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# SELFDUAL SPACES WITH COMPLEX STRUCTURES, EINSTEIN-WEYL GEOMETRY AND GEODESICS

by D.M.J. CALDERBANK and H. PEDERSEN

## 1. Introduction.

Selfdual conformal 4-manifolds play a central role in low dimensional differential geometry. The selfduality equation is integrable, in the sense that there is a twistor construction for solutions, and so one can hope to find many explicit examples [2], [23]. One approach is to look for examples with symmetry. Since the selfduality equation is the complete integrability condition for the local existence of orthogonal (and antiselfdual) complex structures, it is also natural to look for solutions equipped with such complex structures. Our aim herein is to study the geometry of this situation in detail and present a framework unifying the theories of hypercomplex structures and scalar-flat Kähler metrics with symmetry [7], [12], [19]. Within this framework, there are explicit examples of hyperKähler, selfdual Einstein, hypercomplex and scalar-flat Kähler metrics parameterised by arbitrary functions.

The key tool in our study is the Jones and Tod construction [16], which shows that the reduction of the selfduality equation by a conformal vector field is given by the Einstein-Weyl equation together with the linear equation for an abelian monopole. This correspondence between a selfdual space $M$ with symmetry and an Einstein-Weyl space $B$ with a monopole is remarkable for three reasons:

(i) It provides a geometric interpretation of the symmetry reduced equation for an arbitrary conformal vector field.

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(ii) It is a constructive method for building selfdual spaces out of solutions to a linear equation on an Einstein-Weyl space.

(iii) It can be used in the other direction to construct Einstein-Weyl spaces from selfdual spaces with symmetry.

We add to this correspondence by proving that invariant antiselfdual complex structures on $M$ correspond to shear-free geodesic congruences on $B$, *i.e.*, foliations of $B$ by oriented geodesics, such that the transverse conformal structure is invariant along the leaves. This generalises Tod's observation [29] that the Einstein-Weyl spaces arising from scalar-flat Kähler metrics with Killing fields [19] admit a shear-free geodesic congruence which is also twist-free (*i.e.*, surface-orthogonal).

In order to explain how the scalar-flat Kähler story and the analogous story for hypercomplex structures [7], [12] fit into our more general narrative, we begin, in Section 2, by reviewing, in a novel way, the construction of a canonical "Kähler-Weyl connection" on any conformal Hermitian surface [9], [32]. We give a representation theoretic proof of the formula for the antiselfdual Weyl tensor on such a surface [1] and discuss its geometric and twistorial interpretation when the antiselfdual Weyl tensor vanishes. We use twistor theory throughout the paper to explain and motivate the geometric constructions, although we find it easier to make these constructions more general, explicit and precise by direct geometric arguments.

Having described the four dimensional context, we lay the three dimensional foundations for our study in Section 3. We begin with some elementary facts about congruences, and then go on to show that the Einstein-Weyl equation is the complete integrability condition for the existence of shear-free geodesic congruences in a three dimensional Weyl space. As in Section 2, we discuss the twistorial interpretation, this time in terms of the associated "minitwistor space" [14], and explain the minitwistor version of the Kerr theorem, which has only been discussed informally in the existing literature (and usually only in the flat case). We also show that at any point where the Einstein-Weyl condition does not hold, there are at most two possible directions for a shear-free geodesic congruence. The main result of our work in this section, however, is a reformulation of the Einstein-Weyl equation in the presence of a shear-free geodesic congruence. More precisely, we show in Theorem 3.8 that the Einstein-Weyl equation is equivalent to the fact that the divergence and twist of this congruence are both monopoles of a special kind. These "special" monopoles play a crucial role in the sequel.

We end Section 3 by giving examples. We first explain how the Einstein-Weyl spaces arising as quotients of scalar-flat Kähler metrics and hypercomplex structures fit into our theory: they are the cases of vanishing twist and divergence respectively. In these cases it is known that the remaining nonzero special monopole (*i.e.*, the divergence and twist respectively) may be used to construct a hyperKähler metric [3], [7], [12], motivating some of our later results. We also give some new examples: indeed, in Theorem 3.10, we classify explicitly the Einstein-Weyl spaces admitting a geodesic congruence generated by a conformal vector field preserving the Weyl connection. We call such spaces Einstein-Weyl *with a geodesic symmetry.* They are parameterised by an arbitrary holomorphic function of one variable.

The following section contains the central results of this paper, in which the four and three dimensional geometries are related. We begin by giving a new differential geometric proof of the Jones and Tod correspondence [16] between oriented conformal structures and Weyl structures, which reduces the selfduality condition to the Einstein-Weyl condition (see 4.1). Although other direct proofs can be found in the literature [12], [17], [19], they either only cover special cases, or are not sufficiently explicit for our purposes. Our next result, Theorem 4.2, like the Jones and Tod construction, is motivated by twistor theory. Loosely stated, it is as follows.

THEOREM. — *Suppose $M$ is an oriented conformal 4-manifold with a conformal vector field, and $B$ is the corresponding Weyl space. Then invariant antiselfdual complex structures on $M$ correspond to shear-free geodesic congruences on $B$.*

In fact we show explicitly how the Kähler-Weyl connection may be constructed from the divergence and twist of the congruence. This allows us to characterise the hypercomplex and scalar-flat Kähler cases of our correspondence, reobtaining the basic constructions of [3], [7], [12], [19], as well as treating quotients of hypercomplex, scalar-flat Kähler and hyperKähler manifolds by more general holomorphic conformal vector fields. As a consequence, we show in Theorem 4.3 that every Einstein-Weyl space is locally the quotient of some scalar-flat Kähler metric and also of some hypercomplex structure, and that it is a local quotient of a hyperKähler metric (by a holomorphic conformal vector field) if and only if it admits a shear-free geodesic congruence with linearly dependent divergence and twist.

We clarify the scope of these results in Section 5 where we show that our constructions can be applied to all selfdual Einstein metrics with a conformal vector field. Here, we make use of the fact that a selfdual Einstein metric with a Killing field is conformal to a scalar-flat Kähler metric [31].

The last four sections are concerned exclusively with examples. In Section 6 we show how our methods provide some insight into the construction of Einstein-Weyl structures from $\mathbb{R}^4$ [26]. As a consequence, we observe that there is a one parameter family of Einstein-Weyl structures on $S^3$ admitting shear-free twist-free geodesic congruences. This family is complementary to the more familiar Berger spheres, which admit shear-free divergence-free geodesic congruences [7], [12].

In Section 7, we generalise this by replacing $\mathbb{R}^4$ with a Gibbons-Hawking hyperKähler metric [13] constructed from a harmonic function on $\mathbb{R}^3$. If the corresponding monopole is invariant under a homothetic vector field on $\mathbb{R}^3$, then the hyperKähler metric has an extra symmetry, and hence another quotient Einstein-Weyl space. We first treat the case of axial symmetry, introduced by Ward [33], and then turn to more general symmetries. The Gibbons-Hawking metrics constructed from monopoles invariant under a general Killing field give new implicit solutions of the Toda field equation. On the other hand, from the monopoles invariant under dilation, we reobtain the Einstein-Weyl spaces with geodesic symmetry.

In Section 8 we look at the constant curvature metrics on $\mathcal{H}^3$, $\mathbb{R}^3$ and $S^3$ from the point of view of congruences and use this prism to explain properties of the selfdual Einstein metrics fibering over them. Then in the final section, we consider once more the Einstein-Weyl spaces constructed from harmonic functions on $\mathbb{R}^3$, and use them to construct torus symmetric selfdual conformal structures. These include those of Joyce [17], some of which live on $k\mathbb{C}P^2$, and also an explicit family of hypercomplex structures depending on two holomorphic functions of one variable.

This paper is primarily concerned with the richness of the local geometry of selfdual spaces with symmetry, and we have not studied completeness or compactness questions in any detail. Indeed, the local nature of the Jones and Tod construction makes it technically difficult to tackle such issues from this point of view, and doing so would have added considerably to the length of this paper. Nevertheless, there remain interesting problems which we hope to address in the future.

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## 2. Conformal structures and Kähler-Weyl geometry.

Associated to an orthogonal complex structure $J$ on a conformal manifold is a distinguished torsion-free connection $D$. The conformal structure is preserved by this connection and, in four dimensions, so is $J$. Such a connection is called a *Kähler-Weyl connection* [5]: if it is the Levi-Civita connection of a compatible Riemannian metric, then this metric is Kähler. In this section, we review this construction, which goes back to Lee and Vaisman (see [9], [21], [32]).

It is convenient in conformal geometry to make use of the *density bundles* $L^w$ (for $w \in \mathbb{R}$). On an $n$-manifold $M$, $L^w$ is the oriented real line bundle associated to the frame bundle by the representation $A \mapsto |\det A|^{w/n}$ of $\mathrm{GL}(n)$. The fibre $L^w_x$ may be constructed canonically as the space of maps $\rho\colon (\Lambda^n T_x M) \setminus 0 \to \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{-w/n}\rho(\omega)$ for all $\lambda \in \mathbb{R}^\times$ and $\omega \in (\Lambda^n T_x M) \setminus 0$.

A *conformal structure* $\boldsymbol{c}$ on $M$ is a positive definite symmetric bilinear form on $TM$ with values in $L^2$, or equivalently a metric on the bundle $L^{-1}TM$. (When tensoring with a density line bundle, we generally omit the tensor product sign.)

The line bundles $L^w$ are trivialisable and a nonvanishing (usually positive) section of $L^1$ (or $L^w$ for $w \neq 0$) will be called a *length scale* or *gauge* (of weight $w$). We also say that tensors in $L^w \otimes (TM)^j \otimes (T^*M)^k$ have *weight* $w + j - k$. If $\mu$ is a positive section of $L^1$, then $\mu^{-2}\boldsymbol{c}$ is a Riemannian metric on $M$, which will be called *compatible.* A conformal structure may equally be defined by the associated "conformal class" of compatible Riemannian metrics.

A *Weyl derivative* is a covariant derivative $D$ on $L^1$. It induces covariant derivatives on $L^w$ for all $w$. The curvature of $D$ is a real 2-form $F^D$ which will be called the *Faraday curvature* or *Faraday 2-form.* If $F^D = 0$ then $D$ is said to be *closed.* It follows that there are local length scales $\mu$ with $D\mu = 0$. If such a length scale exists globally then $D$ is said to be *exact.* Conversely, a length scale $\mu$ induces an exact Weyl derivative $D^\mu$ such that $D^\mu\mu = 0$. Consequently, we sometimes refer to an exact Weyl derivative as a *gauge*. The space of Weyl derivatives on $M$ is an affine space modelled on the space of 1-forms.

Any connection on $TM$ induces a Weyl derivative on $L^1$. Conversely, on a conformal manifold, the Koszul formula shows that any Weyl derivative determines uniquely a torsion-free connection $D$ on $TM$ with $Dc = 0$ (see [5]). Such connections are called *Weyl connections.* Linearising the Koszul formula with respect to $D$ shows that $(D+\gamma)_X Y = D_X Y + \gamma(X)Y + \gamma(Y)X - \langle X, Y\rangle\gamma$, where $\langle.,.\rangle$ denotes the conformal structure, and $X, Y$ are vector fields. Notice that here, and elsewhere, we make free use of the sharp isomorphism $\sharp : T^*M \to L^{-2}TM$. We sometimes write $\gamma \vartriangle X(Y) = \iota_Y(\gamma \wedge X)$ for the last two terms.

**2.1.** DEFINITION. — *A Kähler-Weyl structure on a conformal manifold $M$ is given by a Weyl derivative $D$ and an orthogonal complex structure $J$ such that $DJ = 0$.*

Suppose now that $M$ is a conformal $n$-manifold ($n = 2m > 2$) and that $J$ is an orthogonal complex structure. Then $\Omega_J\colon = \langle J.,.\rangle$ is a section of $L^2\Lambda^2T^*M$, called the *conformal Kähler form.* It is a nondegenerate weightless 2-form. (In general, we identify bilinear forms and endomorphism by $\Phi(X,Y) = \langle \Phi(X), Y\rangle$.)

**2.2.** PROPOSITION (*cf.* [21]). — *Suppose that $\Omega$ is a nondegenerate weightless 2-form. Then there is a unique Weyl derivative $D$ such that $d^D\Omega$ is trace-free with respect to $\Omega$, in the sense that $\sum d^D\Omega(e_i, e'_i, .) = 0$, where $e_i, e'_i$ are frames for $L^{-1}TM$ with $\Omega(e_i, e'_j) = \delta_{ij}$.*

*Proof.* — Pick any Weyl derivative $D^0$ and set $D = D^0 + \gamma$ for some 1-form $\gamma$. Then $d^D\Omega = d^{D^0}\Omega + 2\gamma \wedge \Omega$ and so the traces differ by

$$\begin{aligned}2(\gamma\wedge\Omega)(e_i, e'_i, .) &= 2\gamma(e_i)\Omega(e'_i, .) + 2\gamma(e'_i)\Omega(., e_i) + 2\gamma\,\Omega(e_i, e'_i)\\ &= 2(n-2)\gamma.\end{aligned}$$

Since $n > 2$ it follows that there is a unique $\gamma$ such that $d^D\Omega$ is trace-free. □

**2.3.** PROPOSITION. — *Suppose that $J$ is an orthogonal complex structure on a conformal manifold $M$ and that $d^D\Omega_J = 0$. Then $D$ defines a Kähler-Weyl structure on $M$, i.e., $DJ = 0$.*

*Proof.* — For any vector field $X$, $D_X J$ anticommutes with $J$ (since $J^2 = -\,\mathrm{id}$) and is skew (since $J$ is skew, and $D$ is conformal).

