ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 50, nº 3 (2000), p. 995-1017

<http://www.numdam.org/item?id=AIF_2000__50_3_995_0>

Ann. Inst. Fourier, Grenoble
50, 3 (2000), 995–1017

ON THE REAL SECONDARY CLASSES OF TRANSVERSELY HOLOMORPHIC FOLIATIONS

by Taro ASUKE

1. Introduction.

Characteristic classes of foliations are studied by many people. It seems however that the relation between the real and the complex secondary classes has been rarely studied, and as far as the author knows, there is only a Rasmussen's paper [24]. In it, some relations between these classes are slightly indicated but not discussed very much. We will first write explicitly a map λ from WO_{2q} to WU_q that induces a map $[\lambda]$ from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ which corresponds to forgetting the transverse holomorphic structure. We will then define a class $[\xi]$ in $H^{2q+1}(\mathrm{WU}_q)$ that represents the imaginary part of the Bott class (Proposition 3.4). As the first fruits we show the following relation between the class $[\xi]$ and the Godbillon-Vey class.

THEOREM A. — *The class $[\xi]$ factors the Godbillon-Vey class as*

$$\mathrm{GV}_{2q} = \frac{(2q)!}{q!q!}[\xi] \cdot \mathrm{Chern}_1(\mathcal{F})^q,$$

where $\mathrm{Chern}_1(\mathcal{F})$ denotes the first Chern class of the complex normal bundle $Q(\mathcal{F})$ of $\mathcal{F}$.

We remark that the Bott class is well-defined only if the first Chern class of $Q(\mathcal{F})$ is trivial. This theorem implies that if the first Chern class

Supported by the JSPS Postdoctoral Fellowships for Reasearch Abroad (1999-2000).
Keywords : Foliations – Transverse holomorphic structure – Characteristic classes.
Math classification: 58H10 – 32S65 – 53C12.

is trivial, then the Godbillon-Vey class of the foliation is trivial. There are some typical cases where this condition arises. One is where the foliation is defined by a $\mathbb{C}$-valued q-form satisfying certain integrability conditions. Another is where the second cohomology group of the ambient manifold is trivial. For example, the Godbillon-Vey class of a transversely holomorphic foliation of $S^1 \times S^4$ is always trivial. This contrasts with the case of real foliations because there is a family of real foliations of $S^1 \times S^4$ that are of real codimension two and whose Godbillon-Vey class varies continuously (see Example 4.5). On the other hand, there is a transversely holomorphic foliation of complex codimension one whose Godbillon-Vey class is non-trivial. We review such an example by following Rasmussen [24]. See Example 4.6 for details. In Example 3.5 by examining Baum-Bott's example [5], [8] we also see that the class $[\xi]$ is non-trivial and varies continuously.

In general the maps λ and $[\lambda]$ are neither injective nor surjective. Hence, in particular, there appear some relations among the elements of $H^*(\mathrm{WO}_{2q})$ when mapped into $H^*(\mathrm{WU}_q)$. In the case where the complex codimension is equal to one, we will show the following:

THEOREM B.— *If $q = 1$, then we have the relation $\lambda(h_1c_1^2) = 2\lambda(h_1c_2)$ as elements of* WU_1. *Moreover,*

(1) *the kernel of the mapping $[\lambda]$ is spanned by the classes $[c_2]$ and $[h_1c_1^2] - 2[h_1c_2]$,*

(2) *the image is spanned by the class $[\tilde{u}_1v_1\overline{v_1}]$*

(3) *the cokernel is spanned by the classes $[v_1]$ and $[\tilde{u}_1(v_1 + \overline{v_1})]$.*

It is worth noting that the class $[\tilde{u}_1v_1\overline{v_1}]$ is the image of the Godbillon-Vey class up to multiplication by a constant. Thus, we can regard the Godbillon-Vey class as the only real secondary class of a transversely holomorphic foliation of complex codimension one.

We will also show a similar relation for higher codimensional cases in Corollary 4.2; moreover, Theorem B can be partially generalized for higher codimensional cases and the image of the mapping $[\lambda]$ can be determined in the case where $q = 2$ or $q = 3$. It is done in the paper [3].

The existence of a linear relation between the classes $[h_1c_1^2]_\lambda$ and $[h_1c_2]_\lambda$, where $[h_1c_1^2]_\lambda = [\lambda]([h_1c_1^2])$ and $[h_1c_2]_\lambda = [\lambda]([h_1c_2])$, is not unexpected. The following explanation is due to S. Morita [21], who kindly allowed the author to quote the details. The author would like to express his gratitude to him.

The continuous cohomology of $B\overline{\Gamma}_1^{\mathbb{C}}$ is identified with that of real homotopy type of $S^3 \times S^3$ via the Bott class. The latter space is the complex homotopy type of $K(\mathbb{C}, 3) = S^3$, and its real part corresponds to the Chern-Simons class, which transgresses to v_1^2 in the following fibration by the Bott vanishing:

$$B\overline{\Gamma}_1^{\mathbb{C}} \to B\Gamma_1^{\mathbb{C}} \to \mathrm{BGL}(1, \mathbb{C}) = K(\mathbb{Z}, 2).$$

Thus the first approximation of this fibration is the real homotopy type of

$$S^3 \times S^3 \to S^2 \times S^3 \to K(\mathbb{Z}, 2)$$

tensored by $\mathbb{Q}$. Consequently the continuous cohomology of $B\Gamma_1^{\mathbb{C}}$ is the same as that of $S^2 \times S^3$. Here S^2 corresponds to v_1 and S^3 corresponds to the imaginary part of the Bott class. In particular $H_c^5(B\Gamma_1^{\mathbb{C}}; \mathbb{C}) = \mathbb{C}$, where we denote by H_c the continuous cohomology. On the other hand, it is known that $H_c^5(B\overline{\Gamma}_2; \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$. Since we are considering the map $B\Gamma_1^{\mathbb{C}} \to B\Gamma_2$, we can expect that there is a linear dependence between $[h_1 c_1^2]_\lambda$ and $[h_1 c_2]_\lambda$, which is determined by Theorem B.

It is an important question whether the secondary classes of foliations are C^1-invariant or not. It is known that the Godbillon-Vey class is C^1-invariant [23], [1]. Though we know nothing about the remaining classes, we have the following.

COROLLARY C.— *Let $(M_i, \mathcal{F}_i)$, $i = 1, 2$, be transversely holomorphic foliations of complex codimension one. Suppose that there is a C^1-diffeomorphism f from M_1 to M_2 which preserves the foliations. Then*

$$f^*(h_1 c_2(\mathcal{F}_2)) = h_1 c_2(\mathcal{F}_1).$$

A similar question can be posed when we consider the complex secondary classes. It is easy to see that such classes are invariant under transversely holomorphic concordance (resp. cobordism, foliation preserving diffeomorphism) of transversely holomorphic foliations. We can ask now if the complex secondary classes defined by $H^*(\mathrm{WU}_q)$ are invariant under C^r-concordance (resp. cobordism, foliation preserving diffeomorphism) of transversely holomorphic foliations. Example 3.5 is related to this question. See also Question 4.10.

It is also important to determine whether the secondary classes under consideration vary continuously or not. After Heitsch [11], we show the following.

THEOREM D. — *Let $\mathcal{F}_s$, $s \in [0,1]$ be a differentiable family of transversely holomorphic foliations of complex codimension q. We denote by $[\chi_0]$ and $[\chi_1]$ the characteristic mappings defined by $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. Then $[\chi_0]([\alpha]) = [\chi_1]([\alpha])$ if $[\alpha]$ is one of the secondary classes in $H^*(\mathrm{WU}_q)$ defined by a cocycle $\alpha = \tilde{u}_I v_J \overline{v_K}$ of WU_q with $i_1 + |J| > q+1$ and $i_1 + |K| > q + 1$, where i_1 is the smallest entry of I.*

There is a similar result for $H^*(\mathrm{W}_q^{\mathbb{C}})$ (see Theorem 3.6).

The meaning of the mapping $[\lambda]$ can be explained as follows. First, the image of the mapping $[\lambda]$ consists of the complex secondary classes which are in fact the real secondary classes. Secondly, the cokernel of the mapping $[\lambda] : H^*(\mathrm{WO}_{2q}) \to H^*(\mathrm{WU}_q)$ plays the role of purely complex secondary characteristic classes. Finally, the elements of the kernel of the mapping $[\lambda]$ can be viewed as obstructions for foliations to be transversely holomorphic. In this paper, by showing that certain real secondary classes do not vanish, we will show that some examples can never be transversely holomorphic (Corollary 3.8 and Example 4.3).

