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Annales de l’institut Fourier, tome 14, n° 2 (1964), p. 221-225

<http://www.numdam.org/item?id=AIF_1964__14_2_221_0>


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ON THE EXISTENCE OF EXCEPTIONAL LEAVES IN FOLIATIONS OF CO-DIMENSION ONE

by Richard SACKSTEDER

1. Introduction.

Let $M$ be a compact $n$-manifold ($n \geq 2$) with a foliated structure of co-dimension one. A leaf of such a foliation is said to be exceptional if it is nowhere dense in $M$, but its topology as a subset of $M$ is not the same as its topology as an $(n-1)$-manifold. Reeb [2] has asked if it is possible for exceptional leaves to exist in sufficiently smooth foliations, and he showed in [2] that, under certain conditions, exceptional leaves do not exist. The author proved other theorems of this type in [3] and [4]. Here we shall answer Reeb's question by giving an example of a 3-manifold with a $C^\infty$ foliated structure of co-dimension one in which there are exceptional leaves. Moreover, these leaves will be contained in a minimal set of the foliation.

2. Diffeomorphisms of $S^1$.

We shall first construct a group of $C^\infty$ diffeomorphisms of $S^1$ which has a perfect, nowhere dense, minimal set $C$. Let $S^1$ be represented as the interval $[0, 2]$ with its endpoints identified. The set $C$ is defined as follows: At the first step the intervals $(1/3, 2/3)$, $(1, 4/3)$, and $(5/3, 2)$ are removed from $[0, 2]$. At the $k$'th step the middle third of each closed interval which remains after the $(k-1)$st step is removed, as in the
usual construction of a Cantor set. The set \( C \) is the set which remains after all of the steps have been completed. \( C \) is perfect and nowhere dense.

The group of diffeomorphisms of \( S^1 \) will be the group generated by the diffeomorphisms \( f \) and \( g \) defined by

\[
\begin{align*}
f(x) &= x + 2/3 \pmod{2} \quad \text{for } x \in [0, 2] \\
g(x) &= x/3 \quad \text{if } 0 \leq x \leq 1 \\
g(x) &= 3x - 10/3 \quad \text{if } 4/3 \leq x \leq 5/3 
\end{align*}
\]

\( g(x) \) is defined elsewhere in \([0, 2]\) so that \( g \) is of class \( C^\infty \), \( g(2) = 2 \), and \( g^{-1} \) exists and is of class \( C^\infty \) on \( S^1 \). Clearly this can be done. Let \( G \) denote the group generated by \( f \) and \( g \).

**Lemma.** — \( C \) is a minimal set under the action of \( G \).

**Proof.** — It is easy to verify that \( C \) is closed and invariant under \( G \). Let \( C_k \) denote the set which remains after the \( k \)'th step in the construction of \( C \) has been carried out. Then \( C_k \) is the union of \( 3.2^{k-1} \) disjoint closed intervals,

\[
I_k^i, \ i = 1, \ldots, 3.2^{k-1},
\]

and \( C = \cap \{ C_k : k = 1, 2, \ldots \} \). To verify that \( C \) is minimal it suffices to prove that any interval \( I_k^i \) is mapped onto \([0, 1/3]\) by an element of \( G \). This is proved by induction on \( k \). For \( k = 1 \), either \( f \) or \( f^2 \) will work. If \( k > 1 \), some power of \( f \) will map \( I_k^i \) into the interval \([0, 1/3]\), hence it can be assumed that \( I_k^i \subset [0, 1/3] \). But then \( g^{-1}(I_k^i) = I_{k-1}^j \) for some \( j \), hence the induction hypothesis shows that \( I_k^i \) is mapped onto \([0, 1/3]\) by an element of \( G \). This proves the lemma.

3. The Example.

In the example, \( M \) is the product manifold \( M = S^1 \times M_2 \), where \( M_2 \) is the sphere \( S^2 \) with two handles attached. \( M_2 \) is a disjoint union of three sets \( A, B, \) and \( C \), where \( A \) is a « band » diffeomorphic to \( S^1 \times [0, 1] \) passing around a handle once, and \( B \) is another such band, disjoint from \( A \) and passing around the other handle. The foliated structure of \( M \) will be defined separately on the sets \( T_A = S^1 \times A, T_B = S^1 \times B, \) and \( T_C = S^1 \times C. \)
Let \( \varphi \) be a function of \( \nu \) defined for \( \nu \in [0, 1] \) with the properties that: (a) \( \varphi \) is increasing and of class \( C^\infty \), (b) \( \varphi(0) = 0 \), \( \varphi(1) = 1 \), (c) all derivatives of \( \varphi \) vanish for \( \nu = 0 \) and \( \nu = 1 \). Again regard \( S^1 \) as the interval \([0, 2]\) with its endpoints identified. Define the \( C^\infty \) functions \( h, k \) from \( S^1 \times [0, 1] \) to \( S^1 \) by:

\[
\begin{align*}
    h(x, \nu) &= x + 2/3 \varphi(\nu) \mod. 2 \\
    k(x, \nu) &= x + (g(x) - x)\varphi(\nu) \mod. 2.
\end{align*}
\]

