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CONTINUOUS MEASURES ON COMPACT LIE GROUPS

by M. ANOUSSIS and A. BISBAS

1. Introduction.

In this paper we study questions concerning continuous measures on a compact semisimple Lie group G. Analogous problems for locally compact abelian groups have been studied extensively. Results about measures on nonabelian groups may also be found in the literature. We should mention the papers of D. L. Ragozin [17] and A. H. Dooley and S. K. Gupta [5] which have motivated the present work.

In Section 2 we prove a Wiener type characterization of a continuous measure based on the system of representative functions of unitary irreducible representations of the group G. Analogous results for various classes of groups which are based on different systems of functions may be found in [1], [14], [15], [20]. To prove the results of the next two sections we introduce central measures on G which are related to the well known Riesz products on locally compact abelian groups. To construct such a measure we begin with a Riesz product μ_T on a maximal torus T of the group G and then use a result of A. H. Dooley and S. K. Gupta in [5] to obtain a central measure μ on G whose Fourier coefficients are determined in a relatively simple fashion from the Fourier coefficients of μ_T. In Section 3 we use these measures and the results of Section 2 to show that if C is a compact set of continuous measures on G there exists a singular measure ν such that

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$\nu * \mu$ is absolutely continuous with respect to the Haar measure on G for every μ in C. This result was proved by C. C. Graham and A. MacLean in [8] for locally compact abelian groups. In [1] the second named author and C. Karanikas show that this result holds for a locally compact metrizable group using a Wiener-type characterization of a continuous measure based on a system of Walsh functions. Our approach seems to be more appropriate from the point of view of Harmonic Analysis on compact semisimple Lie groups. In Section 4 we prove a factorization theorem for central functions on G. We show that if f is a finite linear combination of characters then there exist two singular measures μ and ν on G such that $f = \mu * \nu$. This result was proved by C. C. Graham and A. MacLean in [8] for infinite compact abelian groups.

In the final section we consider a symmetric space of compact type G/K. We obtain a Wiener-type characterization of a continuous measure on G/K based on the system of representative functions corresponding to the spherical representations. Our proof is based on the results of D. L. Ragozin in [18]. Taking into account [12], Ch. IV §2.3.V p. 407 and using Theorem 20 of Section 5 we may obtain another proof of Theorem 7 of Section 2. However we have chosen to include the proof of this theorem in the paper because it is constructive and based only on the representation theory of the group G. Moreover the ideas of the proof are used in Section 3 and especially in the proof of Lemma 9.

To obtain the results of Sections 3 and 4 we use in an essential way Theorem 13, which provides a central measure μ on G with predescribed Fourier coefficients. However, this theorem cannot be applied as it stands in order to prove analogous results for a symmetric space G/K since the measure μ is not in general K-invariant.

2. Wiener's theorem for a compact semisimple Lie group.

In this section we obtain a characterization of continuous measures on a compact semisimple Lie group. Our result is analogous to a well known theorem of N. Wiener which characterizes the continuous measures on the unit circle ([9], Theorem A.2.2).

Let G be a compact simply connected semisimple Lie group of rank l with Lie algebra $\mathfrak{g}$. We denote by $\widehat{G}$ the unitary dual of G. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, Φ the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$ and W the

Weyl group of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. We fix a base Γ of Φ. Let Λ be the weight lattice of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$, $\lambda_1, ..., \lambda_l$ the fundamental dominant weights and Λ_r the root lattice. Then Λ is a lattice with basis $\lambda_1, ..., \lambda_l$ ([16], 13.1) and Λ/Λ_r is a finite group. Let s be the order of Λ/Λ_r. We denote by Λ^+ the sublattice of dominant weights and by Λ_r^+ the set $\Lambda_r \cap \Lambda^+$. It is well known ([16], Ch. V, Th. 1.3 and 1.5) that $\widehat{G}$ is parametrized by Λ^+. For $\lambda \in \Lambda^+$ we denote by π_λ the element of $\widehat{G}$ which corresponds to λ, χ_λ the character of π_λ and d_λ the dimension of π_λ.

The group G is the direct product $G_1 \times ... \times G_m$ where G_k, $k = 1, ..., m$, is a compact simply connected simple Lie group ([21], Theorem 2.7.5). We will indicate by the subscribe k the objects related to the group G_k, $k = 1, ..., m$. It follows from [4], Ch. II, Proposition 4.14 that $\widehat{G} = \widehat{G_1} \otimes ... \otimes \widehat{G_m}$ and hence $\Lambda^+ = \prod_{k=1}^m \Lambda_k^+$.

LEMMA 1.

(a) *There exists a system of representatives* $\nu_1, ..., \nu_s$ *of* Λ/Λ_r *in* Λ *such that* $\nu_i \in \Lambda^+$, $i = 1, ..., s$.

(b) *Let* $\nu_1, ..., \nu_s$ *be a system of representatives of* Λ/Λ_r *in* Λ *such that* $\nu_i \in \Lambda^+$, $i = 1, ..., s$. *Then* $\Lambda^+ - \bigcup_{i=1}^s (\nu_i + \Lambda_r^+)$ *is finite.*

Proof.

(a) Let $\kappa_1, ..., \kappa_s$ be a system of representatives of Λ/Λ_r in Λ. It follows from [16], 13.2 Lemma A, that for each i, $i = 1, ..., s$, there exists an element σ_i of the Weyl group such that $\sigma_i(\kappa_i)$ is dominant. We set $\nu_i = \sigma_i(\kappa_i)$. Then κ_i and ν_i differ by an integral combination of roots and hence they are in the same coset of Λ/Λ_r in Λ.

(b) Since $\Lambda^+ = \bigcup_{i=1}^s ((\nu_i + \Lambda_r) \cap \Lambda^+)$, it suffices to show that $(\nu_i + \Lambda_r) \cap \Lambda^+ - (\nu_i + \Lambda_r^+)$ is finite, $i = 1, ..., s$. Let $\lambda \in \Lambda_r$ and $\lambda + \nu_i \in (\nu_i + \Lambda_r) \cap \Lambda^+ - (\nu_i + \Lambda_r^+)$. Then $2(\lambda + \nu_i, \alpha)/(\alpha, \alpha) \geqslant 0$ and hence $2(\lambda, \alpha)/(\alpha, \alpha) \geqslant -2(\nu_i, \alpha)/(\alpha, \alpha)$ for $\alpha \in \Gamma$. Since $2(\lambda, \alpha)/(\alpha, \alpha)$ is an integer for $\alpha \in \Gamma$, we conclude that if the numbers $2(\lambda, \alpha)/(\alpha, \alpha)$ are not all greater than zero then there exist finitely many values for the vector $(2(\lambda, \alpha)/(\alpha, \alpha))_{\alpha \in \Gamma}$. But λ is determined by this vector, and so the set $(\nu_i + \Lambda_r) \cap \Lambda^+ - (\nu_i + \Lambda_r^+)$ is finite. □

Let A be a finite subset of Λ^+. We denote by $|A|$ the cardinality of A. The proof of the following Lemma is straightforward. The symbol $\triangle$ denotes symmetric difference.

LEMMA 2. — *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two sequences of non empty finite subsets of Λ^+ such that $\lim_{n\to\infty} |A_n \triangle B_n|\ |B_n|^{-1} = 0$ and $\omega(\lambda)$ be a bounded function on Λ^+. Then $\lim_{n\to\infty} |A_n \triangle B_n||A_n|^{-1} = 0$ and the sequence $|A_n|^{-1}\sum_{\lambda\in A_n}\omega(\lambda) - |B_n|^{-1}\sum_{\lambda\in B_n}\omega(\lambda)$ converges to 0, as $n\to\infty$.*

DEFINITION 3.

a) *Assume that G is simple. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of Λ^+ is called admissible if there exists a system of representatives $\nu_1, ..., \nu_s$ of Λ/Λ_r in Λ such that $\nu_i \in \Lambda^+$, $i = 1, ..., s$ and an enumeration $\{\beta_1, \beta_2, ...\}$ of Λ_r^+ such that $\lim_{n\to\infty} |A_n \triangle B_n|\ |B_n|^{-1} = 0$, where $B_n = \{\nu_i + \beta_j : 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant n\}$.*

b) *Let G_k, $k = 1, ..., m$ be the simple factors of G. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of Λ^+ is called admissible if there exists an admissible sequence $\{B_{k,n}\}$ of Λ_k^+, $k = 1, ..., m$, such that $\lim_{n\to\infty} |A_n \triangle B_n|\ |B_n|^{-1} = 0$, where $B_n = \prod_{k=1}^m B_{k,n}$.*

We denote by e the neutral element of G and by $Z(G)$ the center of G. Let A be a finite subset of Λ^+. We define the function

$$\phi_A(g) = |A|^{-1}\sum_{\lambda\in A} {d_\lambda}^{-1}\chi_\lambda(g),\ \ g \in G.$$

It is clear that ϕ_A takes the value 1 at e and is bounded by 1. Let T be the maximal torus of G with Lie algebra $\mathfrak{t}$. For $\lambda \in \Lambda$ we denote by ξ_λ the unitary character of T with differential $\lambda_{|\mathfrak{t}}$.