This paper is organized as follows. First we introduce the secondary classes after Bott [8] and we then define a mapping $[\lambda]$ from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$. In the third section we give the proofs of the main results. We also show that the mapping λ has always an extension to a certain subalgebra WO_{2q}^+ of W_{2q} (Corollary 3.11). In the final section we examine some known examples. We will show that Heitsch's and Rasmussen's examples [12], [25] cannot be transversely holomorphic. On the other hand, we show after Rasmussen [24] that there is a complex codimension one transversely holomorphic foliation of a closed manifold whose Godbillon-Vey class is non-trivial.

This paper is based on a part of the author's thesis [2] and the author would like to express his gratitude to Professors T. Tsuboi, S. Morita and D. Lehmann for their comments and helpful suggestions. The author also appreciates the referee's comments which helped him to improve the paper.

2. Definitions.

First we briefly recall the notion of the secondary classes and some facts that can be found for example in Bott [8], Pittie [22] or Godbillon [10] with more details. In the last part of this section we will define the mapping λ mentioned in the introduction.

We denote by M a differential manifold and by $\mathcal{F}$ a transversely holomorphic foliation of M whose codimension over $\mathbb{C}$ is equal to q. We set $Q = Q(\mathcal{F}) = TM/T\mathcal{F}$. Then Q is naturally a complex vector bundle, which we call the complex normal bundle of $\mathcal{F}$. When we regard $\mathcal{F}$ as a real foliation of codimension $2q$, its normal bundle is denoted by $Q_{\mathbb{R}}$. Thus $Q_{\mathbb{R}} \otimes \mathbb{C} = Q \oplus \overline{Q}$. A complex Bott connection ∇ on Q is a connection satisfying $\nabla_X Y = \pi[X, \tilde{Y}]$ for $X \in \Gamma T\mathcal{F}$ and $Y \in \Gamma Q$, where π denotes the natural projection from TM to Q and $\tilde{Y}$ is a lift of Y to TM. We also consider a real Bott connection on $Q_{\mathbb{R}}$, which satisfies the same condition as a complex Bott connection, having Q replaced by $Q_{\mathbb{R}}$. It is easy to see that if ∇_0 is a complex Bott connection on Q, then $\nabla_0 \oplus \overline{\nabla_0}$ defines a complexified real Bott connection on $Q_{\mathbb{R}} \otimes \mathbb{C}$. In this paper, we will always choose Bott connections on Q and $Q_{\mathbb{R}} \otimes \mathbb{C}$ simultaneously in this way.

The most significant property of Bott connections is the following [8].

Bott Vanishing Theorem. — *Consider a transversely holomorphic foliation of complex codimension q and let ∇_B be a complex Bott connection. We denote by $v_i(\nabla_B)$, $i = 1, \cdots, q$, the Chern forms calculated by using the connection ∇_B. Then any monomial of $v_i(\nabla_B)$ whose degree as a differential form is greater than $2q$ vanishes as a differential form. Similarly, if we consider a real foliation of real codimension q and denote by ∇_b a Bott connection, then any monomial of the Pontryagin forms $c_i(\nabla_b)$, $i = 1, \cdots, q$, whose degree is greater than $2q$ vanishes as a differential form.*

By virtue of this theorem, we can define the secondary characteristic classes of the foliations. First of all, we define the degree of v_i in $\mathbb{C}[v_1, \cdots, v_q]$ as $2i$ and set I_q as the ideal of $\mathbb{C}[v_1, \cdots, v_q]$ generated by the monomials of degree greater than $2q$. We set

$$\mathbb{C}_q[v_1, \cdots, v_q] = \mathbb{C}[v_1, \cdots, v_q]/I_q$$

and define $\mathbb{C}_q[\overline{v_1}, \cdots, \overline{v_q}]$ as the quotient of $\mathbb{C}[\overline{v_1}, \cdots, \overline{v_q}]$ by $\overline{I}_q$, where $\overline{I}_q$ is defined in the obvious manner. For real foliations we define the degree of c_i as $2i$, and define $\mathbb{R}_{2q}[c_1, \cdots, c_{2q}]$ as the quotient of $\mathbb{R}[c_1, \cdots, c_{2q}]$ by the ideal generated by the monomials in c_i whose degree is greater than $4q$.

Definition 2.1. — *We define differential graded algebras WU_q and WO_{2q} by setting*

$$\mathrm{WU}_q = \mathbb{C}_q[v_1, \cdots, v_q] \otimes \mathbb{C}_q[\overline{v_1}, \cdots, \overline{v_q}] \otimes \bigwedge[\tilde{u}_1, \cdots, \tilde{u}_q],$$

$$\mathrm{WO}_{2q} = \mathbb{R}_{2q}[c_1, \cdots, c_{2q}] \otimes \bigwedge[h_1, h_3, \cdots, h_{2q-1}].$$

We equip these algebras with an exterior differential determined by $d\tilde{u}_i = v_i - \overline{v_i}$, $dv_i = d\overline{v_i} = 0$, $dh_i = c_i$ *and* $dc_i = 0$, *respectively. We define the degree of the elements* $\tilde{u}_i$ *and* h_i *as* $2i - 1$, *respectively.*

The meanings of WU_q and WO_{2q} are the following. The element v_i of WU_q corresponds to the i-th Chern form calculated by using a complex Bott connection. Since the Chern classes are real, we can find elements $\tilde{u}_i$ such that $d\tilde{u}_i = v_i - \overline{v_i}$ by using the foliation. On the other hand the Bott vanishing theorem shows that any element of $\mathbb{C}[v_1, \cdots, v_q]$ of degree greater than $2q$ vanishes as a differential form. By writing down these two facts we obtain the algebra WU_q.

Definition of the characteristic mapping

The characteristic mapping

$$[\chi_{\mathbb{C}}] : H^*(\mathrm{WU}_q) \to H^*(M),$$

is given as follows. First of all, let ∇ be a connection on the complex normal bundle Q of $\mathcal{F}$ and denote by $R(\nabla)$ its curvature. We define differential forms $v_i(\nabla)$ by the formula

$$\det\left(tI - \frac{R(\nabla)}{2\sqrt{-1}\pi}\right) = t^q + t^{q-1}v_1(\nabla) + t^{q-2}v_2(\nabla) + \cdots + v_q(\nabla).$$

Let ∇_0 be a complex Bott connection associated with $\mathcal{F}$, and let ∇_1 be a Hermitian connection. Then by setting $\chi_{\mathbb{C}}(v_i) = v_i(\nabla_0)$ and $\chi_{\mathbb{C}}(\overline{v_i}) = \overline{v_i(\nabla_0)}$, we obtain a map $\chi_{\mathbb{C}}$ naturally defined on $\mathbb{C}[v_1, \cdots, v_q] \otimes \mathbb{C}[\overline{v_1}, \cdots, \overline{v_q}]$. Now the Bott vanishing theorem for transversely holomorphic foliations asserts that the mapping $\chi_{\mathbb{C}}$ annihilates the ideals I_q and $\overline{I}_q$. Thus the mapping $\chi_{\mathbb{C}}$ defines in fact a mapping from $\mathbb{C}_q[v_1, \cdots, v_q] \otimes \mathbb{C}_q[\overline{v_1}, \cdots, \overline{v_q}]$, which we denote again by $\chi_{\mathbb{C}}$.

Now we set $\nabla_t = \nabla_0 + t(\nabla_1 - \nabla_0)$ and write $v_i(\nabla_t) = \alpha + \beta \wedge dt$, where α does not involve the term dt. We define a differential form $v_i(\nabla_0, \nabla_1)$ by the formula

$$v_i(\nabla_0, \nabla_1) = \int_0^1 \beta dt,$$

then it is easy to see that $dv_i(\nabla_0, \nabla_1) = v_i(\nabla_0) - v_i(\nabla_1)$. Noticing that $v_i(\nabla_1) = \overline{v_i(\nabla_1)}$ (because ∇_1 is Hermitian), we define $\chi_{\mathbb{C}}(\tilde{u}_i) = v_i(\nabla_0, \nabla_1) - \overline{v_i(\nabla_0, \nabla_1)}$. Then $d(\chi_{\mathbb{C}}(\tilde{u}_i)) = \chi_{\mathbb{C}}(v_i) - \chi_{\mathbb{C}}(\overline{v_i})$ and hence we can extend the mapping $\chi_{\mathbb{C}}$ to the whole WU_q. Finally, we denote by $[\chi_{\mathbb{C}}]$ the mapping induced between the cohomology algebras by $\chi_{\mathbb{C}}$. For simplicity we denote the image $[\chi_{\mathbb{C}}]([\alpha])$ also by $\alpha(\mathcal{F})$, where we denote by $[\alpha]$ the class of $H^*(\mathrm{WU}_q)$ determined by a cocycle α of WU_q.

