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# QUIVER VARIETIES AND WEYL GROUP ACTIONS

by George LUSZTIG (\*)

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## Introduction.

Consider a finite graph of type *ADE* with set of vertices  $I$ . Nakajima [N1], [N2] associates to  $\mathbf{v}, \mathbf{w} \in \mathbf{N}^I$  a smooth algebraic variety  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  (“quiver variety”) and shows that the cohomology of  $\sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$  has a natural module structure over the corresponding enveloping algebra; note that for fixed  $\mathbf{w}$ ,  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  is empty for all but finitely many choices of  $\mathbf{v}$ . He also constructs [N1, Sec. 9] a Weyl group action on this cohomology space using techniques of hyper-Kähler geometry. In this paper we give an alternative construction of this Weyl group action, based not on hyper-Kähler geometry, but on techniques of intersection cohomology, analogous to those used in [L1] to construct Springer representations. This gives in fact a refinement of the Weyl group action (see 6.13, 6.14, 6.15). I wish to thank H. Nakajima for interesting conversations.

## 1. A non-linear $W$ -action.

1.1. We fix a graph with finite set of vertices  $I$ . We assume that there is at most one edge joining two vertices of  $I$  and no edge joining a vertex with itself. Let  $H$  be the set of all ordered pairs  $i, j$  of vertices such that  $i, j$  are joined by an edge. For  $h = (i, j)$ , we set  $\bar{h} = (j, i) \in H$ ,  $j = h' \in I$ ,  $i = h'' \in I$ . We fix a function  $\varepsilon : H \rightarrow \{1, -1\}$  such that  $\varepsilon(h) + \varepsilon(\bar{h}) = 0$  for all  $h$ . We often write  $\varepsilon_h$  instead of  $\varepsilon(h)$ .

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Let  $\mathbf{I}$  be the set of all sequences  $i_1, i_2, \dots, i_s$  (with  $s \geq 1$ ) in  $I$  such that  $(i_k, i_{k+1}) \in H$  for any  $k \in [1, s-1]$ . Let  $\tilde{\mathcal{F}}$  be the  $\mathbf{C}$ -vector space spanned by elements  $[i_1, i_2, \dots, i_s]$  corresponding to the various elements of  $\mathbf{I}$  and by the elements  $u_i$  indexed by  $i \in I$ . Let  $\mathcal{F}$  be the subspace of  $\tilde{\mathcal{F}}$  spanned by the elements of the form  $[i_1, i_2, \dots, i_s]$ . We regard  $\mathcal{F}$  as an algebra in which the product  $[i_1, i_2, \dots, i_s][j_1, j_2, \dots, j_{s'}]$  is equal to  $[i_1, i_2, \dots, i_s, j_2, \dots, j_{s'}]$  if  $i_s = j_1$  and is zero, otherwise.

Let  $E$  be a  $\mathbf{C}$ -vector space with basis  $\{\varpi_i | i \in I\}$ . For  $\lambda \in E$  we define  $\lambda_i \in \mathbf{C}$  by  $\lambda = \sum_i \lambda_i \varpi_i$ . For  $i \in I$  we define  $s_i : E \rightarrow E$  by  $s_i(\lambda) = \lambda'$  where  $\lambda'_i = -\lambda_i$ ,  $\lambda'_j = \lambda_j + \lambda_i$  if  $(i, j) \in H$  and  $\lambda'_j = \lambda_j$  if  $j \neq i$  and  $(i, j) \notin H$ . Let  $W$  be the subgroup of  $GL(E)$  generated by the  $s_i : E \rightarrow E$  with  $i \in I$ . It is well known that  $W$  is a Coxeter group with generators  $s_i$  and relations  $s_i^2 = 1$ ,  $s_i s_j s_i = s_j s_i s_j$  if  $(i, j) \in H$ ,  $s_i s_j = s_j s_i$  if  $i \neq j$  and  $(i, j) \notin H$ . For any  $\lambda \in E$  we define a linear map  $s_i^\lambda : \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$  by

$$s_i^\lambda(u_j) = u_j \quad \text{for all } j;$$

$$s_i^\lambda[i] = [i] + \lambda_i u_i;$$

$$s_i^\lambda[i_1, i_2, \dots, i_s] = \sum_{J; J \subset J_0} \prod_{t \in J} (-\varepsilon_{i_t, i_{t+1}} \lambda_i) [i_1, i_2, \dots, i_s; \hat{J}]$$

if  $[i_1, i_2, \dots, i_s] \neq [i]$ .

Here  $J_0 = \{t \in [2, s-1] | i_t = i, i_{t-1} = i_{t+1}\}$ ;  $[i_1, i_2, \dots, i_s; \hat{J}]$  is the element of  $\mathbf{I}$  obtained from  $[i_1, i_2, \dots, i_s]$  by omitting  $i_t, i_{t+1}$  for all  $t \in J$ .

It will be convenient to define  $\bar{s}_i^\lambda : \mathcal{F} \rightarrow \mathcal{F}$  as the composition  $\mathcal{F} \rightarrow \tilde{\mathcal{F}} \xrightarrow{s_i^\lambda} \tilde{\mathcal{F}} \rightarrow \mathcal{F}$  where the first map is the obvious imbedding and the third map is the projection with kernel  $\sum_j \mathbf{C} u_j$ . We define a map  $s_i : E \times \tilde{\mathcal{F}} \rightarrow E \times \tilde{\mathcal{F}}$  by

$$(a) \quad (\lambda, f) \rightarrow (s_i(\lambda), s_i^\lambda(f)).$$

Let  $\hat{s}_i^\lambda = s_i^{-\lambda}$ . We define  $\hat{s}_i : E \times \tilde{\mathcal{F}} \rightarrow E \times \tilde{\mathcal{F}}$

$$(b) \quad (\lambda, f) \rightarrow (s_i(\lambda), \hat{s}_i^\lambda(f)).$$

LEMMA 1.2. — *The map 1.1(a) is an involution.*

Assume first that  $i_1, i_2, \dots, i_s$  in  $\mathbf{I}$  is other than  $i$ . Let  $J_0$  be as in 1.1. We have

$$\begin{aligned} & s_i s_i(\lambda, [i_1, i_2, \dots, i_s]) \\ &= s_i \left( s_i(\lambda), \sum_{J; J \subset J_0} \prod_{t \in J} (-\varepsilon_{i_t, i_{t+1}} \lambda_i) [i_1, i_2, \dots, i_s; \hat{J}] \right) \end{aligned}$$

$$\begin{aligned}
 &= \left( s_i^2 \lambda, \sum_{\substack{J, J' \\ J \subset J' \subset J_0}} \prod_{t \in J} (-\varepsilon_{i_t, i_{t+1}} \lambda_i) \prod_{t \in J' - J} (\varepsilon_{i_t, i_{t+1}} \lambda_i) [i_1, i_2, \dots, i_s; \hat{J}'] \right) \\
 &= \left( \lambda, \sum_{\substack{J' \subset J_0 \\ J, J' \subset J'}} (-1)^{|J|} \prod_{t \in J'} (\varepsilon_{i_t, i_{t+1}} \lambda_i) [i_1, i_2, \dots, i_s; \hat{J}'] \right) \\
 &= (\lambda, [i_1, i_2, \dots, i_s; \hat{\emptyset}]) = (\lambda, [i_1, i_2, \dots, i_s]).
 \end{aligned}$$

Next, we have  $s_i s_i(\lambda, [i]) = s_i(s_i(\lambda), [i] + \lambda_i u_i) = (s_i s_i(\lambda), [i] + \lambda_i u_i - \lambda_i u_i) = (\lambda, [i])$ . Clearly,  $s_i s_i(\lambda, u_j) = (\lambda, u_j)$  for any  $j$ . The lemma is proved.

LEMMA 1.3. — If  $(i, j) \in H$ , then  $s_i s_j s_i = s_j s_i s_j : E \times \tilde{\mathcal{F}} \rightarrow E \times \tilde{\mathcal{F}}$ .

It suffices to show that

$$(a) \quad s_i^{s_j s_i(\lambda)} s_j^{s_i(\lambda)} s_i^\lambda \phi = s_j^{s_i s_j(\lambda)} s_i^{s_j(\lambda)} s_j^\lambda \phi$$

for any  $\phi \in \tilde{\mathcal{F}}$ . The case where  $\phi = u_j$  for some  $j$  is trivial. Hence it suffices to show that (a) holds for  $\phi = [f]$  where  $f \in \mathbf{I}$  is  $i_1, i_2, \dots, i_s$ . Note that (a) for  $\phi = [f]$  implies

$$(a1) \quad \bar{s}_i^{s_j s_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^\lambda [f] = \bar{s}_j^{s_i s_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^\lambda [f].$$

We prove (a) for  $[f]$  by induction on  $s$ . Assume first that  $[f] = [i]$ . Both sides of (a) are in this case equal to  $[i] + (\lambda_i + \lambda_j)u_i$ . The same argument applies to  $[f] = [j]$ . If  $s \leq 2$  and  $[f]$  is not  $[i]$  or  $[j]$ , then (a) is obviously true for  $\phi = [f]$ . We now assume that  $s \geq 3$ .

Assume first that the first three entries of  $f$  are not of the form  $kik$  or  $ljl$ . Let  $f' \in \mathbf{I}$  be  $i_2, i_3, \dots, i_s$ . We have  $[f] = [i_1, i_2][f']$  and from the definition we see that

$$\begin{aligned}
 s_i^{s_j s_i(\lambda)} s_j^{s_i(\lambda)} s_i^\lambda [f] &= [i_1, i_2] \bar{s}_i^{s_j s_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^\lambda [f'], \\
 s_j^{s_i s_j(\lambda)} s_i^{s_j(\lambda)} s_j^\lambda [f] &= [i_1, i_2] \bar{s}_j^{s_i s_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^\lambda [f'].
 \end{aligned}$$

By the induction hypothesis, (a1) holds for  $[f']$ ; hence (a) holds for  $\phi = [f]$ . Thus, we may assume that the first three entries of  $f$  are  $kik$  or  $ljl$ . Since  $i, j$  play a symmetrical role, we may assume that the first three entries are  $kik$ . Let  $u$  be the largest integer  $\geq 3$  such that  $f_1 = (i_1, i_2, \dots, i_u)$  is of the form  $jijij \dots i$  (so  $u$  is even) or of the form  $kikik \dots ik$  where  $k$  may or may not be  $j$  (so  $u$  is odd). If  $u < s$ , we have  $[f] = [f_1][f_2]$  where  $f_2 \in \mathbf{I}$  is  $i_u, i_{u+1}, \dots, i_s$  and from the definitions we have

$$\begin{aligned}
 s_i^{s_j s_i(\lambda)} s_j^{s_i(\lambda)} s_i^\lambda [f] &= (\bar{s}_i^{s_j s_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^\lambda [f_1]) (\bar{s}_i^{s_j s_i(\lambda)} \bar{s}_j^{s_i(\lambda)} \bar{s}_i^\lambda [f_2]), \\
 s_j^{s_i s_j(\lambda)} s_i^{s_j(\lambda)} s_j^\lambda [f] &= (\bar{s}_j^{s_i s_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^\lambda [f_1]) (\bar{s}_j^{s_i s_j(\lambda)} \bar{s}_i^{s_j(\lambda)} \bar{s}_j^\lambda [f_2]).
 \end{aligned}$$

Since the induction hypothesis is applicable to  $f_1$  and  $f_2$ , we see that (a) holds for  $f$ . Thus, we may assume that  $u = s$ . We must consider three cases:

- (b)  $f = f_u$  is  $kik \cdots ik$  where  $k \neq j$  and  $i$  appears  $u$  times;
- (c)  $f = f_u$  is  $jij \cdots ji$  where  $i$  appears  $u + 1$  times;
- (d)  $f = f_u$  is  $jij \cdots ij$  where  $i$  appears  $u$  times.

