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THE ANALYTICITY OF q -CONCAVE SETS OF LOCALLY FINITE HAUSDORFF $(2n-2q)$ -MEASURE

by Viorel VÂJĂITU

1. Introduction.

Let A be a closed subset of a complex space X . The question of finding reasonable assumptions on A which guarantee its analyticity has been studied over the years by various authors.

Hartogs [14] considered a continuous function $f : D \rightarrow \mathbb{C}$, where $D \subset \mathbb{C}^n$ is open, and showed that the graph G_f of f in $D \times \mathbb{C}$ is pseudoconcave (*i.e.*, the complement of G_f in $D \times \mathbb{C}$ is locally Stein) if and only if f is holomorphic, that is G_f is analytic.

Grauert revealed in his thesis [13] a new interesting aspect of the above question bringing into play thin complements of complete Kähler domains. This topic was afterwards thoroughly studied by Diederich and Fornæss ([6], [7]) and Ohsawa [19].

On the other hand, Hirschowitz [15] settled the case when X is non-singular and A is pseudoconcave of locally finite Hausdorff $(2n-2)$ -measure, where n is the complex dimension of X .

In this article, using q -convexity with corners we introduce the notion of q -concavity. (See §2 for definition. Note that for $q = 1$ we recover the usual pseudoconcavity as used in [15] and [18].) For instance, if X is a complex manifold of pure dimension n and $A \subset X$ is an analytic subset

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such that every irreducible of it has dimension $\geq n-q$, then A is q -concave [20]. Two more examples are given at the end of Section 2.

Our main result in this note, which establishes a converse of the above result due to M. Peternell and generalizes Hirschowitz's theorem already quoted above, is the following:

THEOREM 1. — *Let X be a complex space of pure dimension n and q a positive integer less than n . If $A \subset X$ is a q -concave subset such that its Hausdorff $(2n-2q)$ -measure is locally finite, then A is analytic of pure dimension $n-q$.*

As an application (see also Example 2 in Section 2) we have:

COROLLARY 1. — *Let T be a closed positive current of bidimension (q, q) on a complex manifold M . If the Hausdorff $2q$ -measure of $\text{Supp}(T)$ is locally finite, then $\text{Supp}(T)$ is an analytic subset of M of pure dimension q .*

On the other hand, using [16], Theorem 1 yields the following removability theorem. (For $q = 1$ we recover the main result in [2].)

THEOREM 2. — *Let M be a complex manifold of pure dimension n , q a positive integer less than n , $E \subset M$ a closed subset of locally finite Hausdorff $(2n-2q)$ -measure, and f a meromorphic mapping from $M \setminus E$ into a complex space Y . If E does not contain any $(n-q)$ -dimensional analytic subset of M and Y possesses the meromorphic extension property in bidimension $(q, n-q)$ (e.g., if Y is q -complete), then f is continued to a meromorphic mapping from M into Y .*

The organization of this paper is as follows. After a preliminary section, we give in §3 the proofs of Theorems 1 and 2. The last section, §4, establishes connections with the q -pseudoconcavity notion introduced by M. Peternell [20].

2. Preliminaries.

(•) Let T be a metric space and S a subset of T . For $p > 0$ and $\varepsilon > 0$ let $h_\varepsilon^p(S)$ denote the infimum of all (infinite) sums of the form $\sum \delta(S_n)^p$ where $S = \cup S_n$ is an arbitrary decomposition of S with $\delta(S_n) < \varepsilon$ for all n (δ = diameter). For $p > 0$ the Hausdorff p -measure h^p is defined by

$h^p(S) = \sup_{\varepsilon > 0} h_\varepsilon^p(S) \leq +\infty$. We define $h^0(S)$ to be equal to the cardinality of S . The usual notion of k -dimensional volume in a Riemannian manifold agrees with h^k up to a constant factor depending only on n (for positive integers k). Thus, if A is a pure k -dimensional analytic set in a domain in \mathbb{C}^n , then $h^{2k}(A)$ is equal to a universal constant (depending on k) times the Riemannian volume of the set of regular points of A . For a detailed discussion on Hausdorff measure, see [11].

