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FEFFERMAN'S SAK PRINCIPLE IN ONE DIMENSION

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1. Introduction and result.

We give in this paper a complete proof in one dimension of Fefferman's SAK principle, as stated in [4, (†) page 130]. We illustrate in details the levels I, II and III of microlocalization as suggested in [6] and [4]. Naturally, the simplifications due to the one-dimensional situation are important, nevertheless it seems interesting to provide a thorough investigation of Fefferman-Phong's strategy in this case.

First let us recall some basic features of pseudo-differential calculus. In dimension n of space, for any a hamiltonian – or symbol – in $\mathcal{S}(\mathbb{R}^{2n})$, we can associate operators acting on $L^2(\mathbb{R}^n)$, via a procedure called *quantization.* Throughout this article we chose the so-called *Weyl quantization*

$$(1) \qquad (a^w u)(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y,\xi)} a(\frac{x+y}{2}, \xi) u(y) dy d\xi.$$

For $m \in \mathbb{R}$ we also recall that $a \in \mathcal{C}^\infty(\mathbb{R}^2)$ is in the Hörmander class $S^m_{1,0}$ (see chapter XVIII in [11]) if a satisfies $\forall \alpha,\ \beta \in \mathbb{N}$, $\exists C_{\alpha,\beta}$ such that

$$(2) \qquad \forall x, \xi \in \mathbb{R}, \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta} < \xi >^{m-|\beta|}$$

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where $<\xi>=(1+|\xi|^2)^{1/2}$. We have then the following result:

THEOREM 0.1. — *We consider a and $b \in S^2_{1,0}$ such that $|a| \leqslant b$. Then $\forall \varepsilon > 0$, $s \in \mathbb{R}$, $\exists C_{\varepsilon,s}$ such that $\forall u \in \mathcal{S}(\mathbb{R})$ and $s \in \mathbb{R}$,*

$$\|a^w u\|_s^2 \leqslant C_{\varepsilon,s}\left(\|b^w u\|_s^2 + \|u\|_{s+\varepsilon}^2\right), \tag{3}$$

where $\|.\|_s$ is the usual Sobolev norm.

This question is closely related to the uncertainty principle. In the present paper, we use the strong tools of microlocalization, and in particular the proper metric introduced by Fefferman and Phong in [5], [6]. We also assume that $|a| \leqslant b$ everywhere. Other assumptions can be proposed, for instance that a is smaller than the average of b on symplectic cubes.

We want to stress the fact that this result is not only due to a series of successive microlocalizations involving an increasing sequence of metrics (see e.g. [2]). As noticed in [6], this is sometimes not sufficient to obtain sharp results, and we have to bent the phase space using Egorov theorem [3] before refining the localization in the phase space.

A priori estimates have also been studied in the case of polynomials, by Helffer and Nourrigat [9] in the general context of Lie algebra of operators, by Mohamed and Nourrigat [14], by Shen [15] using singular integrals, and Guibourg [8] in the spirit of Nourrigat.

We want now to give the main result. In fact Theorem 0.1 can be stated in the semi-classical case. First recall that a semi-classical symbol a of order $m \in \mathbb{R}$ is a family of symbol a_Λ satisfying for all α, β multi-indices of length $|\alpha|$, $|\beta|$, if we write $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n}$

$$\sup_{\Lambda \geqslant 1,\ (x,\xi)\in\mathbb{R}^{2n}} \left|\partial_x^\alpha \partial_\xi^\beta a_\Lambda(x,\xi)\right| \Lambda^{-m+\frac{|\alpha|+|\beta|}{2}} = \gamma_{\alpha,\beta} < \infty.$$

This is a Fréchet space with semi-norms $\gamma_{\alpha,\beta}$. We have the following statement:

THEOREM 0.2 (Main). — *For all $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that for any semi-classical symbols a, b of order 2 satisfying $|a(x,\xi)| \leqslant b(x,\xi)$ for $(x,\xi) \in \mathbb{R}^2$, there exists $C_{\varepsilon,a,b}$ such that $\forall u \in \mathcal{S}(\mathbb{R})$,*

$$\|a^w u\|^2 \leqslant C_\varepsilon \|b^w u\|^2 + C_{\varepsilon,a,b}\Lambda^{4\varepsilon}\|u\|^2. \tag{4}$$

Here $C_{\varepsilon,a,b}$ only depends on the first N_ε derivatives of a and b, and $\|.\|$ is the L^2 norm.

We can illustrate the procedure employed in the proof with the following picture, in the spirit of those presented in the didactic paper of Fefferman [4]. We will especially comment below the way of cutting and bending the phase space $\mathbb{R}^2$ in function of the symbol b:

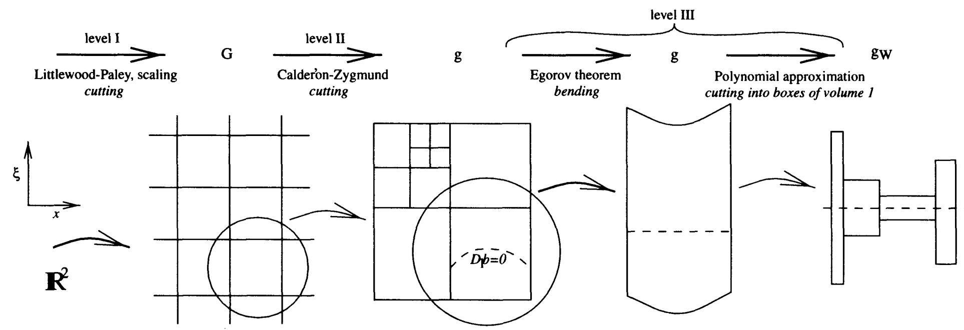

Fig. 1. – Cutting and bending the phase space

The **level I** of microlocalization (the algorithm of the 70's according to [4]) is the one that permits us to reduce the problem in constant metrics. The Littlewood-Paley decomposition is an example of such a procedure. Using a scaling, we can then state the problem in the semi-classical metric $G = (|dx|^2 + |d\xi|^2)/\Lambda$. This is standard and we shall only prove the semi-classical Theorem 0.2.

The **level II** of microlocalization consists of cutting the phase space thanks to a Calderón-Zygmund procedure on the symbol b. We introduce the so-called *proper metric* and if the symbol is non-negative we obtain three types of boxes in the phase space: "Negligible" boxes are boxes in which the operator b^w has the same behavior as a L^2-bounded operator. In "elliptic" boxes, the corresponding operator is bounded from below by a very large constant. These two cases are easy to deal with. In fact, "convexity" boxes concentrate all the difficulties.

In the last type of boxes, we can solve an implicit equation. There is a vector U in the phase space such that $U.\nabla b(x,\xi) = 0$ is represented by the dashed line in the picture. Through a Taylor expansion we get the decomposition $b(x,\xi) = f^2(x,\xi)+g(x,\xi)$ where $U.\nabla g(x,\xi) = 0$ everywhere. This decomposition is the core of the proof of the well-known Fefferman-Phong inequality [5]: For $a \in S^2_{1,0}$

$$a \geqslant 0 \implies \exists C \text{ s.t. } a^w + C \geqslant 0. \tag{5}$$

In our case, we use two striking facts corresponding to this level II. On the one hand, for a and b semi-classical symbols of order 2, we prove that $|a(x,\xi)| \leqslant b(x,\xi)$ is a sufficient condition for a to be in the proper class of b. This allows us to cut *simultaneously* a and b using the proper metric of b, without annihilating the symbolic properties of a. On the other hand, in the operatorial point of view, we prove that we can microlocalize a priori inequalities.

The **level III** deals with convexity boxes and can be splited into two different steps. The first one consists of bending the phase space in order to make b look like the Schrödinger operator. In fact, after a canonical transformation, we can write that

$$b^w = e^w(D_y^2 + V(y)).$$

This bending is possible thanks to the Egorov theorem. The second part of the level III is the introduction of the crucial metric g_W which we call *metric of level III* and which allows us to do a third order microlocalization, involving polynomial approximation. This tool gives symbolic estimates and a priori inequalities for Schrödinger operator. In the context of microlocalization, this is the ultimate step for cutting the phase space.

In this article, we shall closely follow the three steps described above. In the first section, we give two lemmas of level II, the first one for symbols and the second one for operators. The next section is devoted to the proof of the main theorem, which essentially consists of bending the phase space. In the last section, we establish the result for the Schrödinger operator using the level III metric. Eventually an appendix is devoted to two technical lemmas. The first one is a simplified version of the already quoted Egorov theorem, and the second one is a preparation lemma.

Contents.

1. Microlocalization of level I and II.

1.1. Definitions.

Let us first recall some definitions and basic tools about pseudo-differential calculus in dimension n, which can be found in [10] or [2]. Consider the symplectic space $\mathbb{R}^{2n}$ equipped with the symplectic form $\sigma = \sum_{1=1}^{n} d\xi_i \wedge dx_i$. For g a positive definite quadratic form, we define

$$g^{\sigma}(T) = \sup_{g(Y)=1} \sigma(T,Y)^2, \tag{6}$$

which is also a positive definite quadratic form. If Γ is a quadratic form such that $\Gamma^{\sigma} = \Gamma$, we say that Γ is symplectic. We also introduce a family of metric:

DEFINITION 1.1.— *An admissible metric is a Riemannian metric on $\mathbb{R}^{2n}$ satisfying the following conditions:*

uncertainty Principle: $\forall X \in \mathbb{R}^{2n}$, $g_X \leqslant g_X^{\sigma}$;

slowness: $\exists C_0 > 0$ *s.t.* $g_X(X-Y) \leqslant C_0^{-1} \Longrightarrow (g_X/g_Y)^{\pm 1} \leqslant C_0$;

temperance: $\exists C_1, N_1 > 0$, *s.t.* $g_X/g_Y \leqslant C_1 \left(1 + g_X^{\sigma}(X-Y)\right)^{N_1}$.

The constants are respectively called constants of slowness and temperance.

A g-admissible weight is positive function m on the phase space $\mathbb{R}^{2n}$, for which there exists $\tilde{C}_0, \tilde{C}_1, \tilde{N}_1 > 0$ such that

$$g_X(X-Y) \leqslant \tilde{C}_0 \Longrightarrow (m(X)/m(Y))^{\pm 1} \leqslant \tilde{C}_0 \tag{7}$$

$$(m(X)/m(Y))^{\pm 1} \leqslant \tilde{C}_1 \left(1 + g_X^{\sigma}(X-Y)\right)^{\tilde{N}_1}. \tag{8}$$

We define next the *uncertainty function* λ, which is a special weight for g,

$$\lambda(X)=\inf_{T\in\mathbb{R}^{2n}}\left(g^\sigma_X(T)/g_X(T)\right)^{1/2}. \tag{9}$$

Let us now introduce some spaces of symbols:

DEFINITION 1.2. — *Let g be a metric, and m be a weight for g. We say that a function a is a symbol in $S(m,g)$ if $a\in C^\infty(\mathbb{R}^{2n})$, and if the following semi-norms are finite:*

$$\sup_{l\leqslant k,X\in\mathbb{R}^{2n},g_X(T_j)\leqslant 1}\left|a^{(l)}(X)T_1...T_l\right| m^{-1}(X):=\|a\|_{k,S(m,g)}\,. \tag{10}$$

If m is of the form λ^μ, we say that a is of order μ and we note the family $S^\mu(g)$. We also note $S^{-\infty}(g)=\cap_{\mu\in\mathbb{R}}S^\mu(g)$.

Such families are given for example by the classes of symbols $S^\mu_{\rho,\delta}$, with $0\leqslant\delta\leqslant\rho\leqslant 1$ and $\delta<1$, equipped with the metric $g_X=\langle\xi\rangle^{2\delta}|dx|^2+\langle\xi\rangle^{-2\rho}|\xi|^2$, and the weight $m(X)=\langle\xi\rangle^\mu$, where $\langle\xi\rangle=\left(1+|\xi|^2\right)^{1/2}$. For good functions (in $S^\mu(g)$ classes for instance), we define the composition law $\sharp$ such that $(a\sharp b)^w=a^w\circ b^w$ by

$$(a\sharp b)(X)=\pi^{-2n}\iint e^{-2i\sigma(X-Y,X-Z)}a(Y)b(Z)dYdZ, \tag{11}$$

and for $a\in S^{\mu_1}(g)$, $b\in S^{\mu_2}(g)$, if $\{.,.\}$ denotes the Poisson bracket, then there is $r\in S^{\mu_1+\mu_2-2}(g)$ such that

$$a\sharp b=ab+\frac{1}{2i}\{a,b\}+r. \tag{12}$$

In particular for $\varphi\in S^0(g)$, $a\in S^2(g)$, since $\varphi\{a,\varphi\}+\{\varphi,a\}\varphi=0$, there is $r'\in S^0(g)$ such that

$$\varphi\sharp a\sharp\varphi=a\varphi^2+r'. \tag{13}$$

1.2. A symbolic property of the proper class.

Recall first some results about the so-called proper metric, introduced by Fefferman and Phong in [5] and which is the main tool for the proof of the Fefferman-Phong inequality (5).