Note that \( h(x, 0) = k(x, 0) = x \) and \( h(x, 1) = f(x) \),

\[
k(x, 1) = g(x).
\]

Let \( (u, \nu) \), \( u \in S^1 \), \( \nu \in [0, 1] \) represent a point of \( A \), hence \( (x, u, \nu) \) represents a point in \( T_A \) if \( x \in S^1 \). We define the foliation on \( T_A \) by agreeing that the leaf passing through \( (x, u, 0) \) will contain all points \( (h(x, \nu), u', \nu) \). The foliation of \( T_B \) is defined similarly except that \( k \) replaces \( h \). The foliation on \( T_C \) is defined by the condition that \( x = \text{const.} \) on each leaf.

It is easy to check that the foliations defined on \( T_A, T_B, T_C \) fit together to define a \( C^\infty \) foliation of \( M = T_A \cup T_B \cup T_C \). It is also clear that the leaves of the foliation are transversal to \( S^1 \) in product \( M = S^1 \times M_2 \). This transversality property implies that an arc in \( M_2 \) beginning at \( b \in M_2 \) — \( \text{A} \cup \text{B} \) be «lifted» to the leaf through any point \( (x, b) \in M, x \in S^1 \). The lifted arc is uniquely determined by the initial point \( (x, b) \).

If \( \gamma \) is a closed curve parameterized by \( t(0 \leq t \leq 1) \) such that \( \gamma(0) = \gamma(1) = b \), the lifted curve will end at a point \( (T(x, \gamma), b) \in M \). It is easy to verify that the map \( x \to T(x, \gamma) \) is of class \( C^\infty \) and depends only on the homotopy class of \( \gamma \).

Suppose that the closed curve \( \gamma_A \) has the property that \( \gamma_A \) does not intersect \( \text{A} \cup \text{B} \), except for one sub-arc of \( \gamma_A \) which is mapped homeomorphically on to the arc in \( A \) which corresponds to \( u = \text{const.} \) in terms of the \( (u, \nu) \) coordinates established above. Then if \( \gamma_A \) begins at \( b \in M_2 \) — \( \text{A} \cup \text{B} \) and the mapping on the sub-arc is such that increasing \( t \) corresponds to increasing \( \nu \), \( T(x, \gamma_A) = f(x) \). Similar considerations lead to a closed curve \( \gamma_B \) beginning at \( b \) and such that

\[
T(x, \gamma_B) = g(x).
\]

Finally, if \( \gamma_1 \) and \( \gamma_2 \) are closed curves which begin at \( b \) and
do not meet $A \cup B$ at all, $T(x, \gamma_1) = x$. Arcs $\gamma_A, \gamma_B, \gamma_1, \gamma_2$ with properties described can be chosen in such a way that their homotopy classes generate the fundamental group, $\pi_1(M_2)$. The map from $\gamma$ to $T(\gamma, \gamma)$ induces a homorphism from $\pi_1(M_2)$ to a group of $C^\infty$ diffeomorphisms of $S^1$, and it is now clear that this group is just the group $G$ defined above.

These considerations show that if $y = Gx \subset S^1$, ($Gx$ is the orbit of $x$ under $G$), then $(x, b)$ and $(y, b)$ are on the same leaf of the foliation. The converse is also easy to check, that is, if $(x, b)$ and $(y, b)$ are on the same leaf, $y \in Gx$.

Now if one takes $x$ to be a point of $C$, the lemma implies that the closure of the points $(y, b)$ on the leaf containing $(x, b)$ is just $C \times \{b\}$. It follows easily that the leaf through $(x, b)$ is exceptional and its closure is a minimal set.

4. The fundamental group of an exceptional leaf.

It was remarked in [4] that Lemma 12.1 of [4] suggests that the fundamental group of a nowhere dense leaf might be finitely generated. However, this is not the case, as will now be shown. In fact, the exceptional leaf just constructed has a fundamental group which is not finitely generated. To see this, let $\gamma_1$ be, as above, a generator of the fundamental group of $M_2$ which does not intersect the set $A \cup B$. Let $F$ be an exceptional leaf. There are infinitely many points of $F$ which project onto the initial point of $\gamma_1$. One can show that the lifts of $\gamma_1$ through these points are closed curves, which when connected to a base point, represent elements of the fundamental group of $F$. They cannot be represented in terms of any finite number of generators. We omit the details.
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Manuscrit reçu en avril 1964.

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