The real characteristic mapping $[\chi_{\mathbb{R}}] : H^*(\mathrm{WO}_{2q}) \to H^*(M;\mathbb{R})$ is defined in a similar way, namely, for a connection ω of the real normal bundle of $\mathcal{F}$ we denote by $R(\omega)$ its connection and set

$$\det\left(tI - \frac{R(\omega)}{2\pi}\right) = t^q + t^{q-1}c_1(\omega) + t^{q-2}c_2(\omega) + \cdots + c_q(\omega).$$

Then we choose a real Bott connection ω_0 on the real normal bundle $Q_{\mathbb{R}}$ and a Riemannian connection ω_1. We define a mapping $\chi_{\mathbb{R}}$ by setting $\chi_{\mathbb{R}}(c_i) = c_i(\omega_0)$ and $\chi_{\mathbb{R}}(h_i) = c_i(\omega_0, \omega_1)$, respectively. Then $d(\chi_{\mathbb{R}}(h_i)) = \chi_{\mathbb{R}}(c_i)$ when i is an odd integer. Thus by the Bott vanishing theorem we have a mapping $\chi_{\mathbb{R}}$ from WO_{2q} to the space of differential forms on M. We denote by $[\chi_{\mathbb{R}}]$ the mapping induced on cohomology and we denote $[\chi_{\mathbb{R}}]([\beta])$ also as $\beta(\mathcal{F})$, where $[\beta]$ is the class of $H^*(\mathrm{WO}_{2q})$ defined by a cocycle β of WO_{2q}.

In the following, we consider only the complexification $Q_{\mathbb{R}}\otimes\mathbb{C} = Q\oplus\overline{Q}$ of $Q_{\mathbb{R}}$ and we denote again by $\chi_{\mathbb{R}}$ and $[\chi_{\mathbb{R}}]$ the complexified mapping.

Now we give a homomorphism λ from WO_{2q} to WU_q which will induce a homomorphism $[\lambda]$ from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ such that $[\chi_{\mathbb{R}}] = [\chi_{\mathbb{C}}] \circ [\lambda]$. First we choose a complex Bott connection ∇_0 and a Hermitian connection ∇_1, and then define a (complexified) real Bott connection ω_0 by $\omega_0 = \nabla_0 \oplus \overline{\nabla_0}$ and a Riemannian connection ω_1 by $\omega_1 = \nabla_1 \oplus \overline{\nabla_1}$, respectively. Then by comparing the above formulae defining $c_i(\omega_0)$ and $v_i(\nabla_0)$, we see that

$$c_k(\omega_0) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}(\nabla_0)\overline{v_j(\nabla_0)}$$

as differential forms. Here we set $v_0(\mathcal{F}) = \overline{v_0}(\mathcal{F}) = 1$ and used the fact that $v_j(\overline{\nabla_0}) = (-1)^j \overline{v_j(\nabla_0)}$.

In view of this we introduce the following definition.

DEFINITION 2.2. — *Define a mapping λ from WO_{2q} to WU_q by the formulae*

$$\lambda(c_k) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}\overline{v_j},$$

$$\lambda(h_{2k+1}) = \frac{(-1)^k}{2}\sqrt{-1} \sum_{j=0}^{2k+1} (-1)^j \tilde{u}_{2k-j+1}(v_j + \overline{v_j}).$$

The mapping λ induces a mapping $[\lambda]$ between cohomology algebras. For an element α of WO_{2q}, we denote by $[\alpha]$ the class in $H^*(\mathrm{WO}_{2q})$ defined by $[\alpha]$, and by $[\alpha]_\lambda$ the class in $H^*(\mathrm{WU}_q)$ defined by $\lambda(\alpha)$, namely, $[\alpha]_\lambda = [\lambda(\alpha)]$.

By abuse of notation, we will also denote by $[\beta]$ the class in $H^*(\mathrm{WU}_q)$ defined by a cocycle β of WU_q.

We will show in the next section that the induced map $[\lambda]$ satisfies $[\chi_{\mathbb{R}}] = [\chi_{\mathbb{C}}] \circ [\lambda]$ and therefore corresponds to forgetting the transverse holomorphic structure.

In the suitable categories we can define some of the elements such as h_{2i}, u_j and $\overline{u_j}$, so that we can extend λ to certain subalgebras of W_{2q}. Here the elements u_j and $\overline{u_j}$ are considered as transgressions of v_i and $\overline{v_i}$. They then satisfy the relation $u_j - \overline{u_j} = \tilde{u}_j$. The most typical case is when the complex normal bundle of the foliation is trivial. In this case, all the elements such as h_{2i}, u_j and $\overline{u_j}$ are well-defined, and we set

$$\mathrm{W}_q^{\mathbb{C}} = \mathbb{C}_q[v_1, \cdots, v_q] \otimes \mathbb{C}_q[\overline{v_1}, \cdots, \overline{v_q}] \otimes \wedge[u_1, u_2, \cdots, u_q] \wedge \bigwedge[\overline{u_1}, \overline{u_2}, \cdots, \overline{u_q}]$$

$$\mathrm{W}_{2q} = \mathbb{R}_{2q}[c_1, \cdots, c_{2q}] \otimes \bigwedge[h_1, h_2, h_3, \cdots, h_{2q}].$$

Notice that there are natural mappings from WU_q to $\mathrm{W}_q^{\mathbb{C}}$, and from WO_{2q} to W_{2q}, respectively. We can define the characteristic mappings $[\widehat{\chi}_{\mathbb{R}}]$ from $H^*(\mathrm{W}_{2q})$ to $H^*(M)$ and $[\widehat{\chi}_{\mathbb{C}}]$ from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M)$, respectively, in a similar way as $[\chi_{\mathbb{R}}]$ and $[\chi_{\mathbb{C}}]$.

We extend the map λ to a map $\widehat{\lambda}$ from W_{2q} to $\mathrm{W}_q^{\mathbb{C}}$ by setting

$$\widehat{\lambda}(h_{2k}) = (-1)^k \frac{1}{2} \sum_{j=0}^{2k} (-1)^j (u_{2k-j}\overline{v_j} + \overline{u_j} v_{2k-j}),$$

where v_0, $\bar{v}_0$ are regarded as 2 and u_0, $\bar{u}_0$ are regarded as 0, respectively. The map $\widehat{\lambda}$ induces a map $[\widehat{\lambda}]$ from $H^*(\mathrm{W}_{2q})$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ which satisfies the equation $[\widehat{\chi}_{\mathbb{C}}] \circ [\widehat{\lambda}] = [\widehat{\chi}_{\mathbb{R}}]$. It is clear that $[\widehat{\lambda}]$ is an extension of $[\lambda]$.

Though the map λ might not be extended to the whole W_{2q} in general, we can always extend the mapping λ to the subalgebra WO_{2q}^{+} of W_{2q} defined by

$$\mathrm{WO}_{2q}^{+} = \mathbb{R}_{2q}[c_1, \cdots, c_{2q}] \otimes \bigwedge[h_1, h_3, \cdots, h_{q'}, h_{q+1}, \cdots, h_{2q}],$$

where q' denotes the greatest odd integer which is not greater than q (Corollary 3.11). It is worth mentioning that $\mathrm{WO}_2^{+} = \mathrm{W}_2$.

Now we give the definition of the secondary classes.

DEFINITION 2.3. — *The classes in the image of* $[\chi_{\mathbb{C}}]$ *or* $[\widehat{\chi}_{\mathbb{C}}]$ *(resp.* $[\chi_{\mathbb{R}}]$ *or* $[\widehat{\chi}_{\mathbb{R}}]$*) which involve* $\tilde{u}_i$, u_i *or* $\overline{u_i}$ *(resp.* h_i*) are called the complex secondary classes (resp. real secondary classes). In particular the class* $\mathrm{GV}_{2q}(\mathcal{F}) = [\chi_{\mathbb{R}}]([h_1 c_1^{2q}])$ *is called the Godbillon-Vey class and similarly the class* $\mathrm{Bott}_q(\mathcal{F}) = [\chi_{\mathbb{C}}]([u_1 v_1^q])$ *is called the Bott class when it is defined. We call the classes* $\mathrm{GV}_{2q} = [h_1 c_1^{2q}]$ *and* $\mathrm{Bott}_q = [u_1 v_1^q]$ *themselves again the Godbillon-Vey class and the Bott class, respectively. Here* $[h_1 c_1^q]$ *means the cohomology class in* $H^*(\mathrm{WO}_{2q})$ *defined by the cocycle* $h_1 c_1^q$ *and* $[u_1 v_1^q]$ *means the cohomology class defined by the cocycle* $u_1 v_1^q$.