Let $\mathbb{R}^{2n}$ be equipped with the symplectic form σ. Let G be an admissible metric (see definition 1.1) and consider $b\in S(\Lambda^2,G)$, where

Λ is the uncertainty function of G (see (9)). In the following, we denote by $\|.\|_{G_X}$ the multi-linear application norm in the metric G_X, *i.e.*,

$$\|A\|_{G_X} = \sup_{G_X(T_j)=1} |A(T_1, ..., T_k)|.$$

Note that for A multilinear symetric, $\|\!|A|\!\|_{G_X} = \sup_{G_X(T)=1} |AT^k|$ is an uniformly equivalent norm. Define next the new quantity

$$(14) \qquad \lambda(X) = \max_{k<4} \left\{ 1, \left(\left\| a^{(k)}(X) \right\|_{G_X} \Lambda(X)^{-k/2} \right)^{2/(4-k)} \right\}.$$

Then we have the classical result (see for example [12, prop. 3.3]):

LEMMA 1.3. — *Let G be an admissible metric, b be in $S(\Lambda^2, G)$, and λ be given relative to b as in (14). The metric $g : X \longmapsto \Lambda(X)\lambda(X)^{-1}G_X$ is called the proper metric. It is admissible, of uncertainty function λ, and b is in the class $S(\lambda^2, g)$.*

In order to do a simultaneous microlocalization on a and b, we need the following lemma:

LEMMA 1.4. — *Let G be an admissible metric, a, b be in $S(\Lambda^2, G)$, and g be the proper metric of b. If $|a| \leqslant b$ then $a \in S(\lambda^2, g)$, where λ is given relative to b as in (14).*

Proof. — Multiplying b by a positive constant, we can suppose that the first four semi-norms of b are smaller than 1. We note α_k the semi-norms of a in $S(\Lambda^2, G)$. To obtain the lemma, we have to show that $\forall k \in \mathbb{N}$ we have

$$\sup_X \left\| a^{(k)}(X) \right\|_{g_X} \lambda(X)^{-2} < \infty.$$

Let $X \in \mathbb{R}^{2n}$ be fixed for the rest of the proof. We first notice from (14) that $\lambda(X) \leqslant \Lambda(X)$. Therefore we can write $\forall k \geqslant 4$,

$$(15) \quad \left\| a^{(k)}(X) \right\|_{g_X} \lambda(X)^{-2} = \left\| a^{(k)}(X) \right\|_{G_X} \Lambda(X)^{-2} \left(\frac{\Lambda(X)}{\lambda(X)} \right)^{2-k/2} \leqslant \alpha_k.$$

We must now evaluate the first derivatives of a. Let's take $h \in \mathbb{R}^{2n}$, such as $g_X(h) \leqslant C_0^{-1}$, where C_0 is the slowness constant of g. The slowness of the proper metric implies that

$$(16) \qquad b(X+h) \leqslant \lambda^2(X+h) \leqslant C_0^2 \lambda^2(X).$$

Besides we can notice from (15) that for all $\theta \in [0,1]$,

$$|a^{(4)}(X+\theta h)h^4/24| \leqslant \lambda^2(X)C_0^{-2}\alpha_4/24. \tag{17}$$

We know that $|a(X+h)| \leqslant b(X+h)$. Making a fourth-order Taylor expansion of a, and estimating the integral remainder using (17), we obtain with a new constant $C_2 > 0$

$$|a(X) + a'(X)h + a"(X)h^2/2 + a^{(3)}(X)h^3/6| \leqslant C_2\lambda^2(X). \tag{18}$$

Dealing with this inequality for $\pm h/2$ and $\pm h$, we obtain that there exists $\tilde{\alpha}_0, ..., \tilde{\alpha}_3$ such that for all h satisfying $g_X(h) \leqslant C_0^{-1}$ and $0 \leqslant k \leqslant 3$ we have

$$|a^{(k)}(X)h^k| \leqslant \tilde{\alpha}_k\lambda^2(X),$$

which gives $|a^{(k)}(X)T^k| \leqslant \tilde{\alpha}_k\lambda^2(X)g_X(T)^{k/2}C_0^{k/2}$ for all $T \in \mathbb{R}^{2n}$, yielding the lemma. □

1.3. Microlocalization of a priori inequalities.

Let us now recall the fundamental properties of any admissible metric g. We denote by $B_{Y,r}$ the ball of radius r and center Y for g_Y. We first introduce the space of confined symbols (see [2]).

DEFINITION 1.5. — *Let g be an admissible metric on $\mathbb{R}^{2n}$ and m be a weight for g. We say that a function a is a symbol in* conf(m, g, Y, r) *if $a \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$ and if the following semi-norms are finite:*
(19)
$$\sup_{l \leqslant k, X \in \mathbb{R}^{2n}, g_Y(T_j) \leqslant 1} \left| a^{(l)}(X)T_1...T_l \right| m^{-1}(Y)\left(1 + g_Y^\sigma(X - B_{Y,r})\right)^{k/2} < \infty,$$

where $g_Y^\sigma(X - B_{Y,r}) = \inf_{Z \in B_{Y,r}} g_Y^\sigma(X - Z)$. In the case a is supported in $B_{Y,r}$ we note supp(m, g, Y, r).

With these notations, we have the following lemma of partition, (see for example [11, section 18.4], or [2])

LEMMA 1.6. — *Let g be an admissible metric and $r^2 < C_0^{-1}$, where C_0 is the constant of slowness of g. Then there exists a sequence of points $\{X_\nu\}$, of functions $\{\varphi_\nu\}$ uniformly in* supp$(1, g_{X_\nu}, X_\nu, r)$, *such that $\sum_\nu \varphi_\nu^2 = 1$.*

Moreover we have a property of finite overlap: for all r^* such that $r^2 \leqslant r^{*2} < C_0^{-1}$, if we note $B_\nu^* = B_{X_\nu, r^*}$, there exists $\mathcal{N}^*$ such that $\forall \mathcal{E} \subset \mathbb{N}$, $\sharp\mathcal{E} > \mathcal{N}^* \Longrightarrow \bigcap_{\nu \in \mathcal{E}} \mathbb{B}_\nu^* = \emptyset$. Furthemore if we define $\Delta_{\mu,\nu}^* =$

$\max\{1, g^\sigma_{X_\mu}(B^*_\mu - B^*_\nu), g^\sigma_{X_\nu}(B^*_\mu - B^*_\nu)\}^{1/2}$ *which we call the distance function, then there exists* N^*, C^* *such that* $\sup_\mu \sum_\nu \Delta^{*-N^*}_{\mu,\nu} < C^*$.

We can now introduce some lemmas involving these partitions in the context of a priori inequalities. We denote by $\|.\|_{\mathcal{L}(L^2)}$ the operator norm on $L^2(\mathbb{R}^n)$. We first recall (see [2]) that for g an admissible metric, $r < C_0^{-1/2}$, where C_0 is the constant of slowness of g, and $X \in \mathbb{R}^{2n}$,

$$\text{(20)} \qquad \exists k, C, \ \ \forall a \in \text{conf}(1, g, X, r)\,, \qquad \|a^w\|_{\mathcal{L}(L^2)} \leqslant C\,\|a\|_{k,conf(1,g,X,r)}\,;$$

we also recall the well-known embedding

$$\text{(21)} \qquad \exists C, \ \ \forall a \in S(1, g), \qquad \|a^w\|_{\mathcal{L}(L^2)} \leqslant C\,\|a\|_{k,S(1,g)}\,,$$

and if m is a weight for g, then $\forall k, \exists C, l$ s. t. $\forall a \in S(m, g)$, $b \in \text{conf}(1, g, X, r)$,

$$\text{(22)} \qquad \|a\sharp b\|_{k,conf(1,g,X,r)} \leqslant Cm(X)\,\|a\|_{l,S(m,g)}\,\|b\|_{l,conf(1,g,X,r)}\,.$$

Let us keep the notations of Lemma 1.6 and write $\Delta_{\mu,\nu} = \max\{1, g^\sigma_{X_\mu}(B_\mu - B_\nu), g^\sigma_{X_\nu}(B_\mu - B_\nu)\}^{1/2}$, where $B_\nu = B_{X_\nu, r}$. Then $\forall k, N$, $\exists C_{k,N}, l$ s. t. $\forall \mu, \nu \in \mathbb{N}$, $a \in \text{conf}(1, g, X_\mu, r)$, $b \in \text{conf}(1, g, X_\nu, r)$,

$$\text{(23)} \qquad \|a\sharp b\|_{k,\text{conf}(1,g,X_\mu,r)} + \|a\sharp b\|_{k,\text{conf}(1,g,X_\nu,r)} \\ \leqslant C_{k,N}\,\|a\|_{l,\text{conf}(1,g,X_\mu,r)}\,\|b\|_{l,\text{conf}(1,g,X_\nu,r)}\,\Delta^{-N}_{\mu,\nu}\,.$$

LEMMA 1.7.— *Let* g *be an admissible metric,* $\{X_\nu\}$ *a family of points as introduced in Lemma 1.6, and for* $r^{*2} < C_0^{-1}$, $\{r_\nu\}$ *a family of functions uniformly in* $\text{conf}(1, g_{X_\nu}, X_\nu, r^*)$. *Then* $\sum r^w_\nu \in \mathcal{L}(L^2)$.

Proof.— The result is immediate using Cotlar lemma and the third point of Lemma 1.6. We can notice that we only suppose the functions to be confined and not supported. □

The next lemma will be very useful in the context of microlocalization of maximal inequalities. A proof is given in [1, Lemme 7.9].

LEMMA 1.8.— *Let* g *be an admissible metric,* $\{X_\nu\}$ *a family of points introduced in Lemma 1.6, and for* $r^2 < C_0^{-1}$, $\{\varphi_\nu\}$ *a family of functions uniformly in* $\text{conf}(1, g_{X_\nu}, X_\nu, r)$ *satisfying* $\sum \varphi_\nu = 1$. *Then there exists* C *such that for all* $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\text{(24)} \qquad C^{-1}\,\|u\|^2_{L^2(\mathbb{R}^n)} \leqslant \sum_\nu \|\varphi^w_\nu u\|^2_{L^2(\mathbb{R}^n)} \leqslant C\,\|u\|^2_{L^2(\mathbb{R}^n)}\,.$$

We also need two lemmas concerning microlocalization of symbols of order higher than zero (namely 2). The following one deals with symbols of finite order (see Definition 1.2).

LEMMA 1.9.— *Let g be an admissible metric, λ its uncertainty function, $\{X_\nu\}$ a family of points introduced in Lemma 1.6, and for $r^2 < C_0^{-1}$, $\{\varphi_\nu\}$ a family of functions uniformly in* $\mathrm{conf}(1, g_{X_\nu}, X_\nu, r)$ *satisfying $\sum \varphi_\nu = 1$. Then for all $m \in \mathbb{N}$ and $a \in S(\lambda^m, g)$, there exists C such that for all $u \in \mathcal{S}(\mathbb{R}^{2n}, \mathbb{C})$,*

$$\|a^w u\|^2_{L^2(\mathbb{R}^n)} \leqslant C \left(\sum_\nu \|a^w \varphi_\nu^w u\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \right). \tag{24}$$

Proof.— Let a be in $S(\lambda^m, g)$, and let us denote by $(.,.)$ the scalar product on $L^2(\mathbb{R}^n)$, and $\|.\|$ the associated norm. Then for all $u \in L^2(\mathbb{R}^n)$, we get

$$\|a^w u\|^2 = \sum_{\mu,\nu} (a^w \varphi_\nu^w u, a^w \varphi_\mu^w u). \tag{26}$$

For $r^2 < r^{*2} < C$, consider the distance function defined in Lemma 1.6 for which there exists N^* such that $\sup_\mu \sum_\nu {\Delta^*_{\mu,\nu}}^{-N^*} < \infty$. We shall split the sum (26) depending on the size of $\Delta^*_{\mu,\nu}$ with respect to $\rho > 1$. For all $u \in \mathcal{S}(\mathbb{R}^n)$ we can write

$$\|a^w u\|^2 = \sum_{\mu,\nu,\ \Delta^*_{\mu,\nu} < \rho} (a^w \varphi_\nu^w u, a^w \varphi_\mu^w u) + \sum_{\mu,\nu,\ \Delta^*_{\mu,\nu} \geqslant \rho} (a^w \varphi_\nu^w u, a^w \varphi_\mu^w u). \tag{27}$$

Let us first deal with the first term. Since $1 \leqslant \rho (\Delta^*_{\mu,\nu})^{-1}$ we get

$$\sum_{\mu,\nu,\ \Delta^*_{\mu,\nu} < \rho} \left|(a^w \varphi_\nu^w u, a^w \varphi_\mu^w u)\right| \leqslant \sum_{\mu,\nu} \|a^w \varphi_\nu^w u\| \left\|a^w \varphi_\mu^w u\right\| \rho^{N^*} \left(\Delta^*_{\mu,\nu}\right)^{-N^*}.$$