Assume that  $f = f_u$  is as in (b) and  $u \geq 1$ . We must show

$$\begin{aligned} \sum_{u', u''; 0 \leq u'' \leq u' \leq u} \binom{u}{u'} \binom{u'}{u''} (-\varepsilon_{ij} \lambda_j)^{u' - u''} (-\varepsilon_{ij} \lambda_i)^{u - u'} f_{u''} \\ = \sum_{u''; 0 \leq u'' \leq u} \binom{u}{u''} (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u - u''} f_{u''} \end{aligned}$$

or that

$$\sum_{u'; u'' \leq u' \leq u} \binom{u - u''}{u - u'} (-\varepsilon_{ij} \lambda_j)^{u' - u''} (-\varepsilon_{ij} \lambda_i)^{u - u'} = (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u - u''}$$

which is clear.

Assume that  $f = f_u$  is as in (c) and  $u \geq 1$ . We must show that  $A = B$  where

$$\begin{aligned} A &= \sum_{\substack{u_1, u_2, u_3 \\ 0 \leq u_3 \leq u_2 \leq u_1 \leq u}} \binom{u}{u_1} \binom{u_1}{u_2} \binom{u_2}{u_3} (-\varepsilon_{ij} \lambda_j)^{u_2 - u_3} (-\varepsilon_{ji} (\lambda_i + \lambda_j))^{u_1 - u_2} \\ &\quad \times (-\varepsilon_{ij} \lambda_i)^{u - u_1} f_{u_3}, \\ B &= \sum_{\substack{u_1, u_2, u_3 \\ 0 \leq u_3 \leq u_2 \leq u_1 \leq u}} \binom{u}{u_1} \binom{u_1}{u_2} \binom{u_2}{u_3} (-\varepsilon_{ji} \lambda_i)^{u_2 - u_3} (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u_1 - u_2} \\ &\quad \times (-\varepsilon_{ji} \lambda_j)^{u - u_1} f_{u_3}. \end{aligned}$$

We have

$$\begin{aligned} A &= \sum_{u_3 \in [0, u]} \sum_{\substack{a, b, c \\ a + b + c = u - u_3}} \frac{u!}{a!b!c!u_3!} (-\varepsilon_{ij} \lambda_j)^a (-\varepsilon_{ji} (\lambda_i + \lambda_j))^b (-\varepsilon_{ij} \lambda_i)^c f_{u_3} \\ &= \sum_{u_3 \in [0, u]} \sum_{b \in [0, u - u_3]} \frac{u!}{(u - u_3 - b)!b!u_3!} (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u - u_3 - b} \\ &\quad \times (-\varepsilon_{ji} (\lambda_i + \lambda_j))^b f_{u_3} \\ &= \sum_{u_3 \in [0, u]} \sum_{b \in [0, u - u_3]} \frac{u!}{(u - u_3 - b)!b!u_3!} (-1)^b (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u - u_3} f_{u_3} \\ &= \sum_{u_3 \in [0, u]} \frac{u!}{u_3!} \delta_{u, u_3} (-\varepsilon_{ij} (\lambda_i + \lambda_j))^{u - u_3} f_{u_3} = f_0, \end{aligned}$$

$$\begin{aligned}
 B &= \sum_{u_3 \in [0, u]} \sum_{\substack{a, b, c \\ a+b+c=u-u_3}} \frac{u!}{a!b!c!u_3!} (-\varepsilon_{ji}\lambda_i)^a (-\varepsilon_{ij}(\lambda_i + \lambda_j))^b (-\varepsilon_{ji}\lambda_j)^c f_{u_3} \\
 &= \sum_{u_3 \in [0, u]} \sum_{b \in [0, u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!} (-\varepsilon_{ji}(\lambda_i + \lambda_j))^{u-u_3-b} \\
 &\quad \times (-\varepsilon_{ij}(\lambda_i + \lambda_j))^b f_{u_3} \\
 &= \sum_{u_3 \in [0, u]} \sum_{b \in [0, u-u_3]} \frac{u!}{(u-u_3-b)!b!u_3!} (-1)^b (-\varepsilon_{ji}(\lambda_i + \lambda_j))^{u-u_3} f_{u_3} \\
 &= \sum_{u_3 \in [0, u]} \frac{u!}{u_3!} \delta_{u, u_3} (-\varepsilon_{ji}(\lambda_i + \lambda_j))^{u-u_3} f_{u_3} = f_0.
 \end{aligned}$$

Thus,  $A = B$  as desired. Next we assume that  $f = f_u$  is as in (d) and  $u \geq 1$ . We must show that  $A = B$  where

$$\begin{aligned}
 A &= \sum_{\substack{u_1, u_2, u_3 \\ 0 \leq u_3 \leq u_2 \\ 1 \leq u_2 \leq u_1 \leq u}} \binom{u}{u_1} \binom{u_1}{u_2} \binom{u_2}{u_3} (-\varepsilon_{ij}\lambda_j)^{u_2-u_3} (-\varepsilon_{ji}(\lambda_i + \lambda_j))^{u_1-u_2} \\
 &\quad \times (-\varepsilon_{ij}\lambda_i)^{u-u_1} f_{u_3} + (-\varepsilon_{ij}\lambda_i)^u (f_0 + (\lambda_i + \lambda_j)u_j), \\
 B &= \sum_{\substack{u_1, u_2, u_3 \\ 1 \leq u_3 \leq u_2 \leq u_1 \leq u}} \binom{u}{u_1} \binom{u_1}{u_2} \binom{u_2-1}{u_3-1} (-\varepsilon_{ji}\lambda_i)^{u_2-u_3} (-\varepsilon_{ij}(\lambda_i + \lambda_j))^{u_1-u_2} \\
 &\quad \times (-\varepsilon_{ji}\lambda_j)^{u-u_1} f_{u_3} + \sum_{\substack{u_1 \\ 1 \leq u_1 \leq u}} \binom{u-1}{u_1-1} (-\varepsilon_{ij}(\lambda_i + \lambda_j))^{u_1} (-\varepsilon_{ji}\lambda_j)^{u-u_1} (f_0 + \lambda_i u_j).
 \end{aligned}$$

We have  $A = \sum_{u_3 \in [0, u]} A_{u_3} f_{u_3} + \alpha u_j$ ,  $B = \sum_{u_3 \in [0, u]} B_{u_3} f_{u_3} + \beta u_j$ , where

$$\begin{aligned}
 A_{u_3} &= \sum_{\substack{a, b, c \\ a+b+c=u-u_3}} \frac{u!}{a!b!c!u_3!} \frac{u_3+a}{u_3+a+b} (-\varepsilon_{ij}\lambda_j)^a (-\varepsilon_{ji}(\lambda_i + \lambda_j))^b (-\varepsilon_{ij}\lambda_i)^c, \\
 B_{u_3} &= \sum_{\substack{a, b, c \\ a+b+c=u-u_3}} \frac{(u-1)!}{a!b!c!(u_3-1)!} \frac{u_3+a+b}{u_3+a} (-\varepsilon_{ji}\lambda_i)^a (-\varepsilon_{ij}(\lambda_i + \lambda_j))^b \\
 &\quad \times (-\varepsilon_{ji}\lambda_j)^c
 \end{aligned}$$

for  $u_3 \in [1, u]$ ,

$$A_0 = \sum_{\substack{a, b, c \\ a+b+c=u \\ a \geq 1}} \frac{u!}{a!b!c!} \frac{a}{a+b} (-\varepsilon_{ij}\lambda_j)^a (-\varepsilon_{ji}(\lambda_i + \lambda_j))^b (-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u,$$

$$B_0 = (-\varepsilon_{ij}(\lambda_i + \lambda_j))(-\varepsilon_{ij}\lambda_i)^{u-1},$$

$$\alpha = (-\varepsilon_{ij}\lambda_i)^u(\lambda_i + \lambda_j),$$

$$\beta = \sum_{u_1; 1 \leq u_1 \leq u} \binom{u-1}{u_1-1} (-\varepsilon_{ij}(\lambda_i + \lambda_j))^{u_1} (-\varepsilon_{ji}\lambda_j)^{u-u_1} \lambda_i.$$

It is enough to show that

$$(e) \quad A_{u_3} = B_{u_3}$$

for  $u_3 \in [0, u]$ . (The equality  $\alpha = \beta$  is obvious.) We have

$$\begin{aligned} A_0 &= (-\varepsilon_{ij}\lambda_j) \sum_{\substack{a,b,c \\ a+b+c=u \\ a \geq 1}} \frac{u!}{(a-1)!b!c!} \frac{1}{a+b} (-\varepsilon_{ij}\lambda_j)^{a-1} \\ &\quad \times (-\varepsilon_{ji}(\lambda_i + \lambda_j))^b (-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u \\ &= (-\varepsilon_{ij}\lambda_j) \sum_{c \in [0, u-1]} \frac{u!}{(u-c)!c!} (-\varepsilon_{ji}\lambda_i)^{u-c-1} (-\varepsilon_{ij}\lambda_i)^c + (-\varepsilon_{ij}\lambda_i)^u \\ &= (-\varepsilon_{ij}\lambda_j) \sum_{c \in [0, u-1]} \frac{u!}{(u-c)!c!} (-\varepsilon_{ji}\lambda_i)^{u-1} (-1)^c + (-\varepsilon_{ij}\lambda_i)^u \\ &= (-\varepsilon_{ij}\lambda_j)(-\varepsilon_{ji}\lambda_i)^{u-1}(-1)^{u-1} + (-\varepsilon_{ij}\lambda_i)^u \\ &= (-\varepsilon_{ij}\lambda_i)^{u-1}(-\varepsilon_{ij}(\lambda_i + \lambda_j)) = B_0. \end{aligned}$$