(•) The definition of q -convexity is the same as in [1], namely; a function $\varphi \in C^\infty(D, \mathbb{R})$, where $D \subset \mathbb{C}^n$ is an open subset, said to be q -convex if its *Levi form*

$$\mathcal{L}_\varphi(z)(\xi) := \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j, \quad \xi \in \mathbb{C}^n,$$

has at least $n-q+1$ positive (> 0) eigenvalues for every $z \in D$. This definition can be carried over to complex spaces by local restriction.

Let X be a complex space. X is said to be q -complete if there exists a q -convex function $\varphi \in C^\infty(X, \mathbb{R})$ which is *exhaustive*, i.e., the sublevel sets $\{x \in X; \varphi(x) < c\}$, $c \in \mathbb{R}$, are relatively compact in X . We choose the normalization such that 1-complete spaces correspond to Stein spaces.

Following [8] and [20] a function $\varphi \in C^0(X, \mathbb{R})$ is said to be q -convex with corners on X if every point of X admits an open neighborhood U on which there are finitely many q -convex functions f_1, \dots, f_k such that $\varphi|_U = \max(f_1, \dots, f_k)$. Denote by $F_q(X)$ the set of all functions q -convex with corners on X .

We say that X is q -complete with corners if there exists an exhaustion function $\varphi \in F_q(X)$.

DEFINITION 1. — Let X be a complex space. A subset A of X is said to be q -concave (in X) if A is closed and every point of A has an open neighborhood Ω such that $\Omega \setminus A$ is q -complete with corners.

From [24] (see also [25]) we deduce immediately:

COROLLARY 2. — Let $\pi : X \rightarrow Y$ be a finite surjective holomorphic map of complex spaces and $A \subset Y$ a closed subset. Then A is q -concave in Y if and only if $\pi^{-1}(A)$ is q -concave in X .

Subsequently we give some facts on q -completeness with corners which allow us to reduce the proof of Theorem 1 to the case when X is a domain in \mathbb{C}^n .

PROPOSITION 1. — *Let Y be an analytic set in a complex space X . If Y is q -complete with corners, then Y has a neighborhood system of open sets which are q -complete with corners.*

Proof. — By ([3], Lemma 3) if $\varphi \in F_q(Y)$ and $\eta \in C^0(Y, \mathbb{R})$, $\eta > 0$, then there exists an open neighborhood V of Y in X and $\psi \in F_q(V)$ such that $|\psi - \varphi| < \eta$ on Y . The method of Coltoiu ([4], Theorem 2) or Demailly ([5], the proof of Theorem 1, p. 287) can easily be adapted to our case. \square

PROPOSITION 2. — *Let X be a complex space and φ, ψ be continuous exhaustion functions on X such that there is an open neighborhood Ω of the set $\{\varphi = \psi\}$ in X with $\varphi \in F_p(\Omega \cup \{\varphi < \psi\})$ and $\psi \in F_q(\Omega \cup \{\psi < \varphi\})$. Then X is $(p + q)$ -complete with corners.*

Proof. — Let $\Lambda := \{\lambda \in C^\infty(\mathbb{R}, \mathbb{R}); \lambda' > 0, \lambda'' \geq 0\}$. For $\lambda \in \Lambda$ define $\Phi_\lambda : X \rightarrow \mathbb{R}$ by

$$\Phi_\lambda := 1/(\exp(-\lambda(\varphi)) + \exp(-\lambda(\psi))).$$

It is straightforward to see that Φ_λ is exhaustive for X and it is $(p + q)$ -convex with corners on Ω . Now we let $\varepsilon > 0$ be continuous on X such that $\{|\varphi - \psi| \leq \varepsilon\} \subset \Omega$; define $W_- = \{\varphi - \psi \leq -\varepsilon\}$ and $W_+ = \{\varphi - \psi \geq \varepsilon\}$. Clearly W_-, W_+ are closed subsets of X and $W_- \cup W_+ \cup \Omega = X$. The proof is concluded if we show the next

CLAIM. — *There is $\lambda \in \Lambda$ such that Φ_λ is p -convex with corners on W_- and q -convex with corners on W_+ .*

But this follows by adjusting the arguments in [22]. We omit the details. \square

PROPOSITION 3. — *Let U, V be open subsets of a complex space X such that U is p -complete with corners and V is q -complete with corners. Then $U \cup V$ is $(p + q)$ -complete with corners.*

Proof. — Consider exhaustion functions $f \in F_q(U)$ and $g \in F_q(V)$ for U and V respectively. Let $a \in C^\infty(U, \mathbb{R})$ with $0 \leq a \leq 1$, $a(x) = 1$ if $x \in U \setminus V$ or $x \in U \cap V$ and $f(x) \leq g(x) + 1$; $a(x) = 0$ if $x \in U \cap V$ and $f(x) > g(x) + 2$. Set $D := U \cup V$. Define φ on D by setting

$$\varphi = \begin{cases} f & \text{on } U \setminus V, \\ af + (1 - a)(1 + g) & \text{on } U \cap V, \\ 1 + g & \text{on } V \setminus U. \end{cases}$$

Then φ is continuous and exhaustive for D .