LEMMA 4. — *Let $\{A_n\}_{n\in\mathbb{N}}$ be an admissible sequence of subsets of Λ^+ and $z \in Z(G)$, $z \neq e$. Then $\lim_{n\to\infty}\phi_{A_n}(z) = 0$.*

Proof. — Assume first that G is simple. The space of the representation π_λ is the highest weight module V_λ with highest weight λ. It is well known that $z \in T$ and if v_λ is a highest weight vector in V_λ, we have $\pi_\lambda(z)v_\lambda = \xi_\lambda(z)v_\lambda$. Since, by Schur's Lemma, $\pi_\lambda(z)$ is a scalar multiple of the identity we conclude that $\pi_\lambda(z) = \xi_\lambda(z)\,\mathrm{Id}$ and $\chi_\lambda(z) = d_\lambda\xi_\lambda(z)$. Since $\{A_n\}_{n\in\mathbb{N}}$ is an admissible sequence of subsets of Λ^+ there exists a system of representatives $\nu_1, ..., \nu_s$ of Λ/Λ_r in Λ such that $\nu_i \in \Lambda^+$, $i = 1, ..., s$ and an enumeration $\{\beta_1, \beta_2, ...\}$ of Λ_r^+ such that $\lim_{n\to\infty} |A_n \triangle B_n|\ |B_n|^{-1} = 0$, where $B_n = \{\nu_i + \beta_j : 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant n\}$. By Lemma 2 it suffices to show that $\lim_{n\to\infty}\phi_{B_n}(z) = 0$.

The mapping ξ_z defined by $\xi_z(\lambda) \longmapsto \xi_\lambda(z)$ is a unitary character of Λ and its kernel contains Λ_r. Note that since $z \neq e$ it follows from [7], IX 3.9, that ξ_z is non-trivial. The character ξ_z is trivial on Λ_r and induces a unitary character ξ'_z of Λ/Λ_r. The image of ξ'_z is a finite subgroup of the group of complex numbers of modulus 1 and hence it is cyclic. Since $\xi'_z(\Lambda/\Lambda_r) = \xi_z(\{\nu_1, ..., \nu_s\})$ it follows that $\sum_{i=1}^{s} \xi_{\nu_i}(z) = 0$. So

$$\phi_{B_n}(z) = |B_n|^{-1} \sum_{\lambda \in B_n} d_\lambda{}^{-1} \chi_\lambda(z) = |B_n|^{-1} \sum_{\lambda \in B_n} \xi_\lambda(z)$$
$$= (sn)^{-1}(n \sum_{i=1}^{s} \xi_{\nu_i}(z)) = 0.$$

We consider now the general case. Let G_k, $k = 1, ..., m$ be the simple factors of G. There exists an admissible sequence $\{B_{k,n}\}$ of Λ_k^+, $k = 1, ..., m$ such that $\lim_{n\to\infty} |A_n \triangle B_n|\, |B_n|^{-1} = 0$, where $B_n = \prod_{k=1}^{m} B_{k,n}$. It follows from Lemma 2 that it suffices to show that $\lim_{n\to\infty} \phi_{B_n}(z) = 0$. Since $Z(G) = Z(G_1) \times ... \times Z(G_m)$, there exist $z_k \in Z(G_k)$, $k = 1, ..., m$ such that $z = (z_1, .., z_m)$. We have $\phi_{B_n}(z) = \prod_{k=1}^{m} \phi_{B_{k,n}}(z_k)$ and it follows from above that $\lim_{n\to\infty} \phi_{B_n}(z) = 0$. □

LEMMA 5. — *Let $\{A_n\}_{n\in\mathbb{N}}$ be an admissible sequence of subsets of Λ^+ and $g \in G - Z(G)$. Then $\lim_{n\to\infty} \phi_{A_n}(g) = 0$.*

Proof. — Let G_k, $k = 1, ..., m$ be the simple factors of G. There exists an admissible sequence $\{B_{k,n}\}$ of Λ_k^+, $k = 1, ..., m$, such that $\lim_{n\to\infty} |A_n \triangle B_n||B_n|^{-1} = 0$, where $B_n = \prod_{k=1}^{m} B_{k,n}$. It follows from Lemma 2 that it suffices to show that $\lim_{n\to\infty} \phi_{B_n}(g) = 0$. Let $g = (g_1, .., g_m)$ with $g_k \in G_k$, $k = 1, ..., m$. We have $\phi_{B_n}(g) = \prod_{k=1}^{m} \phi_{B_{k,n}}(g_k)$. Since $Z(G) = Z(G_1) \times ... \times Z(G_m)$, there exists an k_0 such that $g_{k_0} \notin Z(G_{k_0})$. It follows from [19], Lemma 11, that $\lim_{n\to\infty} \phi_{B_{k_0,n}}(g_{k_0}) = 0$ and hence $\lim_{n\to\infty} \phi_{B_n}(g) = 0$. □

Remark 1. — D. Rider proves Lemma 11 in [19] based on the results of D. L. Ragozin on central measures [17]. Recently K. E. Hare has given a proof of this result using the structure theory of compact simple Lie groups [10].

Let $M(G)$ be the algebra of regular Borel measures on G. We denote by $\|\mu\|$ the total variation of a measure $\mu \in M(G)$. For $\pi \in \widehat{G}$ and $\mu \in M(G)$ we define $\widehat{\mu}(\pi) = \int_G \pi(g^{-1}) d\mu(g)$.

PROPOSITION 6. — *Let $\mu \in M(G)$ and $\{A_n\}_{n\in\mathbb{N}}$ an admissible sequence of subsets of Λ^+. Then*

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} d_\lambda{}^{-1} \operatorname{Tr}\widehat{\mu}(\pi_\lambda) = \mu(\{e\}).$$

Proof. — Let $\nu = \mu - \mu(\{e\})\delta_e$, where δ_e is the Dirac measure at e. It follows from Lemmas 4 and 5 and Lebesgue's Dominated Convergence Theorem that $\lim_{n\to\infty} \int_G \phi_{A_n}(g^{-1})d\nu(g) = 0$. Hence

$$\lim_{n\to\infty} \int_G \phi_{A_n}(g^{-1})d\mu(g) = \mu(\{e\}).$$

Since $\int_G \phi_{A_n}(g^{-1})d\mu(g) = |A_n|^{-1}\sum_{\lambda\in A_n} d_\lambda{}^{-1} \operatorname{Tr}\widehat{\mu}(\pi_\lambda)$, the proposition follows. □

Let $\mu \in M(G)$. We denote by $\mu^\sim$ the measure on G defined by $\int_G f(g)d\mu^\sim(g) = \int_G \overline{f(g^{-1})}d\mu(g)$. If X is an operator on a finite dimensional Hilbert space we denote by $\|X\|_2$ the Hilbert-Schmidt norm of X.

THEOREM 7. — *Let $\mu \in M(G)$ and $\{A_n\}_{n\in\mathbb{N}}$ be an admissible sequence of subsets of Λ^+. Then*

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} d_\lambda{}^{-1} \|\widehat{\mu}(\pi_\lambda)\|_2^2 = \sum_{g\in G} |\mu(\{g\})|^2.$$

(The sum on the righthand side of the above equality is taken over the set $\{g \in G : \mu(\{g\}) \neq 0\}$ which is of course a countable set).

Proof. — Apply the formula of Proposition 6 to the measure $\mu^\sim * \mu$. □

Let $\mathcal{R}_p$, $1 \leqslant p \leqslant \infty$ be the set of measures $\mu \in M(G)$ such that $\lim_{\lambda\to\infty} d_\lambda{}^{-1/p}\|\widehat{\mu}(\pi_\lambda)\|_p = 0$, where $\|.\|_p$ is the p-norm. The following corollary is proved by M. Blümlinger in [2] in the more general context of compact groups. For $p = \infty$ it follows from earlier results of C. F. Dunkl and D. E. Ramirez in [6].