We can then apply the discrete Schur criterion: we consider the operator on l^2 with kernel $(K_{\mu,\nu}) = \left({\Delta^*_{\mu,\nu}}^{-N^*}\right)$. It satisfies $\sup_\nu \sum_\mu |K_{\mu,\nu}| \leqslant C^*$, and $\sup_\nu \sum_\mu |K_{\nu,\mu}| \leqslant C^*$, and we obtain

$$\sum_{\mu,\nu,\ \Delta^*_{\mu,\nu} < \rho} \left|(a^w \varphi_\nu^w u, a^w \varphi_\mu^w u)\right| \leqslant C^* \rho^{N^*} \sum_\nu \|a^w \varphi_\nu^w u\|^2. \tag{28}$$

We write the second term of the sum (27) as

$$(R_\rho u, u) = \sum_{\mu,\nu,\ \Delta^*_{\mu,\nu} \geqslant \rho} (\bar{\varphi}_\mu^w \bar{a}^w a^w \varphi_\nu^w u, u). \tag{29}$$

We shall use Cotlar lemma again in order to show that R_ρ is bounded on L^2. We set $A_{\mu,\nu} = \bar{\varphi}_\mu^w \bar{a}^w a^w \varphi_\nu^w$, $I_\rho = \{(\mu,\nu)/\Delta^*_{\mu,\nu} \geqslant \rho\}$ and write for $(\mu_0,\nu_0) \in I_\rho$,

(30)
$$\sum_{(\mu,\nu)\in I_\rho} \|A^*_{\mu_0,\nu_0} A_{\mu,\nu}\|^{1/2}_{\mathcal{L}(L^2)} = \sum_{(\mu,\nu)\in I_\rho} \|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w \varphi^w_{\mu_0}\bar{\varphi}^w_\mu \bar{a}^w a^w \varphi^w_\nu\|^{1/2}_{\mathcal{L}(L^2)}.$$

We can take advantage of the confinement properties of φ_ν via inequalities (20-22). We can write

$$\forall k, \exists C_k, \text{ s.t. } \forall \nu, \|a\sharp\varphi_\nu\|_{k,\mathrm{conf}(1,g_\nu,X_\nu,r)} \leqslant C_k \lambda_\nu^m, \tag{31}$$

where we recall the fact that m is the order of a. Writing

$$\begin{aligned}(32)\quad \|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w \varphi^w_{\mu_0}\bar{\varphi}^w_\mu \bar{a}^w a^w \varphi^w_\nu\|^{1/2}_{\mathcal{L}(L^2)} &\\ \leqslant \|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w \varphi^w_{\mu_0}\|^{1/4}_{\mathcal{L}(L^2)} &\|\bar{\varphi}^w_\mu \bar{a}^w a^w \varphi^w_\nu\|^{1/4}_{\mathcal{L}(L^2)}\\ \|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w\|^{1/4}_{\mathcal{L}(L^2)} &\|\varphi^w_{\mu_0}\bar{\varphi}^w_\mu\|^{1/4}_{\mathcal{L}(L^2)} \|\bar{a}^w a^w \varphi^w_\nu\|^{1/4}_{\mathcal{L}(L^2)},\end{aligned}$$

we get using (23) that for all N there exists C' such that

$$\|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w \varphi^w_{\mu_0}\bar{\varphi}^w_\mu \bar{a}^w a^w \varphi^w_\nu\|^{1/2}_{\mathcal{L}(L^2)} \leqslant C'\lambda^m_{\nu_0}\Delta^{-N}_{\nu_0,\mu_0}\Delta^{-N}_{\mu_0,\mu}\Delta^{-N}_{\mu,\nu}\lambda^m_\nu. \tag{33}$$

For all $(\mu,\nu) \in I_\rho$, $\Delta^*_{\mu,\nu}$ is greater than $\rho > 1$. This implies that B^*_ν and B^*_μ are disjoint, therefore there exists $c_0 > 0$ such that for all $(\mu,\nu) \in I_\rho$,

$$g_\mu(B_\nu - B_\mu) \geqslant c_0, \qquad g_\nu(B_\nu - B_\mu) \geqslant c_0. \tag{34}$$

Using the fact that $\forall X \in \mathbb{R}^{2n}$ $\lambda(X)^2 g_X \leqslant g^\sigma_X$, we write

$$\begin{aligned}(25)\ \max\{\lambda_\nu^2 g_\nu(B_\nu - B_\mu), \lambda_\mu^2 g_\mu(B_\nu - B_\mu)\}^{1/2}&\\ \leqslant \max\{g^\sigma_\nu(B_\nu - B_\mu), g^\sigma_\mu(B_\nu - B_\mu)\}^{1/2} &\leqslant \Delta_{\mu,\nu}.\end{aligned}$$

We get then from (34) that for $(\mu,\nu) \in I_\rho$, $\max\{\lambda_\mu,\lambda_\nu\} \leqslant c_0^{-1/2}\Delta_{\mu,\nu}$. Putting this inequality in (33) yields for all (μ,ν), $(\mu_0,\nu_0) \in I_\rho$,

$$\|\bar{\varphi}^w_{\nu_0}\bar{a}^w a^w \varphi^w_{\mu_0}\bar{\varphi}^w_\mu \bar{a}^w a^w \varphi^w_\nu\|^{1/2}_{\mathcal{L}(L^2)} \leqslant C' c_0^{-m}\Delta^{m-N}_{\nu_0,\mu_0}\Delta^{-N}_{\mu_0,\mu}\Delta^{m-N}_{\mu,\nu}. \tag{36}$$

Taking $N \geqslant N^* + m$, using the fact that $\Delta^*_{\mu,\nu} \leqslant \Delta_{\mu,\nu}$ and applying the last property in Lemma 1.6, we get

$$\sup_{(\mu_0,\nu_0)\in I_\rho} \sum_{(\mu,\nu)\in I_\rho} \|A^*_{\mu,\nu}A_{\mu_0,\nu_0}\|^{1/2}_{\mathcal{L}(L^2)} < \infty. \tag{37}$$

The adjoint term $A_{\mu,\nu}A^*_{\mu_0,\nu_0}$ can be treated in a similar way and we get using Cotlar lemma that R_ρ defined in (29) is uniformly in $\mathcal{L}(L^2)$. □

We give now a second result of microlocalization for a non-negative symbol b of order 2. This is in a way the reverse inequality of the one in Lemma 1.9.

LEMMA 1.10. — *Let g be an admissible metric, λ its uncertainty function, $\{X_\nu\}$ a family of points as introduced in Lemma 1.6, and for $r^2 < C_0^{-1}$, $\{\varphi_\nu\}$ a family of functions uniformly in* $\mathrm{conf}(1, g_{X_\nu}, X_\nu, r)$ *satisfying $\sum \varphi_\nu = 1$. Then for $0 \leqslant b \in S(\lambda^2, g)$ there exists C such that for all $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$\sum_\nu \|b^w \varphi_\nu^w u\|^2 \leqslant C \left(\|b^w u\|^2 + \|u\|^2\right). \tag{38}$$

Proof. — We first notice that applying Lemma 1.8 to the function $b^w u$ leads to the existence of a constant C_2 such that for all $u \in \mathcal{S}(\mathbb{R})$,

$$\sum_\nu \|\varphi_\nu^w b^w u\|^2 \leqslant C_2 \|b^w u\|^2.$$

But for all ν we can write

$$\|b^w \varphi_\nu^w u\|^2 \leqslant 2 \|\varphi_\nu^w b^w u\|^2 + 2 \|[b^w, \varphi_\nu^w] u\|^2.$$

In order to get inequality (38) it will be enough to prove the following lemma.

LEMMA 1.11. — *With b as in Lemma 1.10, there is a constant C_3 such that for all $u \in \mathcal{S}$*

$$\sum \|[b^w, \varphi_\nu^w] u\|^2 \leqslant C_3 \left((b^w u, u) + \|u\|^2\right).$$

Proof. — For all ν we set again $g_\nu = g_{X_\nu}$, $\lambda_\nu = \lambda(X_\nu)$, and B_ν the g_ν-ball of center X_ν and radius r for g_ν. Let us consider r^* such that $r^2 < r^{*2} < C_0^{-1}$, and choose a family of functions ψ_ν uniformly in $\mathrm{supp}(1, g_\nu, X_\nu, r^*)$ with value 1 on B_ν. We also set $b_\nu = \psi_\nu b$. For all ν the symbolic calculus gives

$$[b^w, \varphi_\nu^w] = (1/i) \{b_\nu, \varphi_\nu\}^w + r_{1,\nu}^w,$$

where $\{.\}$ is the Poisson bracket and where $r_{1,\nu} \in \mathrm{conf}(1, g_\nu, X_\nu, r)$ uniformly w.r.t. ν. Lemma 1.7 implies that there is a constant C_4 such that

$$\sum \left\|r_{1,\nu}^{w} u\right\|^{2} \leqslant C_{4}\|u\|^{2}.$$

It is therefore enough to study the Poisson bracket. We write that there is $r_{2,\nu}$ uniformly w.r.t. ν in $\operatorname{conf}(1, g_\nu, X_\nu, r)$ such that $\forall u \in \mathcal{S}$,

$$\|(\{b_\nu, \varphi_\nu\})^{w} u\|^{2} = \left(\{b_\nu, \varphi_\nu\}^{2^{w}} u, u\right) + (r_{2,\nu}^{w} u, u), \tag{39}$$

and we observe that there is C_5 such that $\sum_\nu (r_{2,\nu}^{w} u, u) \leqslant C_5 \|u\|^2$. Now for all ν, in a symplectic basis $e_1..e_n, \varepsilon_1..\varepsilon_n$ diagonalizing g_ν, we can write

$$\{b_\nu, \varphi_\nu\}^{2} \leqslant 2n \left(\sum_{j} (\partial_{e_j} b_\nu)^2 (\partial_{\varepsilon_j} \varphi_\nu)^2 + (\partial_{\varepsilon_j} b_\nu)^2 (\partial_{e_j} \varphi_\nu)^2\right). \tag{40}$$

We shall use the standard inequality for non-negative function:

LEMMA 1.12.— $f \geqslant 0, \quad f \in W^{2,\infty}(\mathbb{R}) \Longrightarrow (f'(t))^2 \leqslant 2f(t)\, \|f''\|_\infty$.

This inequality applied to b_ν, and the uniformity of the estimates on the ψ_ν's imply that there is a constant C_6 such that for all ν, j,

$$(\partial_{e_j} b_\nu)^2 + (\partial_{\varepsilon_j} b_\nu)^2 \leqslant C_6 \lambda_\nu b_\nu. \tag{41}$$

But $\varphi_\nu \in S(1, g_\nu)$ uniformly in ν also implies that there exists C_7 such that for all ν, j, $\lambda_\nu (\partial_{e_j} \varphi_\nu)^2 + \lambda_\nu (\partial_{\varepsilon_j} \varphi_\nu)^2 \leqslant C_7$, which yields with (40) and (41)

$$\{b_\nu, \varphi_\nu\}^2 \leqslant C_8 b_\nu,$$

where $C_8 = 4n^2 C_6 C_7$. Then we get that for all ν, there is $r_{3,\nu}, r_{4,\nu}, r_{5,\nu}$ uniformly in $\operatorname{conf}(1, g_\nu, X_\nu, r)$ such that

$$\left(\left(\{b_\nu, \varphi_\nu\}^2\right)^{w} u, u\right) \leqslant \left(\left(\{b_\nu, \varphi_\nu\}^2\right)^{w} \psi_\nu^{w} u, \psi_\nu^{w} u\right) + (r_{3,\nu}^{w} u, u),$$

then by the Fefferman-Phong inequality (5) applied to $C_8 b_\nu - \{b_\nu, \varphi_\nu\}^2$, we get

$$\begin{aligned}\left(\left(\{b_\nu, \varphi_\nu\}^2\right)^{w} u, u\right) &\leqslant C_8\, (b_\nu^{w} \psi_\nu^{w} u, \psi_\nu^{w} u) + (r_{4,\nu}^{w} u, u) \\ &\leqslant C_8 \left(\left(b \psi_\nu^3\right)^{w} u, u\right) + (r_{5,\nu}^{w} u, u).\end{aligned}$$

The finite overlap property (see Lemma 1.6) implies that there exists C_9 such that $\sum_\nu b\psi_\nu^3 \leqslant C_9 b$. Summing up and using the Fefferman-Phong inequality again yield the existence of C_3 such that

$$\sum_\nu \left(\left(\{b_\nu, \varphi_\nu\}^2\right)^{w} u, u\right) \leqslant C_3 \left((b^{w} u, u) + \|u\|^2\right),$$

which is the conclusion of the lemma. □

2. Proof of the main theorem.

Let $\varepsilon > 0$ be given and a, b be symbols satisfying the assumptions of Theorem 0.2 that is to say $|a| \leqslant b$ for a, $b \in S^2(G)$ where

$$G = \frac{|dx|^2 + |d\xi|^2}{\Lambda}, \quad \Lambda \geqslant 1. \tag{42}$$

We can of course assume that a is real-valued. We have to prove that there is a constant C_ε depending only on ε and a finite number of semi-norms of a and b such that the following inequality is verified for all $u \in \mathcal{S}(\mathbb{R})$

$$\|a^w u\|^2 \leqslant C_\varepsilon \left(\|b^w u\|^2 + \Lambda^{4\varepsilon} \|u\|^2 \right). \tag{43}$$

The proof of the main theorem will be done in several steps.