Let $b \in C^\infty(V, \mathbb{R})$ with $0 \leq b \leq 1$, $b(x) = 1$ if $g(x) \leq \varphi(x) + 1$ and $b(x) = 0$ if $g(x) > \varphi(x) + 2$. Define ψ on D by setting

$$\psi = \begin{cases} bg + (1 - b)(1 + \varphi) & \text{on } V, \\ 1 + \varphi & \text{on } U \setminus V. \end{cases}$$

Then ψ is continuous and exhaustive for D .

Finally, it is easy to see that $S := \{\psi < 1 + \varphi\} \subset V$ and $\psi = g$ on S ; hence $\psi \in F_q(S)$. Similarly, $T := \{\varphi < 1 + \psi\} \subset U$ and $\varphi = f$ on T ; so $\varphi \in F_p(T)$. The conclusion then follows from Proposition 2. \square

COROLLARY 3. — *Let A and B be p -concave and q -concave sets in the complex spaces X and Y respectively. Then $A \times B$ is $(p + q)$ -concave in $X \times Y$.*

Proof. — Since the assertion is local, we may assume that X and Y are Stein spaces, $X \setminus A$ is p -complete with corners, and $Y \setminus B$ is q -complete with corners. Then $X \times Y \setminus A \times B = X \times (Y \setminus B) \cup (X \setminus A) \times Y$ is $(p + q)$ -complete with corners by Proposition 3. \square

For a complex space X we introduce [20] the set $G_q(X)$ as follows: For $x_o \in X$ let $G_q(x_o)$ be the set of all functions $g : X \rightarrow \mathbb{R}$ such that there are: an open neighborhood U of x_o (which may depend on g) and $f \in F_q(U)$ with $f(x_o) = g(x_o)$ and $f \leq g|_U$. Then put

$$G_q(X) := C^0(X, \mathbb{R}) \cap \bigcap_{x \in X} G_q(x).$$

Clearly $F_q(X) \subseteq G_q(X) \subset C^0(X, \mathbb{R})$.

Note that given an open set $D \Subset X$, an $\varepsilon > 0$, and a function $g \in G_q(X)$, there is a function $h \in F_q(D)$ such that $|h - g| < \varepsilon$ on D . See [20], Lemma 1. But we cannot use this fact and the classical perturbation procedure (see for instance [8]) to get a globally defined h since we do not know that given $v \in G_q(X)$ and $\theta \in C_o^\infty(X, \mathbb{R})$ there is $\varepsilon_o > 0$ such that $v + \lambda\theta \in G_q(X)$ for every $\lambda \in \mathbb{R}$, $|\lambda| < \varepsilon_o$. However we can avoid this difficulty since we show:

LEMMA 1. — *The set $F_q(X)$ is dense in $G_q(X)$ in the sense that given an arbitrary $g \in G_q(X)$ and $\eta \in C^0(X, \mathbb{R})$, $\eta > 0$, there is $f \in F_q(X)$ such that $|f - g| < \eta$.*

Proof. — We do this in three steps.

Step 1). Fix $x \in X$ and $\varepsilon > 0$. By definition there is an open neighborhood Ω of x and $\varphi \in F_q(\Omega)$ with $\varphi(x) = g(x)$ and $\varphi \leq g$ on Ω . Let W, U be open neighborhoods of x , $W \subseteq U \subseteq \Omega$, such that $\varphi \geq g - \varepsilon$ on U ; then let $\theta \in C_0^\infty(U, \mathbb{R})$, $\theta = -1$ on ∂W and $\theta(x) = 1$. If $c > 0$ is small enough, then $\psi := \varphi + c\theta \in F_q(U)$, $\psi < g$ on ∂W , $\psi > g$ on a neighborhood V of x in W , and $|\psi - g| < 2\varepsilon$ on U .