Remark 2.4. — A basis for $H^*(\mathrm{WO}_{2q})$ is given by Vey [10]. First $I = \{i_1, \ldots, i_t\}$ denotes an index set consisting of odd integers such that $1 \leqslant i_1 < \cdots < i_t \leqslant 2q$, and $J = (j_1, \cdots, j_{2q})$ an index set consisting of nonnegative integers. We set $|J| = j_1 + 2j_2 + \cdots + (2q)j_{2q}$. Then the Vey basis of $H^*(\mathrm{WO}_{2q})$ is given by

$$\begin{aligned} &\{[c_J]\,;\ 1 \leqslant |J| \leqslant 2q,\ j_k = 0 \text{ for all odd integers } k\} \\ \cup &\{[y_I c_J]\,;\ |J| \leqslant 2q,\ i_1 + |J| > 2q,\ i_1 \leqslant k \text{ if } k \text{ is odd and } j_k > 0\}, \end{aligned}$$

where $y_I c_J = c_1^{j_1} \cdots c_{2q}^{j_{2q}} \otimes h_{i_1} \cdots h_{i_t}$.

It seems that an explicit description of a basis for $H^*(\mathrm{WU}_q)$ is unknown. We show a procedure to compute a basis for $H^*(\mathrm{WU}_q)$ in [3] and compute it in the case where $q \leqslant 3$.

3. Proofs of the main results.

First we show that the mapping $[\lambda]$ induced by λ defined in the previous section corresponds to forgetting the transverse holomorphic structure.

LEMMA 3.1.— *Let Q be the normal bundle of the foliation $\mathcal{F}$. Let ∇_B and ∇_H be a Bott and a Hermitian connection on Q, respectively. We define a Bott connection ∇_b and a Riemannian connection ∇_r on $Q \oplus \overline{Q}$ by setting $\nabla_b = \nabla_B \oplus \overline{\nabla_B}$ and $\nabla_r = \nabla_H \oplus \overline{\nabla_H}$, respectively. If we denote by $[\chi_{\mathbb{R}}]$ the characteristic mapping from $H^*(\mathrm{WO}_{2q})$ to $H^*(M)$ defined by using ∇_b and ∇_r, and by $[\chi_{\mathbb{C}}]$ the characteristic mapping from $H^*(\mathrm{WU}_q)$ to $H^*(M)$ defined by using ∇_B and ∇_H, then we have*

$$[\chi_{\mathbb{R}}] = [\chi_{\mathbb{C}}] \circ [\lambda],$$

where $[\lambda]$ denotes the mapping from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ induced by λ.

Proof.— We retain the notations of the previous section. Then for $P \in \mathbb{C}_q[v_1, \cdots, v_q]$ or $P \in \mathbb{R}_{2q}[c_1, \cdots, c_q]$, the equation $dP(\nabla_0, \nabla_1) = P(\nabla_0) - P(\nabla_1)$ holds. Since the relation

$$c_i(\nabla \oplus \overline{\nabla}) = (\sqrt{-1})^i \sum_{j=0}^{i} (-1)^j v_{i-j}(\nabla)\overline{v_j(\nabla)}$$

holds for any connection ∇ on $Q(\mathcal{F})$, we see that $c_i(\nabla_b, \nabla_r) = \lambda(c_i)(\nabla_B, \nabla_H)$. In order to show the lemma, we show that the differential form

$$(*) \quad \big((v_i - \overline{v_i})\,(v_j + \overline{v_j})\big)(\nabla_B, \nabla_H) - (v_i - \overline{v_i})\,(\nabla_B, \nabla_H)\,(v_j + \overline{v_j})\,(\nabla_B)$$

is exact. Once this is established, we can show the lemma as follows. We see easily from the exactness of the above differential form that $\lambda(c_i)(\nabla_B, \nabla_H) = \chi_{\mathbb{C}}(\lambda(h_i))$ modulo exact forms. Suppose then that $[h_I c_J]$ is a member of the Vey basis and set $I' = I \setminus \{i_1\}$. Since the differential form $\chi_{\mathbb{R}}(h_{I'} c_J)$ is closed, we have the equation

$$\begin{aligned}
[\chi_{\mathbb{R}}(h_{i_1} h_{I'} c_J)] &= [c_{i_1}(\nabla_b, \nabla_r)\chi_{\mathbb{R}}(h_{I'} c_J)] \\
&= [\lambda(c_{i_1})(\nabla_B, \nabla_H)\chi_{\mathbb{R}}(h_{I'} c_J)] \\
&= [\chi_{\mathbb{C}}(\lambda(h_{i_1}))\chi_{\mathbb{R}}(h_{I'} c_J)]
\end{aligned}$$

in $H^*(M)$. Thus we can inductively show that $[\chi_{\mathbb{R}}] = [\chi_{\mathbb{C}}] \circ [\lambda]$.

We show now that the differential form $(*)$ is exact. We set $\nabla_0 = \nabla_B$ and $\nabla_1 = \nabla_H$, respectively. Then we write $v_i(\nabla_t)\,(v_j + \overline{v_j})\,(\nabla_t) = \alpha_1 + \beta_1 \wedge dt$ and $v_i(\nabla_t)\,(v_j + \overline{v_j})\,(\nabla_0) = \alpha_2 + \beta_2 \wedge dt$. We set $T_t = (v_j + \overline{v_j})\,(\nabla_t, \nabla_0)$ and write $v_i(\nabla_t) T_t = \zeta + \xi \wedge dt$. Note that $T_0 = 0$ and $\overline{T_t} = T_t$. Since the differential form $v_i(\nabla_t)$ is closed,

$$d(v_i(\nabla_t) T_t) = v_i(\nabla_t)\,(v_j + \overline{v_j})\,(\nabla_t) - v_i(\nabla_t)\,(v_j + \overline{v_j})\,(\nabla_0).$$

It follows that $\frac{\partial\zeta}{\partial t} \wedge dt + (d_M\xi) \wedge dt = (\beta_1 - \beta_2) \wedge dt$. We set $P = \int_0^1 \xi dt$, then

$$\begin{aligned} d_M P &= \int_0^1 d_M \xi dt \\ &= \int_0^1 (\beta_1 - \beta_2)\, dt - \int_0^1 \frac{\partial\zeta}{\partial t} dt \\ &= \left(v_i\left(v_j + \overline{v_j}\right)\right)(\nabla_0, \nabla_1) - v_i(\nabla_0, \nabla_1)\left(v_j + \overline{v_j}\right)(\nabla_0) - \zeta(1) + \zeta(0). \end{aligned}$$

Finally since $\overline{\zeta(1)} = \zeta(1)$, $\zeta(0) = v_i(\nabla_0)T_0 = 0$ and $\overline{v_i(\nabla_H)} = v_i(\nabla_H)$,

$$\begin{aligned} & d_M\left(P - \overline{P}\right) \\ &= \left(\left(v_i - \overline{v_i}\right)\left(v_j + \overline{v_j}\right)\right)(\nabla_B, \nabla_H) - \left(v_i - \overline{v_i}\right)(\nabla_B, \nabla_H)\left(v_j + \overline{v_j}\right)(\nabla_B). \end{aligned}$$

This completes the proof. □

Since $[\chi_{\mathbb{R}}]$, $[\chi_{\mathbb{C}}]$ and $[\lambda]$ are independent of the choise of connections, we have a commutative diagram

$$\begin{array}{ccc} H^*(\mathrm{WO}_{2q}) & \xrightarrow{[\lambda]} & H^*(\mathrm{WU}_q) \\ {\scriptstyle [\chi^0_{\mathbb{R}}]} \downarrow & & \downarrow {\scriptstyle [\chi^0_{\mathbb{C}}]} \\ H^*(B\Gamma_{2q}) & \longrightarrow & H^*(B\Gamma^{\mathbb{C}}_q) \\ {\scriptstyle \gamma^*_{\mathbb{R}}} \downarrow & & \downarrow {\scriptstyle \gamma^*_{\mathbb{C}}} \\ H^*(M;\mathbb{R}) & \longrightarrow & H^*(M;\mathbb{C}) \end{array}$$

where $[\chi^0_{\mathbb{R}}]$ and $[\chi^0_{\mathbb{C}}]$ denote the universal characteristic mappings, $\gamma_{\mathbb{C}}$ is the classifying map of the given transversely holomorphic foliation, and $\gamma_{\mathbb{R}}$ is the classifying map of the foliation viewed as a real foliation. Of course, the mapping $H^*(B\Gamma_{2q}) \to H^*(B\Gamma^{\mathbb{C}}_q)$ is the one induced by the natural mapping $B\Gamma^{\mathbb{C}}_q \to B\Gamma_{2q}$ obtained by forgetting the transverse holomorphic structure. Thus the homomorphism $[\lambda]$ corresponds to forgetting the transverse holomorphic structure.