2.1. Change of metric.

The fact that $|a| \leqslant b$ and Lemmas 1.3, 1.4 imply that a and b are in the proper metric of b, which besides is not constant anymore. We can suppose that the first four semi-norms of a and b are less than or equal to 1. We denote by g the new metric, and λ its uncertainty function.

$$g : X \longmapsto g_X = \frac{|dx|^2 + |d\xi|^2}{\lambda(X)}. \tag{44}$$

2.2. Microlocalization of level II.

Consider $r > 0$ such that $0 < r^2 < C_0^{-1}$, where C_0 is the slowness constant of g. Let $\{X_\nu\}$ be a family of points as introduced in Lemma 1.6, and $\{\varphi_\nu\}$ a family of real functions such that

$$\varphi_\nu \in \operatorname{supp}(1, g_\nu, X_\nu, r) \text{ uniformly }, \quad \sum \varphi_\nu = 1, \tag{45}$$

associated with the proper metric g, where for all ν, $g_\nu = g_{X_\nu}, B_\nu = B_{g_\nu}(X_\nu, r)$. We have the following lemma of level II,

LEMMA 2.1. — *In order to prove the main theorem, it is sufficient to prove that there exist a constant $C'_\varepsilon > 0$ and a family of symbols $\{r_\nu\}$ uniformly in* $\operatorname{conf}(1, g_\nu, X_\nu, r)$, *such that for all $\nu \in \mathbb{N}$, $u \in \mathcal{S}(\mathbb{R})$.*

$$\|a^w \varphi_\nu^w u\|^2 \leqslant C'_\varepsilon \left(\|b^w \varphi_\nu^w u\|^2 + \Lambda^{4\varepsilon} (r_\nu^w u, u) \right). \tag{46}$$

Proof. — Let us take $u \in \mathcal{S}(\mathbb{R})$. Lemma 1.9 implies that there is a constant C such that

$$\|a^w u\|^2 \leqslant C\left(\sum_\nu \|a^w \varphi_\nu^w u\|^2 + \|u\|^2\right).$$

Inequality (46) implies then that for all ν,

$$\|a^w \varphi_\nu^w u\|^2 \leqslant C'_\varepsilon\left(\|b^w \varphi_\nu^w u\|^2 + \Lambda^{4\varepsilon}\left(r_\nu^w u, u\right)\right).$$

Now b is non-negative and of order 2. Lemma 1.10 implies that there exists C' such that

$$\sum_\nu \|b^w \varphi_\nu^w u\|^2 \leqslant C'\left(\|b^w u\|^2 + \|u\|^2\right).$$

From Lemma 1.7 we get that there is C'' such that $\sum \Lambda^{4\varepsilon}\left(r_\nu^w u, u\right) \leqslant C''\Lambda^{4\varepsilon}\|u\|^2$. We conclude that there is a constant C_ε such that

$$\|a^w u\|^2 \leqslant C_\varepsilon\left(\|b^w u\|^2 + \Lambda^{4\varepsilon}\|u\|^2\right),$$

that is to say inequality (43). The proof of the lemma is complete. □

2.3. Reduction of the cases.

We therefore only need to prove the following inequality, where C'_ε is a constant uniform in ν to be defined later,

$$\forall \nu,\quad \|a^w \varphi_\nu^w u\|^2 \leqslant C'_\varepsilon\left(\|b^w \varphi_\nu^w u\|^2 + \Lambda^{4\varepsilon}\left(r_\nu^w u, u\right)\right). \tag{47}$$

Following closely Fefferman-Phong's strategy, we first notice that b is non-negative, so that we can sort out the points in the phase space in three different types, "negligible", "elliptic", or "of convexity" as summarized in Lemma 4.2 of the appendix. It is important to stress that this classification is independent of the radius r of the partition, provided it is small enough with respect to the slowness constant of g. Let $R > 0$ be the radius of validity of the decomposition given by Lemma 4.2.

2.3.1. Negligible case.

In this case λ_ν is bounded above by a uniform constant. We get

$$(b\sharp\varphi_\nu),\ (a\sharp\varphi_\nu) \in \operatorname{conf}(1, g_\nu, X_\nu, r).$$

The associated operators are therefore uniformly in $\mathcal{L}(L^2)$. So if we set $r_\nu = b\sharp\varphi_\nu - a\sharp\varphi_\nu$ we obtain

$$\|a^w\varphi_\nu^w u\|^2 \leqslant 2\left(\|b^w\varphi_\nu^w u\|^2 + \Lambda^{4\varepsilon}\|r_\nu^w u\|^2\right).$$

This implies (47) and in this particular case, the error term is bigger than the others.

2.3.2. Elliptic case.

If $b|_{B_{g_\nu}(X_\nu,r)}$ is elliptic – that is to say X_ν is elliptic in lemma 4.2's terminology – then there is a constant C_0 uniform in ν such that

$$b(X) \geqslant C_0\lambda_\nu^2,\ \forall X \in B_{g_\nu}(X_\nu, r).$$

There is no restriction to suppose from now on $r < R/4$, which will be needed later. Let $\tilde{b} \in S^2(g_\nu)$ be equal to b on B_ν and such that $\tilde{b} \geqslant (C_0/2)\lambda_\nu^2$ elsewhere. We get $\tilde{b}^{-1}\lambda_\nu^2 \in S^0(g_\nu)$, and using (12) we get that there is a symbol $r_{1,\nu}$ uniformly in $\operatorname{conf}(1, g_\nu, X_\nu, r)$ such that

$$\tilde{b}^{-1}\lambda_\nu^2\sharp b\sharp\varphi_\nu = \varphi_\nu\lambda_\nu^2 + r_{1,\nu}$$

since $\tilde{b}^{-1}b = 1$ and $\{\tilde{b}^{-1}, b\} = 0$ on $\operatorname{supp}(\varphi_\nu)$. Therefore there exists C_1 such that

$$\lambda_\nu^2\|\varphi_\nu^w u\| \leqslant \left\|(\tilde{b}^{-1}\lambda_\nu^2)^w b^w\varphi_\nu^w u\right\| + \left\|r_{1,\nu}^w u\right\| \leqslant C_1\|b^w\varphi_\nu^w u\| + \left\|r_{1,\nu}^w u\right\|.$$

and we obtain then that there is an other symbol $r_{2,\nu} \in \operatorname{conf}(1, g_\nu, X_\nu, r)$ and C_2 such that

$$\lambda_\nu^4\|\varphi_\nu^w u\|^2 \leqslant C_2\|b^w\varphi_\nu^w u\|^2 + \left(r_{2,\nu}^w u, u\right).$$

On the other hand there exists $C_3 \geqslant 0$ such that $|a| \leqslant C_3\lambda_\nu^2$ on $B_{g_\nu}(X_\nu, r)$. Using $S^0(g_\nu) \hookrightarrow \mathcal{L}(L^2)$ again yields a new symbol $r_{3,\nu} \in \operatorname{conf}(1, g_\nu, X_\nu, r)$ such that for all $u \in \mathcal{S}$

$$\|a^w\varphi_\nu^w u\|^2 \leqslant C_3'\lambda_\nu^4\|\varphi_\nu^w u\|^2 + \left(r_{3,\nu}^w u, u\right)$$

for a new constant C_3'. These two inequalities together give (47).

2.3.3. Case of convexity.

This is the important case. We get from Lemma 4.2 the following decomposition on $B(X_\nu, R)$, g_ν-ball of radius R:

$$b(x,\xi) = e_\nu(x,\xi)\Big(\lambda_\nu(\xi - \alpha_\nu(x))^2 + V_\nu(x)\Big), \tag{48}$$

where e_ν, α_ν, V_ν are uniformly respectively in $S^0(g_\nu)$, $S^{1/2}(g_\nu)$ and $S^2(g_\nu)$, and $e_\nu \geqslant C^{-1}$ with C a uniform constant. Let χ_ν be the following canonical transformation globally defined on $\mathbb{R}^2$ provided α_ν in (48) is suitably extended in $\mathbb{R}^2$:

$$\chi_\nu : (y,\eta) \longmapsto (y, \eta + \alpha_\nu(y)) = (x,\xi)$$

which components are also in $S^{1/2}(g_\nu)$ on $\chi_\nu^{-1}(B(X_\nu,R))$. With these notations we have, for all $(y,\eta) \in \chi_\nu^{-1}(B(X_\nu,R))$,

$$\left|\left(\frac{a}{e_\nu} \circ \chi_\nu\right)(y,\eta)\right| \leqslant \lambda_\nu \eta^2 + V_\nu(y). \tag{49}$$

2.4. Translation of non-negativity for operators.

We stick from now on to the convexity case. Let us define a new real-valued function $\psi_\nu \in S^0(g_\nu)$ such that

$$\psi_\nu = 1 \text{ on } B(X_\nu, R/2), \quad \psi_\nu = 0 \text{ on } B(X_\nu,R)^c. \tag{50}$$

We first notice that $\psi_\nu a \circ \chi_\nu$ and $\psi_\nu a \circ \chi_\nu$ are in $S^2(g_\nu)$. We get

LEMMA 2.2. — *There is a constant $C_\varepsilon > 0$, such that $\forall v \in \mathcal{S}$,*

$$\left\|\left(\frac{\psi_\nu a}{e_\nu} \circ \chi_\nu\right)^w v\right\|^2 \leqslant C_\varepsilon \left(\left\|(\lambda_\nu \eta^2 + V_\nu)^w v\right\|^2 + \lambda_\nu^{4\varepsilon}\left\|v\right\|^2\right). \tag{51}$$

Proof. — The symbol ψ_ν is supported in $B(X_\nu,R)$, therefore (49) becomes $\left|(e_\nu^{-1}\psi_\nu a)\circ\chi_\nu(y,\eta)\right| \leqslant \lambda_\nu\eta^2 + V_\nu(y)$ everywhere provided $\lambda_\nu\eta^2 + V_\nu(y)$ is extended in the whole of $\mathbb{R}^2$. The symplectic change of variables

$$(t,\tau) = (\lambda_\nu^{-1/2} y,\ \lambda_\nu^{1/2}\eta)$$

allows us to apply Proposition 3.1 given in the next section. The proof of the lemma is complete. □

2.5. Cutting Schrödinger symbols.

We now want to go back to $b\circ\chi_\nu$. Let us first define a new real-valued function $\beta_\nu \in S^0(g_\nu)$ such that

$$\beta_\nu = 1 \text{ on } B(X_\nu, R/4), \quad \beta_\nu = 0 \text{ on } B(X_\nu, R/2)^c. \tag{52}$$

We can write

LEMMA 2.3. — *There is a constant C'_ε such that $\forall v \in \mathcal{S}(\mathbb{R})$,*

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 \leqslant C'_\varepsilon \left(\left\| \left(\frac{\psi_\nu b}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 + \lambda_\nu^{4\varepsilon} \|v\|^2 \right).$$

Proof. — Applying Lemma 2.2 to the function $(\beta_\nu \circ \chi_\nu)^w v$ leads to the following estimate for all $v \in \mathcal{S}(\mathbb{R})$:

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 \leqslant C_\varepsilon \left(\left\| (\lambda_\nu \eta^2 + V_\nu)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 + \lambda_\nu^{4\varepsilon} \|(\beta_\nu \circ \chi_\nu)^w v\|^2 \right).$$

Since we have $1 = \psi_\nu + (1 - \psi_\nu)$ and $(e_\nu^{-1}\psi_\nu b)\circ\chi_\nu = (\psi_\nu \circ \chi_\nu)(\lambda_\nu \eta^2 + V_\nu)$ we obtain

$$\left\| (\lambda_\nu \eta^2 + V_\nu)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 \leqslant 2 \left\| \left(\frac{\psi_\nu b}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 + 2 \|c_\nu^w v\|^2,$$

where c_ν is defined by $c_\nu = \big((1 - \psi_\nu \circ \chi_\nu)(\lambda_\nu \eta^2 + V_\nu)\big) \sharp (\beta_\nu \circ \chi_\nu)$ (see 11). In order to get Lemma 2.3, it is sufficient to prove:

LEMMA 2.4. — *$c_\nu^w \in \mathcal{L}(L^2)$, uniformly in ν.*

Proof. — We introduce the following metric on $\mathbb{R}^2_Y$:

$$\tilde{g}_\nu = \frac{|dY|^2}{\lambda_\nu + |Y - Y_\nu|^2}, \qquad \text{where } Y_\nu = \chi_\nu^{-1}(X_\nu),$$

whose uncertainty function is $\tilde{\lambda}_\nu(Y) = \lambda_\nu + |Y - Y_\nu|^2$ (see (9)). Using the fact that the components of χ_ν are in $S^{1/2}(g_\nu)$ and that $\psi_\nu \in S^0(g_\nu)$, $V_\nu \in S^2(g_\nu)$ uniformly in ν, we get that

$$\Big((1 - \psi_\nu)\circ\chi_\nu\Big)(\lambda_\nu \eta^2 + V_\nu) \in S^2(\tilde{g}_\nu), \qquad (\beta_\nu \circ \chi_\nu) \in S^0(\tilde{g}_\nu).$$

Moreover these two symbols have disjoint supports. This implies that $c_\nu \in S^{-\infty}(\tilde{g}_\nu)$ (see Definition 1.2) with semi-norms independent of ν. The proof of the lemma is complete. □

2.6. Bending

We now go back to the initial coordinates (x, ξ). We first use the weak version of the Egorov theorem given in Theorem 4.1 of the appendix. We first define a new function $A_\nu(x) = \int_0^x \alpha_\nu(s)ds$, and we note U_ν the unitary operator (on $L^2(\mathbb{R})$) of multiplication by $e^{iA_\nu(x)}$. We can then write that there exists three symbols $r_{1,\nu}$, $r_{2,\nu}$, $r_{3,\nu} \in S^0(g_\nu)$ with semi-norms independent of ν such that since $\psi_\nu a$, $\psi_\nu b$ and $\lambda_\nu^2\beta_\nu$ are real-valued of $S^2(g_\nu)$,

$$\left(\frac{\psi_\nu a}{e_\nu}\circ\chi_\nu\right)^w = U_\nu\left(\frac{\psi_\nu a}{e_\nu}\right)^w U_\nu^* + r_{1,\nu}^w, \tag{53}$$

$$\left(\frac{\psi_\nu b}{e_\nu}\circ\chi_\nu\right)^w = U_\nu\left(\frac{\psi_\nu a}{e_\nu}\right)^w U_\nu^* + r_{2,\nu}^w, \tag{54}$$

$$(\beta_\nu\circ\chi_\nu)^w = U_\nu\beta_\nu^w U_\nu^* + \lambda_\nu^{-2} r_{3,\nu}^w. \tag{55}$$

This conjugating process is a way of seeing the inequality of Lemma 2.3 in other coordinates. We notice that

$$f_\nu = (e_\nu)^{1/2} \in S^0(g_\nu)$$

since e_ν is elliptic of order 0. We can then write:

LEMMA 2.5. — *There is a constant C_ε and for all ν a symbol $r_\nu \in$ conf$(1, g_\nu, X_\nu, r)$ with uniform semi-norms such that for all $u \in \mathcal{S}(\mathbb{R})$ we have*

$$\left\|\left(\frac{\psi_\nu a}{e_\nu}\right)^w f_\nu^w\varphi_\nu^w u\right\|^2 \leqslant C_\varepsilon\left(\left\|\left(\frac{\psi_\nu b}{e_\nu}\right)^w f_\nu^w\varphi_\nu^w u\right\|^2 + \lambda_\nu^{4\varepsilon}\left(r_\nu^w u, u\right)\right).$$

Proof. — We generically write r_ν for symbols in the class conf$(1, g_\nu, X_\nu, r)$. We suppose from now on that the partition radius r satisfies

$$r = R/4,$$

so that $\beta_\nu = 1$ on the support of φ_ν. Therefore we have

$$\left\|\left(\frac{\psi_\nu a}{e_\nu}\right)^w f_\nu^w\varphi_\nu^w u\right\|^2 \leqslant 2\left\|\left(\frac{\psi_\nu a}{e_\nu}\right)^w \beta_\nu^w f_\nu^w\varphi_\nu^w u\right\|^2 + \|r_\nu^w u\|^2.$$

The fact that U_ν is unitary implies

$$\left\|\left(\frac{\psi_\nu a}{e_\nu}\right)^w \beta_\nu^w f_\nu^w\varphi_\nu^w u\right\|^2 = \left\|U_\nu^* U_\nu\left(\frac{\psi_\nu a}{e_\nu}\right)^w U_\nu^* U_\nu\beta_\nu^w U_\nu^* U_\nu f_\nu^w\varphi_\nu^w u\right\|^2.$$

We obtain

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \right)^w \beta_\nu^w f_\nu^w \varphi_\nu^w u \right\|^2 \leqslant 2 \left\| U_\nu^* \left(\frac{\psi_\nu a}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w U_\nu f_\nu^w \varphi_\nu^w u \right\|^2 + \|r_\nu^w u\|^2 .$$

Let us use again the fact that U_ν^* is unitary and set $v = U_\nu f_\nu^w \varphi_\nu^w u$. We get

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \right)^w \beta_\nu^w f_\nu^w \varphi_\nu^w u \right\|^2 \leqslant 2 \left\| \left(\frac{\psi_\nu a}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 + (r_\nu^w u, u) .$$

We can now apply Lemma 2.3 which implies that there exists a uniform constant C_ε such that

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \right)^w \beta_\nu^w f_\nu^w \varphi_\nu^w u \right\|^2 \leqslant C_\varepsilon \left(\left\| \left(\frac{\psi_\nu b}{e_\nu} \circ \chi_\nu \right)^w (\beta_\nu \circ \chi_\nu)^w v \right\|^2 + \lambda_\nu^{4\varepsilon} (r_\nu^w u, u) \right) . \tag{56}$$

We have here used the immediate property $\|v\| \leqslant C \|\varphi_\nu^w u\|$. From equalities (53-55) and $v = U_\nu f_\nu^w \varphi_\nu^w u$ we get

$$\left\| \left(\frac{\psi_\nu a}{e_\nu} \right)^w \beta_\nu^w f_\nu^w \varphi_\nu^w u \right\|^2 \leqslant C_\varepsilon \left(\left\| U_\nu \left(\frac{\psi_\nu a}{e_\nu} \right)^w U_\nu^* U_\nu \beta_\nu^w U_\nu^* U_\nu f_\nu^w \varphi_\nu^w u \right\|^2 + \lambda_\nu^{4\varepsilon} (r_\nu^w u, u) \right) .$$

The fact that U_ν is bounded on L^2 completes the proof of the lemma. □

2.7. End of proof.

We first notice that $f_\nu{}^{-1} = (e_\nu)^{-1/2} \in S^0(g_\nu)$ since e_ν is elliptic of order 0. Moreover we have three immediate symbolic properties (see (13))

$$\left\{ \frac{1}{f_\nu}, f_\nu \right\} = 0, \qquad \frac{\psi_\nu a}{e_\nu} = \frac{1}{f_\nu} \sharp (\psi_\nu a) \sharp \frac{1}{f_\nu} + r_\nu, \qquad \frac{\psi_\nu b}{e_\nu} = \frac{1}{f_\nu} \sharp (\psi_\nu b) \sharp \frac{1}{f_\nu} + s_\nu \tag{57}$$

with $r_\nu, s_\nu \in S^0(g_\nu)$. We keep the generic notation r_ν for symbols in $\mathrm{conf}(1, g_\nu, X_\nu, r)$ and from Lemma 2.5 we obtain that there are uniform constants all denoted by C_ε, such that

$$\begin{aligned}(58)\qquad \|(\psi_\nu a)^w \varphi_\nu^w u\|^2 &\leqslant 2\left\| f_\nu^w \left(\frac{\psi_\nu a}{e_\nu}\right)^w f_\nu^w \varphi_\nu^w u\right\|^2 + 2\|r_\nu^w u\|^2 \\ &\leqslant C_\varepsilon \left\|\left(\frac{\psi_\nu a}{e_\nu}\right)^w f_\nu^w \varphi_\nu^w u\right\|^2 + 2\|r_\nu^w u\|^2 \\ &\leqslant C_\varepsilon \left(\left\|\left(\frac{\psi_\nu b}{e_\nu}\right)^w f_\nu^w \varphi_\nu^w u\right\|^2 + \lambda_\nu^{4\varepsilon}(r_\nu^w u, u)\right) \\ &\leqslant C_\varepsilon \left(\left\|\left(\frac{1}{f_\nu}\right)^w f_\nu^w \left(\frac{\psi_\nu b}{e_\nu}\right)^w f_\nu^w \varphi_\nu^w u\right\|^2 + \lambda_\nu^{4\varepsilon}(r_\nu^w u, u)\right).\end{aligned}$$

The fact that $({f_\nu}^{-1})^w$ is bounded on L^2 and the third equality in (57) imply that

$$(59)\qquad \|(\psi_\nu a)^w \varphi_\nu^w u\|^2 \leqslant C_\varepsilon \left(\|(\psi_\nu b)^w \varphi_\nu^w u\|^2 + \lambda_\nu^{4\varepsilon}(r_\nu^w u, u)\right).$$

Recall that φ_ν is supported in $B(X_\nu, R/4)$, and that ψ_ν is equal to 1 on $B(X_\nu, R/2)$. We can then write

$$(60)\qquad \|a^w \varphi_\nu^w u\|^2 \leqslant C_\varepsilon \left(\|b^w \varphi_\nu^w u\|^2 + \lambda_\nu^{4\varepsilon}(r_\nu^w u, u)\right).$$

Using the fact that $\lambda_\nu \leqslant \Lambda$ we conclude the proof of (47) which induces Lemma 2.1. The proof of the main Theorem 0.2 will be complete when Proposition 3.1 below is proved. □

3. Level III metric and Schrödinger estimates.

One of the most important step in the proof of Theorem 0.2 is to get an inequality when b^w is the Schrödinger operator. This is possible thanks to a special metric, called *metric of level III*, which we give below. Let $\mathbb{R}_T^2$ where $T = (t, \tau)$ be equipped with the new (family of) semi-classical metrics

$$G_\Lambda = |dt|^2 + \Lambda^{-2}|d\tau|^2 \quad (\Lambda \geqslant 1).$$

We shall prove the following:

PROPOSITION 3.1.— *For $\varepsilon > 0$ there exists N_ε such that for all symbols $a \in S(\Lambda^2, G_\Lambda)$ and $0 \leqslant V \in S(1, |dt|^2)$ satisfying $|a| \leqslant \tau^2 + \Lambda^2 V(t)$,* *there is a constant $C_{\varepsilon.a,V}$ such that for all $u \in \mathcal{S}(\mathbb{R})$,*

$$\|a^w u\|^2 \leqslant C_{\varepsilon,a,V}\left(\left\|(D_t^2+\Lambda^2 V)u\right\|^2+\Lambda^{4\varepsilon}\|u\|^2\right) \tag{61}$$

Moreover $C_{\varepsilon.a,V}$ depends only on a finite number of semi-norms of a and V.

3.1. Level III metric.

Consider a smooth function $V \geqslant 0$ of one real variable, bounded as well as all its derivatives.

DEFINITION 3.2.— *Let $\varepsilon > 0$, We call level III metric the metric on $\mathbb{R}^2_{t,\tau}$ defined by*

$$g_W = \left(W(t)+\tau^2\right)|dt|^2 + \left(W(t)+\tau^2\right)^{-1}|d\tau|^2, \tag{62}$$

where W is defined by $W(t) = r(t)^{-2}$, with $r(t)$ the unique solution of

$$\frac{1}{2} r \int_{t-r}^{t+r} \left(\Lambda^2 V(s) + \Lambda^{2\varepsilon}\right) ds = 1. \tag{63}$$

This metric is well defined according to the fact that the left-hand side of (63) is a strictly increasing function of r from 0 to infinity. It is not hard to prove its slowness (see Definition 1.1).

LEMMA 3.3.— *The metric g_W above is slow, of slowness constant equal to 1/4.*

Proof.— Let t_1, t_2 be in $\mathbb{R}$ and set $r_1 = r(t_1)$ and $r_2 = r(t_2)$. If $|t_1 - t_2| \leqslant r_1$ then $[t_1 - r_1, t_1 + r_1] \subset [t_2 - 2r_1, t_2 + 2r_1]$ therefore

$$\frac{1}{2} r_1 \int_{t_1-r_1}^{t_1+r_1} \left(\Lambda^2 V(t) + \Lambda^{2\varepsilon}\right) dt = 1 \Longrightarrow r_1 \int_{t_2-2r_1}^{t_2+2r_1} \left(\Lambda^2 V(t) + \Lambda^{2\varepsilon}\right) dt \geqslant 1,$$

which implies $r_2 \leqslant 2r_1$. In a similar way from $|t_1 - t_2| \leqslant r_1/2$ we get $[t_2 - r_1/2, t_2 + r_1/2] \subset [t_1 - r_1, t_1 + r_1]$. We have then

$$\frac{1}{2} r_1 \int_{t_1-r_1}^{t_1+r_1} \left(\Lambda^2 V(t) + \Lambda^{2\varepsilon}\right) dt = 1 \Longrightarrow \frac{1}{4} r_1 \int_{t_2-r_1/2}^{t_2+r_1/2} \left(\Lambda^2 V(t) + \Lambda^{2\varepsilon}\right) dt \leqslant 1,$$

and we obtain $r_2 \geqslant r_1/2$. Eventually we can write $|t_1 - t_2| \leqslant r_1/2 \Longrightarrow \frac{1}{2} \leqslant \frac{r_1}{r_2} \leqslant 2$. The lemma is then easily obtained. □

Remark.— We don't know much about the temperance of g_W, even if we restrict it on balls for the metric G_Λ, (that would suffice to obtain a good microlocalization via the procedure described in [2]). We overcome this difficulty by a polynomial approximation. We want also to point out that g_W is symplectic, *i.e.*, $g_W = g_W^\sigma$. Consequently the g_W balls in the phase space have a symplectic area equivalent to 1.