This verifies (e) for  $u_3 = 0$ . Assume now that  $u_3 \in [1, u]$ . The identities

$$(f) \quad \sum_{p_1, p_2; p_1+p_2=p} \frac{(-1)^{p_1}}{p_1!p_2!(x-p_2)} = \frac{1}{x(x-1)(x-2)\cdots(x-p)},$$

$$(h) \quad \sum_{p_1, p_2; p_1+p_2=p} \frac{(-1)^{p_1}(x+p_2)}{p_1!p_2!} = \delta_{p,0}x + \delta_{p,1},$$

for  $p \in \mathbb{N}$ , are easily verified. We set  $X = -\varepsilon_{ij}\lambda_j$ ,  $Y = -\varepsilon_{ij}\lambda_i$ . Then  $A_{u_3}$  equals

$$\begin{aligned} &\sum_{\substack{a,b_1,b_2,c \\ a+b_1+b_2+c=u-u_3}} \frac{u!}{a!b_1!b_2!c!u_3!} \frac{u_3+a}{u_3+a+b_1+b_2} (-1)^{b_1+b_2} X^a X^{b_1} Y^{b_2} Y^c \\ &= \sum_{\substack{a,b_1,b_2,c \\ a+b_1+b_2+c=u-u_3}} \frac{u!}{a!b_1!b_2!c!u_3!} \frac{u_3+a}{u_3+a+b_1+b_2} (-1)^{b_1+b_2} X^{a+b_1} Y^{c+b_2} \\ &= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=u-u_3}} \frac{u!}{u_3!} \sum_{a+b_1=\alpha} \frac{u_3+a}{a!b_1!} (-1)^{b_1} \sum_{c+b_2=\beta} \frac{(-1)^{b_2}}{b_2!c!(u-c)} X^\alpha Y^\beta \\ &= \sum_{\substack{\alpha,\beta \\ \alpha+\beta=u-u_3}} \frac{u!}{u_3!} (u_3\delta_{\alpha,0} + \delta_{\alpha,1}) \frac{1}{u(u-1)(u-2)\cdots(u-\beta)} X^\alpha Y^\beta \end{aligned}$$

$$\begin{aligned}
 &= \frac{u!}{u_3!} \frac{u_3}{u(u-1)(u-2) \cdots u_3} Y^{u-u_3} \\
 &\quad + \frac{u!}{u_3!} \frac{1}{u(u-1)(u-2) \cdots (u_3+1)} XY^{u-u_3-1} \\
 &= Y^{u-u_3} + XY^{u-u_3-1} = (X+Y)Y^{u-u_3-1},
 \end{aligned}$$

and  $B_{u_3}$  equals

$$\begin{aligned}
 &\sum_{\substack{a, b_1, b_2, c \\ a+b_1+b_2+c=u-u_3}} \frac{(u-1)!}{a!b_1!b_2!c!(u_3-1)!} \frac{u_3+a+b_1+b_2}{u_3+a} (-Y)^a Y^{b_1} X^{b_2} (-X)^c \\
 &= \sum_{\substack{a, b_1, b_2, c \\ a+b_1+b_2+c=u-u_3}} \frac{(u-1)!}{a!b_1!b_2!c!(u_3-1)!} \frac{u_3+a+b_1+b_2}{u_3+a} (-1)^{a+c} Y^{a+b_1} X^{b_2+c} \\
 &= \sum_{\substack{\alpha, \beta \\ \alpha+\beta=u-u_3}} \frac{(u-1)!}{(u_3-1)!} \sum_{a+b_1=\alpha} \frac{(-1)^a}{a!b_1!(u_3+\alpha-b_1)} \\
 &\quad \times \sum_{c+b_2=\beta} \frac{(-1)^c (u_3+\alpha+b_2)}{c!b_2!} Y^\alpha X^\beta \\
 &= \sum_{\substack{\alpha, \beta \\ \alpha+\beta=u-u_3}} \frac{(u-1)!}{(u_3-1)!} \frac{1}{(u_3+\alpha)(u_3+\alpha-1) \cdots u_3} (\delta_{\beta,0}(u_3+\alpha) + \delta_{\beta,1}) Y^\alpha X^\beta \\
 &= Y^{u-u_3} + XY^{u-u_3-1} = (X+Y)Y^{u-u_3-1}.
 \end{aligned}$$

Thus,  $A_{u_3} = B_{u_3}$ . The lemma is proved.

LEMMA 1.4. — If  $i \neq j$ ,  $(i, j) \notin H$ , then  $s_i s_j = s_j s_i : E \times \tilde{\mathcal{F}} \rightarrow E \times \tilde{\mathcal{F}}$ .

As in the proof of 1.3 it is enough to show that  $s_j^{s_i(\lambda)} s_i^\lambda[f] = s_i^{s_j(\lambda)} s_j^\lambda[f]$  for  $f \in \mathbf{I}$  of the form  $kik \cdots ik$  where  $i$  appears  $u$  times. This follows immediately from the definitions.

1.5. From Lemmas 1.2, 1.3, 1.4 we see that there is a unique  $W$ -action on  $E \times \tilde{\mathcal{F}}$  in which the generators  $s_i$  acts by 1.1(a). This  $W$ -action is not a linear one.

Now 1.2, 1.3, 1.4 remain true if the  $s_i^\lambda$  are replaced by  $\hat{s}_i^\lambda$ ; these new statements are obtained from 1.2, 1.3, 1.4 with  $u_i, \varepsilon_{ij}$  replaced by  $-u_i, -\varepsilon_{ij}$ . Hence there is a unique  $W$ -action on  $E \times \tilde{\mathcal{F}}$  in which  $s_i$  acts by 1.1 (b).



## 2. The set $Z_{\mathbf{D}}$ .

2.1. Let  $\mathcal{C}^0$  be the category whose objects are  $I$ -graded  $\mathbf{C}$ -vector spaces  $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$  with  $\dim \mathbf{V}_i < \infty$  for all  $i$ . For  $\mathbf{V} \in \mathcal{C}^0$ , we set  $G_{\mathbf{V}} = \prod_i GL(\mathbf{V}_i)$ .

We fix  $\mathbf{D} \in \mathcal{C}^0$ . Let  $\tilde{\mathcal{F}}_{\mathbf{D}}$  be the vector space of all linear maps  $\tilde{\mathcal{F}} \rightarrow \text{End}(\mathbf{D})$ ; here  $\text{End}(\mathbf{D})$  is understood in the ungraded sense.

For  $i \in I$  and  $\lambda \in E$  we define a map  $\tilde{\mathcal{F}}_{\mathbf{D}} \rightarrow \tilde{\mathcal{F}}_{\mathbf{D}}$  by associating to  $\pi : \tilde{\mathcal{F}} \rightarrow \text{End}(\mathbf{D})$  the composition  $\tilde{\mathcal{F}} \xrightarrow{s_i^\lambda} \tilde{\mathcal{F}} \xrightarrow{\pi} \text{End}(\mathbf{D})$ . This map  $\tilde{\mathcal{F}}_{\mathbf{D}} \rightarrow \tilde{\mathcal{F}}_{\mathbf{D}}$  is denoted again by  $s_i^\lambda$ . We now define  $s_i : E \times \tilde{\mathcal{F}}_{\mathbf{D}} \rightarrow E \times \tilde{\mathcal{F}}_{\mathbf{D}}$  by  $s_i(\lambda, \pi) = (s_i(\lambda), s_i^\lambda(\pi))$ . Since  $s_i^\lambda = (\hat{s}_i^\lambda)^{-1}$ , from 1.5 it follows that there is a unique action of  $W$  on  $E \times \tilde{\mathcal{F}}_{\mathbf{D}}$  such that for any  $i \in I$ ,  $s_i \in W$  acts in the way just described. Following [L4, 2.4], we define a subset  $Z_{\mathbf{D}}$  of  $E \times \tilde{\mathcal{F}}_{\mathbf{D}}$  as follows. An element  $(\lambda, \pi) \in E \times \tilde{\mathcal{F}}_{\mathbf{D}}$  is said to be in  $Z_{\mathbf{D}}$  if it satisfies conditions (a), (b), (c) below. (For  $\phi \in \tilde{\mathcal{F}}$  we write  $\pi_\phi$  instead of  $\pi(\phi) : \mathbf{D} \rightarrow \mathbf{D}$ ).

- (a) If  $f \in \mathbf{I}$  is  $i_1, \dots, i_s$ , then  $\pi_{[f]}$  maps  $\mathbf{D}_{i_s}$  into  $\mathbf{D}_{i_1}$  and maps  $\mathbf{D}_j$  to 0, for  $j \neq i_s$ .
- (b) For any  $i \in I$ ,  $\pi_{u_i}$  is the identity map on  $\mathbf{D}_i$  and is zero on  $\mathbf{D}_j$ , for  $j \neq i$ .
- (c) For any  $f, f' \in \mathbf{I}$  such that  $f$  ends and  $f'$  begins with the same  $i \in I$ , we have

$$\pi_{[f]}\pi_{[f']} = \sum_k \varepsilon_{ik} \pi_{[f][ik i][f']} - \lambda_i \pi_{[f][f']}$$

where  $k$  runs over the set of vertices such that  $(i, k) \in H$ .

PROPOSITION 2.2. —  $Z_{\mathbf{D}}$  is a  $W$ -stable subset of  $E \times \tilde{\mathcal{F}}_{\mathbf{D}}$ .

Let  $(\lambda, \pi) \in Z_{\mathbf{D}}$ . Let  $i \in I$  and let  $(\lambda', \pi') = s_i(\lambda, \pi) \in E \times \tilde{\mathcal{F}}_{\mathbf{D}}$ . It is enough to show that  $(\lambda', \pi') \in Z_{\mathbf{D}}$ . It is clear that  $(\lambda', \pi')$  satisfies conditions 2.1(a),(b). To verify condition 2.1(c), we consider nine cases. In the other cases, the result is trivial. In the following formulas, an expression like

$$\begin{aligned} \pi_{[**ui]}\pi_{[iu**]} - \sum_j \varepsilon_{ij} \pi_{[**uij iu**]} - 2\varepsilon_{iu}(\varepsilon_{iu} \lambda'_i) \pi_{[**ui iu**]} \\ - \varepsilon_{iu}(\varepsilon_{iu} \lambda'_i)^2 \pi_{[**ui**]} + \lambda'_i \pi_{[**ui iu**]} + \lambda'_i(\varepsilon_{iu} \lambda'_i) \pi_{[**ui**]} \end{aligned}$$

should be interpreted as follows: each of

$$\pi_{[**ui]}\pi_{[iu**]}, \pi_{[**uij iu**]}, \pi_{[**ui**]}, \pi_{[**ui iu**]}$$

stands for a linear combination of  $\pi_{[\dots ui]}\pi_{[iu\dots]}$ ,  $\pi_{[\dots uijiu\dots]}$ ,  $\pi_{[\dots u\dots]}$ ,  $\pi_{[\dots uiu\dots]}$  over the same index set, with the same coefficients.