There exists a version of Lemma 3.1 when we consider real foliations.

LEMMA 3.2. — *Let ∇_b be a Bott connection and ∇_h be a Riemannian connection, then*

$$c_i c_j(\nabla_b, \nabla_r) = c_i(\nabla_b, \nabla_r) c_j(\nabla_b)$$

modulo exact forms if i is an odd integer.

Proof. — The proof is almost the same as that of the exactness of $(*)$ in the proof of the previous lemma. The only difference is that we would have $\zeta(1) = 0$, which still gives the result. □

Now we define an important cocycle in WU_q.

DEFINITION 3.3. — *We define a cocycle ξ in WU_q by the formula*

$$\xi = \sqrt{-1}\,\tilde{u}_1(v_1^q + v_1^{q-1}\overline{v_1} + \cdots + \overline{v_1}^q).$$

We denote by $[\xi]$ the induced class in $H^*(\mathrm{WU}_q)$.

It seems that this class $[\xi]$ of $H^*(\mathrm{WU}_q)$ is more fundamental than the Godbillon-Vey class when we consider transversely holomorphic foliations. First of all, we have the following easy but important property of $[\xi]$.

PROPOSITION 3.4. — *If the Bott class is well-defined, namely, the first Chern class of the complex normal bundle is trivial, the class $[\xi]$ coincides with the imaginary part of the Bott class multiplied by -2.*

Proof. — It follows from the simple fact that the equation

$$\xi = \sqrt{-1}(u_1 v_1^q - \overline{(u_1 v_1^q)}) - \sqrt{-1}d\left(u_1\overline{u_1}(v_1^{q-1} + v_1^{q-2}\overline{v_1} + \cdots + \overline{v_1}^{q-1})\right)$$

holds if u_1 and $\overline{u_1}$ are well-defined. □

As we will see, it is known that the Bott class is non-trivial and varies continuously. In fact it takes all values in the complex numbers. Hence the class $[\xi]$ varies continuously and takes all values in the real numbers. Note that the class $[\xi]$ can be non-trivial even if the first Chern class is trivial, while in this case the Godbillon-Vey class is trivial by Theorem A.

We give now the proof of Theorem A as an immediate conclusion of the definitions we have given.

Proof of Theorem A. — Direct calculations show that

$$\lambda(h_1 c_1^{2q}) = \frac{(2q)!}{q!q!}\sqrt{-1}\,\tilde{u}_1 v_q^q \overline{v_q}^q = \frac{(2q)!}{q!q!}\xi \cdot \left(\frac{v_1 + \overline{v_1}}{2}\right)^q.$$

Since v_1 and $\overline{v_1}$ are cohomologous and they correspond to the first Chern class of the complex normal bundle $Q(\mathcal{F})$ of $\mathcal{F}$, Theorem A is proved. □

It is worth noticing that the equation holds in fact at the level of differential forms. We will study more properties of the class $[\xi]$ in [4].

Example 3.5 (Baum-Bott [5], Bott [8]).— Let $M = \mathbb{C}^{q+1} \setminus \{0\}$ and consider a holomorphic vector field

$$X(\lambda_0, \cdots, \lambda_q) = \lambda_0 z_0 \frac{\partial}{\partial z_0} + \lambda_1 z_1 \frac{\partial}{\partial z_0} + \cdots + \lambda_q z_q \frac{\partial}{\partial z_q},$$

where $(z_0, \cdots, z_q)$ is the natural coordinate of $\mathbb{C}^{q+1}$, and suppose that all λ_i's are non-zero and none of the numbers $\frac{\lambda_i}{\lambda_j}$ are negative real numbers. We denote by $\mathcal{F}$ the holomorphic flow given by X, then $\mathcal{F}$ restricts to a transversely holomorphic (real) flow of the unit sphere S^{2q+1} of $\mathbb{C}^{q+1}$. It turns out that in this case the first Chern class is trivial and hence we can define the Bott class $\mathrm{Bott}(\mathcal{F})$ and other classes of the form $u_1\varphi(\mathcal{F}) = [\chi_{\mathbb{C}}(u_1\varphi)]$, where $\varphi \in \mathbb{C}_q[v_1, \cdots, v_q]$ of degree $2q$. According to a formula which appeared in [5],

$$\int_{S^{2q+1}} u_1\varphi(\mathcal{F}) = \frac{\lambda_0 + \lambda_1 + \cdots + \lambda_q}{\lambda_0\lambda_1 \cdots \lambda_q} \varphi(\lambda_0, \lambda_1, \cdots, \lambda_q).$$

In particular

$$\int_{S^{2q+1}} \mathrm{Bott}(\mathcal{F}) = \frac{(\lambda_0 + \lambda_1 + \cdots + \lambda_q)^{q+1}}{\lambda_0\lambda_1 \cdots \lambda_q}.$$

Since $\xi(\mathcal{F})$ is the imaginary part of the Bott class multiplied by -2, we can easily find that the class $\xi(\mathcal{F})$ is non-trivial and in fact, it admits a continuous deformation.

We do not know whether there is a pair of the foliations of S^3 obtained as in the above example such that they have different values of $\xi(\mathcal{F})$ but they are cobordant as real foliations of codimension two. This question is a special case of the question we posed in the introduction (see Question 4.10).

We show in Example 4.6 that there are transversely holomorphic foliations of complex codimension one whose Godbillon-Vey class is non-trivial. Of course in these examples the class $\xi(\mathcal{F})$ is also non-trivial. On the other hand, it appears to be unknown whether the Godbillon-Vey class can be non-trivial if the complex codimension of the foliation is greater than one.

Note that Theorem A shows that if the Godbillon-Vey class is non-trivial, then the cohomology of the ambient manifold must be similar to that of $S^{2q+1} \times \mathbb{C}P^q$. In particular, its second cohomology group must be non-trivial. This is in contrast to the case of real foliations. See Example

4.5 for details. We remark also that if the complex normal bundle $Q(\mathcal{F})$ is trivial, then $\mathrm{GV}(\mathcal{F})$ is trivial.

It is well-known that the Godbillon-Vey class varies continuously in the category of real foliations [12], [25], [26]. If we can find a family of transversely holomorphic foliations whose Godbillon-Vey class varies continuously, the variation comes exactly from the variation of the class $[\xi]$ because the Chern class is rigid under deformations. Unfortunately, at the present we do not know whether there is such a family, neither. On the other hand, it is known that certain elements of $H^*(\mathrm{WO}_q)$ and $H^*(\mathrm{W}_q)$ are rigid under differentiable deformations of the foliations. Here we mean by a differentiable deformation a family of foliations whose associate family of normal bundles is differentiable.

Now we show Theorem D after Heitsch [11].

Proof of Theorem D. — The proof is almost the same as in [11] for $H^*(\mathrm{WO}_q)$. We may identify the normal bundles $Q(\mathcal{F}_s)$ and fix a Hermitian connection $\nabla_1 = \nabla_H$. Then we choose a differential family of complex Bott connections $\nabla_0^s = \nabla_B^s$ of $Q(\mathcal{F}_s)$ and set $\nabla_t^s = \nabla_0^s + t(\nabla_1 - \nabla_0^s)$. We set $R_t^s = R(\nabla_t^s) = d\nabla_t^s + \nabla_t^s \wedge \nabla_t^s$, $\psi_s = \frac{\partial}{\partial s}\nabla_0^s$, and define $P(\nabla_0^s, \nabla_1)$ as in the second section, where P is an element of $\mathbb{C}_q[v_1, \cdots, v_q]$. Finally we set

$$V = \int_0^1 tP(\psi_s, \nabla_0^s - \nabla_1, R_t^s, \cdots, R_t^s)dt.$$

Although our conventions and definitions are slightly different from the original ones, we have still the following formulae:

$$\begin{aligned}
&\frac{\partial}{\partial s}P(\nabla_0^s, \nabla_1) = k(k-1)dV + kP(\psi_s, R_0^s, \cdots, R_0^s),\\
&dP(\nabla_0^s, \nabla_1) = P(R_0^s, \cdots, R_0^s) - P(R_1, \cdots, R_1),\\
&\frac{\partial}{\partial s}P(R_0^s, \cdots, R_0^s) = kdP(\psi_s, R_0^s, \cdots, R_0^s),
\end{aligned}$$

where k is the degree of P.