Hence $\langle (D_{JX}J - JD_XJ)Y, Z\rangle$, which is symmetric in $X, Y$ because $J$ is integrable and $D$ is torsion-free, is also skew in $Y, Z$. It must therefore vanish for all $X, Y, Z$. If we now impose $d^D\Omega_J = 0$ we obtain:

$$\begin{aligned} 0 &= d^D\Omega(X,Y,Z) - d^D\Omega(X,JY,JZ) \\ &= \big\langle (D_XJ)Y,Z\big\rangle + \big\langle (D_YJ)Z,X\big\rangle + \big\langle (D_ZJ)X,Y\big\rangle \\ &\quad - \big\langle (D_XJ)JY,JZ\big\rangle - \big\langle (JD_YJ)JZ,X\big\rangle - \big\langle (JD_ZJ)X,JY\big\rangle \\ &= 2\big\langle (D_XJ)Y,Z\big\rangle. \end{aligned}$$

Hence $DJ = 0$. □

Now if $n = 4$ and $D$ is the unique Weyl derivative such that $d^D\Omega_J$ is trace-free, then in fact $d^D\Omega_J = 0$ since wedge product with $\Omega_J$ is an isomorphism from $T^*M$ to $L^2\Lambda^3T^*M$. Hence, by Proposition 2.3, $DJ = 0$. To summarise:

**2.4.** THEOREM (*cf.* [32]). — *Any Hermitian conformal structure on any complex surface $M$ induces a unique Kähler-Weyl structure on $M$. The Weyl derivative is exact if and only if the conformal Hermitian structure admits a compatible Kähler metric.*

On an oriented conformal 4-manifold, orthogonal complex structures are either selfdual or antiselfdual, in the sense that the conformal Kähler form is either a selfdual or an antiselfdual weightless 2-form. In this paper we shall be concerned primarily with antiselfdual complex structures on selfdual conformal manifolds, *i.e.*, conformal manifolds $M$ with $W^- = 0$, where $W^-$ is the antiselfdual part of the Weyl tensor. In this case, as is well known (see [2]), there is a complex 3-manifold $Z$ fibering over $M$, called the *twistor space* of $M$. The fibre $Z_x$ given by the 2-sphere of orthogonal antiselfdual complex structures on $T_xM$, and the antipodal map $J \mapsto -J$ is a real structure on $Z$. The fibres are called the (real) *twistor lines* of $Z$ and are holomorphic rational curves in $Z$. The canonical bundle $K_Z$ of $Z$ is easily seen to be of degree $-4$ on each twistor line. As shown in [10], [25], any Weyl derivative on $M$ whose Faraday 2-form is selfdual induces a holomorphic structure on $L^1_{\mathbb{C}}$, the pullback of $L^1 \otimes \mathbb{C}$, and (up to reality conditions) this process is invertible; this is the Ward correspondence for line bundles, or the Penrose correspondence for selfdual Maxwell fields.

The Kähler-Weyl connection arising in Theorem 2.4 can be given a twistor space interpretation. Any antiselfdual complex structure $J$ defines divisors $\mathcal{D}, \overline{\mathcal{D}}$ in $Z$, namely the sections of $Z$ given by $J, -J$. Since the divisor $\mathcal{D}+\overline{\mathcal{D}}$ intersects each twistor line twice, the holomorphic line bundle $[\mathcal{D}+\overline{\mathcal{D}}]K_Z^{1/2}$ is trivial on each twistor line: more precisely, by viewing $J_x$ as a constant vector field on $L^2\Lambda^2_-T^*_xM$, its orthogonal projection canonically defines a vertical vector field on $Z$ holomorphic on each fibre and vanishing along $\mathcal{D}+\overline{\mathcal{D}}$. Therefore $[\mathcal{D}+\overline{\mathcal{D}}]$ is a holomorphic structure on the vertical tangent bundle of $Z$. In fact the vertical bundle of $Z$ is $L_{\mathbb{C}}^{-1}K_Z^{-1/2}$ and so $J$ determines a holomorphic structure on $L_{\mathbb{C}}^{-1}$, which, since $[\mathcal{D}+\overline{\mathcal{D}}]$ is real, gives a Weyl derivative on $M$ with selfdual Faraday curvature [11].

Similarly, by projecting each twistor line stereographically onto the orthogonal complement of $J$ in $L^2\Lambda^2_-T^*M$, which we denote $L^2K_J$, we see that the pullback of $L^2K_J$ to $Z$ has a section $s$ meromorphic on each fibre with a zero at $J$ and a pole at $-J$. Therefore the divisor $\mathcal{D}-\overline{\mathcal{D}}$ defines a holomorphic structure on this pullback bundle and hence a covariant derivative with (imaginary) selfdual curvature on $L^2K_J$. This curvature may be identified with the *Ricci form*, since if it vanishes, $[\mathcal{D}-\overline{\mathcal{D}}]$ is trivial, and so $s$, viewed as a meromorphic function on $Z$, defines a fibration of $Z$ over $\mathbb{C}P^1$; that is, $M$ is hypercomplex.

The selfduality of the Faraday and Ricci forms may be deduced directly from the selfduality of the Weyl tensor. To see this, we need a few basic facts from Weyl and Kähler-Weyl geometry.

First of all, let $D$ be a Weyl derivative on a conformal $n$-manifold and let $R^{D,w}$ denote the curvature of $D$ on $L^{w-1}TM$. Then it is well known that

$$R^{D,w}_{X,Y}=W_{X,Y}+wF^D(X,Y)\,\mathrm{id}-r^D(X)\vartriangle Y+r^D(Y)\vartriangle X. \tag{2.1}$$

Here $W$ is the Weyl tensor and $r^D$ is the normalised Ricci tensor, which decomposes under the orthogonal group as

$$r^D=r^D_0+\frac{1}{2n(n-1)}\,\mathrm{scal}^D\,\mathrm{id}-\frac{1}{2}F^D,$$

where $r^D_0$ is symmetric and trace-free, and the trace part defines the scalar curvature of $D$.

**2.5.** Proposition. — *On a Kähler-Weyl $n$-manifold ($n>2$) with Weyl derivative $D$, $F^D\wedge\Omega_J$ and the commutator $[R^{D,w}_{X,Y},J]$ both vanish. If $n>4$ it follows that $F^D=0$, while for $n=4$, $F^D$ is orthogonal to $\Omega_J$.*

*Also if $R^D = R^{D,1}$ then the symmetric Ricci tensor is given by the formula*

$$\frac{1}{2}\langle R^D_{Je_i,e_i}X,JY\rangle = (n-2)r^D_0(X,Y) + \frac{1}{n}\operatorname{scal}^D\langle X,Y\rangle,$$

*where on the left we are summing over a weightless orthonormal basis $e_i$. Consequently the symmetric Ricci tensor is $J$-invariant.*

*Proof.* — The first two facts are immediate from $d^D\Omega_J = 0$ and $DJ = 0$ respectively. If $n > 4$ then wedge product with $\Omega_J$ is injective on 2-forms, while for $n = 4$, $F^D \wedge \Omega_J$ is the multiple $\pm\langle F^D, \Omega_J\rangle$ of the weightless volume form, since $\Omega_J$ is antiselfdual. The final formula is a consequence of the first Bianchi identity:

$$\begin{aligned}\frac{1}{2}\langle R^D_{Je_i,e_i}X, JY\rangle &= \langle R^D_{X,e_i}Je_i, JY\rangle = \langle R^D_{X,e_i}e_i, Y\rangle \\ &= F^D(X,e_i)\langle e_i,Y\rangle - \langle r^D(X)\vartriangle e_i\, e_i, Y\rangle + \langle r^D(e_i)\vartriangle X e_i, Y\rangle \\ &= (n-2)r^D_0(X,Y) + \frac{1}{n}\operatorname{scal}^D\langle X,Y\rangle - \frac{1}{2}(n-4)F^D(X,Y)\end{aligned}$$

and the last term vanishes since $F^D = 0$ for $n > 4$. □

Now suppose $n = 4$. Then $W^+_{X,Y}$ commutes with $J$, and so

$$\begin{aligned}J \circ W^-_{X,Y} - W^-_{X,Y} \circ J = J \circ \big(r^D(X)\vartriangle Y - r^D(Y)\vartriangle X\big) \\ - \big(r^D(X)\vartriangle Y - r^D(Y)\vartriangle X\big)\circ J.\end{aligned}$$

The bundle of antiselfdual Weyl tensors may be identified with the rank 5 bundle of symmetric trace-free maps $L^2\Lambda^2_-T^*M \to \Lambda^2_-T^*M$, where $W^-(U\wedge V)(X\wedge Y) = \langle W^-_{U,V}X, Y\rangle$ and we identify $L^2\Lambda^2_-T^*M$ with $L^{-2}\Lambda^2_-TM$. Under the unitary group $L^2\Lambda^2_-T^*M$ decomposes into the span of $J$ and the weightless canonical bundle $L^2K_J$. This bundle of Weyl tensors therefore decomposes into three pieces: the Weyl tensors acting by scalars on $\langle J\rangle$ and $L^2K_J$; the symmetric trace-free maps $L^2K_J \to K_J$ (acting trivially on $\langle J\rangle$); and the Weyl tensors mapping $\langle J\rangle$ into $K_J$ and vice versa. These subbundles have ranks $1, 2$ and $2$ respectively. Since no nonzero Weyl tensor acts trivially on $K_J$, it follows that the above formula determines $W^-$ uniquely in terms of $r^D$. Now this is an invariant formula which is linear in $r^D$, so $r^D_0$ and $F^D_+$ cannot contribute: they are sections of (isomorphic) irreducible rank 3 bundles. Thus the first and third components of $W^-$ are given by $\operatorname{scal}^D$ and $F^D_-$ respectively, and the second component must vanish. The numerical factors can now be found by taking a trace.

**2.6.** Proposition (*cf.* [1]). — *On a Kähler-Weyl 4-manifold with Weyl derivative* $D$,

$$W^- = \frac{1}{4}\,\mathrm{scal}^D\Big(\frac{1}{3}\,\mathrm{id}_{\Lambda^2_-} - \frac{1}{2}\Omega_J\otimes\Omega_J\Big) - \frac{1}{2}\big(JF^D_-\otimes\Omega_J + \Omega_J\otimes JF^D_-\big),$$

*where* $JF^D_- = F^D_-\circ J$. *In particular* $W^- = 0$ *if and only if* $F^D_- = 0$ *and* $\mathrm{scal}^D = 0$.

The *Ricci form* $\rho^D$ on $M$ is defined to be the curvature of $D$ on the weightless canonical bundle $L^2K_J$. Therefore

$$\begin{aligned}\rho^D(X,Y) &= -\frac{i}{2}\langle R^D_{X,Y}e_k, Je_k\rangle\\ &= -\frac{i}{2}\big(\langle R^D_{X,e_k}e_k, JY\rangle - \langle R^D_{Y,e_k}e_k, JX\rangle\big)\\ &= i\Big(2r^D_0(JX,Y) + \frac{1}{4}\,\mathrm{scal}^D\langle JX,Y\rangle + 2F^D_-(JX,Y)\Big).\end{aligned}$$

Thus $W^- = 0$ if and only if $\rho^D$ and $F^D$ are selfdual 2-forms.

## 3. Shear-free geodesic congruences and Einstein-Weyl geometry.

On a conformal manifold, a foliation with oriented one dimensional leaves may be described by a weightless unit vector field $\chi$. (If $K$ is any nonvanishing vector field tangent to the leaves, then $\chi = \pm K/|K|$.) Such a foliation, or equivalently, such a $\chi$, is often called a *congruence*.

If $D$ is any Weyl derivative, then $D\chi$ is a section of $T^*M\otimes L^{-1}TM$ satisfying $\langle D\chi,\chi\rangle = 0$, since $\chi$ has unit length. Let $\chi^\perp$ be the orthogonal complement of $\chi$ in $L^{-1}TM$. Under the orthogonal group of $\chi^\perp$ acting trivially on the span of $\chi$, the bundle $T^*M\otimes\chi^\perp$ decomposes into four irreducible components: $L^{-1}\Lambda^2(\chi^\perp)$, $L^{-1}S^2_0(\chi^\perp)$, $L^{-1}$ (multiples of the identity $\chi^\perp\mapsto\chi^\perp$), and $L^{-1}\chi^\perp$ (the $\chi^\perp$-valued 1-forms vanishing on vectors orthogonal to $\chi$).

The first three components of $D\chi$ may be found by taking the skew, symmetric trace-free and tracelike parts of $D\chi - \chi\otimes D_\chi\chi$, while the final component is simply $D_\chi\chi$. These components are respectively called the *twist, shear, divergence, and acceleration* of $\chi$ with respect to $D$. If any of these vanish, then the congruence $\chi$ is said to be *twist-free, shear-free, divergence-free, or geodesic* accordingly.

**3.1.** Proposition. — *Let $\chi$ be a unit section of $L^{-1}TM$. Then the shear and twist of $\chi$ are independent of the choice of Weyl derivative $D$. Furthermore there is a unique Weyl derivative $D^\chi$ with respect to which $\chi$ is divergence-free and geodesic.*

This follows from the fact that $(D+\gamma)\chi = D\chi + \gamma(\chi)\,\mathrm{id} - \chi \otimes \gamma$.

The twist is simply the Frobenius tensor of $\chi^\perp$ (*i.e.*, the $\chi$ component of the Lie bracket of sections of $\chi^\perp$), while the shear measures the Lie derivative of the conformal structure of $\chi^\perp$ along $\chi$ (which makes sense even though $\chi$ is weightless).

**3.2.** *Remark.* — If $D^\chi$ is exact, with $D^\chi\mu = 0$, then $K = \mu\chi$ is a geodesic divergence-free vector field of unit length with respect to the metric $g = \mu^{-2}\mathbf{c}$. If $\chi$ is also shear-free, then $K$ is a Killing field of $g$. Note conversely that any nonvanishing conformal vector field $K$ is a Killing field of constant length $a$ for the compatible metric $a^2|K|^{-2}\mathbf{c}$: $\chi = K/|K|$ is then a shear-free congruence, and $D^\chi$ is the exact Weyl derivative $D^{|K|}$, which we call the *constant length gauge* of $K$.

We now turn to the study of geodesic congruences in three dimensional Weyl spaces and their relationship to Einstein-Weyl geometry and minitwistor theory (see [14], [20], [26]). We discuss the "mini-Kerr theorem" which is rather a folk theorem in the existing literature, and rewrite the Einstein-Weyl condition in a novel way by finding special monopole equations associated to a shear-free geodesic congruence.

The *minitwistor space* of an oriented geodesically convex Weyl space is its space of oriented geodesics. We assume that this is a manifold (*i.e.*, we ignore the fact that it may not be Hausdorff), as we shall only be using minitwistor theory to probe the local geometry of the Weyl space. The minitwistor space is four dimensional, and has a distinguished family of embedded 2-spheres corresponding to the geodesics passing through given points in the Weyl space.

Now let $\chi$ be a geodesic congruence on an oriented Weyl space $B$ with Weyl connection $D^B$. Then

$$D^B\chi = \tau(\mathrm{id} - \chi \otimes \chi) + \kappa *\chi + \Sigma, \tag{3.1}$$

where the divergence and twist, $\tau$ and $\kappa$, are sections of $L^{-1}$ and $\Sigma$ is the shear. Note that $D^\chi = D^B - \tau\chi$.