Case 1.

$$\begin{aligned} \pi'_{[i]}\pi'_{[i]} - \sum_j \varepsilon_{ij}\pi'_{[ij i]} + \lambda'_i\pi'_{[i]} \\ = (\pi_{[i]} - \lambda'_i\pi_{u_i})(\pi_{[i]} - \lambda'_i\pi_{u_i}) - \sum_j \varepsilon_{ij}\pi_{[ij i]} + \lambda'_i(\pi_{[i]} - \lambda'_i\pi_{u_i}) \\ = -\lambda_i\pi_{[i]} - \lambda'_i\pi_{[i]} = 0. \end{aligned}$$

Case 2. Assume that  $(u, i) \in H$ .

$$\begin{aligned} \pi'_{[u]}\pi'_{[u]} - \sum_j \varepsilon_{uj}\pi'_{[uj u]} + \lambda'_u\pi'_{[u]} \\ = \pi_{[u]}\pi_{[u]} - \sum_j \varepsilon_{uj}\pi_{[uj u]} - \varepsilon_{ui}(\varepsilon_{iu}\lambda'_i)\pi_{[u]} + \lambda'_u\pi_{[u]} \\ = -\lambda_u\pi_{[u]} + \lambda'_i\pi_{[u]} + \lambda'_u\pi_{[u]} = 0. \end{aligned}$$

Case 3.

$$\begin{aligned} \pi'_{[i]}\pi'_{[ik\dots]} - \sum_j \varepsilon_{ij}\pi'_{[ij ik\dots]} + \lambda'_i\pi'_{[ik\dots]} \\ = (\pi_{[i]} - \lambda'_i\pi_{u_i})\pi_{[ik\dots]} - \sum_j \varepsilon_{ij}\pi_{[ij ik\dots]} - \varepsilon_{ik}(\varepsilon_{ik}\lambda'_i)\pi_{[ik\dots]} + \lambda'_i\pi_{[ik\dots]} \\ = -\lambda_i\pi_{[ik\dots]} - \lambda'_i\pi_{[ik\dots]} = 0. \end{aligned}$$

Case 4. Assume that  $(u, i) \in H$ .

$$\begin{aligned} \pi'_{[u]}\pi'_{[u\dots]} - \sum_j \varepsilon_{uj}\pi'_{[uj u\dots]} + \lambda'_u\pi'_{[u\dots]} \\ = \pi_{[u]}\pi_{[u\dots]} - \sum_j \varepsilon_{uj}\pi_{[uj u\dots]} - \varepsilon_{ui}\varepsilon_{iu}\lambda'_i\pi_{[u\dots]} + \lambda'_u\pi_{[u\dots]} \\ = -\lambda_u\pi_{[u\dots]} + \lambda'_i\pi_{[u\dots]} + \lambda'_u\pi_{[u\dots]} = 0. \end{aligned}$$

Case 5.

$$\begin{aligned} \pi'_{[\dots ki]}\pi'_{[i]} - \sum_j \varepsilon_{ij}\pi'_{[\dots kiji]} + \lambda'_i\pi'_{[\dots ki]} \\ = \pi_{**ki}(\pi_{[i]} - \lambda'_i\pi_{u_i}) - \sum_j \varepsilon_{ij}\pi_{[\dots kiji]} - \varepsilon_{ik}(\varepsilon_{ik}\lambda'_i)\pi_{[\dots ki]} + \lambda'_i\pi_{[\dots ki]} \\ = -\lambda_i\pi_{[\dots ki]} - \lambda'_i\pi_{[\dots ki]} = 0. \end{aligned}$$

Case 6. Assume that  $(u, i) \in H$ .

$$\begin{aligned}
 \pi'_{[\dots u]} \pi'_{[u]} - \sum_j \varepsilon_{uj} \pi'_{[\dots uju]} + \lambda'_u \pi'_{[\dots u]} \\
 = \pi_{**u} \pi_{[u]} - \sum_j \varepsilon_{uj} \pi_{[**uju]} - \varepsilon_{ui} (\varepsilon_{iu} \lambda'_i) \pi_{[**u]} + \lambda'_u \pi_{[**u]} \\
 = -\lambda_u \pi_{[**u]} + \lambda'_i \pi_{[**u]} + \lambda'_u \pi_{[**u]} = 0.
 \end{aligned}$$

Case 7.

$$\begin{aligned}
 \pi'_{[\dots ui]} \pi'_{[iu\dots]} - \sum_j \varepsilon_{ij} \pi'_{[\dots uijiu\dots]} + \lambda'_i \pi'_{[\dots uiu\dots]} \\
 = \pi_{[**ui]} \pi_{[iu**]} - \sum_j \varepsilon_{ij} \pi_{[**uijiu**]} - 2\varepsilon_{iu} (\varepsilon_{iu} \lambda'_i) \pi_{[**uiu**]} \\
 \quad - \varepsilon_{iu} (\varepsilon_{iu} \lambda'_i)^2 \pi_{[***u**]} + \lambda'_i \pi_{[**uiu**]} + \lambda'_i (\varepsilon_{iu} \lambda'_i) \pi_{[***u**]} \\
 = -\lambda_i \pi_{[**uiu**]} - \lambda'_i \pi_{[**uiu**]} = 0.
 \end{aligned}$$

Case 8. Assume that  $u \neq v$ .

$$\begin{aligned}
 \pi'_{[\dots ui]} \pi'_{[iv\dots]} - \sum_j \varepsilon_{ij} \pi'_{[\dots uijiv\dots]} + \lambda'_i \pi'_{[\dots uiv\dots]} \\
 = \pi_{[**ui]} \pi_{[iv**]} - \sum_j \varepsilon_{ij} \pi_{[**uijiiv**]} - \varepsilon_{iu} (\varepsilon_{iu} \lambda'_i) \pi_{[**uiv**]} \\
 \quad - \varepsilon_{iv} (\varepsilon_{iv} \lambda'_i) \pi_{[**uiv**]} + \lambda'_i \pi_{[**uiv**]} \\
 = -\lambda_i \pi_{[**uiv**]} - \lambda'_i \pi_{[**uiv**]} = 0.
 \end{aligned}$$

Case 9. Assume that  $(u, i) \in H$ .

$$\begin{aligned}
 \pi'_{[\dots u]} \pi'_{[u\dots]} - \sum_j \varepsilon_{uj} \pi'_{[\dots uju\dots]} + \lambda'_u \pi'_{[\dots u\dots]} \\
 = \pi_{[**u]} \pi_{[u**]} - \sum_j \varepsilon_{uj} \pi_{[**uju**]} - \varepsilon_{ui} (\varepsilon_{iu} \lambda'_i) \pi_{[**u**]} + \lambda'_u \pi_{[**u**]} \\
 = -\lambda_u \pi_{[**u**]} + \lambda'_i \pi_{[**u**]} + \lambda'_u \pi_{[**u**]} = 0.
 \end{aligned}$$

The proposition is proved.

2.3. Consider the action of  $\mathbf{C}^*$  on  $E \times \tilde{\mathcal{F}}_{\mathbf{D}}$  given by

$$t : (\lambda, \pi) \mapsto (t^2 \lambda, \pi')$$

where  $\pi'_{[f]} = t^{s+1} \pi_{[f]}$  for  $f \in \mathbf{I}$  of form  $i_1, \dots, i_s$  and  $\pi'_{u_i} = \pi_{u_i}$  for  $i \in I$ . It is easy to check that this restricts to an action of  $\mathbf{C}^*$  on  $Z_{\mathbf{D}}$  which commutes with the  $W$ -action.

2.4. Consider the action of  $G_{\mathbf{D}}$  on  $Z_{\mathbf{D}}$  given by

$$(g_i) : (\lambda, \pi) \mapsto (\lambda, \pi')$$

where  $\pi'_{[f]} = g_{i_1} \pi_{[f]} g_{i_s} \subset$  for  $f \in \mathbf{I}$  of form  $i_1, \dots, i_s$  and  $\pi'_{u_i} = \pi_{u_i}$  for  $i \in I$ . It is easy to check that this action commutes with the  $\mathbf{C}^*$ -action and with the  $W$ -action.

### 3. The varieties $\Lambda_{\mathbf{D}, \mathbf{V}}$ .

3.1. Given  $\mathbf{D}, \mathbf{V} \in \mathcal{C}^0$ , let  $M_{\mathbf{D}, \mathbf{V}}$  be the vector space consisting of all triples  $(x, p, q)$  where

$$\begin{aligned} x &= (x_h)_{h \in H}, x_h \in \text{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}), \\ p &= (p_j)_{j \in I}, p_j \in \text{Hom}(\mathbf{D}_j, \mathbf{V}_j), \\ q &= (q_j)_{j \in I}, q_j \in \text{Hom}(\mathbf{V}_j, \mathbf{D}_j). \end{aligned}$$

Following [N1], let  $\Lambda_{\mathbf{D}, \mathbf{V}}$  be the affine variety consisting of all  $((x, p, q), \lambda)$  in  $M_{\mathbf{D}, \mathbf{V}} \times E$  such that

$$(a) \quad \sum_{h; h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_i q_i - \lambda_i = 0 : \mathbf{V}_i \rightarrow \mathbf{V}_i$$

for all  $i \in I$ . For any  $\lambda \in E$  let  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  be the affine variety consisting of all  $(x, p, q) \in M_{\mathbf{D}, \mathbf{V}}$  such that (a) holds; this may be naturally identified with the fibre at  $\lambda$  of the fourth projection  $\Lambda_{\mathbf{D}, \mathbf{V}} \rightarrow E$ .

The group  $G_{\mathbf{V}}$  acts on  $M_{\mathbf{D}, \mathbf{V}}$  in a natural way (see [L4, 1.2]); this induces an action of  $G_{\mathbf{V}}$  on  $\Lambda_{\mathbf{D}, \mathbf{V}}$  and on  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  for any  $\lambda \in E$ .

3.2. In the remainder of this section we fix  $i \in I, \mathbf{D}, \mathbf{V}, \mathbf{V}' \in \mathcal{C}^0, \lambda, \lambda' \in E$  such that  $\lambda' = s_i(\lambda)$  and  $\mathbf{V}_j = \mathbf{V}'_j$  for  $j \in I - \{i\}$ ;  $\dim \mathbf{V}_i + \dim \mathbf{V}'_i = \dim \mathbf{D}_i + \sum_{h; h'=i} \dim \mathbf{V}_{h''}$ . Let  $U = \mathbf{D}_i \oplus \oplus_{h; h'=i} \mathbf{V}_{h''}$ . Let  $F$  be the affine variety whose points are the pairs  $((x, p, q); (x', p', q')) \in M_{\mathbf{D}, \mathbf{V}} \times M_{\mathbf{D}, \mathbf{V}'}$  such that conditions (a)–(d2) below are satisfied:

(a) the sequence  $0 \rightarrow \mathbf{V}'_i \xrightarrow{a} U \xrightarrow{b} \mathbf{V}_i \rightarrow 0$  is exact; here,  $a = (q'_i, (x'_h)_{h; h'=i})$  and  $b = (p_i, (\varepsilon_{\bar{h}} x_h)_{h; h''=i})$ ;

(b1) we have  $\tilde{b}b - a\tilde{a} = \lambda'_i : U \rightarrow U$  where  $\tilde{a} = (p'_i, (\varepsilon_{\bar{h}} x'_h)_{h; h''=i}) : U \rightarrow \mathbf{V}'_i$  and  $\tilde{b} = (q_i, (x_h)_{h; h'=i}) : \mathbf{V}_i \rightarrow U$ ;

(b2)  $\varepsilon_{\bar{h}}(x_{\bar{h}} x_h - x'_{\bar{h}} x'_h) = \delta_{\bar{h}, \tilde{h}} \lambda'_i : \mathbf{V}_{h'} \rightarrow \mathbf{V}_{\tilde{h}'}$  for any  $h, \tilde{h}$  such that  $h'' = i, \tilde{h}' = i$ ;  $q_i p_i - q'_i p'_i = \lambda'_i : \mathbf{D}_i \rightarrow \mathbf{D}_i$ ;  $x_h p_i - x'_h p'_i = 0$  for any  $h$  such that  $h' = i$ ;  $q_i x_h - q'_i x'_h = 0$  for any  $h$  such that  $h'' = i$ ;

- (c)  $x_h = x'_h$  if  $h' \neq i, h'' \neq i; p_j = p'_j, q_j = q'_j$  if  $j \neq i$ ;  
 (d1)  $\sum_{h;h'=j} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_j q_j = \lambda_j : \mathbf{V}_j \rightarrow \mathbf{V}_j$  if  $j \neq i$ ;  
 (d2)  $\sum_{h;h'=j} \varepsilon_{\bar{h}} x'_h x'_h - p'_j q'_j = \lambda'_j : \mathbf{V}_j \rightarrow \mathbf{V}_j$  if  $j \neq i$ .