Step 2). The above step shows that for all compact subsets K, L of X , L a neighborhood of K and $\varepsilon > 0$, there are: a finite set of indices I (which depends on K and L), open sets $V_i \subseteq W_i \subseteq U_i \subset L$ such that $\{V_i\}_{i \in I}$ cover K , functions $f_i \in F_q(U_i)$ with $|f_i - g| < 2\varepsilon$ on W_i , $f_i > g$ on V_i and $f_i < g$ on ∂W_i .

Step 3). Let $\{K_\nu\}_{\nu \in \mathbb{N}}$ be an exhaustion sequence for X by compact sets, $K_0 = \emptyset$ (by convention set $K_{-1} = \emptyset$), and $K_\nu \subset \text{int}(K_{\nu+1})$ for all ν . For each ν apply Step 2 to $K = K_\nu \setminus \text{int}(K_{\nu-1})$, $L = K_{\nu+1} \setminus \text{int}(K_{\nu-2})$, and $\varepsilon = (\min_L \eta)/2$. We therefore obtain open sets $V_{i\nu} \subseteq W_{i\nu} \subseteq U_{i\nu}$ such that the family $\{W_{i\nu}\}$ is locally finite, $\{V_{i\nu}\}_{i\nu}$ is a covering of X , and functions $f_{i\nu} \in F_q(U_{i\nu})$ as in Step 2 from above. Then define $f : X \rightarrow \mathbb{R}$ by $f(x) = \max\{f_{i\nu}(x); x \in W_{i\nu}\}$, where the maximum is taken over all indices i, ν such that $W_{i\nu} \ni x$. It is straightforward to see that f is continuous, $f \in F_q(X)$, and $g < f < g + \eta$. \square

Remark. — It can be shown that for $q > \dim(X)$ the set $F_q(X)$ is dense in the above sense even in $C^0(X, \mathbb{R})$.

From ([20], Lemma 4) we quote:

LEMMA 2. — *Let U be a complex space, V a complex manifold of pure dimension r , and $f \in F_{q+r}(U \times V)$ such that $\sup f < \infty$. Consider $g : U \rightarrow \mathbb{R}$ defined by*

$$g(x) = \sup\{f(x, y); y \in V\}, \quad x \in U.$$

Assume that for some $x_o \in U$ there is $y_o \in V$ with $g(x_o) = f(x_o, y_o)$. Then $g \in G_q(x_o)$.

The key proposition for the proof of Theorem 1 is:

PROPOSITION 4. — *Let X and Y be complex manifolds such that Y is of pure dimension r and p -complete with corners. Let A be a $(q+r)$ -concave subset in $X \times Y$ such that the natural projection $\pi : A \rightarrow X$ is*

proper. Then $\pi(A)$ is $(q + p - 1)$ -concave in X . In particular, if Y is Stein (i.e. $p = 1$), then $\pi(A)$ is q -concave.

Proof. — Set $m := q + p - 1$. We may assume without any loss in generality that X is Stein. The statement of the proposition follows from the next claim.

CLAIM. — For every relatively compact Stein open subset U of X , the set $U \setminus \pi(A)$ is m -complete with corners.

In order to show this, consider a relatively compact open subset V of Y which is p -complete with corners and such that $\pi^{-1}(\overline{U} \times \pi(A)) \subset \overline{U} \times V$. Then $K := \overline{U} \times \partial V$ is compact and disjoint from A . Now, since $U \times Y \setminus A$ is $(m + r)$ -complete with corners by [20], there exists an exhaustion function $\psi \in F_{m+r}(U \times Y \setminus A)$.

Let $\lambda := \max_K \psi$ and define $\sigma : U \setminus \pi(A) \rightarrow \mathbb{R}$ by setting

$$\sigma(x) = \max\{\psi(x, y), y \in \overline{V}\}, x \in U \setminus \pi(A).$$

Clearly σ is continuous. Consider θ be a 1-convex exhaustion function on U and then define $\varphi : U \setminus \pi(A) \rightarrow \mathbb{R}$ by setting

$$\varphi = \theta + \max(\lambda, \sigma).$$

Then φ is continuous and exhaustive. To conclude the proof, in view of Lemma 1, it suffices to show that $\varphi \in G_m(x)$ for ever $x \in U \setminus \pi(A)$. Indeed, two cases may occur:

a) If $\sigma(x) > \lambda$, then $\sigma \in G_m(x)$ by Lemma 2. Since $\varphi = \sigma + \theta$ on a neighborhood of x , we get $\varphi \in G_m(x)$.

b) If $\sigma(x) \leq \lambda$, then $\theta(x) + \lambda = \varphi(x)$ and since $\lambda + \theta \leq \varphi$ on $U \setminus \pi(A)$, $\varphi \in G_1(x)$, *a fortiori*, $\varphi \in G_m(x)$.