By using these formulae and the assumptions $i_p + |J| > q+1$ and $i_p + |K| > q+1$ for all p, we can show as in [11] that the differential form $\frac{\partial}{\partial s}\chi_s(\tilde{u}_I v_J \overline{v_K})$ is exact. Thus Theorem D is proved. □

If we work in the category of transversely holomorphic foliations with trivial normal bundles, we can consider $H^*(\mathrm{W}_q^{\mathbb{C}})$ and we have the following.

THEOREM 3.6.— *Let $\mathcal{F}_s$, $s \in [0,1]$ be a differentiable family of foliations of complex codimension q. We denote by $[\widehat{\chi}_0]$ and $[\widehat{\chi}_1]$ the characteristic mappings defined by $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. Then $[\widehat{\chi}_0](\alpha) = [\widehat{\chi}_1](\alpha)$ if α is one of the secondary classes in $H^*(\mathrm{W}_q^{\mathbb{C}})$ defined by a cocycle $u_I\overline{u_L}v_J\overline{v_K}$ of $\mathrm{W}_q^{\mathbb{C}}$ with $i_1 + |J| > q+1$ and $l_1 + |K| > q+1$, where i_1 and l_1 are the smallest entries of I and L, respectively.* □

For some classes we have formulae that are similar to the ones in Theorem A, in particular we have the following. The proof is easy and left to the readers.

PROPOSITION 3.7.— *Suppose that $|J| = q$, then there is a polynomial $P(J)$ of v_i and $\overline{v_i}$ such that $P(J)$ is of degree $2q$ as an element of WU_q and that the cocycle ξ factors the cocycles of the form $\lambda(h_1c_1^qc_J)$ as $\lambda(h_1c_1^qc_J) = \xi \cdot P(J)$.* □

Now we give the proof of Theorem B.

Proof of Theorem B.— First of all, it is easy to see that $\lambda(c_1^2) = 2\lambda(c_2)$ in WU_1. It follows that $\lambda(h_1c_1^2) = 2\lambda(h_1c_2)$ as elements of WU_1.

As we explained in the previous section, as a vector space, $H^*(\mathrm{WO}_2)$ is spanned by the classes $[c_2]$, $[h_1c_1^2]$ and $[h_1c_2]$ over $\mathbb{R}$. On the other hand, it is easy to see that $H^*(\mathrm{WU}_1)$ is spanned by the classes $[\overline{v_1}]$, $[\tilde{u}_1(v_1 + \overline{v_1})]$ and $[\tilde{u}_1v_1\overline{v_1}]$ over $\mathbb{C}$ (see also [3]). Finally, noticing that $[h_1c_1^2]_\lambda = -2\sqrt{-1}[\tilde{u}_1v_1\overline{v_1}]$ we complete the proof. □

Note that the linear dependence of $[h_1c_1^2]_\lambda$ and $[h_1c_2]_\lambda$ can be expected by using an argument of continuous cohomology as in the introduction.

We can deduce from Theorem B that certain foliations cannot be transversely holomorphic by seeing that the relation $h_1c_1^2(\mathcal{F}) = 2h_1c_2(\mathcal{F})$ fails to hold.

COROLLARY 3.8.— *The examples given in the fifth part of Rasmussen [25] do not admit any transverse holomorphic structure.*

It is often convenient to regard the c_i's as the Pontryagin characters and the v_i's as the Chern characters. From Lemmas 3.1 and 3.2 we see that even under this convention, the image of $H^*(\mathrm{WO}_{2q})$ and $H^*(\mathrm{WU}_q)$ in $H^*(M)$ remains the same when we consider a foliated manifold $(M, \mathcal{F})$. To avoid confusion we adopt the symbols H_i, $\tilde{U}_i$, C_i, V_i, and $\overline{V_i}$ instead of h_i,

$\tilde{u}_i$, c_i, v_i, and $\overline{v_i}$, respectively. Then the mapping λ is rewritten as follows:

$$\lambda(C_i) = \begin{cases} (\sqrt{-1})^i \left(V_i + (-1)^i \overline{V_i}\right) & \text{for } i \leqslant q \\ 0 & \text{for } i > q \end{cases}$$

$$\lambda(H_i) = \begin{cases} (\sqrt{-1})^i \tilde{U}_i & \text{for } i \leqslant q \\ (\sqrt{-1})^i \tilde{h}_i & \text{for } i > q \end{cases}$$

where $\tilde{h}_i$ denotes a certain element obtained by using Newton Polynomials [3, Definition 3.8].

This is just the formula mentioned in [24]. Note that the relation $\lambda(c_1^2) = 2\lambda(c_2)$ is nothing but the relation $\lambda(C_2) = 0$ when $q = 1$. We will make use of this kind of relations to study the higher codimensional cases in [3].

As we mentioned in the introduction, we always have an extension of λ to a subalgebra WO_{2q}^+ of W_q even though the cocycles c_J are not exact in general if $|J|$ is even. This follows from the following.

Proposition 3.9. — *Suppose that $|J| > q$ and $|J|$ is even, then there is a well-defined element η_J of WU_q such that $d\eta_J = c_J$. The element η_J has naturality, and if the complex normal bundle is trivial, then $\lambda(h_{2j})$ coincides with η_{2j}.*

To show Proposition 3.9 we prepare the following lemma.

Proposition 3.10. — *There is always a well-defined element $\tilde{u}_J$ of WU_q such that $d\tilde{u}_J = v_J - \overline{v_J}$.*

Proof. — The statement is true if J is just a single number. Suppose that the lemma holds if $|J| < r_0$. Let J be a given index with $|J| = r_0$ and let s be the number such that $j_s \neq 0$ and such that $j_t = 0$ if $t > s$. We set $J' = J \setminus \{j_s\}$ and

$$\tilde{u}_J = \tilde{u}_{J'} \frac{(v_s + \overline{v_s})}{2} + \frac{(v_{J'} + \overline{v_{J'}})}{2} \tilde{u}_s.$$

It is easy to see that $d\tilde{u}_J = v_J - \overline{v_J}$. □

In this paper we always choose the elements $\tilde{u}_J$ in the above way.

Proof of Proposition 3.9. — From the fact that c_J is invariant under complex conjugation and that $|J|$ is even, we can write $\lambda(c_J)$ as

$$\lambda(c_J) = \sum_{J_1 \cup J_2 = J} a_{J_1,J_2} (v_{J_1} \overline{v_{J_2}} + v_{J_2} \overline{v_{J_1}}),$$

where a_{J_1,J_2} are real numbers such that $a_{J_2,J_1} = a_{J_1,J_2}$.

We set

$$\eta_J = -\sum_{J_1 \cup J_2 = J} a_{J_1,J_2} \tilde{u}_{J_1} \left(v_{J_2} - \overline{v_{J_2}}\right),$$

then it is easy to see that $d\eta_J = c_J$. The naturality is obvious.

Since $\lambda(c_{2j}) = (-1)^{q-j} \left(v_q \overline{v_k} - v_{q-1} \overline{v_{k+1}} + \cdots + v_k \overline{v_q}\right)$, where $k = 2q - j$,

$$\eta_{2j} = -\frac{1}{2} \sum_{l=1}^{q-j} (-1)^l \left(\tilde{u}_{j+l} \left(v_{j-l} - \overline{v_{j-l}}\right) + \tilde{u}_{j-l} \left(v_{j+l} - \overline{v_{j+l}}\right)\right) - \frac{1}{2} \tilde{u}_j \left(v_j - \overline{v_j}\right).$$

From this equation the last statement follows. □

COROLLARY 3.11. — *The mapping* $[\lambda]$ *has always an extension to the subalgebra* WO_q^+ *of* W_q.

Proof. — The extension is obtained by setting $\lambda(h_{2i}) = \eta_{2i}$ if $2i > q$.

□

Since $\mathrm{WO}_2^+ = \mathrm{W}_2$, we can say from the cohomological point of view that transversely holomorphic foliations of complex codimension one behave as the foliations with trivial normal bundle when viewed as real foliations.