COROLLARY 8. — *Let $\mu \in \mathcal{R}_p, 1 \leqslant p \leqslant \infty$. Then μ is continuous.*

Proof. — It follows from the fact that $\mathcal{R}_p \subseteq \mathcal{R}_2$, for $1 \leqslant p \leqslant \infty$ [2] and Theorem 7. □

Remark 2. — We recall that a measure $\mu \in M(G)$ is central if $\mu(gXg^{-1}) = \mu(X)$ for all Borel subsets X of G and for all $g \in G$. Note that if μ is central it follows from Schur's Lemma that we have $\widehat{\mu}(\pi_\lambda) = {d_\lambda}^{-1}\operatorname{Tr}\widehat{\mu}(\pi_\lambda)\operatorname{Id}$ and hence

$$\|\widehat{\mu}(\pi_\lambda)\|_2^2 = d_\lambda|{d_\lambda}^{-1}\operatorname{Tr}\widehat{\mu}(\pi_\lambda)|^2 = {d_\lambda}^{-1}|\operatorname{Tr}\widehat{\mu}(\pi_\lambda)|^2.$$

Assume that G is simple and let μ be a central measure on G. Consider the following conditions:

i) $\lim_{\lambda\to\infty} |{d_\lambda}^{-1}\operatorname{Tr}\widehat{\mu}(\pi_\lambda)| = 0.$

ii) $\lim_{n\to\infty} |A_n|^{-1}\sum_{\lambda\in A_n} {d_\lambda}^{-2}|\operatorname{Tr}\widehat{\mu}(\pi_\lambda)|^2 = 0.$

Condition (i) appears to be stronger than condition (ii). However, it follows from [17], Corollary 3.5, and Theorem 7 that they are in fact equivalent. Hence a central measure μ on G is continuous if ond only if $\lim_{\lambda\to\infty} |{d_\lambda}^{-1}\operatorname{Tr}\widehat{\mu}(\pi_\lambda)| = 0$, in contrast with what happens in the abelian case. In fact, there exist continuous measures on the unit circle whose Fourier-Stieltjes coefficients do not converge to 0 ([9], 7.1).

3. The multiplier theorem.

Let dg be the Haar measure on G of total mass 1. We denote by $M_c(G)$ the subalgebra of continuous measures of $M(G)$. In this section we are interested in the following problem:

Given a compact subset C of $M_c(G)$, construct a measure $\nu \in M_c(G)$, singular with respect to the Haar measure on G, such that $\nu*\mu$ is absolutely continuous for every $\mu \in C$.

We recall that $\delta = (1/2)\sum_{\alpha\in\Gamma}\alpha$. We set $\Gamma^- = \{-\alpha : \alpha \in \Gamma\}$. Let W be the Weyl group of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. The group W acts simply transitively on the set of bases of the root system Φ ([16], Theorem 10.3). It follows that there exists a unique element $\sigma \in W$ such that $\sigma(\Gamma) = \Gamma^-$ and $\sigma^2 = \operatorname{Id}$. Let $\lambda_1, ..., \lambda_l$ be the fundamental dominant weights. Since σ is a bijection from Γ onto Γ^- we have $\{\sigma\lambda_1, ..., \sigma\lambda_l\} = \{-\lambda_1, ..., -\lambda_l\}$. We set $F_n = \{\sum_{i=1}^l m_i\lambda_i : m_i = 0, 1, ..., n\}$, $n \in \mathbb{N}$, $F_1^0 = \emptyset$, $F_n^0 = \{\sum_{i=1}^l m_i\lambda_i : m_i = 1, ..., n-1\}$, $n \in \mathbb{N}$, $n \geqslant 2$. It is clear that $\sigma(F_n) = -F_n$, $\sigma(F_n^0) = -F_n^0$.

LEMMA 9. — *Let $\mu \in M_c(G)$. Then*

(a) $\lim_{n\to\infty} |F_n|^{-1}\sum_{\lambda\in F_n} {d_\lambda}^{-1}\|\widehat{\mu}(\pi_\lambda)\|_2^2 = 0.$

(b) $\lim_{n\to\infty} |F_n|^{-1}\sum_{\lambda\in F_n} (d_{-\sigma\lambda})^{-1}\|\widehat{\mu}(\pi_{-\sigma\lambda})\|_2^2 = 0.$

Proof.

(a) Assume first that G is simple. We consider a system of representatives $\nu_1, ..., \nu_s$ of Λ/Λ_r in Λ such that $\nu_i \in \Lambda^+$, $i = 1, ..., s$. Then by Lemma 1(b) the set $\Omega = \Lambda^+ - \bigcup_{i=1}^{s}(\nu_i + \Lambda_r^+)$ is finite. Let $d_n = |F_n \cap \Lambda_r^+|$ and choose an enumeration $\{\beta_1, \beta_2, ...\}$ of Λ_r^+ such that $\{\beta_1, ...\beta_{d_n}\} = F_n \cap \Lambda_r^+$, $n \in \mathbb{N}$. The sequence $\{B_n\}_{n\in\mathbb{N}}$ defined by $B_n = \{\nu_i + \beta_j : 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant n\}$ is an admissible sequence of subsets of Λ^+. We show that there exists $q \in \mathbb{N}$ such that $\lim_{n\to\infty} |F_{n+q} \triangle B_{d_n}||F_{n+q}|^{-1} = 0$. We have $\nu_i = \sum_{j=1}^{l} m_{ij}\lambda_j$, where $m_{ij} \in \mathbb{N} \cup \{0\}$, $i = 1, ..., s$, $j = 1, ..., l$. Let $K_j = \max\{m_{ij} : i = 1, ..., s\}$, $L_j = \min\{m_{ij} : i = 1, ..., s\}$, $j = 1, ..., l$ and

$$\Phi_n = \Big\{\lambda \in \Lambda : \lambda = \sum_{j=1}^{l} m_j\lambda_j : m_j \geqslant 0, K_j \leqslant m_j \leqslant n + L_j, j = 1, ..., l\Big\},$$

$n \in \mathbb{N}$, $n \geqslant \max\{K_j - L_j : j = 1, ..., l\}$. Assume that $\lambda \in \Phi_n - \Omega$. This means that $\lambda \in \nu_i + F_n$ for every i and since $\lambda \notin \Omega$ we see that $\lambda \in \nu_i + \Lambda_r^+$ for some i. We conclude that $\lambda \in \nu_i + (F_n \cap \Lambda_r^+) \subseteq B_{d_n}$ and so $\Phi_n - \Omega \subseteq B_{d_n}$. If we set $q = \max\{K_j : j = 1, ..., l\}$ then $B_{d_n} \subseteq F_{n+q}$. Hence

$$\begin{aligned}
|F_{n+q} \triangle B_{d_n}|\ |F_{n+q}|^{-1} &= |F_{n+q} - B_{d_n}|\ |F_{n+q}|^{-1} \\
&\leqslant |F_{n+q} - (\Phi_n - \Omega)|\ |F_{n+q}|^{-1} \\
&= (|F_{n+q}| - |(\Phi_n - \Omega)|)|F_{n+q}|^{-1} \\
&\leqslant (|F_{n+q}| - (|\Phi_n| - |\Omega|))|F_{n+q}|^{-1} \\
&\leqslant (n+q+1)^{-l}((n+q+1)^l - \prod_{j=1}^{l}(n+L_j-K_j+1) \\
&\qquad + |\Omega|) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Let $g \in G$, $g \neq e$. It follows from Lemmas 4 and 5 that $\lim_{n\to\infty} \phi_{B_n}(g) = 0$. Hence $\lim_{n\to\infty} \phi_{f_{d_n}}(g) = 0$ and it follows from Lemma 2 that $\lim_{n\to\infty} \phi_{F_n}(g) = \lim_{n\to\infty} \phi_{F_{n+k}}(g) = 0$.

We consider now the general case. Let G_k, $k = 1, ..., m$ be the simple factors of G and $g \in G$, $g \neq e$. Then $g = (g_1, .., g_m)$ with $g_k \in G_k$, $k = 1, ..., m$. We have $\phi_{F_n}(g) = \prod_{k=1}^{m} \phi_{F_{k,n}}(g_k)$. Since $g \neq e$ there exists an k_0 such that $g_{k_0} \neq e$. It follows from above that $\lim_{n\to\infty} \phi_{F_{k_0,n}}(g_{k_0}) = 0$ and consequently $\lim_{n\to\infty} \phi_{F_n}(g) = 0$.