The proof is complete. \square

(\bullet) Denotes by $\Delta^k(t)$ the open polydisc in \mathbb{C}^k of polyradius (t, \dots, t) centered at the origin. Let n and q be positive integers such that $q < n$. We define the $(q, n - q)$ Hartogs figure in $\mathbb{C}^n = \mathbb{C}^q \times \mathbb{C}^{n-q}$ to be the open set $H_q \subset \mathbb{C}^n$ given by

$$H_q := \left((\Delta^q(1) \setminus \overline{\Delta^q(t)}) \times \Delta^{n-q}(1) \right) \cup \left(\Delta^q(1) \times \Delta^{n-q}(s) \right)$$

where $0 < t, s < 1$. Put $\widehat{H}_q := \Delta^n(1)$, i.e. the envelope of holomorphy of H_q .

Following [16] we say that a complex space Y possesses the meromorphic extension property (in bidimension $(q, n-q)$) if every meromorphic map $f : H_q \rightarrow Y$ extends to a meromorphic map $\hat{f} : \hat{H}_q \rightarrow Y$.

By [16] every q -complete complex space possesses a meromorphic extension property in bidimension $(q, n-q)$ for every integer $n > q$.

DEFINITION 2. — M be a complex manifold of pure dimension n . We say that a closed subset $A \subset M$ is pseudoconcave of order q if for every injective holomorphic map $f : \hat{H}_q \rightarrow M$ such that $f(H_q) \cap A$ is empty, the set $f(\hat{H}_q) \cap A$ is also empty.

In this set-up, a variant of Proposition 4 for $Y = \mathbb{C}^r$ is straightforward. See ([10], Lemma 3.6).

Also by ([24], Corollary 5) one has: *A closed subset A of a pure dimensional complex manifold is pseudoconcave of order q if and only if A is q -concave.*

Pseudoconcavity of order q is easier to handle; though it does not suit to complex spaces. One has the next examples:

- 1) Let M be a Stein manifold of pure dimension n and $K \subset M$ a compact set. Then $\hat{K} \setminus K$ is $(n-1)$ -concave in $X \setminus K$. (See [23].)
- 2) The support of a closed positive current of bidegree (q, q) on a pure dimensional complex manifold is q -concave. (This follows by [12], Corollary 2.6 and the above remark.)

3. Proof of Theorems 1 and 2.

Proof of Theorem 1.

We remark that it suffices to show that A is analytic and for this we distinguish three steps.

Step 1). — Here we reduce the proof to the case when $X \subset \mathbb{C}^n$ is open. For this we need:

LEMMA 3. — *Let Z be a complex space, $X \subset Z$ an analytic subset, and $A \subset X$ a closed subset (not necessarily analytic). If A is q -concave in X and X is r -concave in Z , then A is $(q+r)$ -concave in Z .*

Proof. — Let $x_o \in A$ and U be a Stein open neighborhood of x_o in Z such that $U \setminus X$ is r -complete with corners and $(U \setminus A) \cap X$ is q -complete with corners. Since $(U \setminus A) \cap X$ is analytic in $U \setminus A$, there is by Proposition 1 an open subset Ω of $U \setminus A$ which is q -complete with corners and contains $(U \setminus A) \cap X$. Therefore $U \setminus A = (U \setminus X) \cup \Omega$ is $(q+r)$ -complete with corners by Proposition 3. \square

To complete Step 1, we let $x \in A$, then take a coordinate patch $\iota : U \rightarrow D \subset \mathbb{C}^N$ around $x \in X$ with D Stein; hence U is isomorphic to the closed analytic subset $\iota(U)$ of D , hence $\iota(A \cap U)$ is q -concave in $\iota(U)$. Put $p := q + N - n$. Note that $N - p = n - q$. Therefore $\iota(A \cap U)$ is p -concave in D by Lemma 3 since $\iota(U)$ is $(N - n)$ -concave in D . On the other hand, $\iota(A \cap U)$ as a closed subset of D has its Hausdorff $(2N - 2p)$ -measure locally finite.