Equation (3.1) admits a natural complex interpretation, which we give in order to compare our formulae to those in the literature [15], [26]. Let $\mathcal{H} = \chi^\perp \otimes \mathbb{C}$ in the complexified weightless tangent bundle. Then $\mathcal{H}$ has a complex bilinear inner product on each fibre and the orientation of $B$ distinguishes one of the two null lines: if $e_1, e_2$ is an oriented real orthonormal basis, then $e_1 + ie_2$ is null. Let $Z$ be a section of this null line with $\langle Z, \overline{Z}\rangle = 1$. Such a $Z$ is unique up to pointwise multiplication by a unit complex number: at each point it is of the form $(e_1 + ie_2)/\sqrt{2}$. Now $D^B\chi = \rho\, Z \otimes \overline{Z} + \bar\rho\, \overline{Z} \otimes Z + \sigma\, Z \otimes Z + \bar\sigma\, \overline{Z} \otimes \overline{Z}$, where $\rho = \tau + i\kappa$ and $\sigma = \Sigma(\overline{Z}, \overline{Z})$ are sections of $L^{-1} \otimes \mathbb{C}$. Note that $\sigma$ depends on the choice of $Z$: the ambiguity can partially be removed by requiring that $D^B_\chi Z = 0$, but we shall instead work directly with $\Sigma$.

**3.3.** *Conventions.* — There are two interesting sign conventions for the Hodge star operator of an oriented conformal manifold. The first satisfies $\alpha \wedge \tilde{*}\beta = \langle \alpha, \beta\rangle \underline{\mathrm{or}}$, where $\underline{\mathrm{or}}$ is the unit section of $L^n\Lambda^n T^*M$ given by the orientation. This is convenient when computing the star operator of an explicit example. The second satisfies $*1 = \underline{\mathrm{or}}$ and $\iota_X *\alpha = *(X \wedge \alpha)$, which is a more useful property in many theoretical calculations. Also $*^2 = (-1)^{\frac{1}{2}n(n-1)}$ depends only on the dimension of the manifold, not on the degree of the form. If $\alpha$ is a $k$-form, then $*\alpha = (-1)^{\frac{1}{2}k(k-1)}\tilde{*}\alpha$.

**3.4.** Proposition. — *The curvature of $D^B$ applied to the geodesic congruence $\chi$ is given by*

$$\begin{aligned} R^{B,0}_{X,Y}\chi = \iota_\chi\big[&D^B_X\tau\, \chi \wedge Y - D^B_Y\tau\, \chi \wedge X - D^B_X\kappa\, *Y + D^B_Y\kappa\, *X \\ &+ (\tau^2 - \kappa^2)X \wedge Y - 2\tau\kappa\, \chi \wedge *(X \wedge Y)\big] + (D^B_X\Sigma)(Y) \\ &- (D^B_Y\Sigma)(X) - \tau\big(\Sigma(X)\langle\chi,Y\rangle - \Sigma(Y)\langle\chi,X\rangle\big) \\ &+ \kappa\, *\big(Y \wedge \Sigma(X) - X \wedge \Sigma(Y)\big) \end{aligned}$$

*and also by its decomposition*

$$\begin{aligned} &R^{B,0}_{X,Y}\chi \\ &= r^B_0(Y,\chi)X - \frac{1}{2}F^B(Y,\chi)X + \Big(r^B_0(X) + \frac{1}{6}\,\mathrm{scal}^B X - \frac{1}{2}F^B(X)\Big)\langle\chi,Y\rangle \\ &- r^B_0(X,\chi)Y + \frac{1}{2}F^B(X,\chi)Y - \Big(r^B_0(Y) + \frac{1}{6}\,\mathrm{scal}^B Y - \frac{1}{2}F^B(Y)\Big)\langle\chi,X\rangle. \end{aligned}$$

The first formula follows from $R^{B,0}_{X,Y}\chi = D^B_X(D^B\chi)_Y - D^B_Y(D^B\chi)_X$, using

$$\begin{aligned} D^B_X(D^B\chi) = &D^B_X\tau(\mathrm{id} - \chi \otimes \chi) + D^B_X\kappa\, *\chi \\ &- \tau(D^B_X\chi \otimes \chi + \chi \otimes D^B_X\chi) + \kappa\, *D^B_X\chi + D^B_X\Sigma. \end{aligned}$$

The second formula follows easily from $R^{B,0}_{X,Y} = -r^B(X) \vartriangle Y + r^B(Y) \vartriangle X$ where $r^B = r^B_0 + \frac{1}{12}\operatorname{scal}^B - \frac{1}{2}F^B$.

In order to compare the rather different formulae in Proposition 3.4, we shall first take $Y$ parallel to $\chi$ and $X$ orthogonal to $\chi$. The formulae reduce to

$$\begin{aligned}D^B_\chi\tau\, X + D^B_\chi\kappa\, JX + (\tau^2-\kappa^2)X + 2\tau\kappa\, JX \\ + \Sigma(D^B_X\chi) + (D^B_\chi\Sigma)(X) + \tau\Sigma(X) - \kappa\Sigma(JX) \\ = -R^{B,0}_{X,\chi}\chi \\ = -r^B_0(X) + r^B_0(X,\chi)\chi + \frac{1}{2}\big(F^B(X) - F^B(X,\chi)\chi\big) \\ - r^B_0(\chi,\chi)X - \frac{1}{6}\operatorname{scal}^B X,\end{aligned}$$

where $JX\colon = \iota_X *\chi$ and we have used the fact that

$$(D^B_X\Sigma)(\chi) + \Sigma(D^B_X\chi) = 0.$$

If we contract with another vector field $Y$ orthogonal to $\chi$, then we obtain

$$\begin{aligned}D^B_\chi\tau\,\langle X,Y\rangle + D^B_\chi\kappa\,\langle JX,Y\rangle + \langle (D^B_\chi\Sigma)(X),Y\rangle \\ + (\tau^2-\kappa^2)\langle X,Y\rangle + 2\tau\kappa\langle JX,Y\rangle + 2\tau\big\langle\Sigma(X),Y\big\rangle + \big\langle\Sigma(X),\Sigma(Y)\big\rangle \\ = -r^B_0(X,Y) + \frac{1}{2}F^B(X,Y) - \Big(r^B_0(\chi,\chi) - \frac{1}{6}\operatorname{scal}^B\Big)\langle X,Y\rangle.\end{aligned}$$

Decomposing this into irreducibles gives the equations

$$D^B_\chi\tau + \tau^2 - \kappa^2 + \frac{1}{2}|\Sigma|^2 + \frac{1}{2}r^B_0(\chi,\chi) + \frac{1}{6}\operatorname{scal}^B = 0, \tag{3.2}$$

$$D^B_\chi\kappa + 2\tau\kappa + \frac{1}{2}\langle\chi, *F^B\rangle = 0, \tag{3.3}$$

$$D^B_\chi\Sigma + 2\tau\Sigma + \operatorname{sym}_0^{\chi^\perp} r^B_0 = 0, \tag{3.4}$$

which may, assuming $D^B_\chi Z = 0$, be rewritten as

$$D^B_\chi\rho + \rho^2 + \sigma\bar\sigma + \frac{1}{2}r^B_0(\chi,\chi) + \frac{1}{6}\operatorname{scal}^B + \frac{i}{2}\langle\chi, *F^B\rangle = 0, \tag{3.5}$$

$$D^B_\chi\sigma + (\rho+\bar\rho)\sigma + r^B_0(\overline{Z},\overline{Z}) = 0. \tag{3.6}$$

Along a single geodesic, these formulae describe the evolution of nearby geodesics in the congruence and therefore may be interpreted infinitesimally (*cf.* [26]). We say that a vector field $X$ along an oriented geodesic $\Gamma$ with weightless unit tangent $\chi$ is a *Jacobi field* if and only if $(D^B)^2_{\chi,\chi}X = R^{B,0}_{\chi,X}\chi$. The space of Jacobi fields orthogonal to $\Gamma$ is four dimensional, since the initial data for the Jacobi field equation is $X, D^B_\chi X$. In fact this is the tangent space to the minitwistor space at $\Gamma$. If we now consider a two dimensional family of Jacobi fields spanning (at each point on an open subset of $\Gamma$) the plane orthogonal to $\Gamma$, then we may write $D^B_\chi X = \tau X + \kappa JX + \Sigma(X)$ for the Jacobi fields $X$ in this family. If we differentiate again with respect to $\chi$, we reobtain the equations (3.2)–(3.6).

A geodesic congruence gives rise to such a two dimensional family of Jacobi fields along each geodesic in the congruence. We define the Lie derivative $\mathcal{L}_\chi X$ of a vector field $X$ along $\chi$ to be the horizontal part of $D^B_\chi X - D^B_X\chi$. Then if $\mathcal{L}_\chi X = 0$, $X$ is a Jacobi field, and such Jacobi fields are determined along a geodesic by their value at a point. Next note that $\mathcal{L}_\chi J = 0$ (*i.e.*, $\mathcal{L}_\chi(JX) = J\mathcal{L}_\chi X$) if and only if $\chi$ is shear-free. However, equation (3.4) shows that if $\chi$ is a shear-free, then $r^B_0(X,Y) = -\frac{1}{2}r^B_0(\chi,\chi)\langle X,Y\rangle$ for all $X, Y$ orthogonal to $\chi$. More generally, this equation shows that $J$ is a well defined complex structure on the space of Jacobi fields orthogonal to a geodesic $\Gamma$ if and only if $r^B_0(X,Y) = -\frac{1}{2}r^B_0(\chi,\chi)\langle X,Y\rangle$ for all $X, Y$ orthogonal to $\Gamma$. The Jacobi fields defined by a congruence are then invariant under $J$ if and only if the congruence is shear-free.

**3.5.** DEFINITION (*cf.* [14]). — *A Weyl space $B, D^B$ is said to be Einstein-Weyl if and only if $r^B_0 = 0$.*

As mentioned above, the space of orthogonal Jacobi fields along a geodesic is the tangent space to the minitwistor space at that geodesic. Therefore, if $B$ is Einstein-Weyl, the minitwistor space admits a natural almost complex structure. This complex structure turns out to be integrable, and so the minitwistor space of an Einstein-Weyl space is a complex surface $\mathcal{S}$ containing a family of rational curves, called *minitwistor lines*, parameterised by points in $B$ [14]. These curves have normal bundle $\mathcal{O}(2)$ and are invariant under the real structure on $\mathcal{S}$ defined by reversing the orientation of a geodesic. Conversely, any complex surface with real structure, containing a real (*i.e.*, invariant) rational curve with normal bundle $\mathcal{O}(2)$, determines an Einstein-Weyl space as the real points in the Kodaira moduli space of deformations of this curve. We therefore have a twistor construction for Einstein-Weyl spaces, called the *Hitchin correspondence*. We note that the canonical bundle $K_{\mathcal{S}}$ of $\mathcal{S}$ has degree $-4$ on each minitwistor line.

Since geodesics correspond to points in the minitwistor space, a geodesic congruence defines a real surface $\mathcal{C}$ intersecting each minitwistor line once. By the definition of the complex structure on $\mathcal{S}$, the surface $\mathcal{C}$ is a holomorphic curve if and only if the geodesic congruence is shear-free. This may be viewed as a minitwistor version of the Kerr theorem: *every shear-free geodesic congruence in an Einstein-Weyl space is obtained locally from a holomorphic curve in the minitwistor space.* In particular, we have the following.

**3.6.** PROPOSITION. — *Let $B, D^B$ be a three dimensional Weyl space. Then the following are equivalent:*

(i) *$B$ is Einstein-Weyl.*

(ii) *Given any point $b \in B$ and any unit vector $v \in L^{-1}T_bB$, there is a shear-free geodesic congruence $\chi$ defined on a neighbourhood of $b$ with $\chi_b = v$.*

(iii) *Given any point $b \in B$ there are three shear-free geodesic congruences defined on a neighbourhood of $b$ which are pairwise non-tangential at $b$.*

*Proof.* — Clearly (ii) implies (iii). It is immediate from (3.4) that (ii) implies (i); to obtain the stronger result that (iii) implies (i) suppose that $B$ is not Einstein-Weyl, *i.e.*, at some point $b \in B$, $r_0^B \neq 0$. If $\chi$ is a shear-free geodesic congruence near $b$ then by equation (3.4), $r_0^B$ is a multiple of the identity on $\chi^\perp$, and one easily sees that this multiple must be the middle eigenvalue $\lambda_0 \in L_b^{-2}$ of $r_0^B$ at $b$. Now at $b$, $r_0^B$ may be written $\alpha \otimes \sharp\beta + \beta \otimes \sharp\alpha + \lambda_0 \,\mathrm{id}$ where $\alpha, \beta \in T_b^*B$ with $\langle \alpha, \beta \rangle = -\frac{3}{2}\lambda_0$. The directions of $\sharp\alpha$ and $\sharp\beta$ are uniquely determined by $r_0^B$ and $\chi$ must lie in one of these directions. Hence if $B$ is not Einstein-Weyl at $b$, there are at most two possible directions at $b$ (up to sign) for a shear-free geodesic congruence. (Note that the linear algebra involved here is the same as that used to show that there are at most two principal directions of a nonzero antiselfdual Weyl tensor in four dimensions; see, for instance [1]. Our result is just the symmetry reduction of this fact.)

Finally, to see that (i) implies (ii), we simply observe that given any minitwistor line and any point on that line, we can find, in a neighbourhood of that point, a transverse holomorphic curve. This curve will also intersect nearby minitwistor lines exactly once. □

We now want to study Einstein-Weyl spaces with a shear-free geodesic congruence in more detail. As motivation for our main result, notice that the curve $\mathcal{C}$ in the minitwistor space given by $\chi$ defines divisors $\mathcal{C}+\overline{\mathcal{C}}$ and $\mathcal{C}-\overline{\mathcal{C}}$ such that the line bundles $[\mathcal{C}+\overline{\mathcal{C}}]K_S^{1/2}$ and $[\mathcal{C}-\overline{\mathcal{C}}]$ are trivial on each minitwistor line. It is well known [16] that such line bundles correspond to solutions $(w, A)$ of the abelian monopole equation $*D^B w = dA$, where $w$ is a section of $L^{-1}$ and $A$ is a 1-form. Therefore, we should be able to find two special solutions of this monopole equation, one real and one imaginary, associated to any shear-free geodesic congruence.