As in Theorem 7 we see that $\lim_{n\to\infty} |F_n|^{-1} \sum_{\lambda\in F_n} d_\lambda{}^{-1}\|\widehat{\mu}(\pi_\lambda)\|_2^2 = 0$.

(b) Since $\{\sigma\lambda_1, ..., \sigma\lambda_l\} = \{-\lambda_1, ..., -\lambda_l\}$ we have $\{\pi_{-\sigma\lambda} : \lambda \in F_n\} = \{\pi_\lambda : \lambda \in F_n\}$. The assertion now follows from (a). $\square$

LEMMA 10. — *Let C be a compact subset of $M_c(G)$, $\varepsilon > 0$ and $m \in \mathbb{N}$. There exists $\xi \in \Lambda^+$ such that the following conditions are satisfied:*

i) $(\xi + F_m^0) \cap F_m = \emptyset$.

ii) $d_\lambda{}^{-1}\|\widehat{\mu}(\pi_\lambda)\|_2^2 < \varepsilon$ *for* $\lambda \in \xi + F_m^0$ *and* $\mu \in C$.

iii) $(d_{-\sigma\lambda})^{-1}\|\widehat{\mu}(\pi_{-\sigma\lambda})\|_2^2 < \varepsilon$ *for* $\lambda \in \xi + F_m^0$ *and* $\mu \in C$.

Proof. — There exist continuous measures $\mu_1, ..., \mu_r$ in C such that for any $\mu \in C$ there exists μ_i, $i = 1, ..., r$, with $\|\mu - \mu_i\| < \varepsilon$. Since $d_\lambda{}^{-1/2}\|\widehat{\mu}(\pi_\lambda)\|_2 \leqslant \|\mu\|$, it suffices to show that there exists $\xi \in \Lambda^+$ such that the following conditions are satisfied:

i) $(\xi + F_m^0) \cap F_m = \emptyset$.

ii) $d_\lambda{}^{-1}\|\widehat{\mu_i}(\pi_\lambda)\|_2^2 < \varepsilon$ for $\lambda \in \xi + F_m^0$ and $i = 1, ..., r$.

iii) $(d_{-\sigma\lambda})^{-1}\|\widehat{\mu_i}(\pi_{-\sigma\lambda})\|_2^2 < \varepsilon$ for $\lambda \in \xi + F_m^0$ and $i = 1, ..., r$.

Let $k \in \mathbb{N}$. There exist $\xi_j \in F_{km}$, $j = 1, ..., k^l - 1$ such that $\xi_j + F_m^0 \subseteq F_{km}$, $(\xi_j + F_m^0) \cap F_m = \emptyset$ for every j with $1 \leqslant j \leqslant k^l - 1$ and $(\xi_j + F_m^0) \cap (\xi_{j'} + F_m^0) = \emptyset$ for every pair (j, j') with $1 \leqslant j < j' \leqslant k^l - 1$. Then $-\sigma(\xi_j + F_m^0) \subseteq F_{km}$, $(-\sigma(\xi_j + F_m^0)) \cap (-\sigma(F_m)) = \emptyset$ for every j with $1 \leqslant j \leqslant k^l - 1$ and $(-\sigma(\xi_j + F_m^0)) \cap (-\sigma(\xi_{j'} + F_m^0) = \emptyset$ for every pair (j, j') with $1 \leqslant j < j' \leqslant k^l - 1$. Assume that for every j with $1 \leqslant j \leqslant k^l - 1$, there exists i, $1 \leqslant i \leqslant r$, such that either $\xi_j + F_m^0$ contains a λ with $d_\lambda{}^{-1}\|\widehat{\mu_i}(\pi_\lambda)\|_2^2 \geqslant \varepsilon$ or $\xi_j + F_m^0$ contains a λ with $(d_{-\sigma\lambda})^{-1}\|\widehat{\mu_i}(\pi_{-\sigma\lambda})\|_2^2 \geqslant \varepsilon$. We get

$$\begin{aligned}
&\sum_{i=1}^{r} |F_{km}|^{-1} \sum_{\lambda \in F_{km}} (d_\lambda{}^{-1}\|\widehat{\mu_i}(\pi_\lambda)\|_2^2 + (d_{-\sigma\lambda})^{-1}\|\widehat{\mu_i}(\pi_{-\sigma\lambda})\|_2^2) \\
&\quad\geqslant |F_{km}|^{-1} \sum_{i=1}^{r} \sum_{j=1}^{k^l-1} \sum_{\lambda \in \xi_j + F_m^0} (d_\lambda{}^{-1}\|\widehat{\mu_i}(\pi_\lambda)\|_2^2 + (d_{-\sigma\lambda})^{-1}\|\widehat{\mu_i}(\pi_{-\sigma\lambda})\|_2^2) \\
&\quad\geqslant |F_{km}|^{-1} \sum_{j=1}^{k^l-1} \sum_{i=1}^{r} \sum_{\lambda \in \xi_j + F_m^0} (d_\lambda{}^{-1}\|\widehat{\mu_i}(\pi_\lambda)\|_2^2 + (d_{-\sigma\lambda})^{-1}\|\widehat{\mu_i}(\pi_{-\sigma\lambda})\|_2^2) \\
&\quad\geqslant (km+1)^{-l}(k^l - 1)\varepsilon.
\end{aligned}$$

But $(km+1)^{-l}(k^l - 1)\varepsilon \to m^{-l}\varepsilon$ as $k \to \infty$. This is in contradiction with Lemma 9. □

LEMMA 11. — *Let C be a compact subset of $M_c(G)$. There exists a sequence $(\xi_j, m_j, k_j, \psi_j)_{j \in \mathbb{N}}$ such that*

(i) $\xi_j \in \Lambda^+$, $\psi_j \in \Lambda^+$, $m_j \in \mathbb{N}, (m_j)_{j\in\mathbb{N}}$ *is strictly increasing,* $k_j \in \mathbb{N}$, k_j *is the smallest number such that* F_{k_j} *contains* 0 *and* $\xi_j + \delta + F_{m_j}$.

(ii) $\psi_j \in \xi_j + F_{m_j}$.

(iii) $\psi_{j+1} \pm F_{k_j} \subseteq \xi_{j+1} + F_{m_{j+1}}$.

(iv) $F_{k_j} \cap (\xi_{j+1} + F_{m_{j+1}}) = \emptyset$.

(v) $d_{\pi_\lambda}{}^{-1}\|\widehat{\mu}(\pi_\lambda)\|_2^2 < 3^{-m_j}$ *and* $(d_{-\sigma_\lambda})^{-1}\|\widehat{\mu}(\pi_{-\sigma\lambda})\|_2^2 < 3^{-m_j}$ *for* $\lambda \in (\xi_j + F_{m_j}^0)$ *and* $\mu \in C$.

Proof. — We prove the lemma by induction. We set $m_1 = 4$. It follows from Lemma 10 that there exists $\xi_1 \in \Lambda^+$ such that condition (v) is satisfied. Let ψ_1 be the center of $\xi_1 + F_{m_1}$ and k_1 be the smallest number such that F_{k_1} contains 0 and $\xi_1+\delta+F_{m_1}$. Then conditions (i), (ii) and (v) are satisfied. Suppose that we have constructed $(\xi_j, m_j, k_j, \psi_j)$ for $j = 1, ..., n$. Put $m_{n+1} = 4k_n$. It follows from Lemma 10 that there exists $\xi_{n+1} \in \Lambda^+$ such that (iv) and (v) are satisfied. Take $\psi_{n+1}$to be the center of $\xi_{n+1}+F_{m_{n+1}}$ and let k_{n+1} be the smallest number such that $F_{k_{n+1}}$ contains 0 and $\xi_{n+1} + \delta + F_{m_{n+1}}$. Then (i), (ii) and (iii) are satisfied. □

We recall that for $\lambda \in \Lambda$ we denote by ξ_λ the unitary character of T with differential $\lambda_{|\mathfrak{t}}$. The mapping $\lambda \mapsto \xi_\lambda$ is a bijection from Λ onto the dual group of T. We identify the group Λ with the dual group of T via this mapping. For the definition of dissociate sets the reader is referred to [9], 7.1.