*Remarks.*

(i) Conditions (b1), (b2) are equivalent.

(ii) In the presence of (b2), (c), conditions (d1), (d2) are equivalent. Indeed, let  $\delta$  be the difference of the left hand sides of the equalities in (d1), (d2). We have  $\delta = \sum_{h;h'=j} \varepsilon_{\bar{h}} (x_{\bar{h}} x_h - x'_h x'_h) - p_j q_j + p'_j q'_j$ . Using (c) we see that  $\delta = \sum_{h;h'=j, h''=i} \varepsilon_{\bar{h}} (x_{\bar{h}} x_h - x'_h x'_h)$  and, by (b2), this is  $\lambda'_i \sharp(h; h' = j, h'' = i)$ , which equals  $\lambda_j - \lambda'_j$ .

(iii) For a point in  $F$  we have automatically  $\sum_{h;h'=i} \varepsilon_{\bar{h}} x'_h x'_h - p'_i q'_i = \lambda'_i$ . Indeed, since  $a$  in (a) is injective, it is enough to show that

$$(*) \quad \sum_{h;h'=i} \varepsilon_{\bar{h}} x'_h x'_h - x'_h p'_i q'_i - \lambda'_i x'_h = 0 : \mathbf{V}'_i \rightarrow \mathbf{V}_{h''}$$

for any  $\tilde{h}$  such that  $\tilde{h}' = i$  and

$$(**) \quad \sum_{h;h'=i} \varepsilon_{\bar{h}} q'_i x'_h x'_h - q'_i p'_i q'_i - \lambda'_i q'_i = 0 : \mathbf{V}'_i \rightarrow \mathbf{D}'_i.$$

By (b2), the left hand side of (\*) is

$$\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\tilde{h}} x_{\bar{h}} x'_h - x_{\tilde{h}} p_i q'_i$$

and this is 0, by (a). Again by (b2), the left hand side of (\*\*) is

$$\sum_{h;h'=i} \varepsilon_{\bar{h}} q_i x_{\bar{h}} x'_h - q_i p_i q'_i$$

and this is 0, by (a).

(iv) For a point in  $F$  we have automatically  $\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h - p_i q_i = \lambda_i$ . Indeed, since  $b$  in (a) is surjective, it is enough to show that

$$(*) \quad \sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h x_{\tilde{h}} - p_i q_i x_{\tilde{h}} - \lambda_i x_{\tilde{h}} = 0 : \mathbf{V}_{h'} \rightarrow \mathbf{V}_i$$

for any  $\tilde{h}$  such that  $\tilde{h}'' = i$  and

$$(**) \quad \sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h p_i - p_i q_i p_i - \lambda_i p_i = 0 : \mathbf{D}_i \rightarrow \mathbf{V}_i.$$

By (b2), the left hand side of (\*) is

$$\sum_{h;h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x'_h x'_{\tilde{h}} - \lambda'_i x'_{\tilde{h}} - p_i q'_i x'_{\tilde{h}} - \lambda_i x'_{\tilde{h}}$$

which by (a) equals  $-\lambda'_i x_h - \lambda_i x_h = 0$ . Again by (b2), the left hand side of (\*\*) is

$$\sum_{h,h'=i} \varepsilon_h x_h x'_h p'_i - p_i q'_i p'_i - \lambda'_i p_i - \lambda_i p_i$$

which by (a) equals  $-\lambda'_i p_i - \lambda_i p_i = 0$ .

3.3. From Remarks (iii), (iv) in 3.2, we see that the first (resp. second) projection is a well defined map  $r : F \rightarrow \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  (resp.  $r' : F \rightarrow \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$ ).

3.4. In the remainder of this section we assume that  $\lambda_i \neq 0$ . In this case, for any  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ , the map  $b : U \rightarrow \mathbf{V}_i$  (as in 3.2(a)) is surjective. Indeed, the identity map of  $\mathbf{V}_i$  is equal to  $-\lambda_i^{-1} \subset \tilde{b}\tilde{b}$  with  $\tilde{b}$  as in 3.2(b1).

Similarly, for any  $(x', p', q') \in \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$ , the map  $a : \mathbf{V}'_i \rightarrow U$  (as in 3.2(a)) is injective. Indeed, the identity map of  $\mathbf{V}'_i$  is equal to  $-\lambda_i'^{-1} \subset \tilde{a}\tilde{a}$  with  $\tilde{a}$  as in 3.2(b1). The group

$$G = GL(\mathbf{V}_i) \times GL(\mathbf{V}'_i) \times \prod_{j:j \neq i} GL(\mathbf{V}_j)$$

acts naturally on  $F, \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}, \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$  compatibly with the maps  $r, r'$ , so that the  $G$  action on  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  factors through the  $G_{\mathbf{V}}$ -action in 3.1 and the  $G$  action on  $\Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$  factors through the analogous  $G_{\mathbf{V}'}$ -action.

PROPOSITION 3.5.

- (a)  $r$  is a principal  $GL(\mathbf{V}'_i)$ -bundle.
- (b)  $r'$  is a principal  $GL(\mathbf{V}_i)$ -bundle.

We prove (a). We fix  $x = (x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ . Then  $r^{-1} \subset (x)$  may be identified with the set of all pairs  $(a, \tilde{a})$  where

$$a = (q'_i, (x'_h)_{h,h'=i}) \in \text{Hom}(\mathbf{V}'_i, U), \quad \tilde{a} = (p'_i, (\varepsilon_h x'_h)_{h,h''=i}) \in \text{Hom}(U, \mathbf{V}'_i)$$

are such that 3.2(a), (b1) hold. (Then 3.2(d2) holds automatically by 3.2(ii).) We show that the first projection establishes a bijection

$$(c) \quad \{(a, \tilde{a}) \mid 3.2(a), (b1) \text{ hold}\} \xrightarrow{\sim} \{a \mid 3.2(a) \text{ holds}\}.$$

Let  $\phi = \tilde{b}\tilde{b} - \lambda'_i : U \rightarrow U$  where  $\tilde{b} = (q_i, (x_h)_{h,h'=i}) : \mathbf{V}_i \rightarrow U$ . Assume that  $a$  satisfies 3.2(a). We must show that there is a unique linear map  $\tilde{a} : U \rightarrow \mathbf{V}'_i$  such that  $a\tilde{a} = \phi$ , or equivalently, that the image of  $\phi$  is contained in the image of the imbedding  $a$  (that is, in the kernel of  $b$ ). Thus, it is enough

to show that  $b\phi = 0$ , or that  $\widetilde{b}b - \lambda'_i b = 0$ . It is also enough to show that  $\widetilde{b}b - \lambda'_i = 0$ . This is clear.

We see that  $r^{-1} \subset (x)$  may be identified with the set of all  $a : \mathbf{V}'_i \rightarrow U$  such that 3.2(a) holds, or equivalently (since  $b$  is surjective) with the set of all isomorphisms of  $\mathbf{V}'_i$  onto  $\text{Ker } b$ . This is clearly isomorphic to  $GL(\mathbf{V}'_i)$ . This proves (a).

We prove (b). We fix  $(x', p', q') \in \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$ . Then  $r'^{-1} \subset (x')$  may be identified with the set of all pairs  $(b, \widetilde{b})$  where

$$b = (p_i, (\varepsilon_{\bar{h}} x_h)_{h; h''=i}) \in \text{Hom}(U, \mathbf{V}_i), \quad \widetilde{b} = (q_i, (x_h)_{h; h'=i}) \in \text{Hom}(\mathbf{V}_i, U)$$

are such that 3.2(a), (b) hold. (Then 3.2(d1) holds automatically by 3.2(ii).) We show that the first projection establishes a bijection

$$(d) \quad \{(b, \widetilde{b}) | 3.2(a), (b1) \text{ hold}\} \xrightarrow{\sim} \{b | 3.2(a) \text{ holds}\}.$$

Let  $\psi = a\widetilde{a} + \lambda'_i : U \rightarrow U$  where  $\widetilde{a} = (p'_i, (\varepsilon_{\bar{h}} x'_h)_{h; h''=i})$ . Assume that  $b$  satisfies 3.2(a). We must show that there is a unique linear map  $\widetilde{b} : \mathbf{V}_i \rightarrow U$  such that  $\widetilde{b}b = \psi$ , or equivalently, that the kernel of  $b$  (that is, the image of  $a$ ) is contained in  $\text{Ker } \psi$ . Thus, it is enough to show that  $\psi a = 0$  or that  $a\widetilde{a}a + \lambda'_i a = 0$ . It is enough to show that  $a\widetilde{a} + \lambda'_i = 0$ . This is clear. We see that  $r'^{-1} \subset (x')$  may be identified with the set of all  $b : U \rightarrow \mathbf{V}_i$  such that 3.2(a) holds, or equivalently (since  $a$  is injective) with the set of all isomorphisms of the cokernel of  $a$  onto  $\mathbf{V}_i$ . This is clearly isomorphic to  $GL(\mathbf{V}_i)$ . This proves (b). The proposition is proved.

COROLLARY 3.6. — *The maps  $r, r'$  in 3.3 induce bijections*

$$(a) \quad \Lambda_{\mathbf{D}, \mathbf{V}, \lambda} / G_{\mathbf{V}} \xleftarrow{\sim} F/G \xrightarrow{\sim} \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'} / G_{\mathbf{V}'}$$

*where orbit spaces are taken in the set theoretical sense, and isomorphisms of affine algebraic varieties*

$$(b) \quad \Lambda_{\mathbf{D}, \mathbf{V}, \lambda} // G_{\mathbf{V}} \xleftarrow{\sim} F // G \xrightarrow{\sim} \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'} // G_{\mathbf{V}'}$$

*where the orbit spaces are taken in the algebraic geometric sense.*

COROLLARY 3.7. — *Assume that  $\mathbf{D} = 0$  and let*

$$d_{\mathbf{V}} = (1/2) \sum_h \dim \mathbf{V}_{h'} \dim \mathbf{V}_{h''}.$$

*Then  $\Lambda_{0, \mathbf{V}, \lambda}$  has pure dimension  $d_{\mathbf{V}}$  if and only if  $\Lambda_{0, \mathbf{V}', \lambda'}$  has pure dimension  $d_{\mathbf{V}'}$ .*

We set  $\nu_j = \dim \mathbf{V}_j, \nu'_j = \dim \mathbf{V}'_j$ . The two conditions above are equivalent to the condition that  $F$  has pure dimension  $d_{\mathbf{V}} + \nu_i'^2$ , (resp.  $d_{\mathbf{V}'} + \nu_i^2$ ). So it is enough to prove that  $d_{\mathbf{V}} - d_{\mathbf{V}'} = \nu_i^2 - \nu_i'^2$ . We have

$$\begin{aligned} d_{\mathbf{V}} - d_{\mathbf{V}'} &= \sum_{h; h'=i} \nu_{h''} \nu_i - \sum_{h; h'=i} \nu_{h''} \nu'_i = \sum_{h; h'=i} \nu_{h''} (\nu_i - \nu'_i) \\ &= (\nu_i + \nu'_i)(\nu_i - \nu'_i) = \nu_i^2 - \nu_i'^2. \end{aligned}$$

The lemma is proved.