4. Examples.

Before introducing the examples, we prepare a proposition. Let $J = (j_1, \cdots, j_{2q})$. We write $J < j$ if $j_l = 0$ for $l \geqslant j$.

PROPOSITION 4.1. — *Suppose that* i *is an odd integer greater than* q, *then the secondary class* $[h_i c_j]_\lambda$ *can be written as a linear combination of the classes of the form* $[h_{i'} c_{J'}]_\lambda$ *where* $i' < j$ *or* $J' < j$.

Proof. — It suffices to show that $[h_i C_j]_\lambda$ is trivial in $H^*(\mathrm{WU}_q)$, where C_j is the Chern character of degree $2i$. First we set $l = q - \frac{i+1}{2}$, then the element $\lambda(h_i)$ is a linear combination of the elements of the form $\alpha_t = \tilde{u}_t \left(v_s + \overline{v_s}\right) - \tilde{u}_s \left(v_t + \overline{v_t}\right)$, where $s = i - t$ and $t \geqslant s$. Note that $0 \leqslant t \leqslant q$.

Now we set $\beta_t = -(\sqrt{-1})^q \tilde{u}_t \tilde{u}_s \left(V_j - (-1)^j \overline{V_j}\right)$, then $d\beta_t = \alpha_t$. This follows from the fact that $(v_s - \overline{v_s})\left(V_j - (-1)^j \overline{V_j}\right) = -(v_s + \overline{v_s})(V_j + (-1)^j \overline{V_j})$ because $s + j = i + j - t \geqslant 2q + 1 - q > q$. □

We apply this proposition in the case where $i = 2q - 1$, and we obtain a generalization of Theorem B.

COROLLARY 4.2. — *We have the relation*

$$2[h_{2q-1}c_2]_\lambda = [h_1 c_1 c_{2q-1}]_\lambda$$

as elements of $H^{4q+1}(\mathrm{WU}_q)$. □

Notice that the class $[h_{2q-1}c_2]_\lambda$ is the only class coming from $H^{4q+1}(\mathrm{WO}_{2q})$ which involves h_{2q-1}.

From this corollary we see that there are many foliations which cannot be transversely holomorphic, we have in fact the following.

Example 4.3 [12]. — Heitsch's example given in [12, Example 2] cannot be transversely holomorphic. This can be seen in the following ways. First we can see that the relation shown in Corollary 4.2 does not hold by using the residue formula. On the other hand, we can also show that there is a leaf holonomy which cannot be holomorphic.

The following are examples of transversely holomorphic foliations.

Example 4.4. — Consider the natural foliation of $\mathbb{R}^p \times \mathbb{C}^q$ by subspaces $\mathbb{R}^p \times \{z\}$, $z \in \mathbb{C}^q$, which is transversely holomorphic. Now restrict the foliation to $(\mathbb{R}^p \times \mathbb{C}^q) \setminus \{(0,0)\}$. Since the foliation is invariant under the mapping ρ defined by $\rho(x, z) = (2x, 2z)$, we have a foliation of $S^1 \times S^{p+2q-1}$, which is still transversely holomorphic. It is easy to see that in this case the Godbillon-Vey class is trivial either by direct calculation or by using Theorem A.

Example 4.5. — We retain the notations of the previous example. If we set $p = 3$ and $q = 1$, we obtain a complex codimension one transversely holomorphic foliation of $S^1 \times S^4$. We consider this foliation as a real foliation of codimension two and we have a mapping γ from $S^1 \times S^4$ to $B\Gamma_2$. Since there is a family of foliations of a 5-manifold which are of real codimension two and whose Godbillon-Vey class varies continuously, we can find a mapping δ_t from $S^1 \times S^4$ to $B\overline{\Gamma}_2$ which gives the continuous deformation of the Godbillon-Vey class and whose image is contained in a fiber of the fibration $B\overline{\Gamma}_2 \to B\Gamma_2 \to \mathrm{BGL}_2$. Such an example is, for instance, given by Heitsch [12]. By adding δ_t projected to $B\Gamma_2$ and γ we obtain a family of Γ-structures of $S^1 \times S^4$ whose Godbillon-Vey class varies in all the real numbers. We can perturb the family so that we have a family of foliations and we obtain the desired family.

Example 4.6 [24]. — We consider the complexified Anosov foliation of the unit tangent bundle over a closed 3-dimensional real hyperbolic manifold. For this purpose, we consider the foliation $\tilde{\mathcal{F}}$ of $\mathrm{SL}(2,\mathbb{C})$ by the left cosets of a subgroup B given by $B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}; a \in \mathbb{C}^*\right\}$, namely, $\tilde{\mathcal{F}} = \{gB;\, g \in \mathrm{SL}(2,\mathbb{C})\}$. This foliation clearly projects down to the homogeneous space $\mathrm{SL}(2,\mathbb{C})/\mathrm{U}(1)$, where $\mathrm{U}(1) = \left\{\begin{pmatrix} a & 0 \\ 0 & \bar{a}\end{pmatrix}\right\}$. Then we choose a discrete cocompact group Γ of $\mathrm{SL}(2,\mathbb{C})$ such that $E = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})/\mathrm{U}(1)$ is a Hausdorff manifold. Note that E fibers over $M = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ with fibers $\mathrm{SU}(2)/\mathrm{U}(1) = \mathbb{C}P^1$. Notice also that the foliation $\tilde{\mathcal{F}}$ induces a foliation $\widehat{\mathcal{F}}$ of $\widehat{E} = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$, which we will use later.

Now, to compute the Godbillon-Vey class we will make use of the relative cohomology of Lie algebras. We regard $\mathfrak{sl}(2,\mathbb{C})$ as left invariant vector fields on $\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})^*$ as left invariant differential forms on $\mathrm{SL}(2,\mathbb{C})$ and so on. Then $\mathfrak{sl}(2,\mathbb{C}) = \langle X_0, X_1, X_2\rangle_{\mathbb{C}}$, where $X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, and $X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$. Let ω_0, ω_1 and ω_2 be the dual of X_0, X_1 and X_2, respectively. Then $\mathfrak{sl}(2,\mathbb{C})^* = \langle \omega_1, \omega_2, \omega_3\rangle_{\mathbb{C}}$. These forms satisfy the following equations, namely, $d\omega_0 = -\omega_1 \wedge \omega_2$, $d\omega_1 = -2\omega_0 \wedge \omega_1$, and $d\omega_2 = 2\omega_0 \wedge \omega_2$. If we write $\omega_i = \lambda_i + \sqrt{-1}\mu_i$, $i = 0, 1, 2$, then the above equations imply

$$
\begin{aligned}
d\lambda_0 &= -\lambda_1 \wedge \lambda_2 + \mu_1 \wedge \mu_2, & d\mu_0 &= -\mu_1 \wedge \lambda_2 - \lambda_1 \wedge \mu_2,\\
d\lambda_1 &= -2\lambda_0 \wedge \lambda_1 + 2\mu_0 \wedge \mu_1, & d\mu_1 &= -2\mu_0 \wedge \lambda_1 - 2\lambda_0 \wedge \mu_1,\\
d\lambda_2 &= 2\lambda_0 \wedge \lambda_2 - 2\mu_0 \wedge \mu_2, & d\mu_2 &= 2\mu_0 \wedge \lambda_2 + 2\lambda_0 \wedge \mu_2.
\end{aligned}
$$

We denote by $(\wedge^*\mathfrak{sl}(2,\mathbb{C})^*)_{\mathfrak{u}(1)}$ the $\mathfrak{u}(1)$-basic elements of $\wedge^*\mathfrak{sl}(2,\mathbb{C})$, in other words, we set

$$(\wedge^*\mathfrak{sl}(2,\mathbb{C})^*)_{\mathfrak{u}(1)} = \{\omega \in \mathfrak{sl}(2,\mathbb{C})^*; \iota_X\omega = \mathcal{L}_X\omega = 0 \text{ for } \forall X \in \mathfrak{u}(1)\}.$$

We can show that

$$
\begin{aligned}
&H^*(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{u}(1))\\
&= \langle 1, \lambda_1 \wedge \mu_2 + \mu_1 \wedge \lambda_2, \lambda_0 \wedge \lambda_1 \wedge \mu_2 + \lambda_0 \wedge \mu_1 \wedge \lambda_2, \lambda_0 \wedge \lambda_1 \wedge \lambda_2 \wedge \mu_1 \wedge \mu_2\rangle_{\mathbb{R}}\,.
\end{aligned}
$$

Note that $H^k(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{u}(1)) = \{0\}$ for $k \geqslant 6$ and also that $H^*(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{u}(1)) \cong H^*(S^3 \times \mathbb{C}P^1)$.