LEMMA 12. — *Let* C *be a compact subset of* $M_c(G)$ *and* $(\xi_j, m_j, k_j, \psi_j)_{j\in\mathbb{N}}$ *be a sequence which satisfies the conditions (i) up to (iv) of Lemma 11. We set* $\Psi = \{\psi_j + \delta : j \in \mathbb{N}\}$ *and* $\Theta(\Psi) = \{\sum_{j=1}^n \varepsilon_j(\psi_j + \delta) : n \in \mathbb{N}$, $\varepsilon_j = 0, 1$ *or* -1, $j = 1, ..., n\}$. *Then*

(a) *The set* $\Theta(\Psi)$ *is contained in* $\{0\} \bigcup (\bigcup_{j=1}^\infty (\xi_j + \delta + F_{m_j})) \bigcup (- \bigcup_{j=1}^\infty (\xi_j + \delta + F_{m_j}))$.

(b) *The set* Ψ *is a dissociate subset of* Λ.

Proof. — Let $\sum_{j=1}^n \varepsilon_j(\psi_j + \delta) \in \Theta(\Psi)$ with $\varepsilon_n \neq 0$. We show by induction that $\sum_{j=1}^n \varepsilon_j(\psi_j + \delta) \in \varepsilon_n(\xi_n + \delta + F_{m_n})$. If $n = 1$ it follows from condition (ii) that $\varepsilon_1(\psi_1 + \delta) \in \varepsilon_1(\xi_1 + \delta + F_{m_1})$. Assume that the assertion is true for $1, ..., n-1$. Let $n' = \max\{j : 1 \leqslant j \leqslant n-1, \varepsilon_j \neq 0\}$. Then by the induction hypothesis $\sum_{j=1}^{n-1} \varepsilon_j(\psi_j + \delta) = \sum_{j=1}^{n'} \varepsilon_j(\psi_j + \delta) \in \varepsilon_{n'}(\xi_{n'} + \delta + F_{m_{n'}})$. By condition (i) we have $\varepsilon_{n'}(\xi_{n'} + \delta + F_{m_{n'}}) \subseteq \varepsilon_{n'} F_{k_{n'}}$

which is contained in $\varepsilon_{n'}F_{k_{n-1}}$ since by conditions (i) and (iv) the sequence $(k_j)_{j\in\mathbb{N}}$ is increasing. Then $\sum_{j=1}^{n}\varepsilon_j(\psi_j+\delta)\in\varepsilon_n(\psi_n+\delta)+\varepsilon_{n'}F_{k_{n-1}}$ which by condition (iii) is contained in $\varepsilon_n(\xi_n+\delta+F_{m_n})$.

(a) It follows immediately from above.

(b) Consider r,s in $\mathbb{N}$. Let $\varepsilon_j=0,1$ or -1 for $n=0,...,r$, $\varepsilon_r\neq 0$ and $\varepsilon'_j=0,1$ or -1 for $n=0,...,s$, $\varepsilon_s\neq 0$. To show that Ψ is dissociate it suffices to show that if $\sum_{j=1}^{r}\varepsilon_j(\psi_j+\delta)=\sum_{n=1}^{s}\varepsilon'_j(\psi_j+\delta)$ then $r=s$ and $\varepsilon_r=\varepsilon'_s$. It follows from above that $\sum_{j=1}^{r}\varepsilon_j(\psi_j+\delta)\in\varepsilon_r(\xi_r+\delta+F_{m_r})$ and that $\sum_{j=1}^{s}\varepsilon'_j(\psi_j+\delta)\in\varepsilon'_s(\xi_s+\delta+F_{m_s})$. If $\varepsilon_r\varepsilon'_s=-1$ we have $\varepsilon_r(\xi_r+\delta+F_{m_r})\cap\varepsilon'_s(\xi_s+\delta+F_{m_s})=\emptyset$. We conclude that $\varepsilon_r=\varepsilon'_s$. Let $r>s$. Then $\xi_s+F_{m_s}\subseteq F_{k_s}$ and by condition (iv) $(\xi_r+F_{m_r})\cap(\xi_s+F_{m_s})=\emptyset$ and hence $\varepsilon_r(\xi_r+\delta+F_{m_r})\cap\varepsilon'_s(\xi_s+\delta+F_{m_s})=\emptyset$. Similarly we show that we cannot have $s>r$. We conclude that $r=s$. □

The following result is proved in [5], p. 3117.

THEOREM 13. — *Let ν_T be a singular, probability measure on T. There exists a central measure ν on G singular with respect to the Haar measure dg such that*

$$\widehat{\nu}(\pi_\lambda)=M{d_\lambda}^{-1}\sum_{w\in W}\operatorname{sgn}(w)\widehat{\nu_T}(w(\lambda+\delta))\,\mathrm{Id},$$

for $\lambda\in\Lambda^+$ and where M is a constant.

If X is an operator on a finite dimensional Hilbert space we denote by $\|X\|$ the norm of X.

THEOREM 14. — *Let C be a compact subset of $M_c(G)$. There exists a central, singular measure ν on G, $\nu\neq 0$ such that $\nu*\mu$ is absolutely continuous with respect to dg for every $\mu\in C$.*

Proof. — Let $(\xi_j,m_j,k_j,\psi_j)_{j\in\mathbb{N}}$, Ψ, $\Theta(\Psi)$ be as in Lemma 12. Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of complex numbers such that $\sum_{j=1}^{\infty}|a_j|^2=+\infty$, $|a_j|\leqslant 1/2$, $j\in\mathbb{N}$ and α be the function on Ψ defined by $\alpha(\psi_j+\delta)=a_j$, $j\in\mathbb{N}$. We denote by ν_T the Riesz product based on Ψ and α. (For the definition of Riesz products see [9], 7.1). Then ν_T is a probability measure on T and the support of $\widehat{\nu_T}$ is contained in $\Theta(\Psi)$. It follows from [9], 7.1.4 and 7.2.2 that ν_T is a continuous measure singular to the Haar measure on T and from Theorem 13 that there exists a singular, central measure ν on G such that $\widehat{\nu}(\pi_\lambda)=M{d_\lambda}^{-1}\sum_{w\in W}\operatorname{sgn}(w)\widehat{\nu_T}(w(\lambda+\delta))\,\mathrm{Id}$ for $\lambda\in\Lambda^+$ and where M is a constant. Let $W(\Gamma)$ (resp. $W(\Gamma^-)$) be the Weyl chamber which corresponds to the base Γ (resp. Γ^-) of the root system Φ. By Lemma 12, $\widehat{\nu_T}$ is supported in $W(\Gamma) \cup W(\Gamma^-)$ and since the Weyl group W acts simply transitively on the set of Weyl chambers [16], 10.3, $\widehat{\nu_T}(w(\lambda+\delta)) = 0$ for any w in W, $w \notin \{\sigma, \mathrm{Id}\}$. Hence

$$\widehat{\nu}(\pi_\lambda) = M{d_\lambda}^{-1}(\widehat{\nu_T}(\lambda+\delta) + \mathrm{sgn}(\sigma)\widehat{\nu_T}(\sigma(\lambda+\delta)))\,\mathrm{Id}\,.$$

We are going to show that we may choose the sequence $\{a_j\}_{j\in\mathbb{N}}$ in such a way that $\nu \neq 0$. We have $\widehat{\nu}(\pi_{\psi_1}) = M{d_\lambda}^{-1}(\widehat{\nu_T}(\psi_1+\delta) + \mathrm{sgn}(\sigma)\widehat{\nu_T}(\sigma(\psi_1+\delta)))\,\mathrm{Id}$. If $\sigma \neq -\mathrm{Id}$ taking $a_j = (1/2)j^{-(1/2)}$, we have $\widehat{\nu}(\pi_{\psi_1}) \neq 0$. If $\sigma = -\mathrm{Id}$ then if we take $a_1 \neq \pm\overline{a_1}$ we obtain $\widehat{\nu}(\pi_{\psi_1}) \neq 0$.