Another striking point is the fact that the main tools for obtaining level III inequalities are the Bernstein inequalities about polynomials, coming from the equivalence of norms in finite-dimensional spaces. Indeed, let $m \in \mathbb{N}$, then for any $V \in \mathbb{R}_m[X]$

$$\max_{|t|\leqslant 1} |V(t)| \sim \int_{-1}^{1} |V(t)| dt. \tag{64}$$

These inequalities allow us to control V by its average W as given by the procedure used by Fefferman and Phong. Rephrasing this, it means that V is in the symbolic calculus associated with W. The fact that we have $|a| \leqslant \tau^2 + V$ also implies that a is in the same symbolic calculus.

3.2. Symbolic estimates.

First take $\varepsilon > 0$, and set for the rest of the proof

$$N_\varepsilon = [2/\varepsilon] - 2, \tag{65}$$

where [.] is the integer part.

LEMMA 3.4.— *Consider $a \in S(\Lambda^2, G_\Lambda)$, $V \in S(1, |dt|^2)$ such that $|a(t,\tau)| \leqslant \tau^2 + \Lambda^2 V(t)$. If W and g_W are defined according to Definition 3.2, then for all α, β $\in \mathbb{N}$ with $\beta \leqslant 2$,*

$$\left|\partial_t^\alpha \partial_\tau^\beta a(t,\tau)\right| \leqslant C_{\varepsilon,\alpha,\beta} \left(\tau^2 + W(t)\right)^{1+\frac{|\alpha|}{2}-\frac{|\beta|}{2}}, \tag{66}$$

where $C_{\varepsilon,\alpha,\beta}$ only depends on a finite number of semi-norms of a and V. Moreover if a vanishes for $|\tau| \geqslant \Lambda$ we get $a \in S(\tau^2 + W(t), g_W)$.

The cases are to be distinguished, according to the values of the derivatives indices.

Proof in the case $\alpha = \beta = 0$.— We first notice that the non-negativity of V and (63) imply $r(t)^{-2} \geqslant \Lambda^{2\varepsilon}$ *i.e.*, $0 < r(t) \leqslant \Lambda^{-\varepsilon}$. Consider for m integer, $V_{m,t}$ the Taylor expansion of V at t of order m,

$$V_{m,t}(s) = V(t) + V'(t)(s-t) + \ldots + V^{(m)}(t)(s-t)^m/m!.$$

We get for $|s-t| \leqslant r(t) \leqslant \Lambda^{-\varepsilon}$,

$$\Lambda^2 |V_{m,t}(s) - V(s)| \leqslant \beta_{m+1} \Lambda^{2-(m+1)\varepsilon},$$

where β_k is the k-th semi-norm of V. Taking $m = [2/\varepsilon] - 2 = N_\varepsilon$ yields

$$\Lambda^2 |V_{m,t}(s) - V(s)| \leqslant \beta_{m+1} \Lambda^{2\varepsilon}. \tag{67}$$

By hypothesis $|a(t,\tau)| \leqslant \tau^2 + \Lambda^2 V(t)$. We therefore get that there is a constant $C_{0,\varepsilon,V}$ such that

$$|a(t,\tau)| \leqslant \tau^2 + \max_{|t-s| \leqslant r(t)} \left(\Lambda^2 V_{m,t}(s) + C_{0,\varepsilon,V} \Lambda^{2\varepsilon} \right).$$

The Bernstein inequalities then give the existence of a constant C depending on ε via m such that

$$|a(t,\tau)| \leqslant \tau^2 + C \frac{1}{2r(t)} \int_{t-r(t)}^{t+r(t)} \left(\Lambda^2 V_{m,t}(s) + \Lambda^{2\varepsilon} \right) ds.$$

We use again (67) as well as the definitions of r and W given in (63) and we get that there is a constant $C_{0,\varepsilon}$ such that

$$|a(t,\tau)| \leqslant C_{0,\varepsilon} \left(\tau^2 + W(t) \right). \tag{68}$$

This ends the proof in the case $\alpha = \beta = 0$. □

Proof in the case of any α, and $\beta = 0$.— For $t_0 \in \mathbb{R}$ let I_0 be the interval $I_0 = \{t \text{ s.t. } |t - t_0| < \Lambda^{-\varepsilon}\}$. Consider for $m \in \mathbb{N}$

$$a_m(t,\tau) = a(t_0,\tau) + (t-t_0) \frac{\partial a}{\partial t}(t_0,\tau) + \ldots + \frac{(t-t_0)^m}{m!} \frac{\partial^m a}{\partial t^m}(t_0,\tau).$$

If we bound the remainder of the Taylor expansion of a on I_0, we get for all $t \in I_0$, $\tau \in \mathbb{R}$:

$$|a_m(t,\tau) - a(t,\tau)| \leqslant \gamma_{m+1} \Lambda^{2-(m+1)\varepsilon}, \tag{69}$$

$$|\partial_t^\alpha a_m(t,\tau) - \partial_t^\alpha a(t,\tau)| \leqslant \gamma_{m+1+\alpha} \Lambda^{2-(m+1)\varepsilon}, \quad \alpha \in \mathbb{N}. \tag{70}$$

We take again $m = N_\varepsilon$ for controlling the remainders by $C_{\alpha,\varepsilon,a} \Lambda^{2\varepsilon}$. Now consider the Bernstein inequality for the polynomial $t \longrightarrow a_m(t,\tau)$ (see (64)). For all $\Delta > 0$ we get

$$|\partial_t^\alpha a_m(t,\tau)| \leqslant C_m \frac{1}{\Delta^\alpha} \max_{|t-s| \leqslant \Delta} |a_m(s,\tau)|.$$

According to the former estimates (69-70), this inequality can be transposed on a, and we get for all $t \in I_0$, $\Delta \leqslant \Lambda^{-\varepsilon}$,

$$(71)\quad |\partial_t^\alpha a(t,\tau)| \leqslant \gamma_{m+1}\Lambda^{2\varepsilon} \times \frac{1}{\Delta^\alpha} + C_m \frac{1}{\Delta^\alpha} \max_{|t-s|\leqslant\Delta} |a(s,\tau)| + \gamma_{m+1+\alpha}\Lambda^{2\varepsilon}.$$

We first use the fact that $\Delta \leqslant \Lambda^{-\varepsilon}$ in order to bound the first term and estimate the error term in (71), so we can write for $t \in I_0$,

$$(71)\qquad |\partial_t^\alpha a(t,\tau)| \leqslant C_{\alpha,\varepsilon,a}\left(\frac{1}{\Delta^{\alpha+2}} + \frac{1}{\Delta^\alpha}\max_{|t-s|\leqslant\Delta}|a(s,\tau)|\right).$$

Now (68) implies for all $\Delta \leqslant \Lambda^{-\varepsilon}$

$$(73)\qquad |\partial_t^\alpha a(t,\tau)| \leqslant C'_{\alpha,\varepsilon,a}\left(\frac{1}{\Delta^{\alpha+2}} + \frac{1}{\Delta^\alpha}\max_{|t-s|\leqslant\Delta}(\tau^2+W(s))\right).$$

Let's take Δ defined by $(2\Delta)^{-2} = \tau^2 + W(t) \geqslant \Lambda^{2\varepsilon}$, and use the slowness of the metric g_W, which constant of slowness is $1/4$ (see Lemma 3.3). We can now uniformly bound $W(s)$ by $4W(t)$ for all $|t-s| \leqslant \Delta$. We then get the desired result, *i.e.*, for all t, τ, α,

$$(74)\qquad |\partial_t^\alpha a(t,\tau)| \leqslant C''_{\alpha,\varepsilon,a}(\tau^2+W(t))^{1+\frac{\alpha}{2}}.$$

The proof in the case $\beta = 0$ is complete. □

Proof of the case $\alpha = 0$, $\beta = 1$. — Inequality (66) has already been proved in the case $\alpha = \beta = 0$. It gives the estimate

$$0 \leqslant a(t,\tau) + C_{\varepsilon,0,0}\left(\tau^2 + W(t)\right).$$

For fixed t the function $\tau \longmapsto a(t,\tau) + C_{\varepsilon,0,0}\left(\tau^2+W(t)\right)$ is non-negative, smooth, with second-order derivatives uniformly bounded by $\gamma_2 + 2C_{\varepsilon,0,0}$, where γ_k is the k-th semi-norm of a. We can therefore apply to it the classical Lemma 1.12 which leads to the following estimate for all t, τ:

$$(75)\quad \left|\frac{\partial a}{\partial\tau}(t,\tau) + 2C_{\varepsilon,0,0}\tau\right|^2 \leqslant (\gamma_2+2C_{\varepsilon,0,0})\Big(a(t,\tau)+C_{\varepsilon,0,0}\left(\tau^2+W(t)\right)\Big).$$

We use again (66) to bound a by $\tau^2 + W(t)$, and we get that there exists $C_{\varepsilon,1,0}$ such that

$$(76)\qquad \left|\frac{\partial a}{\partial\tau}(t,\tau)\right| \leqslant C_{\varepsilon,1,0}\left(\tau^2+W(t)\right)^{1/2}.$$

The case $\alpha = 0$ and $\beta = 1$ in (66) is done. □

Proof in the case of any α, and $\beta = 1$. — We can use the same tools of approximation by polynomials used in the case $\beta = 0$. Keeping the notation of the case $\beta = 0$, we get again thanks to the Bernstein inequalities that for $\alpha \in \mathbb{N}$, t, $\tau \in \mathbb{R}$,

$$\left|\partial_t^\alpha \frac{\partial a_m}{\partial \tau}(t,\tau)\right| \leqslant C_m \frac{1}{\Delta^\alpha} \max_{|t-s|\leqslant\Delta} \left|\frac{\partial a_m}{\partial \tau}(s,\tau)\right|.$$

Approximation inequalities of type (69-70) are true when replacing a by $\partial a/\partial\tau$, γ_{m+1} by γ_{m+2} and $\gamma_{m+1+\alpha}$ by $\gamma_{m+2+\alpha}$. Choosing again $m = N_\varepsilon$, we obtain that the polynomial remainders are bounded by $C'_{m,\alpha,a}\Lambda^{2\varepsilon}$. If we take Δ defined by $(2\Delta)^{-2} = \tau^2 + W(t) > \Lambda^{2\varepsilon}$, we get the desired estimate

$$\left|\partial_t^\alpha \frac{\partial a}{\partial \tau}(t,\tau)\right| \leqslant C_{\varepsilon,\alpha,1}\left(\tau^2 + W(t)\right)^{\frac{1}{2}+\frac{\alpha}{2}},$$

i.e., (66) in the case $\beta = 1$. □

Proof in the case $\beta = 2$. — For $\beta \geqslant 2$ the symbol inequality is

$$\left|\partial_t^\alpha\partial_\tau^\beta a(t,\tau)\right| \leqslant \gamma_{\alpha+\beta}\Lambda^{2-\beta} \leqslant \gamma_{\alpha+\beta}. \tag{77}$$

Choosing $\beta = 2$ implies that for all α we have $\left(\tau^2 + W(t)\right)^{1-\frac{\beta}{2}+\frac{\alpha}{2}} \geqslant 1$. The proof in that case is complete. □

Remark. — The fact that a is supported in $\{|\tau| \leqslant c\Lambda\}$ would have been sufficient for a to be in $S(\tau^2 + W(t), g_W)$. Indeed in that case $|\tau^2 + W(t)| \leqslant C_c\Lambda^2$ on the support of a. Therefore the first inequality of (77) implies that (66) is satisfied for all derivatives of a. This will often be the case in what will follow below. Anyway we will only use the first two derivatives of a.

3.3. Proof of the result for a Schrödinger operator.

We want now to prove Proposition 3.1. We consider two symbols a and V satisfying the hypothesis. We also define W and g_W according to Definition 3.2. We can therefore use the symbolic results of level III of the previous section, described in Lemma 3.4. We keep the definition $N_\varepsilon = [2/\varepsilon] - 2$ already introduced.

