = D^{(M)}(g, s, \psi^\rho).$$

To prove (A29) we use

$$\lambda(\sigma)g^\sigma = g^\sigma|U(p^2)|_l \begin{pmatrix} -M & 0 \\ 0 & -M \end{pmatrix} = (-1)^{nl}(\gamma(p)^2)^\sigma g^\sigma.$$

Finally, (A30) involves (see [36, Lemma 5])

$$g^{\rho\sigma\rho}|_l W = \overline{\varphi}^\sigma(\eta^n)\,(g|_l W)^{\rho\sigma\rho}.$$

Putting these things together we see that $(\gamma(p)^2)^\sigma$ cancels in (A26) and we get for (A26)

$$(\varphi\chi^{n+1})(-1)\varphi^\sigma(\eta^{n+1})\chi^\sigma(\eta^{n+1})(-1)^{nl}\frac{i^{(n+1)k}\pi^d D^{(M)}(g^\sigma, k-n, \chi^\sigma)}{\langle g^\sigma, g^{\rho\sigma\rho}\rangle}.$$

Using the fact

$$(-1)^{k(n+1)}(-1)^{nl}(\varphi\chi^{n+1})(-1) = 1$$

and the property (A14) of Gauß sums, we obtain part(a) of the theorem.

The proof of (b) goes along the same lines: We take

$$G(z, w, \varphi, \chi) = \left((\pi i)^{n\nu} B_{2n}^{k}\right)^{-1} \mathcal{L}(1-t, \varphi\chi) E(z, w)$$

with

$$E(z, w) = D_{n,k}^{\nu}\left(\mathbb{F}_{2n}^{k}(M, \varphi\chi, s)^{(\chi)}\right)_{s=s_1} |^{z} U(p) |^{w} U(p).$$

Now we proceed in the same way as in case (a) using again (3.24), Corollary (5.2) and (1.30). We omit the details.

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