We are going to show that $\nu * \mu$ is absolutely continuous with respect to dg for every $\mu \in C$. From [13], Theorem 28.43, it suffices to show that

$$\sum_{\lambda\in\Lambda^+} d_\lambda \|\widehat{\nu*\mu}(\pi_\lambda)\|_2^2 < +\infty.$$

We have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda^+} d_\lambda \|\widehat{\nu*\mu}(\pi_\lambda)\|_2^2 \\
&\quad= \sum_{\lambda\in\Lambda^+} d_\lambda \|\widehat{\nu}(\pi_\lambda)\|^2 \|\widehat{\mu}(\pi_\lambda)\|_2^2 \\
&\quad= M^2 \sum_{\lambda\in\Lambda^+} |\widehat{\nu_T}(\lambda+\delta) + \mathrm{sgn}(\sigma)\widehat{\nu_T}(\sigma(\lambda+\delta))|^2 {d_\lambda}^{-1} \|\widehat{\mu}(\pi_\lambda)\|_2^2 \\
&\quad\leqslant 2M^2 \sum_{\lambda\in\Lambda^+} {d_\lambda}^{-1}(|\widehat{\nu_T}(\lambda+\delta)|^2 + |\widehat{\nu_T}(\sigma(\lambda+\delta))|^2)\|\widehat{\mu}(\pi_\lambda)\|_2^2.
\end{aligned}$$

It follows from Lemmas 11 and 12

$$\begin{aligned}
\sum_{\lambda\in\Lambda^+} |\widehat{\nu_T}(\lambda+\delta)|^2 {d_\lambda}^{-1} \|\widehat{\mu}(\pi_\lambda)\|_2^2 &= \sum_{j=1}^{\infty} \sum_{\lambda\in\xi_j+F_{m_j}} |\widehat{\nu_T}(\lambda+\delta)|^2 {d_\lambda}^{-1} \|\widehat{\mu}(\pi_\lambda)\|_2^2 \\
&\leqslant \sum_{j=1}^{\infty} (m_j+1)^l 3^{-m_j} < +\infty.
\end{aligned}$$

The mapping $\lambda \mapsto -\sigma(\lambda)$ is a bijection of Λ^+ and $\sigma(\delta) = -\delta$. Hence

$$\begin{aligned}
&\sum_{\lambda\in\Lambda^+} |\widehat{\nu_T}(\sigma(\lambda+\delta))|^2 {d_\lambda}^{-1} \|\widehat{\mu}(\pi_\lambda)\|_2^2 \\
&\qquad= \sum_{\lambda\in\Lambda^+} |\widehat{\nu_T}(-(\lambda+\delta))|^2 (d_{-\sigma\lambda})^{-1} \|\widehat{\mu}(\pi_{-\sigma\lambda})\|_2^2.
\end{aligned}$$

Since $-(\lambda+\delta) \in -(\xi_i+\delta+F_{m_j})$ if and only if $\lambda \in (\xi_i+F_{m_j})$ we get from

Lemmas 11 and 12

$$\begin{aligned}\sum_{\lambda\in\Lambda^+} &|\widehat{\nu_T}(\sigma(\lambda+\delta))|^2 d_\lambda{}^{-1}\|\widehat{\mu}(\pi_\lambda)\|_2^2 \\ &= \sum_{j=1}^{\infty}\sum_{\lambda\in\xi_j+F_{m_j}} |\widehat{\nu_T}(-(\lambda+\delta))|^2 (d_{-\sigma_\lambda})^{-1}\|\widehat{\mu}(\pi_{-\sigma\lambda})\|_2^2 \\ &\leqslant \sum_{j=1}^{\infty}(m_j+1)^l 3^{-m_j} < +\infty.\end{aligned}$$

We conclude that $\sum_{\lambda\in\Lambda^+} d_\lambda \|\widehat{\nu*\mu}(\pi_\lambda)\|_2^2 < +\infty$. □

4. The factorization theorem.

In [8] C. C. Graham and A. MacLean show that if f is a trigonometric polynomial on the unit circle $\mathbb{T}$ then there exist two singular measures μ and ν on $\mathbb{T}$ such that $f = \mu * \nu$. In this section we prove an analogous factorization result. Let $\lambda \in \Lambda$. Then there exist real numbers $x_1, ..., x_l$ such that $\lambda = \sum_{i=1}^{l} x_i\lambda_i$. We set $|\lambda|_1 = \sum_{i=1}^{l}|x_i|$. Since $\{\sigma\lambda_1, ..., \sigma\lambda_l\} = \{-\lambda_1, ..., -\lambda_l\}$ we have $|\sigma(\lambda)|_1 = |\lambda|_1$ for $\lambda \in \Lambda$.

LEMMA 15. — *Let $q \in \mathbb{N}$, $q \geqslant 3$. There exist singular, central measures μ_0 and ν_0 on G such that*

a) $\int_G d\mu_0(g) = \int_G d\nu_0(g) = 1$.

b) *If κ, η are in Λ^+, $(\kappa,\eta) \neq (0,0)$ and $\widehat{\mu_0}(\pi_\kappa)\widehat{\nu_0}(\pi_\eta) \neq 0$ then $|\kappa-\eta|_1 \geqslant (q-2)l$.*

Proof. — Let $\Psi = \{\delta\} \cup \{q^n\delta : n \in \mathbb{N}\}$. It is easy to see that Ψ is a dissociate set. Let $\{a_n\}_{n\geqslant 0}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty}|a_n|^2 = +\infty$, $|a_n| \leqslant 1/2$, $n = 0, 1, ...$ and α the function on Ψ defined by $\alpha(q^n\delta) = a_n$, $n = 0, 1, ...$ We set $\Psi_1 = \{\delta\} \cup \{q^{2n-1}\delta : n \in \mathbb{N}\}$, $\Psi_2 = \{\delta\} \cup \{q^{2n}\delta : n \in \mathbb{N}\}$. Let μ_T and ν_T be the Riesz products based on Ψ_1 and $\alpha_{|\Psi_1}$ and Ψ_2 and $\alpha_{|\Psi_2}$ respectively. Then μ_T and ν_T are probability measures on T such that $\widehat{\mu_T}$ has support $\Theta(\Psi_1) = \{\sum_{n=1}^{k}\varepsilon_n q^{2n-1}\delta + \varepsilon_0\delta : k \in \mathbb{N},\ \varepsilon_n = 0,\ 1 \text{ or } -1,\ n = 0, 1, ..., k\}$ and $\widehat{\nu_T}$ has support $\Theta(\Psi_2) = \{\sum_{n=1}^{k}\varepsilon_n q^{2n}\delta + \varepsilon_0\delta : k \in \mathbb{N},\ \varepsilon_n = 0,\ 1 \text{ or } -1,\ n = 0, 1, ..., k\}$. It follows from [9], 7.1.4 and 7.2.2, that μ_T and ν_T are continuous measures singular to the Haar measure on T. By Theorem 13 there exist singular, central measures μ_0 and ν_0 on G such that

$$\widehat{\mu_0}(\pi_\lambda) = M{d_\lambda}^{-1} \sum_{w \in W} \operatorname{sgn}(w)\widehat{\mu_T}(w(\lambda+\delta)) \,\mathrm{Id}$$

$$\widehat{\nu_0}(\pi_\lambda) = M{d_\lambda}^{-1} \sum_{w \in W} \operatorname{sgn}(w)\widehat{\nu_T}(w(\lambda+\delta)) \,\mathrm{Id}$$

for $\lambda \in \Lambda^+$ and where M is a constant.

Since by construction $\widehat{\mu_T}$ and $\widehat{\nu_T}$ are supported in $W(\Gamma) \cup W(\Gamma^-)$ we have

$$\widehat{\mu_0}(\pi_\lambda) = M{d_\lambda}^{-1}(\widehat{\mu_T}(\lambda+\delta) + \operatorname{sgn}(\sigma)\widehat{\mu_T}(\sigma(\lambda+\delta)))\,\mathrm{Id}$$

and

$$\widehat{\nu_0}(\pi_\lambda) = M{d_\lambda}^{-1}(\widehat{\nu_T}(\lambda+\delta) + \operatorname{sgn}(\sigma)\widehat{\nu_T}(\sigma(\lambda+\delta)))\,\mathrm{Id}\,.$$

a) Since $\sigma(\delta) = -\delta$, we have $\int_G d\mu_0(g) = \int_G d\nu_0(g) = M(a_0 + \operatorname{sgn}(\sigma)\overline{a_0})$. Hence we may choose a_0 in such a way that $a_0 + \operatorname{sgn}(\sigma)\overline{a_0} \neq 0$, $|a_n| \leqslant 1/2$. Replacing μ_0 and ν_0 by suitable scalar multiples of them we have the assertion.