LEMMA 3.5. — *There is a constant $C_{\varepsilon,a,V}$ such that for all $u \in \mathcal{S}(\mathbb{R})$,*

$$\|a^w u\|^2 \leqslant C_{\varepsilon,a,V}\left(\|W(t)u\|^2 + (W(t)D_t u, D_t u) + \left\|D_t^2 u\right\|^2 + \|u\|^2\right). \tag{78}$$

Proof. — We first write the Taylor expansion of a up to order 2 in $(t,0)$.

$$a(t,\tau) = a(t,0) + \frac{\partial a}{\partial \tau}(t,0)\tau + b_0(t,\tau)\tau^2,$$

with $b_0 \in S(1, G_\Lambda)$. The triangle inequality gives $\forall u \in \mathcal{S}(\mathbb{R})$:
(79)

$$\|a^w u\|^2 \leqslant 3\left(\|a(t,0)u\|^2 + \left\|\left(\frac{\partial a}{\partial \tau}(t,0)\tau\right)^w u\right\|^2 + \left\|\left(b_0(t,\tau)\tau^2\right)^w u\right\|^2\right).$$

As for the third term of the right hand side of (79), we can use the symbolic calculus (see 12):

$$\begin{aligned} b_0\tau^2 &= b_0\sharp\tau^2 - \left\{b_0, \tau^2\right\}/(2i) + b_1 \\ &= b_0\sharp\tau^2 + b'_{0t}\tau/i + b_1 \\ &= b_0\sharp\tau^2 + b'_{0t}\sharp\tau/i + b_2 \end{aligned}$$

with $b_2, b'_{0t} \in S^0(G_\Lambda)$. L^2-continuity for symbols of order 0 implies that there exists a constant $C_a > 0$ such that $\forall u \in \mathcal{S}$,

$$\left\|\left(b_0(t,\tau)\tau^2\right)^w u\right\|^2 \leqslant C_a\left(\left\|D_t^2 u\right\|^2 + \|u\|^2\right). \tag{80}$$

As for the second term of (79), the symbolic calculus also gives

$$\frac{\partial a}{\partial \tau}(t,0)\sharp\tau = \frac{\partial a}{\partial \tau}(t,0)\tau - \frac{1}{2i}\frac{\partial^2 a}{\partial t \partial \tau}(t,0).$$

We can deduce that the second term of (79) satisfies

$$\begin{aligned} \left\|\left(\frac{\partial a}{\partial \tau}(t,0)\tau\right)^w u\right\|^2 &\leqslant 2\left\|\frac{\partial a}{\partial \tau}(t,0)D_t u\right\|^2 + \frac{1}{2}\left\|\frac{\partial^2 a}{\partial t\partial \tau}(t,0)u\right\|^2 \\ &= 2\left(\left|\frac{\partial a}{\partial \tau}(t,0)\right|^2 D_t u, D_t u\right) + \frac{1}{2}\left(\left|\frac{\partial^2 a}{\partial t \partial \tau}(t,0)\right|^2 u, u\right). \end{aligned} \tag{81}$$

The estimates of Lemma 3.4 for $(\alpha,\beta) = (0,1)$ and $(\alpha,\beta) = (1,1)$ give

$$\left\|\left(\frac{\partial a}{\partial \tau}(t,0)\tau\right)^w u\right\|^2 \leqslant C'_{\varepsilon,a,V}\Big((WD_t u, D_t u) + (W^2 u, u)\Big). \tag{82}$$

For the first term of (79), we can use the case $(\alpha,\beta) = (0,0)$, *i.e.*, the fact that a is bounded by W:

$$\|a(t,0)u\|^2 \leqslant C''_{\varepsilon,a,V}\|Wu\|^2. \tag{83}$$

According to the three estimates (80), (82) and (83), we get Lemma 3.5. □

We state now a classical lemma (for a proof see for instance [8], [15], and in a far wider context [9]). It would also be possible to give a direct proof using level III metric.

LEMMA 3.6 (Polynomial estimate). — *For all m there exists $C_m > 0$ s.t. if $\tilde{V}$ is a polynomial of degree at most m and $\tilde{W}$ is defined by*

$$\tilde{W}(t) = \tilde{r}^{-2}(t), \text{ where } \frac{1}{2}\tilde{r}\int_{t-\tilde{r}}^{t+\tilde{r}} |\tilde{V}(s)|ds = 1, \tag{84}$$

then for all $u \in \mathcal{S}(\mathbb{R})$, we have

$$\left\|\tilde{W}u\right\|^2 + \left(\tilde{W}D_t u, D_t u\right) + \left\|D_t^2 u\right\|^2 \leqslant C_m \left\|(D_t^2 + |\tilde{V}|)u\right\|^2, \tag{85}$$

Using this result we get

LEMMA 3.7. — *There is a constant $C_{\varepsilon,a,V}$ such that for all $u \in \mathcal{S}$.*

(86)

$$\|Wu\|^2 + (WD_t u, D_t u) + \left\|D_t^2 u\right\|^2 \leqslant C_{\varepsilon,a,V}\left(\left\|(D_t^2 + \Lambda^2 V)u\right\|^2 + \Lambda^{4\varepsilon}\|u\|^2\right).$$

Proof. — We have to cut the t-space according to the constant metric

$$\Lambda^{2\varepsilon}|dt|^2.$$

For $R > 0$ to be picked below, we can associate a partition of unity, *i.e.*, families of points $\{t_\nu\}$ and of functions $\{\psi_\nu\}$ with $\sum \psi_\nu^2 = 1$, such that for all k, there exists γ_k

$$\forall \nu, \quad |\partial_t^{(k)}\psi_\nu| \leqslant C_k \Lambda^{k\varepsilon}.$$

We can suppose that every ψ_ν is supported in the ball

$$B_\nu = \left\{t, \ |t - t_\nu| \leqslant R\Lambda^{-\varepsilon}\right\}.$$

We first localize the problem. We write

$$\sum_\nu \left\|\left(D_t^2 + \Lambda^2 V(t)\right)\psi_\nu u\right\|^2 \leqslant 2\left\|\left(D_t^2 + \Lambda^2 V(t)\right)u\right\|^2 + 2\sum_\nu \left\|\left[D_t^2, \psi_\nu\right]u\right\|^2. \tag{87}$$

As for the commutator, we say that for all ν,

$$\left\|\left[D_t^2, \psi_\nu\right]u\right\|^2 \leqslant 2\left\|\psi_\nu'' u\right\|^2 + 8\left\|\psi_\nu' D_t u\right\|^2.$$

We next sum over ν, and we get that there is a constant C_1 such that

$$\sum_{\nu} \left\| \left[D_t^2, \psi_\nu\right] u \right\|^2 \leqslant C_1 \left(\Lambda^{4\varepsilon} \|u\|^2 + \Lambda^{2\varepsilon} \|D_t u\|^2 \right).$$

Therefore, using $\|D_t u\|^2 = (D_t^2 u, u)$ we get for all $\mu > 0$ the existence of C_μ such that

$$(88) \qquad 2\sum_{\nu} \left\| \left[D_t^2, \psi_\nu\right] u \right\|^2 \leqslant C_\mu \Lambda^{4\varepsilon} \|u\|^2 + \mu \left\| D_t^2 u \right\|^2.$$

Eventually (87) becomes

$$(89) \quad \sum_{\nu} \left\| \left(D_t^2 + \Lambda^2 V(t) \right) \psi_\nu u \right\|^2 \leqslant 2 \left\| D_t^2 + \Lambda^2 V(t) u \right\|^2 + C_\mu \Lambda^{4\varepsilon} \|u\|^2 + \mu \left\| D_t^2 u \right\|^2.$$

Let us study now the first term of (89). For all $\nu \in \mathbb{N}$, we note $V_{m,\nu}$ the Taylor expansion of V at t_ν of order m

$$V_{m,\nu}(t) = V(t_\nu) + V'(t_\nu)(t - t_\nu) + \ldots + V^{(m)}(t_\nu)(t - t_\nu)^m / m!.$$

We then write $\forall \nu \in \mathbb{N}$,

$$\begin{aligned} &\left\| \left(D_t^2 + \Lambda^2 V_{m,\nu} + \Lambda^{2\varepsilon} \right) \psi_\nu u \right\|^2 \\ (90) \quad &\leqslant 3 \left\| \left(D_t^2 + \Lambda^2 V \right) \psi_\nu u \right\|^2 + 3\Lambda^4 \left\| \left(V - V_{m,\nu} \right) \psi_\nu u \right\|^2 + 3\Lambda^{2\varepsilon} \|\psi_\nu u\|^2. \end{aligned}$$

We know that $\forall \nu \in \mathbb{N}$, $|V(t) - V_{m,\nu}(t)| \leqslant \beta_{m+1} |t - t_\nu|^{m+1} / (m+1)!$ where β_k is the k-th semi-norm of V. If we choose $m = [2/\varepsilon] - 2 = N_\varepsilon$, and R such that

$$(91) \qquad \beta_{m+1} (4R)^{m+1} / (m+1)! \leqslant 1/2,$$

then we get for all t such that $|t - t_\nu| \leqslant 4R\Lambda^{-\varepsilon}$

$$(92) \qquad \Lambda^2 |V(t) - V_{m,\nu}(t)| \leqslant \Lambda^{2\varepsilon}/2.$$

Moreover ψ_ν is supported in $B_\nu = \{|t - t_\nu| \leqslant R\Lambda^{-\varepsilon}\}$. We can a fortriori apply (92) and obtain

$$\begin{aligned} (93) \quad &\left\| \Lambda^2 \left(V - V_{m,\nu} \right) \psi_\nu u \right\|^2 \\ &\leqslant \left(\beta_{m+1} R^{m+1} / (m+1)! \right)^2 \Lambda^{4 - 2(m+1)\varepsilon} \|\psi_\mu u\|^2 \leqslant \Lambda^{4\varepsilon} \|\psi_\mu u\|^2. \end{aligned}$$

Applying this we get that there is a constant C_2 such that for all ν,

(94)
$$\left\|(D_t^2 + \Lambda^2 V_{m,\nu} + \Lambda^{2\varepsilon})\psi_\nu u\right\|^2 \leqslant C_2 \left(\left\|(D_t^2 + \Lambda^2 V)\psi_\nu u\right\|^2 + \Lambda^{4\varepsilon} \left\|\psi_\nu u\right\|^2\right).$$

As for the first term of the right-hand-side of (90), we can use Lemma 3.6 applied to the polynomial $\tilde{V} = \Lambda^2 V_{m,\nu} + \Lambda^{2\varepsilon}$. We get that there is a constant C_3 such that if we set

$$W_{m,\nu}(t) = r_m^{-2}(t),$$

where r_m is the unique solution of

$$\frac{1}{2} r \int_{t-r}^{t+r} |\Lambda^2 V_{m,\nu}(s) + \Lambda^{2\varepsilon}| ds = 1, \tag{95}$$

we have

$$\begin{aligned} &\left\|W_{m,\nu}\psi_\nu u\right\|^2 + (W_{m,\nu} D_t \psi_\nu u, D_t \psi_\nu u) + \left\|D_t^2 \psi_\nu u\right\|^2 \\ &\qquad \leqslant C_3 \left(\left\|(D_t^2 + |\Lambda^2 V_{m,\nu} + \Lambda^{2\varepsilon}|)\psi_\nu u\right\|^2\right). \end{aligned} \tag{96}$$

According to (92) we can write for all s such that $|s - t_\nu| \leqslant 4R\Lambda^{-\varepsilon}$,

$$\Lambda^2 |V_{m,\nu}(s) - V(s)| \leqslant 2^{-1}\Lambda^{2\varepsilon},$$

therefore

$$\Lambda^2 V_{m,\nu}(s) + \Lambda^{2\varepsilon} \geqslant 2^{-1}(\Lambda^2 V(s) + \Lambda^{2\varepsilon}) \geqslant 0.$$