Step 2). — We give here some general facts for further reduction of the proof of Theorem 1.

Let $E \subset \mathbb{C}^n$ be a locally closed set with $h^{2n-2q+1}(E) = 0$ and suppose $0 \in E$. Then there is a complex $(n-q)$ -plane Γ through 0 such that $h^1(E \cap \Gamma) = 0$ ([21], Lemma 2). Hence for a suitable unitary transformation σ of \mathbb{C}^n we have $h^1(\sigma(E) \cap (\mathbb{C}^{n-q} \times \{0\})) = 0$. By ([21], Corollary 2), $\sigma(E) \cap (\partial B(r) \times \{0\})$ is empty for (h^1) -almost all $r > 0$. (Here $B(r)$ denotes the open unit ball in \mathbb{C}^{n-q} of radius r .) Since $\sigma(E)$ is also locally closed in \mathbb{C}^n and $0 \in \sigma(E)$, there is $r > 0$ arbitrary small and a polydisc P in \mathbb{C}^q centered at the origin such that $\sigma(E) \cap (\overline{B(r)} \times \overline{P})$ is closed in $\overline{B(r)} \times \overline{P}$ and $\sigma(E) \cap (\partial B(r) \times \overline{P})$ is empty. In particular, the canonically induced projection map π from $\sigma(E) \cap (B(r) \times P)$ into $B(r)$ is proper.

If furthermore $h^{2n-2q}(E) < \infty$, then $\pi^{-1}(z)$ is finite for (h^{2n-2q}) -almost all $z \in B(r)$ ([21], Corollary 4).

Recall that a set $\Gamma \subset \mathbb{C}^n$ is said to be *locally pluripolar* if for every $a \in \Gamma$ there is a connected neighborhood $U \ni a$ and a plurisubharmonic function φ on U , $\varphi \neq -\infty$, such that $\Gamma \cap U \subset \{\varphi = -\infty\}$. In fact, if Γ is locally pluripolar then by [17] one can take $U = \mathbb{C}^n$, so Γ is pluripolar. Note that for $n = 1$ pluripolarity of a set in \mathbb{C} means that it is of *zero-capacity* as used in [18]. Also it is easy to check that for $U \subset \mathbb{C}^n$ open and $S \subset \mathbb{C}^n$ of zero Lebesgue measure, the set $U \setminus S$ is not pluripolar.

Step 3). — Here we conclude the proof.

By Steps 1, 2, and Proposition 4 it remains to show the next lemma.

LEMMA 4. — *Let $U \subset \mathbb{C}^{n-q}$ be an open set, Δ the open unit disc in \mathbb{C} , and $A \subset U \times \Delta^q$ a closed subset such that the canonical projection $\pi : A \rightarrow U$ is proper. If A is q -concave and $\pi^{-1}(z)$ is finite for z in a non pluripolar subset of U , then A is analytic of pure dimension $n-q$.*

Proof. — For $q = 1$ this is precisely the lemma due to Hartogs-Oka-Nishino [18]. For $q > 1$ we proceed as follows. Notice that it suffices to show the analyticity of A . In order to do this we let $p_j : \Delta^q \rightarrow \Delta$, $j = 1, \dots, q$, denote the projection onto the j^{th} component of Δ^q , then let $\sigma_j : A \rightarrow U \times \Delta$ naturally induced by p_j . Then σ_j is proper and Proposition 4 implies that $\sigma_j(A)$ is 1-concave in $U \times \Delta$ for all indices $j = 1, \dots, q$. Furthermore if we consider $\pi_j : \sigma_j(A) \rightarrow U$ canonically induced, we arrive at the case $q = 1$. So the sets $\sigma_j(A)$ are analytic for all j .

Now, if $\iota : U \times \Delta^q \rightarrow (U \times \Delta) \times \dots \times (U \times \Delta)$ (the product is taken q -times) is given by $\iota(z, t_1, \dots, t_q) = ((z, t_1), \dots, (z, t_q))$, then $A = \iota^{-1}(\sigma_1(A) \times \dots \times \sigma_q(A))$, whence the lemma. Thus the proof of Theorem 1. \square

Proof of Theorem 2.