b) If $\widehat{\mu_0}(\pi_\kappa) \neq 0$ we have $\kappa + \delta \in \Theta(\Psi_1)$ or $\sigma(\kappa+\delta) \in \Theta(\Psi_1)$. Since $\Theta(\Psi_1)$ is invariant by σ we conclude that $\kappa + \delta \in \Theta(\Psi_1)$. In the same way $\widehat{\nu_0}(\pi_\eta) \neq 0$ implies that $\eta + \delta \in \Theta(\Psi_2)$. There exist r, s in $\mathbb{N}$ such that $\kappa + \delta = \sum_{n=1}^r \varepsilon_n q^{2n-1}\delta + \varepsilon_0\delta$, with $\varepsilon_n = 0, 1$ or -1 for $n = 0, ..., r$ and $\eta + \delta = \sum_{n=1}^s \varepsilon'_n q^{2n}\delta + \varepsilon'_0\delta$, with $\varepsilon'_n = 0, 1$ or -1 for $n = 0, ..., s$. We get $\kappa - \eta = \sum_{n=1}^r \varepsilon_n q^{2n-1}\delta - \sum_{n=1}^s \varepsilon'_n q^{2n}\delta + \varepsilon_0\delta - \varepsilon'_0\delta$. If $\sum_{n=1}^r \varepsilon_n q^{2n-1}\delta - \sum_{n=1}^s \varepsilon'_n q^{2n}\delta \neq 0$, it is a non zero integral multiple of $q\delta$. Since $|\delta|_1 = l$ by [16], 13.3 Lemma A, and $|\varepsilon_0 - \varepsilon'_0| \leqslant 2$ we see that $|\kappa - \eta|_1 \geqslant (q-2)l$. If $\sum_{n=1}^r \varepsilon_n q^{2n+1}\delta - \sum_{n=1}^s \varepsilon'_n q^{2n}\delta = 0$, we have $\kappa + \delta = \varepsilon_0\delta$ and $\eta + \delta = \varepsilon'_0\delta$, since Ψ is dissociate. But κ and η belong to Λ^+ and so $\kappa = \eta = 0$. $\square$

For $\pi \in \widehat{G}$ and $\mu \in M(G)$ we denote by $\pi(\mu)$ the operator $\int_G \pi(g) d\mu(g)$.

THEOREM 16. — *Let $f = \sum_{i=1}^n a_i \chi_{\lambda_i}$, where $n \in \mathbb{N}$, a_i are complex numbers and $\lambda_i \in \Lambda^+$, $i = 1, ..., n$. Then there exist two singular central measures μ and ν on G such that $f = \mu * \nu$.*

Proof. — Put $h = \sum_{i=1}^n d_i \chi_{\lambda_i}$. From the orthogonality relations ([4], Ch. II, Proposition 4.16) it follows that $h * f = f$. Let $q \in \mathbb{N}$, $q \geqslant 3$ and μ_0, ν_0 as in Lemma 15. We set $\mu = h\mu_0$ and $\nu = f\nu_0$. The measures μ and ν

are singular, central measures on G. We are going to show that $f = \mu * \nu$. To see this it suffices to show that

$$\operatorname{Tr} \int_G f(g)\pi_\lambda(g^{-1})dg = \operatorname{Tr} \int_G \pi_\lambda(g^{-1})d(\mu * \nu)(g), \lambda \in \Lambda^+.$$

We have

$$\begin{aligned}\operatorname{Tr} \int_G f(g)\pi_\lambda(g^{-1})dg &= \int_G f(g)\chi_\lambda(g^{-1})dg \\ &= \int_G \sum_{i=1}^n a_i\chi_{\lambda_i}(g)\chi_\lambda(g^{-1})dg \\ &= \sum_{i=1}^n a_i(\chi_{\lambda_i} * \chi_\lambda)(e),\end{aligned}$$

which by the orthogonality relations ([4], Ch. II, Proposition 4.16), is equal to a_k if $\lambda = \lambda_k$ for some $k \in \{1, ..., n\}$ and equal to 0 if $\lambda \notin \{\lambda_1, ..., \lambda_n\}$. It follows from [7], IX 5.2, that $\operatorname{Tr} \int_G \pi_\lambda(g^{-1})d(\mu * \nu)(g) = \operatorname{Tr} \pi_{\lambda^*}(\mu)\pi_{\lambda^*}(\nu)$ where π_{λ^*} is the representation adjoint to π_λ. (For the definition of the adjoint representation see [7], IX 3.7). Since $\pi_{\lambda^*}(\mu)$, $\pi_{\lambda^*}(\nu)$ are multiples of the identity,

$$\operatorname{Tr} \pi_{\lambda^*}(\mu)\pi_{\lambda^*}(\nu) = d_{\lambda^*}{}^{-1} \operatorname{Tr} \pi_{\lambda^*}(\mu) \operatorname{Tr} \pi_{\lambda^*}(\nu).$$

We have

$$\operatorname{Tr} \pi_{\lambda^*}(\mu) = \int_G \sum_{i=1}^n d_i\chi_{\lambda_i}(g)\chi_{\lambda^*}(g)d\mu_0(g).$$

Now $\chi_{\lambda_i}(g)\chi_{\lambda^*}(g) = \sum_{\kappa\in\Theta_i} m_i(\kappa)\chi_\kappa(g)$ where Θ_i is the set of the elements $\kappa \in \Lambda^+$ such that π_k is contained in $\pi_{\lambda_i} \otimes \pi_{\lambda^*}$ and $m_i(\kappa)$ the multiplicity of π_k in $\pi_{\lambda_i} \otimes \pi_{\lambda^*}$. We obtain

$$\operatorname{Tr} \pi_{\lambda^*}(\mu) = \int_G \sum_{i=1}^n d_i \sum_{\kappa\in\Theta_i} m_i(\kappa)\chi_\kappa(g)d\mu_0(g).$$

Similarly we get

$$\operatorname{Tr} \pi_{\lambda^*}(\nu) = \int_G \sum_{j=1}^n a_j \sum_{\eta\in\Theta j} m_j(\eta)\chi_\eta(g)d\nu_0(g).$$

Finally

$$\begin{aligned}&\operatorname{Tr} \int_G \pi_\lambda(g^{-1})d(\mu * \nu)(g) \\ &= d_{\lambda^*}{}^{-1}\int_G \sum_{i=1}^n d_i \sum_{\kappa\in\Theta_i} m_i(\kappa)\chi_\kappa(g)d\mu_0(g) \int_G \sum_{j=1}^n a_j \sum_{\eta\in\Theta_j} m_j(\eta)\chi_\eta(g)d\nu_0(g) \\ &= d_{\lambda^*}{}^{-1}\sum_{i=1}^n\sum_{j=1}^n \sum_{\kappa\in\Theta_i} \sum_{\eta\in\Theta_j} d_ia_jm_i(\kappa)m_j(\eta)\int_G \chi_\kappa(g)d\mu_0(g)\int_G \chi_\eta(g)d\nu_0(g).\end{aligned}$$

Assume that $(\kappa,\eta) \neq (0,0)$ and $\int_G \chi_\kappa(g)d\mu_0(g) \int_G \chi_\eta(g)d\nu_0(g) \neq 0$. Then $\widehat{\mu_0}(\pi_{\kappa^*})\widehat{\nu_0}(\pi_{\eta^*}) \neq 0$ and it follows from Lemma 15 that $|\kappa^* - \eta^*|_1 \geqslant (q-2)l$. We have $\kappa^* = -\sigma(\kappa)$, $\eta^* = -\sigma(\eta)$ and hence $|\kappa - \eta|_1 \geqslant (q-2)l$. Let $\Pi(\lambda_i)$ be the set of weights of the module of highest weight λ_i, $1 \leqslant i \leqslant l$. Set $\Pi = \bigcup_{i=1}^{l} \Pi(\lambda_i)$ and $m = \max\{|\zeta|_1 : \zeta \in \Pi\}$. It follows from [16], Exercise 24.12, that $\kappa \in \lambda^* + \Pi$ and $\eta \in \lambda^* + \Pi$ and hence $|\kappa - \eta|_1 \leqslant 2m$. We conclude that if we choose $q > 2ml^{-1} + 2$ we obtain $\int_G \chi_\kappa(g)d\mu_0(g) \int_G \chi_\eta(g)d\nu_0(g) = 0$ if $(\kappa,\eta) \neq (0,0)$. Therefore

$$\begin{aligned}\operatorname{Tr}\int_G \pi_\lambda(g^{-1})d(\mu*\nu)(g) &\\ &= d_{\lambda^*}{}^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} d_i a_j m_i(0)m_j(0)\int_G d\mu_0(g)\int_G d\nu_0(g)\\ &= d_{\lambda^*}{}^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} d_i a_j m_i(0)m_j(0).\end{aligned}$$