It is well-known that the generator of $H^5(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{u}(1))$ can be identified with the volume form of the homogeneous space $\mathrm{SL}(2,\mathbb{C})/\mathrm{U}(1)$. Since the differential forms are assumed to be left invariant, the volume form projects down to the space $E = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})/\mathrm{U}(1)$. So it suffices to show that the Godbillon-Vey class is a generator of $H^5(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{u}(1))$.

First notice that the foliation $\mathcal{F}$ is given by the 1-form $\omega_1 = \lambda_1 + \sqrt{-1}\mu_1$ at the point of E corresponding to the identity element of $\mathrm{SL}(2,\mathbb{C})$.

Then the equation

$$d\begin{pmatrix}\lambda_1\\ \mu_1\end{pmatrix} = -2\begin{pmatrix}\lambda_0 & -\mu_0\\ \mu_0 & \lambda_0\end{pmatrix}\wedge\begin{pmatrix}\lambda_1\\ \mu_1\end{pmatrix}$$

shows that

$$\begin{aligned}
\chi_{\mathbb{R}}(h_1) &= -\frac{2}{\pi}\lambda_0,\\
\chi_{\mathbb{R}}(c_1) &= \frac{2}{\pi}(\lambda_1\wedge\lambda_2 - \mu_1\wedge\mu_2),\\
\chi_{\mathbb{R}}(c_2) &= -\frac{4}{\pi^2}\lambda_1\wedge\lambda_2\wedge\mu_1\wedge\mu_2.
\end{aligned}$$

Hence $\chi_{\mathbb{R}}(h_1c_1^2) = \frac{16}{\pi^3}\lambda_0\wedge\lambda_1\wedge\lambda_2\wedge\mu_1\wedge\mu_2$, and the Godbillon-Vey class of $(E,\mathcal{F})$ is non-trivial in $H^5(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{u}(1))$. On the other hand, we see from a direct calculation, or from Theorem B that $\chi_{\mathbb{R}}(h_1c_2) = \frac{1}{2}\chi_{\mathbb{R}}(h_1c_1^2)$.

Now we lift the foliation $(E,\mathcal{F})$ to $(\widehat{E},\widehat{\mathcal{F}})$. Then $\widehat{\mathcal{F}}$ is globally defined by a 1-form ω_1. This shows that the first Chern class of the normal bundle of $\widehat{\mathcal{F}}$ is trivial, therefore the Godbillon-Vey class is trivial by Theorem A. We can also see the vanishing of the Godbillon-Vey class directly as follows. We still have

$$\begin{aligned}
\chi_{\mathbb{R}}(h_1) &= -\frac{2}{\pi}\lambda_0, \quad \chi_{\mathbb{R}}(c_1) = -\frac{2}{\pi}d\lambda_0 = \frac{2}{\pi}(\lambda_1\wedge\lambda_2 - \mu_1\wedge\mu_2),\\
\chi_{\mathbb{C}}(v_1) &= -\frac{1}{\pi\sqrt{-1}}d\omega_0 = -\frac{1}{\pi\sqrt{-1}}(d\lambda_0 + \sqrt{-1}d\mu_0).
\end{aligned}$$

Thus the Godbillon-Vey class is represented by the differential form $-\frac{8}{\pi^3}\lambda_0\wedge(d\lambda_0)^2$. Now $d(\lambda_0\wedge d\mu_0\wedge\mu_0) = -\lambda_0\wedge(d\lambda_0)^2$ because $d\lambda_0^2 - d\mu_0^2 = d\lambda_0\wedge d\mu_0 = 0$. This shows that the Godbillon-Vey class of $\widehat{\mathcal{F}}$ is trivial though it is non-trivial as a differential form. Note that in this case $\chi_{\mathbb{C}}(u_1)$ is well-defined on $\widehat{E}$ and in fact equal to $-\frac{1}{\pi\sqrt{-1}}\omega_0$, which is not well-defined on E. Note also that the equation $d\lambda_0^2 - d\mu_0^2 = d\lambda_0 \wedge d\mu_0 = 0$ is equivalent to the fact $\chi_{\mathbb{C}}(v_1^2) = 0$. This is just the Bott vanishing theorem [8].

This phenomena reflects the fact that the space $\widehat{E}$, an S^1-bundle over E, can be viewed as the 'fibrewise Hopf fibration of E', namely, $\widehat{E}$ fibers over M as well as E does, and if we denote the fibre, which is naturally identified with S^3, by $\widehat{F}$, then $\widehat{F}$ is a principal S^1-bundle over the fiber of $E \to M$. We can say that the non-triviality of the Godbillon-Vey class comes from the non-triviality of the Euler class of this S^1-bundle. See also Morita [20].

Remark 4.7.

(1) The above argument is valid for any foliation defined by a 1-form.

(2) The class $\xi(\mathcal{F})$ defined at Definition 3.3 is now represented by

$$-\frac{4}{\pi^2}(\lambda_0 \wedge \lambda_1 \wedge \mu_2 + \lambda_0 \wedge \mu_1 \wedge \lambda_2)$$

and corresponds to the generator of $H^3(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{u}(1))$.

The above construction can be partially generalized to higher codimensional cases.

Example 4.8 [24]. — Consider $\mathrm{SL}(q+1,\mathbb{C})$, and define a subgroup $\widetilde{\mathrm{SL}}(q,\mathbb{C})$ of $\mathrm{SL}(q+1,\mathbb{C})$ by setting

$$\widetilde{\mathrm{SL}}(q,\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} ; \; |\det A| = 1, a = \det A^{-1} \right\}.$$

We foliate $\mathrm{SL}(q+1,\mathbb{C})$ by the cosets of $B = \left\{ \begin{pmatrix} a & * \\ 0 & C \end{pmatrix} ; C \in \mathrm{GL}(q,\mathbb{C}) \right\}$. As in the above example, we denote by $\widehat{\mathcal{F}}$ this foliation of $\mathrm{SL}(q+1,\mathbb{C})$. The foliation $\widehat{\mathcal{F}}$ projects down to a foliation $\mathcal{F}$ of the homogeneous space

$$M = \mathrm{SL}(q+1,\mathbb{C})/\widetilde{\mathrm{SL}}(q,\mathbb{C}).$$

The foliation $\mathcal{F}$ is transversely holomorphic and of complex codimension q. We can show that Godbillon-Vey form is a natural volume form of M which is invariant under the left action of $\mathrm{SL}(q+1,\mathbb{C})$.

Though the proof is somewhat complicated, it is very similar to the proof of Example 4.6 and we omit it (see [2]).

Remark 4.9. — If we can find a discrete subgroup Γ of $\mathrm{SL}(q+1,\mathbb{C})$ such that the double coset $\Gamma\backslash\mathrm{SL}(q+1,\mathbb{C})/\widetilde{\mathrm{SL}}(q,\mathbb{C})$ is a closed Hausdorff manifold, then as in Example 4.6 we can obtain an example of a transversely holomorphic foliation of a closed manifold whose Godbillon-Vey class is non-trivial. However, it is known that there is *no* such discrete subgroup Γ of $\mathrm{SL}(q+1,\mathbb{C})$ if $q=2$ (T. Kobayashi [18, Example 7]) or q is even (Y. Benoist [6, Exemple 1]). It seems that even in the case where q is an even integer greater than 2, we do not have such a lattice.

We end up this paper by summing up some questions.

QUESTION 4.10.

(1) *Is there a transversely holomorphic foliation of complex codimension greater than one whose Godbillon-Vey class is non-trivial ?*

(2) *Is there a family of transversely holomorphic foliations whose Godbillon-Vey class varies continuously ?*

(3) *Are the complex secondary classes invariant under C^r-concordance (resp. cobordism, foliation preserving diffeomorphism) ?*

Added in Proof. Recently, we found answers to (1) and (2) of Question 4.10. Namely, we found examples of transversely holomorphic foliations whose Godbillon-Vey classes are non-trivial in any codimensional case. One of the examples is obtained by changing $\widetilde{\mathrm{SL}}(q,\mathbb{C})$ in Example 4.8 to an appropriate compact subgroup. On the other hand, by generalizing Theorem D, we can show that the Godbillon-Vey class is rigid under smooth deformations in any codimensional case. These results will appear in [27] with some more examples.

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Manuscrit reçu le 28 septembre 1998,
accepté le 30 août 1999.

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