#### 4. The subsets $Z_{D, \mathbf{V}}$ and ${}_{\mathbf{V}}Z_D$ of $Z_D$ .

4.1. In the remainder of this paper we assume that our graph is of finite type, that is,  $W$  is a finite group.

Let  $E' = \text{Hom}(E, \mathbf{C})$ . We shall regard  $i \in I$  as an element of  $E'$  by  $i(\varpi_j) = \delta_{ij}$ . For any  $i \in I$ , we define  $\alpha_i \in E$  by  $\alpha_i = 2\varpi_i - \sum_{h; h'=i} \varpi_{h''}$ . Then  $\{\alpha_i, i\}$  form a root datum and  $W$  is its Weyl group. For  $\nu \in E'$  we define  $\nu_i \in \mathbf{C}$  by  $\nu = \sum_i \nu_i i$ . The action of  $W$  on  $E$  induces an action of  $W$  on  $E'$ , given by  $s_i : \xi \mapsto s_i(\xi) = \xi - \xi(\alpha_i)i$ . Let  $\tilde{R}$  be the set of vectors in  $E'$  of the form  $w(i)$  for some  $i \in I, w \in W$ . For  $\lambda \in E$  let

$$\tilde{R}_{\lambda} = \{\tilde{\alpha} \in \tilde{R} | \tilde{\alpha}(\lambda) = 0\}.$$

4.2. Let  $E_0$  be the set of all  $\lambda \in E$  such that for any  $i$  we have either  $\text{Re}(\lambda_i) > 0$ , or  $\text{Re}(\lambda_i) = 0$  and  $\text{Im}(\lambda_i) \geq 0$ .

LEMMA 4.3. — Any  $W$ -orbit in  $E$  meets  $E_0$  in a unique point.

This is well known.

LEMMA 4.4. — Let  $\mathbf{V} \in \mathcal{C}^0, \lambda \in E_0$ . Assume that  $\Lambda_{0, \mathbf{V}, \lambda} \neq \emptyset$ . Then for any  $i \in I$  we have either  $\lambda_i = 0$  or  $\mathbf{V}_i = 0$ .

Let  $((x_h), 0, 0) \in \Lambda_{0, \mathbf{V}, \lambda}$ . We have

$$\begin{aligned} \sum_i \lambda_i \dim \mathbf{V}_i &= \sum_i \text{Tr}(\lambda_i, \mathbf{V}_i) = \sum_i \text{Tr}\left(\sum_{h; h'=i} \varepsilon_{\bar{h}} x_{\bar{h}} x_h, \mathbf{V}_i\right) \\ &= \sum_h \varepsilon_{\bar{h}} \text{Tr}(x_{\bar{h}} x_h, \mathbf{V}_{h'}). \end{aligned}$$

In the last sum the term corresponding to  $h, \bar{h}$  cancel out. Hence the sum is zero and we have

$$\sum_i \lambda_i \dim \mathbf{V}_i = 0.$$



Let  $I' = \{i \in I \mid \mathbf{V}_i \neq 0\}$ . Write  $\operatorname{Re}(\lambda_i) = \lambda'_i, \operatorname{Im}(\lambda_i) = \lambda''_i$ . Then  $\sum_{i \in I'} \lambda_i \dim \mathbf{V}_i = 0$ , hence

$$\sum_{i \in I'} \lambda'_i \dim \mathbf{V}_i = 0, \quad \sum_{i \in I'} \lambda''_i \dim \mathbf{V}_i = 0.$$

Since  $\lambda'_i \in \mathbf{R}_{\geq 0}$  for all  $i$ , we deduce that for any  $i \in I', \lambda'_i = 0$ , hence  $\lambda''_i \geq 0$ . Hence the equality  $\sum_{i \in I'} \lambda''_i \dim \mathbf{V}_i = 0$  implies  $\lambda''_i = 0$  for all  $i \in I'$ . Thus, for  $i \in I'$  we have  $\lambda'_i = \lambda''_i = 0$  hence  $\lambda_i = 0$ . The lemma is proved.

PROPOSITION 4.5. — *Let  $\mathbf{V} \in \mathcal{C}^0, \lambda \in E$  be such that  $\Lambda_{0, \mathbf{V}, \lambda} \neq \emptyset$ . Then*

- (a)  $\Lambda_{0, \mathbf{V}, \lambda}$  has pure dimension  $d_{\mathbf{V}}$ ;
- (b)  $G_{\mathbf{V}}$  has a unique closed orbit in  $\Lambda_{0, \mathbf{V}, \lambda}$ ;
- (c) if  $\tilde{R}_{\lambda} = \emptyset$ , then  $\mathbf{V} = 0$ .

Assume first that  $\lambda \in E_0$ . Let  $I_0 = \{i \in I \mid \lambda_i = 0\}$ . Clearly,  $I_0 \subset \tilde{R}_{\lambda}$ .

We prove (a). We may replace the datum  $(I, H, \dots)$  in 1.1 by  $(I_0, H_0, \dots)$  where  $H_0 = \{h \in H \mid h' \in I_0, h'' \in I_0\}$ . Using Lemma 4.4, we see that  $\Lambda_{0, \mathbf{V}, \lambda}$  may be identified with  $\Lambda_{0, \mathbf{V}', 0}$  defined in terms of  $(I_0, H_0, \dots)$  where  $\mathbf{V}'$  is the  $I_0$ -graded vector space defined by  $\mathbf{V}'_i = \mathbf{V}_i$  for  $i \in I_0$ . By [L3, 12.3],  $\Lambda_{0, \mathbf{V}', 0}$  has pure dimension  $d_{\mathbf{V}'}$ . Since  $\mathbf{V}_i = 0$  for  $i \notin I_0$ , we have  $d_{\mathbf{V}} = d_{\mathbf{V}'}$ . This proves (a).

We prove (b). As in the proof of (a), we are reduced to the case where  $\lambda = 0$ . In that case the result is contained in [L4, 5.9].

We prove (c). Assume that  $\tilde{R}_{\lambda} = \emptyset$  and  $\mathbf{V}_i \neq 0$ . By Lemma 4.4, we have  $\lambda_i = 0$  hence  $i \in \tilde{R}_{\lambda}$ , a contradiction.

This completes the proof of the proposition under the assumption that  $\lambda \in E_0$ . We now consider the general case.

For any  $\lambda \in E$ , let  $r = r_{\lambda}$  be the smallest integer  $\geq 0$  such that there exists a sequence  $\lambda = \lambda^0, \lambda^1, \dots, \lambda^r$  in  $E$  and a sequence  $i_1, i_2, \dots, i_r$  in  $I$  with the following properties:

$$\lambda^r \in E_0, \quad \lambda^1 = s_{i_1}(\lambda^0), \lambda^2 = s_{i_2}(\lambda^1), \dots, \lambda^r = s_{i_r}(\lambda^{r-1}), \quad \lambda^j \neq \lambda^{j+1}$$

for  $j = 0, 1, \dots, r-1$ . Note that  $r_{\lambda}$  is well defined by 4.3. We prove the proposition for  $\lambda \in E$  (and any  $\mathbf{V}$  such that  $\Lambda_{0, \mathbf{V}, \lambda} \neq \emptyset$ ) by induction on  $r_{\lambda}$ . If  $r_{\lambda} = 0$ , the result is clear by the first part of the proof. Assume now that  $r_{\lambda} \geq 1$ . By definition, we can find  $i \in I$  such that  $\lambda' = s_i(\lambda) \neq \lambda$  and  $r_{\lambda'} = r_{\lambda} - 1$ . Since  $\lambda' \neq \lambda$ , we have  $\lambda_i \neq 0$ . We define  $U, b: \mathbf{V}_i \rightarrow U$  as in

3.2 (with  $\mathbf{D} = 0$ ) in terms of some  $(x, 0, 0) \in \Lambda_{0, \mathbf{V}, \lambda}$ . As in 3.4, from  $\lambda_i \neq 0$  we deduce that  $b$  is surjective. Hence  $\dim \mathbf{V}_i \geq \dim U = \sum_{h; h'=i} \dim \mathbf{V}_{h''}$ . Hence there exists  $\mathbf{V}' \in \mathcal{C}^0$  such that

$$\begin{aligned} \mathbf{V}_j &= \mathbf{V}'_j \text{ for } j \in I - \{i\}; \\ \dim \mathbf{V}_i + \dim \mathbf{V}'_i &= \sum_{h; h'=i} \dim \mathbf{V}_{h''}. \end{aligned}$$

By 3.6(a),  $\Lambda_{0, \mathbf{V}', \lambda'} \neq \emptyset$ . By the induction hypothesis, the proposition holds for  $(\mathbf{V}', \lambda')$ . Using 3.7, we see that (a) holds for  $(\mathbf{V}, \lambda)$ . Using 3.6(b), we see that (b) holds for  $(\mathbf{V}, \lambda)$ . Finally, assume that  $\check{R}_\lambda = \emptyset$ . Then  $\check{R}_{\lambda'} = s_i(\check{R}_\lambda) = \emptyset$ . Hence  $\mathbf{V}' = 0$ , by the induction hypothesis. Then the formulas above relating  $\mathbf{V}_j, \mathbf{V}'_j$  show that  $\mathbf{V}_j = 0$  for all  $j$ . The proposition is proved.

4.6. Let  $\mathbf{D} \in \mathcal{C}^0$ . For any  $f \in \mathbf{I}$  of form  $i_1, i_2, \dots, i_s$  and any linear form  $\chi : \text{Hom}(\mathbf{D}_{i_s}, \mathbf{D}_{i_1}) \rightarrow \mathbf{C}$ , we define a function  $b_{f, \chi} : Z_{\mathbf{D}} \rightarrow \mathbf{C}$  by  $b_{f, \chi}(\lambda, \pi) = \chi(\pi_{[f]})$ . For any  $i \in I$ , let  $\xi_i : Z_{\mathbf{D}} \rightarrow \mathbf{C}$  be defined by  $\xi_i(\lambda, \pi) = \lambda_i$ . Let  $B_1$  be the  $\mathbf{C}$ -algebra with 1 of functions  $Z_{\mathbf{D}} \rightarrow \mathbf{C}$  generated by the functions  $b_{f, \chi}$  for various  $f, \chi$  as above and by the functions  $\xi_i$  with  $i \in I$ . An argument almost identical to that in [L4, 5.3] shows that  $B_1$  is a finitely generated algebra and that  $Z_{\mathbf{D}}$  is naturally in bijection with the set of algebra homomorphisms  $B_1 \rightarrow \mathbf{C}$ . Thus,  $Z_{\mathbf{D}}$  has a natural structure of affine variety.

Now let  $\mathbf{V} \in \mathcal{C}^0$ . As in [L4, 2.12], we define a map  $\vartheta' : \Lambda_{\mathbf{D}, \mathbf{V}} \rightarrow Z_{\mathbf{D}}$  by  $(x, p, q, \lambda) \mapsto (\lambda, \pi)$  where  $\pi \in \tilde{\mathcal{F}}_{\mathbf{D}}$  is given by

$$\pi_{[f]} = q_{i_1} x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_{s-1}, i_s} p_{i_s} : \mathbf{D}_{i_s} \rightarrow \mathbf{D}_{i_1}$$

for any  $f \in \mathbf{I}$  of form  $i_1, i_2, \dots, i_s$  with  $s \geq 2$ ,

$$\pi_{[j]} = q_j p_j : \mathbf{D}_j \rightarrow \mathbf{D}_j$$

for any  $j \in I$  and  $\pi_{u_i}$  is as in 2.1(b) for any  $i \in I$ .