These solutions turn out to be $\kappa$ and $i\tau$. To see this, we return to the curvature equations in Proposition 3.4 and look at the horizontal components. If $X, Y$ are orthogonal to a geodesic congruence $\chi$ on any three dimensional Weyl space then

$$\begin{aligned}D^B_X\tau\, Y - D^B_Y\tau\, X + D^B_X\kappa\, JY - D^B_Y\kappa\, JX \\ + (D^B_X\Sigma)(Y) - (D^B_Y\Sigma)(X) + \kappa*\big(Y\wedge\Sigma(X) - X\wedge\Sigma(Y)\big) \\ = r_0^B(Y,\chi)X - \frac{1}{2}F^B(Y,\chi)X - r_0^B(X,\chi)Y + \frac{1}{2}F^B(X,\chi)Y.\end{aligned}$$

If $\chi$ is shear-free this reduces to the equation

$$D^B_X\tau - D^B_{JX}\kappa + r_0^B(\chi, X) + \frac{1}{2}F^B(\chi, X) = 0,$$

where $X\perp\chi$. From this, and our earlier formulae, we have:

**3.7.** Proposition. — *Let $\chi$ be shear-free geodesic congruence with divergence $\tau$ and twist $\kappa$ in a three dimensional Weyl space $B$. Then $\chi$ satisfies the equations*

$$D^B_\chi\tau + \tau^2 - \kappa^2 + \frac{1}{6}\,\mathrm{scal}^B = 0, \tag{3.7}$$

$$D^B_\chi\kappa + 2\tau\kappa + \frac{1}{2}\langle\chi, *F^B\rangle = 0, \tag{3.8}$$

$$(D^B\tau - D^B\kappa\circ J)_{|\chi^\perp} + \frac{1}{2}\iota_\chi F^B = 0, \tag{3.9}$$

*if and only if $B$ is Einstein-Weyl.*

The last equation, like the first two (see (3.5)), admits a natural complex formulation in terms of $\rho$. Instead, however, we shall combine these equations to give the following result.

**3.8.** THEOREM. — *The three dimensional Einstein-Weyl equations are equivalent to the following special monopole equations for a shear-free geodesic congruence $\chi$ with $D^B\chi = \tau(\mathrm{id} - \chi\otimes\chi) + \kappa *\chi$:*

$$(3.10)\qquad *D^B\tau = -\frac{1}{2}*\iota_\chi F^B - \frac{1}{6}\,\mathrm{scal}^B *\chi - (\tau^2+\kappa^2)*\chi + d(\kappa\chi),$$

$$(3.11)\qquad *D^B\kappa = \frac{1}{2}F^B - d(\tau\chi).$$

*(By "monopole equations", we mean that the right hand sides are closed 2-forms. Note also that these equations are not independent: they are immediately equivalent to* (3.7) *and* (3.11), *or to* (3.8) *and* (3.10).*)*

*Proof.* — The equations of the previous proposition are equivalent to the following:

$$D^B\tau = D^B\kappa\circ J + (\kappa^2-\tau^2)\chi - \frac{1}{6}\,\mathrm{scal}^B\,\chi - \frac{1}{2}\iota_\chi F^B,$$

$$D^B\kappa = -D^B\tau\circ J - 2\tau\kappa\chi - \frac{1}{2}*F^B.$$

Applying the star operator readily yields the equations of the theorem. The second equation is clearly a monopole equation, since $F^B$ is closed. It remains to check that the right hand side of the first equation is closed:

$$\begin{aligned}
d\Big(\frac{1}{2}\chi\wedge *F^B &+ \frac{1}{6}\,\mathrm{scal}^B *\chi + (\tau^2+\kappa^2)*\chi\Big)\\
&= \frac{1}{2}d^B\chi\wedge *F^B - \frac{1}{2}\chi\wedge *\delta^B F^B + \frac{1}{6}D^B\,\mathrm{scal}^B\wedge *\chi\\
&\quad + (2\tau D^B\tau + 2\kappa D^B\kappa)\wedge *\chi + \Big(\frac{1}{6}\,\mathrm{scal}^B + \tau^2+\kappa^2\Big)*\delta^B\chi\\
&= \frac{1}{2}\chi\wedge *\Big(\frac{1}{3}D^B\,\mathrm{scal}^B - \delta^B F^B\Big)\\
&\qquad + \Big(\kappa\langle\chi, *F^B\rangle + 2\tau D^B_\chi\tau + 2\kappa D^B_\chi\kappa + 2\tau\Big(\frac{1}{6}\,\mathrm{scal}^B + \tau^2+\kappa^2\Big)\Big)*1.
\end{aligned}$$

Here $\delta^B = \mathrm{tr}\, D^B$ is the divergence on forms, and so the first term vanishes by virtue of the second Bianchi identity. The remaining multiple of the orientation form $*1$ is

$$\kappa\langle\chi, *F^B\rangle + 2\kappa D^B_\chi\kappa + 2\kappa(2\kappa\tau) + 2\tau D^B_\chi\tau + 2\tau\Big(\frac{1}{6}\,\mathrm{scal}^B + \tau^2 - \kappa^2\Big),$$

which vanishes by the previous proposition. □

Two key special cases of this theorem have already been studied.

**LeBrun-Ward geometries**

Suppose an Einstein-Weyl space admits a shear-free geodesic congruence *which is also twist-free.* Then $\kappa = 0$ and so the Einstein-Weyl equations (3.7), (3.11) are:

$$D^B_\chi \tau + \tau^2 = -\frac{1}{6}\,\mathrm{scal}^B, \tag{3.12}$$

$$F^B = 2d(\tau\chi) = 2D^B\tau \wedge \chi. \tag{3.13}$$

As observed by Tod [29], these Einstein-Weyl spaces are the spaces first studied by LeBrun [19], [20] and Ward [33], who described them using coordinates in which the above equations reduce to the SU($\infty$) Toda field equation $u_{xx} + u_{yy} + (e^u)_{zz} = 0$. Consequently these Einstein-Weyl spaces are also said to be *Toda.*

It may be useful here to sketch how this follows from our formulae, since Lemma 4.1 in [29], given there without proof, is only true after making use of the gauge freedom to set $z = f(\tilde{z})$ and rescale the metric by $f'(\tilde{z})^{-2}$. The key point is that since $D^{LW} := D^B - 2\tau\chi$ is locally exact by (3.13), there is locally a canonical gauge (up to homothety) in which to work, which we call the *LeBrun-Ward* gauge $\mu_{LW}$. Since $\chi$ is twist-free and also geodesic with respect to $D^{LW}$, the 1-form $\mu_{LW}^{-1}\chi$ is locally exact. Taking this to be $dz$ and introducing isothermal coordinates $(x,y)$ on the quotient of $B$ by $\chi$, we may write $g_{LW} = \mathrm{e}^u(dx^2 + dy^2) + dz^2$ for some function $u(x,y,z)$, since $\chi$ is shear-free. By computing the divergence of $\chi$ we then find that the Toda monopole is $\tau = -\frac{1}{2}u_z\mu_{LW}^{-1}$, and equation (3.12) reduces easily in this gauge to the Toda equation. One of the reasons for the interest in this equation is that it may be used to construct hyperKähler and scalar-flat Kähler 4-manifolds [3], [19], as we shall see in the next section.

LeBrun [19] shows that these spaces are characterised by the existence of a divisor $\mathcal{C}$ in the minitwistor space with $[\mathcal{C} + \overline{\mathcal{C}}] = K_S^{-1/2}$. This agrees with our assertion that $[\mathcal{C} + \overline{\mathcal{C}}]K_S^{1/2}$ corresponds to the monopole $\kappa$.

In [4], it is shown that an Einstein-Weyl space admits at most a three dimensional family of shear-free twist-free geodesic congruences.

**Gauduchon-Tod geometries**

Suppose an Einstein-Weyl space admits a shear-free geodesic congruence *which is also divergence-free.* Then $\tau = 0$ and so the Einstein-Weyl equations (3.7), (3.11) are:

$$\kappa^2 = \frac{1}{6}\,\mathrm{scal}^B, \tag{3.14}$$

$$*D^B\kappa = \frac{1}{2}F^B. \tag{3.15}$$

It follows that these are the geometries which arose in the work of Gauduchon and Tod [12] and also Chave, Tod and Valent [7] on hypercomplex 4-manifolds with triholomorphic conformal vector fields. Gauduchon and Tod essentially observe the following equivalent formulation of these equations.

**3.9.** PROPOSITION. — *The connection $D^\kappa = D^B - \kappa *1$ on $L^{-1}TB$ is flat.*

*Proof.* — The curvature of $D^\kappa$ is easily computed to be:

$$\begin{aligned} R^\kappa_{X,Y} = -r_0^B(X) \vartriangle Y + r_0^B(Y) \vartriangle X - \frac{1}{6}\,\mathrm{scal}^B\, X \vartriangle Y + \frac{1}{2}F^B(X) \vartriangle Y \\ - \frac{1}{2}F^B(Y) \vartriangle X - D^B_X\kappa *Y + D^B_Y\kappa *X + \kappa^2 X \vartriangle Y. \end{aligned}$$

Now $D^B_X\kappa *Y - D^B_Y\kappa *X = (*D^B\kappa)(X) \vartriangle Y - (*D^B\kappa)(Y) \vartriangle Y$, so equations (3.14) and (3.15) imply that $R^\kappa_{X,Y}$ vanishes if $B$ is Einstein-Weyl. (Conversely if there is a $\chi$ with $R^\kappa_{X,Y}\chi = 0$ for all $X, Y$, then $B$ is Einstein-Weyl.) □

This shows that the existence of a single shear-free divergence-free geodesic congruence gives an entire 2-sphere of such congruences and we say that these Einstein-Weyl spaces are *hyperCR* [6]. There is also a simple minitwistor interpretation of this. The divisor $\mathcal{C}$ corresponding to a shear-free divergence-free geodesic congruence has $[\mathcal{C} - \overline{\mathcal{C}}]$ trivial, *i.e.*, $\mathcal{C} - \overline{\mathcal{C}}$ is the divisor of a meromorphic function. Hence we have a nonconstant holomorphic map from the minitwistor space to $\mathbb{C}P^1$, and its fibres correspond to the 2-sphere of congruences. This argument is the minitwistor analogue of the twistor characterisation of hypercomplex structures discussed in the previous section.

Since the Einstein-Weyl structure determines $\kappa$ up to sign, it follows that an Einstein-Weyl space admits at most two hyperCR structures. If it admits exactly two, then we must have $\kappa \neq 0$ and $F^B = 0$, *i.e.*, the Einstein-Weyl space is the round sphere.

### Einstein-Weyl spaces with a geodesic symmetry

The Einstein-Weyl equation can be completely solved in the case of Einstein-Weyl spaces admitting a shear-free geodesic congruence $\chi$ such that $\chi = K/|K|$ with $K$ a conformal vector field preserving the Weyl connection. In this case $D^\chi = D^B - \tau\chi$ is exact, $|K|$ being a parallel section of $L^1$ (see Remark 3.2). We introduce $g = |K|^{-2}\boldsymbol{c}$ so that $D^\chi = D^g$. Since $K$ preserves the Weyl connection and $\mathcal{L}_K g = 0$, we may write $\tau = \tau_g |K|^{-1}$, $\kappa = \kappa_g |K|^{-1}$, where $\partial_K \tau_g = \partial_K \kappa_g = 0$. Now $\iota_\chi F^B = \iota_\chi d(\tau\chi) = -D^g\tau$ and so equation (3.9) becomes

$$\frac{1}{2} d\tau_g - d\kappa_g \circ J = 0.$$

This is solved by setting $2\kappa_g - i\tau_g = H$, where $H$ is a holomorphic function on the quotient $C$ of $B$ by $K$. Since $D^B_\chi \tau = -\tau^2$ and $D^B_\chi \kappa = -\tau\kappa$, the remaining Einstein-Weyl equations reduce to $\tau\kappa + \frac{1}{2}\langle \chi, *F^B\rangle = 0$ and $\kappa^2 = \frac{1}{6}\,\mathrm{scal}^B$. The first of these is automatic. To solve the second we note that $\mathrm{scal}^B$ can be computed from the scalar curvature of the quotient metric on $C$ using a submersion formula [2], [5]. This gives $\mathrm{scal}^B = \mathrm{scal}^C - 2\tau^2 - 2\kappa^2$ and hence $\mathrm{scal}^C = 2\tau^2 + 8\kappa^2 = 2|2\kappa - i\tau|^2$. If this is zero, then $\tau = \kappa = 0$ and $D^B$ is flat. Otherwise we observe that $\log |H|^2$ is harmonic, and so rescaling the quotient metric by $|H|^2$ gives a metric of constant curvature 1 (*i.e.*, the scalar curvature is 2).

Remarkably, these Einstein-Weyl spaces are also all hyperCR: since $\kappa^2 = \frac{1}{6}\,\mathrm{scal}^B$ and $*D^B\kappa = \frac{1}{2}F^B - d(\tau\chi) = -\frac{1}{2}F^B$, reversing the sign of $\kappa$ (or equivalently, reversing the orientation of $B$) solves the equations of the previous subsection. Thus we have established the following theorem.

**3.10.** THEOREM. — *The three dimensional Einstein-Weyl spaces with geodesic symmetry are either flat with translational symmetry or are given locally by*

$$\begin{aligned} g &= |H|^{-2}(\sigma_1^2 + \sigma_2^2) + \beta^2, \\ \omega &= \frac{i}{2}(H - \overline{H})\beta, \\ d\beta &= \frac{1}{2}(H + \overline{H})|H|^{-2}\sigma_1 \wedge \sigma_2, \end{aligned}$$

*where $\sigma_1^2 + \sigma_2^2$ is the round metric on $S^2$, and $H$ is any nonvanishing holomorphic function on an open subset of $S^2$. The geodesic symmetry $K$ is dual to $\beta$ and the monopoles associated to $K/|K|$ are $\tau = \frac{1}{2}i(H - \overline{H})\mu_g^{-1}$ and $\kappa = \frac{1}{4}(H + \overline{H})\mu_g^{-1}$. These spaces all admit hyperCR structures, with flat connection $D^B + \kappa *1$.*

The equation for $\beta$ can be integrated explicitly. Indeed if $\zeta$ is a holomorphic coordinate such that $\sigma_1 \wedge \sigma_2 = 2i d\zeta \wedge d\bar{\zeta}/(1+\zeta\bar{\zeta})^2$ then one can take

$$\beta = d\psi + \frac{i}{1+\zeta\bar{\zeta}}\Big(\frac{d\zeta}{\zeta H} - \frac{d\bar{\zeta}}{\bar{\zeta}\bar{H}}\Big).$$

Of course, this is not the only possible choice: for instance one can write $d\zeta/(\zeta H) = dF$ with $F$ holomorphic and use

$$\beta = d\psi - i(F - \bar{F})d\Big(\frac{1}{1+\zeta\bar{\zeta}}\Big).$$

Note that $\omega$ is dual to a Killing field of $g$ if and only if $H$ is constant, in which case we obtain the well known Einstein-Weyl structures on the Berger spheres. The Einstein metric on $S^3$ arises when $H$ is real, in which case the connections $D^B \pm \kappa *1$ are both flat: they are the left and right invariant connections. The flat Weyl structure with radial symmetry (which is globally defined on $S^1 \times S^2$) occurs when $H$ is purely imaginary. Gauduchon and Tod [12] prove that these are the only hyperCR structures on compact Einstein-Weyl manifolds.