This implies that for all t such that $|t - t_\nu| \leqslant R\Lambda^{-\varepsilon}$, we have an alternative. First assume that $r(t) \leqslant R\Lambda^{-\varepsilon}$. We can write

$$\begin{aligned} 1 = \frac{1}{2} r(t) \int_{|s-t|\leqslant r(t)} (\Lambda^2 V(s) + \Lambda^{2\varepsilon}) ds &\leqslant r(t) \int_{|s-t|\leqslant r(t)} |\Lambda^2 V_{m,\nu}(s) + \Lambda^{2\varepsilon}| ds \\ &\leqslant r(t) \int_{|s-t|\leqslant 2r(t)} |\Lambda^2 V_{m,\nu}(s) + \Lambda^{2\varepsilon}| ds, \end{aligned} \tag{97}$$

since $\Lambda^2 V_{m,\nu}(s) + \Lambda^{2\varepsilon} \geqslant 0$ on $|s - t| \leqslant 2r(t)$. This implies from (95) that $r_m(t) \leqslant 2r(t)$, and we get for all $t \in B_\nu = \{|t - t_\nu| \leqslant R\Lambda^{-\varepsilon}\}$

$$W(t) \leqslant 4W_{m,\nu}(t).$$

If we suppose that $r(t) > R\Lambda^{-\varepsilon}$ we can write $W(t) \leqslant R^{-2}\Lambda^{2\varepsilon}$. Anyhow we get that there is a constant C_4 uniform in ν, such that for all $t \in \operatorname{supp}(\psi_\nu) \subset B_\nu$, we have

$$W(t) \leqslant C_4 \left(\Lambda^{2\varepsilon} + W_{m,\nu}(t)\right).$$

This implies that there is a constant C_5 such that

$$\begin{aligned}
&\|W\psi_\nu u\|^2 + (WD_t\psi_\nu u, D_t\psi_\nu u) + \left\|D_t^2\psi_\nu u\right\|^2 \\
&\leqslant C_5\Big(\|W_{m,\nu}\psi_\nu u\|^2 + (W_{m,\nu}D_t\psi_\nu u, D_t\psi_\nu u) \\
&\qquad +\Lambda^{2\varepsilon}\left(D_t\psi_\nu u, D_t\psi_\nu u\right) + \Lambda^{4\varepsilon}\left\|\psi_\nu u\right\|^2 + \left\|D_t^2\psi_\nu u\right\|^2\Big) \\
&\leqslant 3C_5\left(\|W_{m,\nu}\psi_\nu u\|^2 + (W_{m,\nu}D_t\psi_\nu u, D_t\psi_\nu u) + \Lambda^{4\varepsilon}\left\|\psi_\nu u\right\|^2 + \left\|D_t^2\psi_\nu u\right\|^2\right).
\end{aligned}$$

According to (96) and (94) we get that there is $C_6 > 0$ such that

$$\begin{aligned}
&\|W\psi_\nu u\|^2 + \left\|W^{1/2}D_t\psi_\nu u\right\|^2 + \left\|D_t^2\psi_\nu u\right\|^2 \\
&\qquad\qquad \leqslant C_6\left(\left\|\left(D_t^2+\Lambda^2 V\right)\psi_\nu u\right\|^2 + \Lambda^{4\varepsilon}\left\|\psi_\nu u\right\|^2\right).
\end{aligned}$$

We can commute as in (87), use the fact that $\sum_\nu \left\|\psi_\nu' u\right\|^2 \leqslant C_7\left\|u\right\|^2$, and use again (88) during summation over $\nu > 0$. We get that there is a constant $C_{7,\mu} > 0$ such that

$$\begin{aligned}
&\|Wu\|^2 + \left\|W^{1/2}D_t u\right\|^2 + \left\|D_t^2 u\right\|^2 \\
&\qquad\qquad \leqslant C_{7,\mu}\left(\left\|\left(D_t^2+\Lambda^2 V\right)u\right\|^2 + \Lambda^{4\varepsilon}\left\|u\right\|^2\right) + 4\mu\left\|D_t^2 u\right\|^2.
\end{aligned}$$

Taking $\mu = 1/8$ completes the proof of the Lemma 3.7. □

Proof of Proposition 3.1. — This is immediate, according to Lemmas 3.5 and 3.7.

4. Appendix.

4.1. Egorov theorem in dimension 1.

We give here a version of the theorem of Egorov ([3]), in the very special case of dimension one of space, and with a gain of two derivatives. This is essentially the version given by Fefferman and Phong in [6]. An homogeneous statement can be found for example in [7].

THEOREM 4.1 (Egorov). — *Let $G = \Lambda^{-1}|dX|^2$ a family of semi-classical metrics on $\mathbb{R}^2$, with $X = (x,\xi)$ symplectic coordinates. Let $x \mapsto \alpha(x)$ be real-valued in $S^{1/2}(G)$ (see def. 1.2), $A(x) = \int_0^x \alpha(s)ds$, and* *let U be the unitary operator (on L^2) of multiplication by $e^{iA(x)}$. Finally let us define the following canonical transformation:*

$$\chi : (y,\eta) \longmapsto (y, \eta + \alpha(y)) = (x,\xi).$$

Then for all real symbols $a \in S^2(G)$, $a \circ \chi \in S^2(G)$ and there exists $a_0 \in S^0(G)$ such that

$$Ua^wU^* = (a \circ \chi)^w + a_0^w.$$

Remark. — This version does not use confinement, and we can notice that the canonical transformation χ is globally defined. A similar statement in dimension n can be written, and the tools for the proof are similar. In fact the unitary operator U is here the simplest example of Fourier integral operator, and the function A is related to the generating function of χ (see for example [7], [10]).

Proof. — We only give a sketch of the proof. Let $a \in S^2(G)$, and denote by K the kernel of the operator Ua^wU^*

$$K(x,y) = \frac{1}{(2\pi)^n} \int e^{i(A(x)-A(y)+x\eta-y\eta)} a\left(\frac{x+y}{2}, \eta\right) d\eta$$

(in the oscillatory integral sense). Let us define the Weyl symbol $\tilde{a}$ a priori in $\mathcal{S}'$ such that K is also the kernel of $\tilde{a}^w$. Then

$$\begin{aligned} \tilde{a}(x,\xi) &= \int e^{-it\xi} K(x+t/2, x-t/2)\, dt \\ &= \frac{1}{(2\pi)^n} \int e^{i\left(A\left(x+\frac{t}{2}\right)-A\left(x-\frac{t}{2}\right)+t\eta-t\xi\right)} a(x,\eta) d\eta dt. \end{aligned} \tag{98}$$

We only need to prove that $\tilde{a} = a \circ \chi + a_0$ around 0. Given any $X = (x,\xi)$, let us set

$$\Phi_X(t,\eta) = A(x+t/2) - A(x-t/2) + t\eta - t\xi,$$

and study the critical points:

$$\partial_{t,\eta}\Phi_X(t,\eta) = 0 \Longleftrightarrow \begin{cases} t = 0 \\ \eta = \xi - \alpha(x) \end{cases}, \quad D^2_{t,\eta}\Phi_X(0, \xi - \alpha(x)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can apply the Morse lemma with parameter X around 0, taking into account the scale Λ inherited from the metric. We get that there is a local diffeomorphism $K_X : (s,\zeta) \mapsto (t,\eta)$ smooth in X around 0, on a G-neighborhood of $(t,\eta) = 0$, such that

$$\Phi_X(K_X(s,\zeta)) = \langle s, \zeta - (\xi - \alpha(x)) \rangle,$$

whose Jacobian at the critical point is 1. Then let us take for all X around 0, a symbol ω_X of order 0, depending smoothly on X in the scale Λ, supported on a G-ball of radius r and center the critical point $(0, \xi - \alpha(x))$, with value 1 on ball of radius $r/2$, where the precedent form is valid. We get that for all X around 0, $\tilde{a} = \tilde{a}_1 + \tilde{a}_2$, where

$$\tag{99} \begin{aligned} \tilde{a}_1(x,\xi) &= (2\pi)^{-n} \int e^{i\Phi_X(t,\eta)} \omega_X(t,\eta) a(x,\eta) d\eta dt, \\ \tilde{a}_2(x,\xi) &= (2\pi)^{-n} \int e^{i\Phi_X(t,\eta)} (1 - \omega_X(t,\eta)) a(x,\eta) d\eta dt. \end{aligned}$$

The second term can be treated thanks to the non-stationary phase formula. We get $\tilde{a}_2 \in S^{-\infty}(G)$ near $X = 0$. As for the first term, the stationary phase formula gives

$$\tag{100} \tilde{a}_1(x,\xi) = (2\pi)^{-n} \int e^{\langle s, \zeta - (\xi - \alpha(x)) \rangle} \underbrace{|K'_X(s,\zeta)| \omega_X(K_X(s,\zeta)) a(x, \eta(s,\zeta))}_{b_X(s,\zeta)} d\zeta ds.$$

We get then according to the quadratic form of the phase

$$\tilde{a}_1(x,\xi) = b_X(0, \xi - \alpha(x)) + i\partial_\zeta \partial_s b_X(0, \xi - \alpha(x)) + a_0(x,\xi).$$

a_0 is then of order 0. The fact that $\tilde{a}_1$ is real implies that the second term vanishes. Moreover the fact that the Jacobian and ω_X have value 1 at the critical point gives $b_X(0, \xi - \alpha(x)) = a(x, \xi - \alpha(x)) = a \circ \chi(x,\xi)$. The proof is complete. □

4.2. A very normal form.

We give here a useful preparation lemma, in the spirit of the Fefferman-Phong classification (see [5]). The proof uses a Calderón-Zygmund decomposition and some tools of differential geometry, such as the Malgrange preparation theorem (see [13], [10, Lemma 7.5.5]).

LEMMA 4.2 (Very normal form). — *Let $G = \Lambda^{-1}|dX|^2$ a family of semi-classical metrics on $\mathbb{R}^{2n}$. Let a be in $S^2(G)$ and let (g, λ) be the proper metric of a. Then there exists $C, R > 0$ depending only on a finite number of semi-norms of a such that the phase space is made of the three following types of g-balls:*

For any X_0, if we pose $(g_0, \lambda_0) = (g_{X_0}, \lambda(X_0))$, then for any X in the g_0-ball of radius R and center X_0.

(1) *either* $\Lambda_0 \leqslant C$,

(2) *or* $a(X) \geqslant C^{-1}\lambda_0^2$,

(3) *or* $e_0^{-1}a(X) = \lambda_0(\xi_1 - \alpha_0(x_1, x', \xi'))^2 + V_0(x_1, x', \xi')$,

where $X = (x_1, x', \xi_1, \xi')$ *is a set of linear symplectic coordinates, and* e_0, α_0, V_0 *are globally defined and respectively in* $S^0(g_0)$, $S^{1/2}(g_0)$ *and* $S^2(g_0)$, *with semi-norms controlled by those of* a, *and* $e_0 \geqslant C^{-1}$. *The point* X_0 *is said to be respectively either negligible or elliptic or of convexity.*

Remark. — The new fact is that from (3) we shall be directly able to get $e_0^{-1}a = \eta^2 + V(y)$, without a non-constant factor attached to the term η^2.

Proof. — We again give only a sketch of proof. The result is essentially an improvement of the following decomposition which is a standard result of level II (see [5], [12], [11, Lemma 18.6.9]):

(1) either $\Lambda_0 \leqslant C$,

(2) or $a(X) \geqslant C^{-1}\lambda_0^2$,

(3) or $a(X) = \lambda_0 e(X)(\xi_1 - \alpha(x_1, x', \xi'))^2 + V(x_1, x', \xi')$,

where e, α, and V have the same properties as e_0, α_0, and V_0. The only difference is that e is factor of the square, and not of the whole right-hand-side. In order to prove the lemma, we first use the rescaling $X \mapsto \lambda_0^{-1}(X - X_0)$. It is sufficient to show it for $X_0 = 0$, and $\lambda_0 = 1$. First study the following function:

$$F(X, y, z) = e(X)(\xi_1 - y)^2 + z,$$

for which we have the following derivatives at 0:

$$F(0,0,0) = 0, \quad \partial F/\partial\xi_1(0,0,0) = 0, \quad \partial^2 F/\partial\xi_1^2(0,0,0) = 2e(0) > 0.$$

From the estimates on e and the rescaling, we get that the constant $e(0)$ can be framed by absolute positive constants. The Malgrange preparation theorem (see [13], [10, Lemma 7.5.5]) then implies that we can find three smooth functions $K > 0$, β, and γ such that for $X = (x_1, \xi_1, X')$ we have

$$\begin{aligned} F(X, y, z) &= K(X, y, z)\big(\xi_1^2 + 2\beta(x_1, X')\xi_1 + \gamma(x_1, X')\big) \\ &= K(X, y, z)\big((\xi_1 + \beta(x_1, X'))^2 + W(x_1, X')\big) \end{aligned} \tag{101}$$

on a (X, y, z)-neighborhood $\mathcal{V}$ of 0, where $W = \gamma - \beta^2$. We can suppose, in the computational proof, $\alpha(0)$ and $V(0)$ to be sufficiently small and so to be in $\mathcal{V}$ (otherwise we get ellipticity). It implies the existence of $R_0 > 0$ such that in the X-ball $B(0, R_0)$,

$$\begin{aligned} e_0(X) &= K(X, \alpha(\xi_1, X'), V(x_1, X')), \\ \alpha_0(\xi_1, X') &= -\beta(\xi_1, X', \alpha(\xi_1, X'), V(x_1, X')), \\ V_0(\xi_1, X') &= W(\xi_1, X', \alpha(\xi_1, X'), V(x_1, X')). \end{aligned} \tag{102}$$

We then obtain $a = e_0((\xi_1 - \alpha_0)^2 + V_0)$. Moreover, for the same reason as before, we can also suppose that $(x_1, \xi_1, \alpha_0(x_1, X'))$ is in $B(0, R_0)$ for X in a smaller ball $B(0, R)$. Taking $\xi_1 = \alpha_0(x_1, X'))$ in $a = e_0((\xi_1 - \alpha_0)^2 + V_0)$ leads to $V_0 \geqslant 0$ on $B(0, R)$. We observe that the estimates on α_0, e_0 and V_0 are inherited from the ones on α, e and V. Using eventually the rescaling gives the result of the lemma. □

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