The map  $\vartheta' : \Lambda_{\mathbf{D}, \mathbf{V}} \rightarrow Z_{\mathbf{D}}$  is easily seen to be a morphism of algebraic varieties. Since this map is constant on the orbits of  $G_{\mathbf{V}}$  on  $\Lambda_{\mathbf{D}, \mathbf{V}}$ , it induces a morphism  $\vartheta : \Lambda_{\mathbf{D}, \mathbf{V}} // G_{\mathbf{V}} \rightarrow Z_{\mathbf{D}}$  of algebraic varieties.

**THEOREM 4.7.** —  *$\vartheta$  is a finite, injective morphism. In particular, it is a homeomorphism onto its image (both in the Zariski and ordinary topology).*

The finiteness of  $\vartheta$  is proved as in [L4, 5.8]. The injectivity is proved in [L4, 5.10] modulo the statement [L4, 5.9] which at the time of writing

[L4] was only known for  $\lambda = 0$  but is now known without restriction, by 4.5(b).

4.8. Let  $\Lambda_{\mathbf{D}, \mathbf{V}}^s$  be the open subvariety of  $\Lambda_{\mathbf{D}, \mathbf{V}}$  consisting of all  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}$  such that the following stability condition holds: if  $\mathbf{V}'$  is an  $I$ -graded subspace of  $\mathbf{V}$  such that  $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$  for all  $h \in H$  and  $p_i(\mathbf{D}_i) \subset \mathbf{V}'_i$  for all  $i$ , then  $\mathbf{V}' = \mathbf{V}$ .

Let  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$  be the open subvariety of  $\Lambda_{\mathbf{D}, \mathbf{V}}$  consisting of all  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}$  such that the following stability condition holds: if  $\mathbf{V}'$  is an  $I$ -graded subspace of  $\mathbf{V}$  such that  $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$  for all  $h \in H$  and  $q_i(\mathbf{V}') = 0$  for all  $i$ , then  $\mathbf{V}' = 0$ .  $\Lambda_{\mathbf{D}, \mathbf{V}}^s, \Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$  were introduced in [N2], in a different notation.

For any  $\lambda \in E$ , let  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s$  (resp.  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*,s}$ ) be the open subvariety of  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  consisting of all triples  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  such that  $(x, p, q, \lambda)$  belongs to  $\Lambda_{\mathbf{D}, \mathbf{V}}^s$  (resp. to  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$ ). Then  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s$  (resp.  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*,s}$ ) may be naturally identified with the fibre of the fourth projection  $\Lambda_{\mathbf{D}, \mathbf{V}}^s \rightarrow E$  (resp.  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*,s} \rightarrow E$ ) at  $\lambda$ .

By [N2], the natural action of  $G_{\mathbf{V}}$  on  $\Lambda_{\mathbf{D}, \mathbf{V}}^s$  or  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$  is free; hence the orbit spaces  $\Lambda_{\mathbf{D}, \mathbf{V}}^s/G_{\mathbf{V}}, \Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}/G_{\mathbf{V}}$  are well defined.

4.9. There is a natural isomorphism  $\Lambda_{\mathbf{D}, \mathbf{V}}^s \xrightarrow{\sim} \Lambda_{\mathbf{D}^*, \mathbf{V}^*}^{*,s}$ ; it is given by  $(x, p, q, \lambda) \mapsto ({}^t\bar{x}, {}^tq, {}^tp, \lambda)$ . (Notation of [L4, 2.27].)

4.10. Let  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s,*,s} = \Lambda_{\mathbf{D}, \mathbf{V}}^s \cap \Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$ . For any  $\lambda \in E$ , let  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{s,*,s} = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s \cap \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*,s}$ .

LEMMA 4.11. — *The following two conditions for  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}$  are equivalent:*

- (i)  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^{s,*,s}$ ;
- (ii)  $(x, p, q, \lambda)$  has trivial isotropy group in  $G_{\mathbf{V}}$  and its  $G_{\mathbf{V}}$ -orbit in  $\Lambda_{\mathbf{D}, \mathbf{V}}$  is closed.

Assume that (ii) holds. Then  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^{*,s}$  by [N2, 3.24]; the same proof shows that  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^s$ , hence (i) holds.

Assume now that (i) holds. Then the first assertion of (ii) is proved in [N2, 3.10]. It remains to prove the second assertion. Let  $\mathcal{O}$  be a  $G_{\mathbf{V}}$ -orbit in the closure of the orbit of  $(x, p, q, \lambda)$ . By Hilbert's theorem, there exists a one parameter subgroup  $\zeta_t$  of  $G_{\mathbf{V}}$  ( $t \in \mathbf{C}^*$ ) such that  $\lim_{t \rightarrow \infty} \zeta_t(x, p, q, \lambda) =$

$(x', p', q', \lambda) \in \mathcal{O}$ . We can write  $\mathbf{V} = \oplus_k \mathbf{V}^k$  where  $\zeta_t v = t^k v$  for all  $v \in \mathbf{V}^k$ . Let  $\mathbf{V}^{(k)} = \oplus_{k' \leq k} \mathbf{V}^{k'}$ . We have  $x_h(v) = \sum_{k'} x_{h;k,k'} v$  for  $v \in \mathbf{V}_{h'}^k$  where  $x_{h;k,k'} : \mathbf{V}_{h'}^k \rightarrow \mathbf{V}_{h''}^{k'}$ . We have  $p(d) = \sum_k p^{(k)}(d)$  for  $d \in \mathbf{D}$  where  $p^k : \mathbf{D} \rightarrow \mathbf{V}^k$ . For  $t \in \mathbf{C}^*$ , write  $\zeta_t(x, p, q) = (x(t), p(t), q(t))$ . We have

$$x(t)_h = \sum_{k'} x_{h;k,k'} t^{k'-k}, \quad p(t) = \sum_k t^k p^{(k)}$$

and  $q(t)(v) = t^{-k} q(v)$  for  $v \in \mathbf{V}^k$ . Since  $\lim_{t \rightarrow \infty} \zeta_t(x, p, q, \lambda)$  exists, it follows that  $x_{h;k,k'} = 0$  for  $k' > k$ ,  $p^{(k)} = 0$  for  $k > 0$ ,  $q|_{\mathbf{V}^k} = 0$  for  $k < 0$ . Hence  $\mathbf{V}^{(-1)}$  is  $x$ -stable and contained in  $\text{Ker}(q)$ . Since  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^{s, ss}$ , it follows that  $\mathbf{V}^{(-1)} = 0$  and  $p(\mathbf{D}) \subset \mathbf{V}^0$ . (Up to this point, the argument is exactly as in [N2, 3.20].) Moreover, for  $k \geq 0$ ,  $\mathbf{V}^{(k)}$  is  $x$ -stable and contains  $\text{Im}(p)$ . Since  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^s$ , it follows that  $\mathbf{V}^{(k)} = \mathbf{V}$  for  $k \geq 0$ . Thus,  $\mathbf{V} = \mathbf{V}^0$  hence  $(x', p', q', \lambda) = (x, p, q, \lambda)$ . This proves that the  $G_{\mathbf{V}}$ -orbit of  $(x, p, q, \lambda)$  is closed. The lemma is proved.

LEMMA 4.12.

- (a) The map  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, ss}/G_{\mathbf{V}} \rightarrow Z_{\mathbf{D}}$  induced by  $\vartheta' : \Lambda_{\mathbf{D}, \mathbf{V}} \rightarrow Z_{\mathbf{D}}$  is injective.
- (b) Its image,  $Z_{\mathbf{D}, \mathbf{V}}$ , depends only on the isomorphism class of  $\mathbf{V}$  in  $\mathcal{C}^0$ .
- (c)  $Z_{\mathbf{D}, \mathbf{V}}$  is a locally closed subvariety of  $Z_{\mathbf{D}}$  and is homeomorphic (both for the Zariski and ordinary topology) to  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, ss}/G_{\mathbf{V}}$ .
- (d) The subsets  $Z_{\mathbf{D}, \mathbf{V}}$  (for  $\mathbf{V}$  running through a set of representatives of the isomorphism classes of objects in  $\mathcal{C}^0$ ) form a partition of  $Z_{\mathbf{D}}$ .

We prove (a). Our map is the composition

$$\Lambda_{\mathbf{D}, \mathbf{V}}^{s, ss}/G_{\mathbf{V}} \rightarrow \Lambda_{\mathbf{D}, \mathbf{V}}//G_{\mathbf{V}} \xrightarrow{\vartheta} Z_{\mathbf{D}}$$

where the first map (the obvious one) is injective by 4.11 and  $\vartheta$  is injective by 4.7. This proves (a). The proof of (b) is trivial.

We prove (c). From 4.11 we see that  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, ss}/G_{\mathbf{V}}$  is an open subvariety of  $\Lambda_{\mathbf{D}, \mathbf{V}}//G_{\mathbf{V}}$  and from 4.7 we see that  $\Lambda_{\mathbf{D}, \mathbf{V}}//G_{\mathbf{V}}$  is mapped by  $\vartheta$  homeomorphically onto a closed subvariety of  $Z_{\mathbf{D}}$ . This proves (c).

We prove (d). Let  $(\lambda, \pi) \in Z_{\mathbf{D}}$ . Let  $\mathbf{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{K}^{\pi}$  (notation of [L4, 2.3, 2.8]). We have  $\mathbf{V} \in \mathcal{C}^0$  by [L4, 5.12]. We define  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  as in [L4, 2.18] (with  $\mathcal{V} = \mathcal{K}^{\pi}$ ). As pointed out in [L4, 2.18], we have  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s$  and  $\vartheta'(x, p, q, \lambda) = (\lambda, \pi)$ . From the definition of  $\mathcal{K}^{\pi}$  we see also that  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{s, ss}$ . Thus,  $(\lambda, \pi) \in Z_{\mathbf{D}, \mathbf{V}}$ . We see that the union of the subsets  $Z_{\mathbf{D}, \mathbf{V}}$  is the whole of  $Z_{\mathbf{D}}$ .

Now let  $(\lambda, \pi) \in Z_{\mathbf{D}, \mathbf{V}} \cap Z_{\mathbf{D}, \mathbf{V}'}$  where  $\mathbf{V}, \mathbf{V}' \in \mathcal{C}^0$ . We want to prove that  $\mathbf{V}, \mathbf{V}'$  are isomorphic in  $\mathcal{C}^0$ . We can find  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s}$  and  $(x', p', q', \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}'}^{s, *s}$  such that  $\vartheta'(x, p, q, \lambda) = \vartheta'(x', p', q', \lambda) = (\lambda, \pi)$ . By [L4, 2.20] we can assume that  $\mathbf{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{V}$ ,  $\mathbf{V}' = \mathcal{E}^{\mathbf{D}}/\mathcal{V}'$  (notation of [L4, 2.3]) where  $\mathcal{V}, \mathcal{V}'$  are  $I$ -graded subspaces of  $\mathcal{E}^{\mathbf{D}}$  containing  $\mathcal{I}^\pi$  and contained in  $\mathcal{K}^\pi$  (notation of [L4, 2.8]), that  $(x, p, q)$  is obtained from  $\mathcal{V}$  as in [L4, 2.18] and that  $(x', p', q')$  is obtained in an analogous way from  $\mathcal{V}'$ . From the definition of  $\mathcal{K}^\pi$  we see that the condition that  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathcal{E}^{\mathbf{D}}/\mathcal{V}}^{*s}$  is equivalent to the condition that  $\mathcal{V} = \mathcal{K}^\pi$ . Similarly, the condition that  $(x', p', q', \lambda) \in \Lambda_{\mathbf{D}, \mathcal{E}^{\mathbf{D}}/\mathcal{V}'}^{*s}$  is equivalent to the condition that  $\mathcal{V}' = \mathcal{K}^\pi$ . Hence we have  $\mathcal{V} = \mathcal{V}' = \mathcal{K}^\pi$ . It follows that  $\mathcal{E}^{\mathbf{D}}/\mathcal{V} = \mathcal{E}^{\mathbf{D}}/\mathcal{V}'$  and our claim follows. The lemma is proved.