The fact that the Einstein-Weyl spaces with geodesic symmetry are hyperCR may equally be understood via minitwistor theory. Indeed, any symmetry $K$ (a conformal vector field preserving the Weyl connection) on a 3-dimensional Einstein-Weyl space induces a holomorphic vector field $X$ on the minitwistor space $\mathcal{S}$. If $K$ is nonvanishing, then on each minitwistor line, $X$ will be tangent at two points (since the normal bundle is $\mathcal{O}(2)$) and if the line corresponds to a real point $x$, then these two tangent points in $\mathcal{S}$ will correspond to the two orientations of the geodesic generated by $K_x$. Hence $X$ vanishes at a point of $\mathcal{S}$ if and only if $K$ is tangent along the corresponding geodesic.

Now if $K$ is a geodesic symmetry then $X$ will be tangent to each minitwistor line precisely at the points at which it vanishes, and the zeroset of $X$ will be a divisor (rather than isolated points). This means that $X$ is a section of a line subbundle $\mathcal{H} = [\operatorname{div} X]$ of $T\mathcal{S}$ transverse to the minitwistor lines ($\mathcal{H}$ must be transverse even where $X$ vanishes, because $K$, being real, is not null, and so the points of tangency are simple): the $\kappa$ monopole of $K$ is therefore $\mathcal{H} \otimes K_{\mathcal{S}}^{1/2}$. Now the integral curves of the distribution $\mathcal{H}$ in the neighbourhood $U$ of some real minitwistor line give a holomorphic map from $U$ to $\mathbb{C}P^1$. Viewing this as a meromorphic function (by choosing conjugate points on $\mathbb{C}P^1$) we obtain a divisor $\mathcal{C} - \bar{\mathcal{C}}$, where $\mathcal{C} + \bar{\mathcal{C}}$ is a divisor for $T\mathcal{S}/\mathcal{H}$, because $T\mathcal{S}/\mathcal{H}$ is isomorphic to $T\mathbb{C}P^1$ over each minitwistor line. Since $K_{\mathcal{S}}^{-1} = \mathcal{H} \otimes T\mathcal{S}/\mathcal{H}$ we find that $[\mathcal{C}+\bar{\mathcal{C}}]K_{\mathcal{S}}^{1/2}$ is dual to $\mathcal{H} \otimes K_{\mathcal{S}}^{1/2}$, which explains (twistorially) why the $\kappa$ monopole of the hyperCR structure is simply the negation of the $\kappa$ monopole of the geodesic symmetry.

Another explanation is that the geodesic symmetry preserves the hyperCR congruences. Indeed, we have the following observation.

**3.11.** Proposition. — *Suppose that $B$ is a hyperCR Einstein-Weyl space with flat connection $D^B + \kappa *1$. Then a vector field $K$ preserves the hyperCR congruences $\chi$ (i.e., $\mathcal{L}_K\chi = 0$ for each $\chi$) if and only if it is a geodesic symmetry with twist $\kappa$.*

*Proof.* — Since $\chi$ is a weightless vector field,

$$\mathcal{L}_K\chi = D^B_K\chi - D^B_\chi K + \frac{1}{3}(\operatorname{tr} D^B K)\chi.$$

This vanishes if and only if $D^B_\chi K = \frac{1}{3}(\operatorname{tr} D^B K)\chi - \kappa *(K \wedge \chi)$. Hence $\mathcal{L}_K\chi = 0$ for all of the hyperCR congruences $\chi$ if and only if $D^B K = \frac{1}{3}(\operatorname{tr} D^B K)\operatorname{id} + \kappa *K$. This formula shows that $K$ is a conformal vector field, and that $K/|K|$ is a shear-free geodesic congruence with twist $\kappa$. Also $K$ preserves the flat connection $D^B + \kappa *1$, since it preserves the parallel sections. Finally, note that the twist of $K$ is determined by the conformal structure from the skew part of $D^{|K|}K$, so it is also preserved by $K$. Hence $K$ preserves $D^B$ and is therefore a geodesic symmetry. □

## 4. The Jones and Tod construction.

In [16], Jones and Tod proved that the quotient of a selfdual conformal manifold $M$ by a conformal vector field $K$ is Einstein-Weyl: the twistor lines in the twistor space $Z$ of $M$ project to rational curves with normal bundle $\mathcal{O}(2)$ in the space $\mathcal{S}$ of trajectories of the holomorphic vector field on $Z$ induced by $K$. Furthermore the Einstein-Weyl space comes with a solution of the monopole equation from which $M$ can be recovered: indeed $Z$ is (an open subset of) the total space of the line bundle over $\mathcal{S}$ determined by this monopole. In other words there is a correspondence between selfdual spaces with symmetry and Einstein-Weyl spaces with monopoles. In this section, we explain the differential geometric constructions involved in the Jones and Tod correspondence, and prove that invariant antiselfdual complex structures on $M$ correspond to shear-free geodesic congruences on $B$. These direct methods, although motivated by the twistor approach, also reveal what happens when $M$ is not selfdual.

Therefore we let $M$ be an oriented conformal manifold with a conformal vector field $K$, and (by restricting to an open set if necessary) we assume $K$ is nowhere vanishing. Let $D^{|K|}$ be the constant length gauge of $K$, so that $\langle D^{|K|}K,.\rangle$ is a weightless 2-form. One can compute $D^{|K|}$ in terms of an arbitrary Weyl derivative $D$ by the formula

$$D^{|K|} = D - \frac{\langle DK, K\rangle}{\langle K,K\rangle} = D - \frac{1}{4}\frac{(\operatorname{tr} DK)K}{\langle K,K\rangle} + \frac{1}{2}\frac{(d^D K)(K,.)}{\langle K,K\rangle},$$

where $(d^D K)(X,Y) = \langle D_X K, Y\rangle - \langle D_Y K, X\rangle$.

The key observation for the Jones and Tod construction is that there is a unique Weyl derivative $D^{\mathrm{sd}}$ on $M$ such that $\langle D^+K,.\rangle$ is a weightless *selfdual* 2-form: let $\omega = -(*d^D K)(K,.)/\langle K,K\rangle$ (which is independent of $D$) and define

$$D^{\mathrm{sd}} = D^{|K|} + \frac{1}{2}\omega = D - \frac{1}{4}\frac{(\operatorname{tr} DK)K}{\langle K,K\rangle} + \frac{1}{2}\frac{(d^D K)(K,.) - (*d^D K)(K,.)}{\langle K,K\rangle}.$$

Since $D$ is arbitrary, we may take $D = D^{\mathrm{sd}}$ to get $(D^{\mathrm{sd}}K - *D^{\mathrm{sd}}K)(K,.) = 0$ from which it is immediate that $D^{\mathrm{sd}}K = *D^{\mathrm{sd}}K$ since an antiselfdual 2-form is uniquely determined by its contraction with a nonzero vector field. The Weyl derivative $D^{\mathrm{sd}}$ plays a central role in the proof that $D^B = D^{|K|} + \omega$ is Einstein-Weyl on $B$. Notice that the conformal structure and Weyl derivatives $D^{|K|}$, $D^{\mathrm{sd}}$, $D^B$ do indeed descend to $B$ because $K$ is a Killing field in the constant length gauge and $\omega$ is a basic 1-form. Since the Lie derivative of Weyl derivatives on $L^1$ is given by $\mathcal{L}_K D = \frac{1}{n} d \operatorname{tr} DK + F^D(K,.)$, it follows that $F^{\mathrm{sd}}(K,.) = F^B(K,.) = 0$.

We call $D^B$ the *Jones-Tod* Weyl structure on $B$.

**4.1.** THEOREM (*cf.* [16]). — *Suppose $M$ is an oriented conformal 4-manifold and $K$ a conformal vector field such that $B = M/K$ is a manifold. Let $D^{|K|}$ be the constant length gauge of $K$ and $\omega = -2(*D^{|K|}K)(K,.)/\langle K,K\rangle$. Then the Jones-Tod Weyl structure $D^B = D^{|K|} + \omega$ is Einstein-Weyl on $B$ if and only if $M$ is selfdual.*

*Note that $*D^B|K|^{-1} = -*\omega|K|^{-1}$ is a closed 2-form. Conversely, if $(B,D^B)$ is an Einstein-Weyl 3-manifold and $w \in C^\infty(B,L^{-1})$ is a non-vanishing solution of the monopole equation $d{*}D^B w = 0$ then there is* *a selfdual 4-manifold $M$ with symmetry over $B$ such that $*D^B w$ is the curvature of the connection defined by the horizontal distribution.*

*Proof.* — The monopole equation on $B$ is equivalent, via the definition of $\omega$, to the fact that $D^{\mathrm{sd}}$ lies midway between $D^B$ and $D^{|K|}$. So it remains to show that under this condition, the Einstein-Weyl equation on $B$ is equivalent to the selfduality of $M$. The space of antiselfdual Weyl tensors is isomorphic to $S^2_0(K^\perp)$ via the map sending $W^-$ to $W^-_{.,K}K$, and so it suffices to show that $r^B_0 = 0$ if and only if $W^-_{.,K}K = 0$.

Since $D^{\mathrm{sd}}$ is basic, as a Weyl connection on $TM$, $0 = (\mathcal{L}_K D^{\mathrm{sd}})_X = R^{\mathrm{sd}}_{K,X} + D^{\mathrm{sd}}_X D^{\mathrm{sd}} K$. Therefore

$$D^{\mathrm{sd}}_X D^{\mathrm{sd}} K = W_{X,K} + r^{\mathrm{sd}}(K) \vartriangle X - r^{\mathrm{sd}}(X) \vartriangle K.$$

If we now take the antiselfdual part of this equation, contract with $K$ and $Y$, and take $\langle X,K\rangle = \langle Y,K\rangle = 0$, then we obtain

$$2\langle W^-_{X,K}K, Y\rangle + r^{\mathrm{sd}}(X,Y)\langle K,K\rangle + r^{\mathrm{sd}}(K,K)\langle X,Y\rangle + *\big(K \wedge r^{\mathrm{sd}}(K) \wedge X \wedge Y\big) = 0.$$

Symmetrising in $X, Y$, we see that $W^- = 0$ if and only if the horizontal part of the symmetric Ricci endomorphism of $D^{\mathrm{sd}}$ is a multiple of the identity. The first submersion formula [2] relates the Ricci curvature of $D^{|K|}$ on $B$ to the horizontal Ricci curvature of $D^{|K|}$ on $M$. If we combine this with the fact that $D^{\mathrm{sd}} = D^{|K|} + \frac{1}{2}\omega$ and $D^B = D^{|K|} + \omega$, then we find that

$$\operatorname{sym}\operatorname{Ric}^{D^B}_B(X,Y) = \operatorname{sym}\operatorname{Ric}^{D^{\mathrm{sd}}}_M(X,Y) + 2\langle D^{|K|}_X K, D^{|K|}_Y K\rangle + \frac{1}{2}\omega(X)\omega(Y) + \mu\langle X,Y\rangle$$

for some section $\mu$ of $L^{-2}$. Since $D^{|K|}_K K = 0$, $\omega$ vanishes on the plane spanned by $D^{|K|}K$, and so, by comparing the lengths of $\omega$ and $D^{|K|}K$, we verify that the trace-free part of $2\langle D^{|K|}_X K, D^{|K|}_Y K\rangle + \frac{1}{2}\omega(X)\omega(Y)$ vanishes. Hence $W^- = 0$ on $M$ if and only if $D^B$ is Einstein-Weyl on $B$. □

The inverse construction of $M$ from $B$ can be carried out explicitly by writing $*_B D^B w = dA$ on $U \subset B$, so that the real line bundle $M$ is locally isomorphic to $U \times \mathbb{R}$ with connection $dt + A$, where $t$ is the fibre coordinate. Then the conformal structure $\boldsymbol{c}_M = \pi^* \boldsymbol{c}_B + w^{-2}(dt + A)^2$ is selfdual and $K = \partial/\partial t$ is a unit Killing field of the representative metric $g_M = \pi^* w^2 c_B + (dt+A)^2$. Note that $w = \pm|K|^{-1}$ and that the orientations on $M$ and $B$ are related by $*(\xi\wedge\alpha) = (*_B\alpha)w|K|$ where $\alpha$ is any 1-form on $B$ and $\xi = K|K|^{-1}$. This ensures that if $D^B = D^{|K|} + \omega$, then the equation $-(*_B\omega)w = *_B D^B w = dA$ is equivalent to $*(\xi\wedge\omega)|K|^{-1} = -d(dt+A)$ and hence $\omega = -\iota_K(*d^D K)/|K|^2$ as above.

Jones and Tod also observe that any other solution $(w_1, A_1)$ of the monopole equation on $B$ corresponds to a selfdual Maxwell field on $M$ with potential $\widetilde{A}_1 = A_1 - (w_1/w)(dt+A)$. Indeed, since $(dt+A) = |K|^{-1}\xi$, one readily verifies that

$$d\widetilde{A}_1 = \big(w^{-1}|K|^{-1}\xi\wedge D^B w_1 + dA_1\big) - \frac{w_1}{w}\big(w^{-1}|K|^{-1}\xi\wedge D^B w + dA\big),$$

which is selfdual by the monopole equations for $w$ and $w_1$, together with the orientation conventions above.

We now want to explain the relationship between invariant complex structures on $M$ and shear-free geodesic congruences on $B$. That these should be related is again clear from the twistor point of view: indeed if $\mathcal{D}$ is an invariant divisor on $Z$, then it descends to a divisor $\mathcal{C}$ in $\mathcal{S}$, which in turn defines, at least locally, a shear-free geodesic congruence. The line bundles $[\mathcal{D}+\overline{\mathcal{D}}]K_Z^{1/2}$ and $[\mathcal{D}-\overline{\mathcal{D}}]$ are the pullbacks of $[\mathcal{C}+\overline{\mathcal{C}}]K_{\mathcal{S}}^{1/2}$ and $[\mathcal{C}-\overline{\mathcal{C}}]$ and so we expect the Faraday and Ricci forms on $M$ to be related to the twist and divergence of the congruence on $B$. In order to see all this in detail, and without the assumption of selfduality, we carry out the constructions directly.

Suppose that $J$ is an antiselfdual complex structure on $M$ with $\mathcal{L}_K J = 0$, so that $K$ is a holomorphic conformal vector field. If $D$ is the Kähler-Weyl connection, then $DK = -\kappa_0\,\mathrm{id} + \frac{1}{2}\tau_0 J + \frac{1}{2}(d^D K)^+$ where $(d^D K)^+$ is a selfdual 2-form and $\kappa_0,\tau_0$ are functions.