Denote by $A^0 :=$ the set of points $x \in A$ such that f extends meromorphically onto a neighborhood of x . Then $A' := A \setminus A^0$ is closed and as the complement to A is locally connected in M these local meromorphic continuations of f in points of A^0 glue together to a unique meromorphic map from $M \setminus A'$ into Y .

Now, we assert that A' is pseudoconcave of order q . For this we let $\Phi : \widehat{H}_q \rightarrow M$ be an injective holomorphic map with $\Phi(H_q) \cap A' = \emptyset$. Then $f \circ \Phi$ is meromorphic from H_q into Y , hence it extends to \widehat{H}_q ; therefore f extends over $\Phi(\widehat{H}_q)$, and by definition $\Phi(\widehat{H}_q) \subset A^0$; whence the desired assertion.

Finally, by Theorem 1, if A' is not the empty set, then A' is analytic of pure dimension $n-q$. But this contradicts the hypothesis, whence the proof. \square

4. A final remark.

Motivated by M. Peternell's work ([20], §7) we give:

DEFINITION 3. — *Let X be a complex space of pure dimension n . A closed subset A of X is said to be q -pseudoconcave if there is an analytic subset $B \subset X$ such that*

$$1) \overline{A \setminus B} = A.$$

2) *For each point $x \in A \setminus B$ there is a locally closed analytic subset Y of X which passes through x , $Y \subset A$, and Y is a complex manifold of dimension $n - q$.*

As an example, if A is analytic and $\dim_x A \geq n - q$, $\forall x \in A$, then A is q -pseudoconcave.

Let now r be a non-negative integer and suppose X is purely dimensional. We say that X has property (E_r) , if there is $\varphi \in F_{n+r}(X \times X \setminus \Delta_X)$, where Δ_X is the diagonal set of $X \times X$, such that $\varphi(x_\nu, x) \rightarrow +\infty$ if $x_\nu \rightarrow x$, $x_\nu \neq x$, $\forall x \in X$. Condition (E_r) holds locally on X if every point of X admits an open neighborhood U which satisfies (E_r) .

The next proposition is an easy consequence of ([20], Lemma 9).

PROPOSITION 5. — *Let X be a pure dimensional complex space such that (E_r) holds locally. Then every q -pseudoconcave subset of X is $(q + r)$ -concave.*

The importance of the condition (E_r) resides in the fact that, for example, if a Stein space X fulfils (E_0) , then every locally Stein open subset of X is Stein. It is easy to check for a Stein manifold that (E_0) holds. However, this fails, in general, if we allow singularities. For example, we let X be the Segre cone in \mathbb{C}^4 , $X = \{xy = zw\}$. Clearly the hypersurface $A = \{x = z = 0\}$ is 1-pseudoconcave. Now, if (E_0) would hold locally on X , then A will be 1-concave; and as X has isolated singularities $X \setminus A$ will be Stein. But this is absurd since $X \setminus A$ is biholomorphic to $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$.

COROLLARY 4. — *If X is a complex manifold, then every q -pseudoconcave subset of X is also q -concave.*

Example 3. — For every positive integer q there is an open subset X of \mathbb{C}^{q+1} and a q -concave subset $A \subset X$ which is **not** q -pseudoconcave.

To do this we consider a compact subset K of \mathbb{C}^2 such that $\widehat{K} \setminus K$ contains no analytic disc. See [26] for the existence of K . Put $X := (\mathbb{C}^2 \setminus K) \times \mathbb{C}^{q-1}$ and $A := (\widehat{K} \setminus K) \times \{0\}$. Then A is **not** q -pseudoconcave in X ; however, by Example 1 in §2 and Corollary 3 it is easily seen that $\widehat{K} \setminus K$ is q -concave in X . \square

The corresponding version of Theorem 1 reads:

THEOREM 3. — *Let A be a closed subset of a pure n -dimensional complex space X such that A is q -pseudoconcave and its Hausdorff $(2n-2q)$ -measure is locally finite. Then A is analytic of pure dimension $n-q$.*

Proof. — If $\iota : U \rightarrow D$ is a local path of X , where D is an open subset of \mathbb{C}^N , then $\iota(A \cap U)$ is $(N-n+q)$ -pseudoconcave in D . Now we conclude by the above corollary and Theorem 1. \square

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