Now it follows from [7], IX 5.6, that $m_i(0) = 0$ if $\lambda_i \neq \lambda$ and $m_i(0) = 1$ if $\lambda_i = \lambda$. Hence if $\lambda \notin \{\lambda_1, ..., \lambda_n\}$ we have $\operatorname{Tr}\int_G \pi_\lambda(g^{-1})d(\mu*\nu)(g) = 0$ and if $\lambda = \lambda_k$ for some $k \in \{1,...,n\}$ we have

$$\operatorname{Tr}\int_G \pi_\lambda(g^{-1})d(\mu*\nu)(g) = d_{\lambda_k^*}{}^{-1}d_{\lambda_k}a_k = a_k$$

since $d_{\lambda_k^*} = d_{\lambda_k}$ by [7] IX 3.7. We conclude that $f = \mu*\nu$. □

5. Wiener's theorem for symmetric spaces of compact type.

Let K be a closed subgroup of G which is the group of fixed points of an involutive automorphism of G. It follows then from [11], Ch. VII Theorem 8.2, that K is connected. We assume that the symmetric pair (G,K) is irreducible. It follows from [12], Ch. IV §3 Theorem 3.1, that the assumptions of Theorem 3.11 in [18] are satisfied and hence we may use the results of Section 3 of that paper. In this section we prove a Wiener-type theorem for the symmetric space G/K. We will use in the sequel some facts about spherical representations and spherical functions. For the definitions and information concerning these notions the reader is referred to [12].

We set $\Lambda_K^+ = \{\lambda \in \Lambda^+ : \pi_\lambda \text{ is spherical}\}$ and we denote by φ_λ the spherical function on G which corresponds to π_λ. Let N be the normalizer of K in G. It follows from [18], Proposition 1.11 and Lemma 2.4, that N/K is finite abelian group. Let r be the order and $\widehat{N/K}$ the dual group of N/K. It follows from [12], Ch. IV §3 Theorem 3.4 and Lemma 3.6, that φ_λ, $\lambda \in \Lambda_K^+$, induces a unitary character $\varphi'_\lambda$of the group N/K. Let $\gamma \in \widehat{N/K}$ and $M_\gamma = \{\lambda \in \Lambda_K^+ : \varphi'_\lambda = \gamma\}$. Then the family $\{M_\gamma\}_{\gamma \in \widehat{N/K}}$ is a partition of Λ_K^+ and it follows from [18], Theorem 3.16, that M_γ is infinite for every $\gamma \in \widehat{N/K}$.

DEFINITION 17. — *A sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite subsets of Λ_K^+ is called K-admissible if* $\lim_{n\to\infty} |A_n \cap M_\gamma|\ |A_n|^{-1} = r^{-1}$, *for $\gamma \in \widehat{N/K}$.*

Let A be a finite subset of Λ_K^+. We define the function

$$\psi_A(g) = |A|^{-1} \sum_{\lambda \in A} \varphi_\lambda(g), g \in G.$$

The function ψ_A takes the value 1 at e and is bounded by 1.

LEMMA 18. — *Let $g \in N - K$ and $\{A_n\}_{n\in\mathbb{N}}$ be a K-admissible sequence of subsets of Λ_K^+. Then* $\lim_{n\to\infty} \psi_{A_n}(g) = 0$.

Proof. — Let q be the canonical projection from N onto N/K. We have

$$\begin{aligned}\psi_{A_n}(g) &= |A_n|^{-1} \sum_{\lambda \in A_n} \varphi_\lambda(g) = |A_n|^{-1} \sum_{\gamma \in \widehat{N/K}} \sum_{\lambda \in A_n \cap M_\gamma} \varphi_\lambda(g)\\ &= |A_n|^{-1} \sum_{\gamma \in \widehat{N/K}} \sum_{\lambda \in A_n \cap M_\gamma} \gamma(q(g))\\ &= \sum_{\gamma \in \widehat{N/K}} |A_n \cap M_\gamma|\ |A_n|^{-1} \gamma(q(g))\end{aligned}$$

and hence $\lim_{n\to\infty} \psi_{A_n}(g) = r^{-1} \sum_{\gamma \in \widehat{N/K}} \gamma(q(g))$. Since $q(g) \neq e$ we have $\sum_{\gamma \in \widehat{N/K}} \gamma(q(g)) = 0$. We conclude that $\lim_{n\to\infty} \psi_{A_n}(g) = 0$. □

Let $M(G/K)$ be the space of regular Borel measures on G/K and $M(G, K)$ the space of regular Borel right K-invariant measures on G. We denote by dk the Haar measure on K of total mass 1 and by p be the canonical projection from G onto G/K. Let f be a continuous function on G. We set: $f^\flat(p(g)) = \int_K f(gk)dk$. If $\mu \in M(G/K)$ we define a measure $\mu^\natural$ on G by the relation: $\mu^\natural(f) = \mu(f^\flat)$. Then $\mu^\natural \in M(G, K)$ and the mapping $\mu \longmapsto \mu^\natural$ is a bijection from $M(G/K)$ onto $M(G, K)$ ([3], Ch. VII, §2, n° 2, Proposition 4).

PROPOSITION 19. — *Let $\mu \in M(G,K)$ and $\{A_n\}_{n\in\mathbb{N}}$ be a K-admissible sequence of subsets of Λ_K^+. Then*

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} \operatorname{Tr}\widehat{\mu}(\pi_\lambda) = \mu(K).$$

Proof. — It follows from [18], Corollary 3.14 and Lemma 18 that $\lim_{n\to\infty}\psi_{A_n}(g) = 0$ for every $g \in G - K$. Hence by Lebesgue's Dominated Convergence Theorem $\lim_{n\to\infty}\int_{G-K}\psi_{A_n}(g)d\mu(g) = 0$. On the other hand $\psi_{A_n}(g) = 1$ for $g \in K$ and hence $\int_K \psi_{A_n}(g)d\mu(g) = \mu(K)$. We conclude that

$$\lim_{n\to\infty}\int_G \psi_{A_n}(g)d\mu(g) = \mu(K).$$

To finish the proof we have to show that $\int_G \varphi_\lambda(g)d\mu(g) = \operatorname{Tr}\widehat{\mu}(\pi_\lambda)$. From [12], Ch. IV §4 Theorem 4.2, we have $\varphi_\lambda(g) = \int_K \chi_\lambda(g^{-1}k)dk$. Hence

$$\int_G \varphi_\lambda(g)d\mu(g) = \int_G\int_K \chi_\lambda(g^{-1}k)dkd\mu(g) = \int_G\int_K \chi_\lambda(kg^{-1})dkd\mu(g)$$

since χ_λ is a central function. Using Fubini's Theorem and the fact that μ is right K-invariant we obtain

$$\int_G \varphi_\lambda(g)d\mu(g) = \int_K\int_G \chi_\lambda(kg^{-1})d\mu(g)dk = \int_G \chi_\lambda(g^{-1})d\mu(g) = \operatorname{Tr}\widehat{\mu}(\pi_\lambda).$$

□

For $\nu \in M(G)$ the measure $\nu^\sim$ is defined in Section 2.

THEOREM 20. — *Let $\mu \in M(G/K)$ and $\{A_n\}_{n\in\mathbb{N}}$ be a K-admissible sequence of subsets of Λ_K^+. Then*

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} \|\widehat{\mu^\natural}(\pi_\lambda)\|_2^2 = \sum_{x\in G/K} |\mu(\{x\})|^2.$$

(The sum on the righthand side of the above equality is taken over the set $\{x \in G/K : \mu(\{x\}) \neq 0\}$ which is of course a countable set).

Proof. — Let $\nu \in M(G,K)$. Then $\nu^\sim * \nu \in M(G,K)$ and applying Proposition 19 to this measure we have

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} \|\widehat{\nu}(\pi_\lambda)\|_2^2 = \nu^\sim * \nu(K).$$

Let A be a system of representatives of G/K in G. Since $\nu^\sim * \nu(K) = \sum_{y\in A}|\nu(yK)|^2$ we obtain

$$\lim_{n\to\infty} |A_n|^{-1} \sum_{\lambda\in A_n} \|\widehat{\nu}(\pi_\lambda)\|_2^2 = \sum_{y\in A} |\nu(yK)|^2.$$

The theorem follows applying this formula to the measure $\mu^\natural$, since $\mu^\natural(yK) = \mu(p(y))$. □

Remark 3. — Using [11], Ch. VIII §5 Proposition 5.5, we can show that Theorem 20 holds without the assumption that G/K is irreducible.

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