Now let $\kappa = \kappa_0|K|^{-1}$, $\tau = \tau_0|K|^{-1}$, $\xi = K|K|^{-1}$, $\chi = J\xi$. Since $d^D K = \tau_0 J + (d^D K)^+$, it follows that

$$d^D K(K,.)|K|^{-2} = \tau\chi + (d^D K)^+(K,.)|K|^{-2},$$

and

$$(*d^D K)(K,.)|K|^{-2} = -\tau\chi + (d^D K)^+(K,.)|K|^{-2}.$$

Therefore $D^{\mathrm{sd}} = D + \kappa\xi + \tau\chi$ and $(d^D K)^+(K,.)|K|^{-2} = \tau\chi - \omega$.

**4.2.** Theorem. — *Let $M$ be an oriented conformal 4-manifold with conformal vector field $K$ and suppose that $J$ is an invariant antiselfdual almost complex structure on $M$. Then $J$ is integrable if and only if $\chi = J\xi = JK/|K|$ is a shear-free geodesic congruence on the Jones-Tod Weyl space $B$. Furthermore, the Kähler-Weyl structure associated to $J$ is given by $D = D^{\mathrm{sd}} - \kappa\xi - \tau\chi$ where $D^B\chi = \tau(\mathrm{id} - \chi\otimes\chi) + \kappa *\chi$ on $B$.*

*Proof.* — Clearly $\chi$ is invariant and horizontal, hence basic. Let $\tau, \kappa$ be invariant sections of $L^{-1}$ and set $D = D^{\mathrm{sd}} - \kappa\xi - \tau\chi$. If $J$ is integrable then we have seen above that the Kähler-Weyl connection is of this form. Therefore it suffices to prove that $DJ = 0$ if and only if $D^B\chi = \tau(\mathrm{id} - \chi\otimes\chi) + \kappa *\chi$ on $B$. Since $J = \xi\wedge\chi - *(\xi\wedge\chi)$ this is a straightforward computation. Let $X$ be any vector field on $M$. Then

$$D_X J = D_X\xi\wedge\chi + \xi\wedge D_X\chi - *(D_X\xi\wedge\chi - \xi\wedge D_X\chi).$$

Now $D = D^{|K|} + \frac{1}{2}\omega - \kappa\xi - \tau\chi$ and so, since $D^{|K|}\xi = -\frac{1}{2}*\xi\wedge\omega$, we have

$$D_X\xi = -\frac{1}{2}*(X\wedge\xi\wedge\omega) - \frac{1}{2}\langle\xi, X\rangle\omega - \kappa\big(X - \langle\xi, X\rangle\xi\big) + \tau\langle\xi, X\rangle\chi.$$

Also $D = D^B - \frac{1}{2}\omega - \kappa\xi - \tau\chi$ and so

$$D_X\chi = D^B_X\chi - \frac{1}{2}\omega(\chi)X + \langle\chi, X\rangle\omega - \tau\big(X - \langle\chi, X\rangle\chi\big) + \kappa\langle\chi, X\rangle\xi.$$

Therefore

$$\begin{aligned} D_X\xi\wedge\chi &= \frac{1}{2}\langle\chi, X\rangle *(\xi\wedge\omega) - \frac{1}{2}\omega(\chi) *(\xi\wedge X) \\ &\qquad - \kappa\big(X - \langle\xi, X\rangle\xi\big)\wedge\chi - \frac{1}{2}\langle\xi, X\rangle\omega\wedge\chi \\ \xi\wedge D_X\chi &= \xi\wedge D^B_X\chi - \frac{1}{2}\omega(\chi)\xi\wedge X + \frac{1}{2}\langle\chi, X\rangle\xi\wedge\omega \\ &\qquad - \tau\xi\wedge\big(X - \langle\chi, X\rangle\chi\big) \end{aligned}$$

and so

$$\begin{aligned} \xi\wedge D_X\chi - *(D_X\xi\wedge\chi) &= \xi\wedge D^B_X\chi - \tau\xi\wedge\big(X - \langle\chi, X\rangle\chi\big) \\ &\qquad + \kappa *\big((X - \langle\xi, X\rangle\xi)\wedge\chi\big) + \frac{1}{2}\langle\xi, X\rangle *(\omega\wedge\chi). \end{aligned}$$

Since the right hand side is vertical, it follows that $D_X J = 0$ if and only if

$$\begin{aligned} D^B_X\chi - \langle D^B_X\chi, \xi\rangle &= \tau\big(X - \langle\chi, X\rangle\chi - \langle\xi, X\rangle\xi\big) + \kappa\,\iota_X *_B\chi \\ &\qquad - \frac{1}{2}\langle\xi, X\rangle *(\xi\wedge\omega\wedge\chi). \end{aligned}$$

If $X$ is parallel to $\xi$, this holds automatically since $\mathcal{L}_K\chi = 0$, and so by considering $X\perp\xi$ we obtain the theorem. □

When $M$ is selfdual, this theorem unifies (the local aspects of) LeBrun's treatment of scalar-flat Kähler metrics with symmetry [19], [20] and the hypercomplex structures with symmetry studied by Chave, Tod and Valent [7] and Gauduchon and Tod [12]. To see this, note that since $D$ is canonically determined by $\Omega_J$, and $\mathcal{L}_K\Omega_J = 0$, it follows that $\mathcal{L}_K D = 0$ on $L^1$, which means that $d\kappa_0 = F^D(K,.)$. Since $K$ is a conformal vector field, it follows that $\mathcal{L}_K D = 0$ on $TM$ as well, which gives:

$$\frac{1}{2}d\tau_0(X)J + \frac{1}{2}D_X(d^D K)^+ + W_{K,X} - r^D(K) \vartriangle X + r^D(X) \vartriangle K = 0.$$

If we contract this with $J$, we obtain $d\tau_0 = 2r_0^D(JK,.) = -i\rho^D(K,.)$. Thus $\rho^D$ and $F^D$ are the selfdual Maxwell fields associated to the monopoles $i\tau$ and $\kappa$ respectively. Since they are selfdual, it follows that $d\kappa_0 = 0$ if and only if $M, J$ is locally scalar-flat Kähler, while $d\tau_0 = 0$ if and only if $M, J$ is locally hypercomplex.

Now suppose that $B$ is Einstein-Weyl and that $w$ is any nonvanishing monopole, and let $M$ be the corresponding selfdual conformal 4-manifold. Then *each* shear-free geodesic congruence $\chi$ induces on $M$ an invariant antiselfdual complex structure $J$. On the other hand if we fix $\chi$, then, as we have seen, its divergence and twist, $\tau$ and $\kappa$, are monopoles on $B$. Using these we can characterise special cases of the construction as follows.

(i) $(M, J)$ is locally scalar-flat Kähler if and only if $\kappa = aw$ for some constant $a$, and if $a$ is nonzero, we may assume $a = 1$, by normalising $w$.

- If $\kappa = 0$ then $M$ is locally scalar-flat Kähler and $K$ is a holomorphic Killing field. If $\tau = bw$, then $M$ is locally hyperKähler [19], [20].
- If $\kappa = w$ then $M$ is locally scalar-flat Kähler and $K$ is a holomorphic homothetic vector field.

(ii) $(M, J)$ is locally hypercomplex if and only if $\tau = bw$ for some constant $b$, and if $b$ is nonzero, we may assume $b = 1$, by normalising $w$.

- If $\tau = 0$ then $M$ is locally hypercomplex and $K$ is a triholomorphic vector field. If $\kappa = aw$, then $M$ is locally hyperKähler (see [7], [12]).
- If $\tau = w$ then $M$ is locally hypercomplex and $K$ is a hypercomplex vector field.

Here we say a conformal vector field on a hypercomplex 4-manifold is *hypercomplex* if and only if $\mathcal{L}_K D = 0$ where $D$ is the Obata connection. It follows that for each of the hypercomplex structures $I$, $\mathcal{L}_K I$ is a $D$-parallel antiselfdual endomorphism anticommuting with $I$. The map $I \mapsto \mathcal{L}_K I \perp I$

is therefore given by $I \mapsto [cJ, I]$ for one of the hypercomplex structures $J$ and a real constant $c$. Consequently $K$ is holomorphic with respect to $\pm J$, and is *triholomorphic* if and only if $c = 0$.

The twistorial interpretation of the above special cases is as follows. Firstly, if $\kappa = 0$ on $B$ then the corresponding line bundle on $\mathcal{S}$ is trivial; hence so is its pullback to $Z$. On the other hand, if $\kappa = w$ then the line bundle on $\mathcal{S}$ is nontrivial, but we are pulling it back to (an open subset of) its total space. Such a pullback has a tautological section, and hence is trivial away from the zero section. The story for $\tau$ is similar.

We now combine these observations with the mini-Kerr theorem.

**4.3.** THEOREM. — *Let $B$ be an arbitrary three dimensional Einstein-Weyl space.*

(i) *$B$ may be obtained (locally) as the quotient of a scalar-flat Kähler 4-manifold by a holomorphic homothetic vector field.*

(ii) *It may also be obtained as the quotient of a hypercomplex 4-manifold by a hypercomplex vector field.*

(iii) *$B$ is locally the quotient of a hyperKähler 4-manifold by a holomorphic homothetic vector field if and only if it admits a shear-free geodesic congruence with linearly dependent divergence and twist.*

*Proof.* — By the mini-Kerr theorem $B$ admits a shear-free geodesic congruence. The divergence $\tau$ and twist $\kappa$ are monopoles on $B$, which may be used to construct the desired hypercomplex and scalar-flat Kähler spaces wherever they are nonvanishing. The hyperKähler case was characterised above by the constancy of $\tau_0$ and $\kappa_0$. On $B$, this implies that $\tau$ and $\kappa$ are linearly dependent, *i.e.*, $c_1\tau + c_2\kappa = 0$ for constants $c_1$ and $c_2$. Conversely given an Einstein-Weyl space with a shear-free geodesic congruence $\chi$ whose divergence and twist satisfy this condition, any nonvanishing monopole $w$ with $\kappa = aw$ and $\tau = bw$ gives rise to a hyperKähler metric (and this $w$ is unique up to a constant multiple unless $\tau = \kappa = 0$). □

Maciej Dunajski and Paul Tod [8] have recently obtained a related description of hyperKähler metrics with homothetic vector fields by reducing Plebanski's equations.

The following diagram (Fig. 1) conveniently summarises the various Weyl derivatives involved in the constructions of this section, together with the 1-forms translating between them.

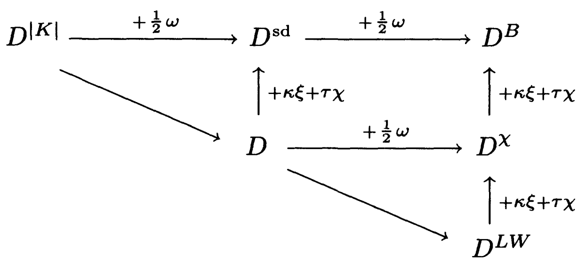


*Figure 1*

The Weyl derivatives in the right hand column are so labelled because on $B$ we have $D^{LW} \xrightarrow{+\tau\chi} D^\chi \xrightarrow{+\tau\chi} D^B$, where $D^B$ is Einstein-Weyl, $D^\chi$ is the Weyl derivative canonically associated to the congruence $\chi$, and, *in the case that* $\kappa = 0$, $D^{LW}$ is the LeBrun-Ward gauge. The central role played by $D^{\mathrm{sd}}$ in these constructions explains the frequent occurrence of the Ansatz $g = Vg_B + V^{-1}(dt + A)^2$ for selfdual metrics with symmetry. In particular, if $g_B$ is the LeBrun-Ward gauge of a LeBrun-Ward geometry and $V$ is a monopole in this gauge, then $g$ is a scalar-flat Kähler metric.

## 5. Selfdual Einstein 4-manifolds with symmetry.

In this section we combine results of Tod [31] and Pedersen and Tod [27] to show that the constructions of the previous section cover essentially all selfdual Einstein metrics with symmetry.

**5.1.** Proposition (*cf.* [27]). — *Let $g$ be a four dimensional Einstein metric with a conformal vector field $K$. Then one of the following must hold:*

(i) *$K$ is a Killing field of $g$;*

(ii) *$g$ is Ricci-flat and $K$ is a homothetic vector field* (*i.e.*, $\mathcal{L}_K D^g = 0$);

(iii) *$g$ is conformally flat.*

Now suppose $g$ is a selfdual Einstein metric with nonzero scalar curvature and a conformal vector field $K$. Then, except in the conformally flat case, $K$ is a Killing field of $g$ and so we may apply the following.

**5.2.** Theorem (*cf.* [31]). — *Let $g$ be a selfdual Einstein metric with nonzero scalar curvature and $K$ a Killing field of $g$. Then the antiselfdual* *part of $D^g K$ is nonzero, and is a pointwise multiple of an integrable complex structure $J$. The corresponding Kähler-Weyl structure is Kähler, and $K$ is also a Killing field for the Kähler metric.*

If, on the other hand, $\mathrm{scal}^g$ is zero, then $g$ itself is (locally) a hyperKähler metric and, unless $g$ is conformally flat, $\mathcal{L}_K D^g = 0$, and so $K$ is a hypercomplex vector field. In the conformally flat case, $K$ may not be a homothety of $g$, but it is at least a homothety with respect to *some* compatible flat metric. Thus, in any case, the conformal vector field $K$ is holomorphic with respect to some Kähler structure on $M$.

We end this section by noting that in the case of selfdual Einstein metrics with Killing fields, Tod's work [31] shows how to recover the Einstein metric from the LeBrun-Ward geometry. More precisely, if $M$ is a selfdual Einstein 4-manifold with a Killing field, and $B$ is the LeBrun-Ward quotient of the corresponding scalar-flat Kähler metric, then either $B$ is flat, or the monopole defining $M$ is of the form

$$\mathrm{w} = \Big(a\Big(1 - \frac{1}{2}zu_z\Big) + \frac{1}{2}bu_z\Big)\mu_{LW}^{-1},$$

where $u(x, y, z)$ is the solution of the $\mathrm{SU}(\infty)$ Toda field equation, and $a, b \in \mathbb{R}$ are not both zero. Conversely, for any LeBrun-Ward geometry (given by $u$), the section $\big(a(1 - \frac{1}{2}zu_z) + \frac{1}{2}bu_z\big)\mu_{LW}^{-1}$ of $L^{-1}$ is a monopole for any $a, b \in \mathbb{R}$, and if $g_K$ is the corresponding Kähler metric, then $(az - b)^{-2}g_K$ is Einstein with scalar curvature $-12a$. When $a = 0$, we reobtain the case of hyperKähler metrics with Killing fields, while if $a \neq 0$, one can set $b = 0$ by translating the $z$ coordinate (although $u$ will be a different function of the new $z$ coordinate).

## 6. Einstein-Weyl structures from $\mathbb{R}^4$.

Our aim in the remaining sections is to unify and extend many of the examples of Kähler-Weyl structures with symmetry studied up to the present, using the framework developed in Sections 2–4. We discuss both the simplest and most well known cases and also more complicated examples which we believe are new. We begin with $\mathbb{R}^4$.

A conformally flat 4-manifold is both selfdual and antiselfdual, so when we apply the Jones and Tod construction we have the freedom to reverse the orientation. Consequently, not only is $D^B = D^{|K|} + \omega$ Einstein-Weyl, but so is $\widetilde{D}^B = D^{|K|} - \omega$. Therefore

$$0 = \mathrm{sym}_0(D^B\omega + \omega \otimes \omega) = \mathrm{sym}_0\, D^{|K|}\omega = \mathrm{sym}_0(\widetilde{D}^B\omega - \omega \otimes \omega).$$

Since $|K|^{-1}$ is a monopole, $g = |K|^{-2}c_B$ (the gauge in which the monopole is constant) is a *Gauduchon metric* in the sense that $\omega$ is divergence-free with respect to $D^g = D^{|K|}$. It follows that $\omega$ is dual to a Killing field of $g$. Furthermore, the converse is also true: that is, if $D^B = D^g + \omega$ is Einstein-Weyl and $\omega$ is dual to a Killing field of $g$, then $\widetilde{D}^B = D^g - \omega$ is also Einstein-Weyl, and therefore the 4-manifold $M$ given by the monopole $\mu_g^{-1}$ is both selfdual and antiselfdual, hence conformally flat.

The condition that an Einstein-Weyl space admits a compatible metric $g$ such that $D = D^g + \omega$ with $\omega$ dual to a Killing field of $g$ is of particular importance because it always holds in the compact case: on any compact Weyl space there is a Gauduchon metric $g$ unique up to homothety [9], and $g$ has this additional property when the Weyl structure is Einstein-Weyl [28]. Consequently, the local quotients of conformally flat 4-manifolds exhaust the possible local geometries of compact Einstein-Weyl 3-manifolds. These geometries were obtained in [26] as local quotients of $S^4$. Now any conformal vector field $K$ on $S^4$ has a zero and is a homothetic vector field with respect to the flat metric on $\mathbb{R}^4$ given by stereographic projection away from any such zero. Hence we can view these Einstein-Weyl geometries as local quotients of the flat metric on $\mathbb{R}^4$ by a homothetic vector field and use the constructions of Section 4 to understand some of their properties.

Suppose first that $K$ vanishes on $\mathbb{R}^4$ and let the origin be such a zero. Then $K$ generates one parameter group of *linear* conformal transformations of the flat metric $g$. This is case 1 of [26] and we may choose coordinates such that

$$g = dr^2 + \frac{1}{4}r^2\big(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2\big),$$
$$K = ar\frac{\partial}{\partial r} - (b+c)\frac{\partial}{\partial\phi} - (b-c)\frac{\partial}{\partial\psi}.$$

Note that $K$ is also a homothety of the flat metric $\tilde{g} = r^{-4}g$ obtained from $g$ by the orientation reversing conformal transformation $r \mapsto \tilde{r} = 1/r$. With a fixed orientation,

$$D^g K = \quad a\ \mathrm{id} + \frac{1}{2}(b+c)J^+ + \frac{1}{2}(b-c)J^-,$$
$$D^{\tilde{g}} K = -a\ \mathrm{id} + \frac{1}{2}(b-c)\widetilde{J}^+ + \frac{1}{2}(b+c)\widetilde{J}^-,$$

where $J^\pm$ are $D^g$-parallel complex structures on $\mathbb{R}^4$, one selfdual, the other antiselfdual, and, similarly, $\widetilde{J}^\pm$ are $D^{\tilde{g}}$-parallel. The Weyl structures

$D^{|K|} \pm \omega$ are Einstein-Weyl on the quotient $B$, where

$$\omega = \frac{(b+c)g(J^+K,.) - (b-c)g(J^-K,.)}{g(K,K)} = \frac{(b-c)\tilde{g}(\widetilde{J}^+K,.) - (b+c)\tilde{g}(\widetilde{J}^-K,.)}{\tilde{g}(K,K)}.$$

Without loss of generality, we consider only $D^B = D^{|K|} + \omega$. By Theorem 4.2, $J^-K$ and $\widetilde{J}^-K$ generate shear-free geodesic congruences with $\tau^- = (b+c)|K|^{-1}$, $\tilde{\tau}^- = (b-c)|K|^{-1}$ and $\kappa^- = a|K|^{-1} = -\tilde{\kappa}^-$.

If $b^2 = c^2$, then $K$ is triholomorphic, and so the quotient geometry is hyperCR: it is the Berger sphere family. If we take $b = c$ then $J^-$ is no longer unique, and the hyperCR structure is given by the congruences associated to $JK$, where $J$ ranges over the parallel antiselfdual complex structures of $g$; $\widetilde{J}^-K$, by contrast, is the geodesic symmetry $\partial/\partial\phi$ of $B$. In addition, the antiselfdual rotations all commute with $K$, so $B$ has a four dimensional symmetry group, locally isomorphic to $S^1 \times S^3$.

If $bc = 0$, then although $K$ is not a Killing field on $\mathbb{R}^4$ unless $a = 0$, it is Killing with respect to the product metric on $S^2 \times \mathcal{H}^2$ which is scalar flat Kähler (where the hyperbolic metric on $\mathcal{H}^2$ has equal and opposite curvature to the round metric on $S^2$) and conformal to $\mathbb{R}^4 \setminus \mathbb{R}$. Hence these quotients are Toda.

If $a = 0$, then $K$ is a Killing field, and so the (local) quotient geometry is also Toda, simply because it is the quotient of a flat metric by a Killing field.

If $b^2 = c^2$ and $bc = 0$ then $b = c = 0$ and the quotient is the round 3-sphere, while if $a = 0$ and $bc = 0$ it turns out to be the hyperbolic metric. If $a = 0$ and $b^2 = c^2$, the quotient geometry is the flat Weyl space: the hyperCR congruences become the translational symmetries, and (for $b = c$) $\widetilde{J}^-K$ is the radial symmetry.

We now briefly consider the case that $K$ does not vanish on $\mathbb{R}^4$ (and so is not linear with respect to any choice of origin). This is case 2 of [26], and we may choose a flat metric $g$ with respect to which $K$ is a transrotation. Since $K$ is a Killing field, the quotient Einstein-Weyl space is Toda. For $b = 0$, it is flat, while for $c = 0$ we obtain $\mathcal{H}^3$.

## 7. HyperKähler metrics with triholomorphic Killing fields.

If $M$ is a hyperKähler 4-manifold and $K$ is a triholomorphic Killing field, then $\tau$ and $\kappa$ both vanish, so the corresponding Einstein-Weyl space is flat and the congruence consists of parallel straight lines. HyperKähler 4-manifolds with triholomorphic Killing fields therefore correspond to nonvanishing solutions of the Laplace equation on an open subset of $\mathbb{R}^3$, or some discrete quotient. This is the Gibbons-Hawking Ansatz for selfdual Euclidean vacua [13].

In [33], Ward used this Ansatz to generate new Toda Einstein-Weyl spaces from axially symmetric harmonic functions. The idea is beautifully simple: since the harmonic function is preserved by a Killing field on $\mathbb{R}^3$, the Gibbons-Hawking metric admits a two dimensional family of commuting Killing fields; one of these is triholomorphic, but the others need not be, and so they have other Toda Einstein-Weyl spaces with symmetry as quotients.

Let us carry out this procedure explicitly. In cylindrical polar coordinates $(\eta, \rho, \phi)$, the flat metric is $d\eta^2 + d\rho^2 + \rho^2 d\phi^2$ and the generator of the axial symmetry is $\partial/\partial\phi$. An invariant monopole (in the gauge determined by the flat metric) is a function $W(\rho, \eta)$ satisfying $\rho^{-1}(\rho W_\rho)_\rho + W_{\eta\eta} = 0$. Note that if $W$ is a solution of this equation, then so is $W_\eta$, and $W_\eta$ determines $W$ up to the addition of $C_1 \log(C_2\rho)$ for some $C_1, C_2 \in \mathbb{R}$. This provides a way of integrating the equation $d{*}dW = 0$ to give ${*}dW = dA$: if we write $W = V_\eta$, then we can take $A = \rho V_\rho d\phi$. This choice of integral determines the lift of $\partial/\partial\phi$ to the 4-manifold. The hyperKähler metric is

$$g = V_\eta(d\eta^2 + d\rho^2 + \rho^2 d\phi^2) + V_\eta^{-1}(d\psi + \rho V_\rho d\phi)^2.$$

In order to take the quotient by $\partial/\partial\phi$, we rediagonalise:

$$g = V_\eta\Big(d\rho^2 + d\eta^2 + \frac{1}{V_\eta^2 + V_\rho^2}\, d\psi^2\Big) + \frac{\rho^2(V_\eta^2 + V_\rho^2)}{V_\eta}\Big(d\phi + \frac{V_\rho}{\rho(V_\eta^2 + V_\rho^2)}\, d\psi\Big)^2.$$

We now recall that the hyperKähler metric lies midway between the constant length gauge of $\partial/\partial\phi$ and the LeBrun-Ward gauge of the quotient.

Consequently we find that $D^B = D^{LW} + \omega$ where

$$\begin{aligned} g_{LW} &= \rho^2(V_\eta^2 + V_\rho^2)(d\rho^2 + d\eta^2) + \rho^2 d\psi^2 \\ &= \rho^2(dV^2 + d\psi^2) + (\rho V_\rho d\eta - \rho V_\eta d\rho)^2, \\ \omega &= \frac{2V_\eta}{\rho^2(V_\eta^2 + V_\rho^2)}(\rho V_\rho d\eta - \rho V_\eta d\rho). \end{aligned}$$

Note that $d(\rho V_\rho d\eta - \rho V_\eta d\rho) = 0$. This can be integrated by writing $V = U_\eta$, with $U(\rho, \eta)$ harmonic. Then $z = \rho U_\rho$ parameterises the hypersurfaces orthogonal to the shear-free twist-free congruence, and isothermal coordinates on these hypersurfaces are given by $x = U_\eta$, $y = \psi$. Hence, although the Einstein-Weyl space is completely explicit, the solution $e^u = \rho^2$ of the SU($\infty$) Toda field equation is only given implicitly. Nevertheless, we *have* found the congruence, the isothermal coordinates and the monopole $u_z \mu_{LW}^{-1}$.

The symmetry $\partial/\partial\psi$, like the axial symmetry $\partial/\partial\phi$ on $\mathbb{R}^3$, generates a congruence which is divergence-free and twist-free, although it is not geodesic. For this reason it is natural to say that Ward's spaces are Einstein-Weyl *with an axial symmetry.* They are studied in more detail in [4].

Ward's construction can be considerably generalised. First of all, one can obtain new Toda Einstein-Weyl spaces by considering harmonic functions invariant under other Killing fields. The general Killing field on $\mathbb{R}^3$ may be taken, in suitably chosen cylindrical coordinates, to be of the form $b\partial/\partial\phi + c\partial/\partial\eta$ for $b, c \in \mathbb{R}$. By introducing new coordinates $\zeta = (b\eta - c\phi)/\sqrt{b^2+c^2}$ and $\theta = (b\phi + c\eta)/\sqrt{b^2+c^2}$, so that the Killing field is a multiple of $\partial/\partial\theta$, one can carry out the same procedure as before to obtain the following Toda Einstein-Weyl spaces:

$$\begin{aligned} g_{LW} &= G(\rho,\zeta)\big(d\rho^2 + F(\rho)d\zeta^2\big) + \rho^2 F(\rho)^{-1}\beta^2 \\ &= \rho^2\Big(dV^2 + \frac{1}{b^2+c^2}\Big(c\big[\rho V_\rho d\zeta - F(\rho)^{-1}\rho V_\zeta d\rho\big] + b\, d\psi\Big)^2\Big) \\ &\qquad + \frac{1}{b^2+c^2}\Big(b\big[\rho V_\rho d\zeta - F(\rho)^{-1}\rho V_\zeta d\rho\big] - c d\psi\Big)^2, \\ \omega &= \frac{2bV_\zeta}{(b^2+c^2)G(\rho,\zeta)}\Big(b\big[\rho V_\rho d\zeta - F(\rho)^{-1}\rho V_\zeta d\rho\big] - c d\psi\Big), \end{aligned}$$

where

$$F(\rho) = \frac{(b^2+c^2)\rho^2}{b^2\rho^2+c^2}, \quad G(\rho,\zeta) = \frac{(b^2\rho^2+c^2)V_\zeta^2 + (b^2+c^2)\rho^2 V_\rho^2}{b^2+c^2},$$

$$\beta = d\psi - \frac{bc(1-\rho^2)}{b^2\rho^2+c^2}\big[\rho V_\rho d\eta - F(\rho)^{-1}\rho V_\eta d\rho\big].$$

Note that the symmetry $\partial/\partial\psi$ is twist-free if and only if $bc = 0$. When $b = 0$, the Toda Einstein-Weyl space is just $\mathbb{R}^3$ (the only Einstein-Weyl space with a parallel symmetry), while $c = 0$ is Ward's case.

A further generalisation of this procedure is obtained by observing that the flat Weyl structure on $\mathbb{R}^3$ is preserved not just by Killing fields, but by homothetic vector fields. Now, for a section $w$ of $L^{-1}$, invariance no longer means that the function $w\mu_{\mathbb{R}^3}$ is constant along the flow of the homothetic vector field, since the length scale $\mu_{\mathbb{R}^3}$ is not invariant. Hence it is better to work in a gauge in which the homothetic vector field is Killing. To do this we may choose spherical polar coordinates $(r,\theta,\phi)$ such that the flat Weyl structure on $\mathbb{R}^3$ is

$$g_0 = r^{-2}dr^2 + d\theta^2 + \sin^2\theta\, d\phi^2, \quad \omega_0 = r^{-1}dr$$

and the homothetic vector field is a linear combination of $r\partial/\partial r$ and $\partial/\partial\phi$. For simplicity, we shall only consider here the case of a pure dilation $X = r\partial/\partial r$. If $w = W\mu_0^{-1}$ is an invariant monopole (where $\mu_0$ is the length scale of $g_0$) then $W_r = 0$ and $W(\theta,\phi)$ is a harmonic function on $S^2$. We write $g_{S^2} = \sigma_1^2 + \sigma_2^2$ and $W = \frac{1}{2}(h+\bar{h})$ with $h$ holomorphic on an open subset of $S^2$. Then the hyperKähler metric is

$$g = \frac{r(h+\bar{h})}{2|h|^2}\big(|h|^2(\sigma_1^2+\sigma_2^2)+\beta^2\big) + \frac{2|h|^2}{(h+\bar{h})r}\big(dr + i(h+\bar{h})r\beta\big)^2,$$

where $\beta$ is a 1-form on $S^2$ with $d\beta = \frac{1}{2}(h+\bar{h})\sigma_1\wedge\sigma_2$. One easily verifies that the quotient space is the Einstein-Weyl space with geodesic symmetry given by the holomorphic function $H = 1/h$.

The computation for the general homothetic vector field is more complicated, but one obtains Gibbons-Hawking metrics admitting holomorphic conformal vector fields which are neither triholomorphic or Killing, and therefore, as quotients, explicit examples of Einstein-Weyl spaces (with symmetry) which are neither hyperCR, nor Toda, yet they admit a shear-free geodesic congruence with linearly dependent divergence and twist.

## 8. Congruences and monopoles on $\mathcal{H}^3$, $\mathbb{R}^3$ and $S^3$.

An important special case of the theory presented in this paper is the case of monopoles on spaces of constant curvature. Since each shear-free geodesic congruence on these spaces induces a complex structure on the selfdual space associated to any monopole, it is interesting to find such congruences.

The twist-free case has been considered by Tod [29]. In this case we have a LeBrun-Ward space of constant curvature, given by a solution $u$ of the Toda field equation with $u_z dz$ exact. This happens precisely when $u(x,y,z) = v(x,y) + w(z)$. The solutions, up to changes of isothermal coordinates, are given by

$$e^u = \frac{4(az^2+bz+c)}{(1+a(x^2+y^2))^2}$$

where $a, b, c$ are constants constrained by positivity. As shown in [29], there are essentially six cases: three on hyperbolic space ($b^2 - 4ac > 0$), two in flat space ($b^2 - 4ac = 0$), and one on the sphere ($b^2 - 4ac < 0$). One of the congruences in each case is a radial congruence, orthogonal to distance spheres. The other two types of congruences on hyperbolic space are orthogonal to horospheres and hyperbolic discs respectively, while the other type of congruence on flat space is translational. Only the radial congruences have singularities, and in the flat case, even the radial congruence is globally defined on $S^1 \times S^2$. We illustrate the congruences in the diagrams of Figure 2.

The congruences on hyperbolic space $\mathcal{H}^3$ have been used by LeBrun (see [19], [20]) to construct selfdual conformal structures on complex surfaces. The first type of congruence gives scalar-flat Kähler metrics on blow ups of line bundles over $\mathbb{C}P^1$. The second type gives asymptotically Euclidean scalar-flat Kähler metrics on blow-ups of $\mathbb{C}^2$ and hence selfdual conformal structures on $k\mathbb{C}P^2$ and closed Kähler-Weyl structures on blow-ups of Hopf surfaces. The final type of congruence descends to quotients by discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and leads to scalar-flat Kähler metrics on ruled surfaces of genus$\geq 2$.

If we look instead for hyperCR structures (*i.e.*, divergence-free congruences), we have, in addition to the translational congruences on $\mathbb{R}^3$, two such structures on $S^3$: the left and right invariant congruences, but this exhausts the examples on spaces of constant curvature. Of course

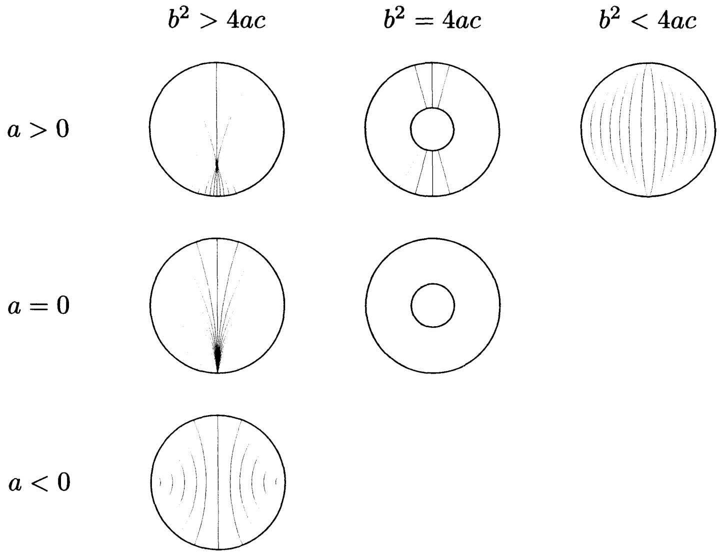


*Figure 2*

there is still an abundance of congruences which are neither twist-free nor divergence-free. For instance, on $\mathbb{R}^3$, a piece of a minitwistor line and its conjugate define a congruence on some open subset: if the line is real then this is a radial congruence, but in general, we get a congruence of rulings of a family of hyperboloids (figure 3).

This congruence is globally defined on the nontrivial double cover of $\mathbb{R}^3 \setminus S^1$. Its divergence and twist are closely related to the Eguchi-Hanson I metric as we shall see below.

In general, a holomorphic curve in the minitwistor space of $\mathbb{R}^3$ corresponds to a null curve in $\mathbb{C}^3$ and the associated congruence consists of the real points in the tangent lines to the null curve. Since null curves may be constructed from their real and imaginary parts, which are conjugate minimal surfaces in $\mathbb{R}^3$, this shows that more complicated congruences are associated with minimal surfaces.

Turning now to monopoles, we have two simple and explicit types of solutions of the monopole equation: the constant solutions and the fundamental solutions. Linear combinations of these give rise to an interesting family of selfdual conformal structures whose properties are

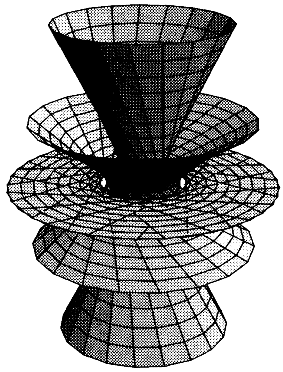

*Figure 3*

given by the above congruences. Since such monopoles are spherically symmetric, these selfdual conformal structures will admit local U(2) or $S^1 \times \mathrm{SO}(3)$ symmetry.

The 3-metric with constant curvature $c$ is

$$g_c = \frac{4}{(1+cr^2)^2}\left(dr^2 + r^2 g_{S^2}\right)$$

and the monopoles of interest are $a + bz$, where $z = (1 - cr^2)/2r$ is the fundamental solution centred at $r = 0$. The fundamental solution is the divergence of the radial congruence, and if we use the coordinate $z$ in place of $r$, we obtain

$$g_c = \Big(\frac{dz}{z^2+c}\Big)^2 + \frac{1}{z^2+c} g_{S^2}.$$

Rescaling by $(z^2+c)^2$ gives the Toda solution

$$g_{LW} = (z^2+c)g_{S^2} + dz^2, \quad \omega_{LW} = -\frac{2z}{z^2+c}\, dz.$$

In the LeBrun-Ward gauge, the monopoles of interest are

$$w = (a+bz)/(z^2+c).$$

If $c \neq 0$ then $w = ac^{-1}\big(1 - \frac{1}{2} z u_z\big) + \frac{1}{2} b u_z$ and so we may apply Tod's prescription for the construction of Einstein metrics with symmetry. Rescaling by $(a^2+c^2)/c$ gives the Einstein metric

$$g = \frac{a^2+c^2}{(az-bc)^2}\Big(\frac{a+bz}{z^2+c}\left(dz^2 + (z^2+c)g_{S^2}\right) + \frac{z^2+c}{a+bz}(d\psi + A)^2\Big)$$

of scalar curvature $-12ac/(a^2+c^2)$, where $dA = *Dw = b\,\mathrm{vol}_{S^2}$. This is easily integrated by $A = -b\cos\theta d\phi$ where $g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2$. These metrics are also well-defined when $c = 0$ when they become Taub-NUT metrics with triholomorphic Killing field $\partial/\partial\psi$. They are also Gibbons-Hawking metrics for $a = 0$, when we obtain the Eguchi-Hanson I and II metrics: this time $\partial/\partial\phi$ (and the other infinitesimal rotations of $S^2$) is a triholomorphic Killing field. To relate the metrics to those of [24], one can set $z = 1/\rho^2$ and rescale by a further factor $\frac{1}{4}$. Then

$$g = \frac{a^2+c^2}{(a-bc\rho^2)^2}\Big(\frac{a\rho^2+b}{1+c\rho^4}d\rho^2 + \frac{1}{4}\rho^2\Big[(a\rho^2+b)g_{S^2} + \frac{1+c\rho^4}{a\rho^2+b}(d\psi - b\cos\theta d\phi)^2\Big]\Big)$$

is Einstein with scalar curvature $-48ac/(a^2+c^2)$. Up to homothety, this is really only a one parameter family of Einstein metrics, since the original constant curvature metric and the monopole $w$ can be rescaled. However, the use of three parameters enables all the limiting cases to be easily found.

These metrics are all conformally scalar-flat Kähler via the radial Toda congruences [18]. The metrics over $\mathcal{H}^3$ are also conformal to other scalar-flat Kähler metrics, via the horospherical and disc-orthogonal congruences. The translational congruences on $\mathbb{R}^3$ correspond to the hyperKähler structures associated with the Ricci-flat $c = 0$ metrics. The metrics coming from $S^3$ admit two hypercomplex structures (coming from the hyperCR structures), which explains an observation of Madsen [22]. In particular when $a = 0$, the Eguchi-Hanson I metric has two additional hypercomplex structures with respect to which $\partial/\partial\psi$ is triholomorphic. On the other hand, although $\partial/\partial\phi$ is triholomorphic with respect to the hyperKähler metric, it only preserves one complex structure from each of these additional families. The corresponding congruences on $\mathbb{R}^3$ are the two rulings of the families of hyperboloids, which have the same divergence but opposite twist. The monopole giving Eguchi-Hanson I must be the divergence of this congruence.

In [27], it is claimed that the above constructions give all the Einstein metrics over $\mathcal{H}^3$. This is not quite true, because we have not yet considered the Einstein metrics associated to the horospherical and disc-orthogonal congruences. These turn out to give Bianchi type $\mathrm{VII}_0$ and VIII analogues of the above Bianchi type IX metrics (by which we mean, the SU(2) symmetry group is replaced by $\mathrm{Isom}(\mathbb{R}^2)$ and $\mathrm{SL}(2,\mathbb{R})$ respectively—see [30]). This omission from [27] was simply due to the nowhere vanishing conformal vector fields on hyperbolic space being overlooked.

## 9. Kähler-Weyl spaces with torus symmetry.

On an Einstein-Weyl space with symmetry, an invariant shear-free geodesic congruence and an invariant monopole together give rise to a selfdual Kähler-Weyl structures, possessing, in general, only two continuous symmetries. Many explicit examples of such Einstein-Weyl spaces with symmetry were given in Section 7. Being quotients of Gibbons-Hawking metrics, these spaces already come with invariant congruences, and solutions of the monopole equation can be obtained by introducing an additional invariant harmonic function on $\mathbb{R}^3$, lifting it to the Gibbons-Hawking space, and pushing it down to the Einstein-Weyl space. Carrying out this procedure in full generality would take us too far afield, so we confine ourselves to the two simplest classes of examples: the Einstein-Weyl spaces with axial symmetry, and the Einstein-Weyl spaces with geodesic symmetry.

We first consider the case of axial symmetry, when the Kähler-Weyl structure is (locally) scalar flat Kähler. In [17], Joyce constructs such torus symmetric scalar-flat Kähler metrics from a linear equation on hyperbolic 2-space. In this way he obtains selfdual conformal structures on $k\mathbb{C}P^2$, generalising (for $k \geq 4$) those of LeBrun [19]. Joyce does not consider the intermediate Einstein-Weyl spaces in his construction, but one easily sees that his linear equation is equivalent to the equation for axially symmetric harmonic functions, and that the associated Einstein-Weyl spaces are precisely the ones with axial symmetry [4].

Let us turn now to the spaces with geodesic symmetry, where a monopole invariant under the symmetry is given by a nonvanishing holomorphic function on an open subset of $S^2$. Indeed, if we write (as before)

$$g = |H|^{-2}(\sigma_1^2 + \sigma_2^2) + \beta^2, \quad \omega = \frac{i}{2}(H - \bar{H})\beta$$

with $\beta$ dual to the symmetry, then an invariant monopole in this gauge is given by the pullback $V$ of a harmonic function on an open subset of $S^2$, as one readily verifies by direct computation. Hence $V = \frac{1}{2}(F + \bar{F})$ for some holomorphic function $F$. The selfdual space constructed from $V$ will admit a Kähler-Weyl structure (coming from the geodesic symmetry) and also a hypercomplex structure (coming from the hyperCR structure). By Proposition 3.11, the geodesic symmetry preserves the hyperCR congruences, and so it lifts to a triholomorphic vector field of the hypercomplex structure. Since 3.11 is a characterisation, we immediately deduce the following result.

**9.1.** THEOREM. — *Let $M$ be a hypercomplex 4-manifold with a two dimensional family of commuting triholomorphic vector fields. Then the quotient of $M$ by any of these vector fields is Einstein-Weyl with a geodesic symmetry, and so the conformal structure on $M$ depends explicitly on two holomorphic functions of one variable.*

There are two special choices of monopole on such an Einstein-Weyl space: the $\kappa$ and $\tau$ monopoles of the geodesic symmetry. The $\kappa$ monopole ($F = H$) leads us back to the Gibbons-Hawking hyperKähler metric, but the $\tau$ monopole ($F = iH$) is more interesting. In this case, the Kähler-Weyl structure given by the geodesic symmetry is hypercomplex and so these torus symmetric selfdual spaces are hypercomplex in two ways. The symmetries are both triholomorphic with respect to the first hypercomplex structure, but only one of them is triholomorphic with respect to the additional hypercomplex structure. If we take the quotient by the bi-triholomorphic symmetry, we obtain an Einstein-Weyl space with two hyperCR structures, which must be $S^3$. Hence the spaces with geodesic symmetry, as well as coming from invariant monopoles on $\mathbb{R}^3$, also come from invariant monopoles on $S^3$.

We end by discussing a third situation in which the spaces with geodesic symmetry occur. This involves some explicit new solutions [6] of the SU($\infty$) Toda field equation generalising the solutions on $S^3$ described earlier. The corresponding LeBrun-Ward geometries are:

$$g = (z+h)(z+\bar{h})(\sigma_1^2+\sigma_2^2) + dz^2, \quad \omega = -\frac{2z+h+\bar{h}}{(z+h)(z+\bar{h})}\,dz,$$

where $h$ is an arbitrary nonvanishing holomorphic function on an open subset of $S^2$. These spaces have no symmetries and so one obtains from them Einstein metrics with a one dimensional isometry group. However, $\partial/\partial z$ does lift to a shear-free congruence on the Einstein space, and a generalised Jones and Tod construction may be used to show that the quotient by this conformal submersion is the Einstein-Weyl space with geodesic symmetry given by $H = 1/h$ (see [6]). In fact this was how these interesting Einstein-Weyl spaces were found.

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