LEMMA 4.13.

- (a) The morphism  $\Lambda_{\mathbf{D}, \mathbf{V}}^s/G_{\mathbf{V}} \rightarrow Z_{\mathbf{D}}$  induced by  $\vartheta' : \Lambda_{\mathbf{D}, \mathbf{V}} \rightarrow Z_{\mathbf{D}}$  is proper. Hence its image,  ${}_{\mathbf{V}}Z_{\mathbf{D}}$ , is a closed subvariety of  $Z_{\mathbf{D}}$ .
- (b)  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  depends only on the isomorphism class of  $\mathbf{V}$  in  $\mathcal{C}^0$ .

Our map is the composition

$$\Lambda_{\mathbf{D}, \mathbf{V}}^s/G_{\mathbf{V}} \rightarrow \Lambda_{\mathbf{D}, \mathbf{V}}/G_{\mathbf{V}} \xrightarrow{\vartheta} Z_{\mathbf{D}}$$

where the first map (the obvious one) is proper by [N2, 3.18] and  $\vartheta$  is proper by 4.7. This proves (a). The proof of (b) is trivial.

## 5. A computation of dimensions.

LEMMA 5.1. — Let  $\mathbf{V}, \mathbf{D} \in \mathcal{C}^0$ . The varieties  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*s}, \Lambda_{\mathbf{D}, \mathbf{V}}^s$  are smooth of pure dimension

$$2d_{\mathbf{V}} + 2 \sum_i \dim \mathbf{V}_i \dim \mathbf{D}_i - \sum_i \dim \mathbf{V}_i^2 + |I|$$

and the fourth projections  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*s} \rightarrow E, \Lambda_{\mathbf{D}, \mathbf{V}}^s \rightarrow E$  are submersions.

The fact that  $\Lambda_{\mathbf{D}, \mathbf{V}, 0}^{*s}$  is smooth is proved in [N2, 3.10]. That proof identifies the tangent space of  $\Lambda_{\mathbf{D}, \mathbf{V}, 0}^{*s}$  at  $(x, p, q)$  with the kernel of a certain linear map  $m : M_{\mathbf{D}, \mathbf{V}} \rightarrow \text{Hom}_{\mathcal{C}^0}(\mathbf{V}, \mathbf{V})$  (obtained by taking the derivative of the equation defining  $\Lambda_{\mathbf{D}, \mathbf{V}, 0}$ ). The main point is that  $m$  is surjective (which follows from the stability condition in the definition of  $\Lambda_{\mathbf{D}, \mathbf{V}, 0}^{*s}$ ). A similar argument shows that  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*s}$  is smooth and that the tangent space of  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*s}$  at  $(x, p, q, \lambda)$  is  $\{(k, \lambda') \in M_{\mathbf{D}, \mathbf{V}} \oplus E | m(k) + \lambda' = 0\}$  where  $m$  is as

above and, in the last equation  $\lambda'$  is regarded as an element of  $\text{Hom}_{\mathcal{C}^0}(\mathbf{V}, \mathbf{V})$  whose  $i$ -component is multiplication by  $\lambda'_i$ . This tangent space maps to the tangent space of  $E$  at  $\lambda$  by  $(k, \lambda') \mapsto \lambda'$ . Using the fact that, as above,  $m$  is surjective, the assertions relative to  $\Lambda_{\mathbf{D}, \mathbf{V}}^{*s}$  follow. These assertions imply the assertions relative to  $\Lambda_{\mathbf{D}, \mathbf{V}}^s$ , by 4.9.

5.2. Let  $\lambda \in E$ . Given  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ , let  $\mathbf{V}'$  be the largest  $I$ -graded subspace of  $\mathbf{V}$  such that  $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$  for all  $h$  and  $q_i(\mathbf{V}'_i) = 0$  for all  $i$ . Clearly,  $\mathbf{V}'$  is well defined. Note that  $(x, p, q)$  induces in an obvious way elements

$$(x', 0, 0) \in \Lambda_{0, \mathbf{V}', \lambda}, \quad (x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{*s}.$$

Conversely, assume that we are given an  $I$ -graded subspace  $\mathbf{V}' \subset \mathbf{V}$  and elements  $(x', 0, 0) \in \Lambda_{0, \mathbf{V}', \lambda}$ ,  $(x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{*s}$ .

Let  $\Phi$  be the set of all  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  which give rise as above to  $\mathbf{V}'$ ,  $(x', 0, 0)$ ,  $(x'', p'', q'')$ .

LEMMA 5.3. — *A choice of an  $I$ -graded complement  $\mathbf{V}''$  of  $\mathbf{V}'$  in  $\mathbf{V}$  defines on  $\Phi$  a structure of vector space of dimension*

$$(a) \quad \sum_i \dim \mathbf{V}'_i \left( \dim \mathbf{D}_i - \dim \mathbf{V}''_i + \sum_{h; h'=i} \dim \mathbf{V}''_{h''} \right).$$

Let  $\mathbf{V}''$  as above. We identify  $\mathbf{V}/\mathbf{V}' = \mathbf{V}''$  in an obvious way. For  $(x, p, q) \in \Phi$  we have

$$x_h(v'') = y_h(v'') + x''_h(v'')$$

for all  $v'' \in \mathbf{V}'_{h'}$ , where

$$\begin{aligned} y &= (y_h)_{h \in H}, \quad y_h : \mathbf{V}''_{h'} \rightarrow \mathbf{V}'_{h''}, \\ x_h(v') &= x'_h(v') \quad \text{for all } v' \in \mathbf{V}'_{h'}, \\ p_i(d) &= p'_i(d) + p''_i(d) \quad \text{for } d \in \mathbf{D}_i, \\ p'_i : \mathbf{D}_i &\rightarrow \mathbf{V}'_i, \quad p''_i : \mathbf{D}_i \rightarrow \mathbf{V}''_i, \\ q_i(v'') &= q''_i(v''), \quad \text{for all } v'' \in \mathbf{V}''_i. \end{aligned}$$

By the change of variable  $(x, p, q) \mapsto (y, p')$  the variety  $\Phi$  becomes the set of all  $(y, p')$  as above such that

$$\sum_{h; h'=i} \varepsilon_{\bar{h}} x'_h y_h + \varepsilon_{\bar{h}} y_{\bar{h}} x''_h - p'_i q''_i = 0 : \mathbf{V}''_i \rightarrow \mathbf{V}'_i$$

for all  $i \in I$ . The solutions of this system of equations (with fixed  $x', x'', q''$ ) form a vector space. It remains to show that this vector space has dimension as in (a). This vector space is the kernel of the linear map

$$T : \oplus_h \text{Hom}(\mathbf{V}''_{h'}, \mathbf{V}'_{h''}) \oplus \oplus_i \text{Hom}(\mathbf{D}_i, \mathbf{V}'_i) \rightarrow \oplus_i \text{Hom}(\mathbf{V}''_i, \mathbf{V}'_i),$$

$$(y, p') \mapsto \left( \sum_{h; h'=i} \varepsilon_{\bar{h}} x'_{\bar{h}} y_h + \varepsilon_{\bar{h}} y_{\bar{h}} x''_h - p'_i q''_i \right)_{i \in I}.$$

We will show that  $T$  is surjective; this implies that  $\dim \operatorname{Ker} T$  is given by (a). To show the surjectivity of  $T$ , we consider the perfect bilinear pairing

$$\oplus_i \operatorname{Hom}(\mathbf{V}'_i, \mathbf{V}''_i) \times \oplus_i \operatorname{Hom}(\mathbf{V}''_i, \mathbf{V}'_i) \rightarrow \mathbf{C}$$

given by  $((a_i), (b_i)) = \sum_i \operatorname{tr}(a_i b_i)$ . It is enough to show that, if  $(a_i)$  is orthogonal to  $\operatorname{Im} T$  under this pairing, then  $(a_i) = 0$ . Thus, we assume that

$$\sum_h \varepsilon_{\bar{h}} \operatorname{tr}(a_{h'} x'_{\bar{h}} y_h) + \varepsilon_{\bar{h}} \operatorname{tr}(a_{h'} y_{\bar{h}} x''_h) - \sum_i \operatorname{tr}(a_i p'_i q''_i) = 0$$

for any  $(y, p')$ . Equivalently,

$$\sum_h \varepsilon_{\bar{h}} \operatorname{tr}((a_{h'} x'_{\bar{h}} - x''_{\bar{h}} a_{h''}) y_h) - \sum_i \operatorname{tr}(q''_i a_i p'_i) = 0$$

for any  $(y, p')$ . It follows that

$$(*) \quad a_{h'} x'_{\bar{h}} - x''_{\bar{h}} a_{h''} = 0 \quad \text{for all } h,$$

$$(**) \quad q''_i a_i = 0 \quad \text{for all } i.$$

(\*) shows that  $\operatorname{Im}(a)$  is an  $x'$ -stable  $I$ -graded subspace of  $\mathbf{V}''$ ; (\*\*) shows that  $\operatorname{Im}(a) \subset \operatorname{Ker}(q'')$ . By the stability condition for  $(x'', p'', q'')$ , we then have  $\operatorname{Im}(a) = 0$  hence  $a = 0$ . (Compare with the argument in the proof of [N2, 3.10].) The lemma is proved.

5.4. Now let  $\mathbf{D}, \mathbf{V}, \tilde{\mathbf{V}} \in \mathcal{C}^0$  and let  $(\lambda, \pi) \in {}_{\mathbf{V}} Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}}$ . By definition,  $(\lambda, \pi)$  is in the image of the map

$$(a) \quad \Lambda_{\mathbf{D}, \mathbf{V}}^s \rightarrow Z_{\mathbf{D}}$$

(restriction of  $\vartheta'$ ) and there is a unique  $(\tilde{x}, \tilde{p}, \tilde{q}, \lambda) \in \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^{s, *s}$  which maps to  $(\lambda, \pi)$  under the map  $\vartheta'$  defined in terms of  $\tilde{\mathbf{V}}$ .

Let  $\Psi$  be the fibre of (a) at  $(\lambda, \pi)$ .

PROPOSITION 5.5. —  $\Psi$  has pure dimension equal to

$$(1/2)(\dim \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} - \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}}) + \dim G_{\mathbf{V}}.$$

If  $(x, p, q, \lambda) \in \Psi$  then, by attaching to it

(a)  $\mathbf{V}', (x', 0, 0) \in \Lambda_{0, \mathbf{V}', \lambda}, \quad (x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{*s}$   
as in 5.2, we have automatically  $(x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{s, *s}$ ; moreover, from

the definitions,  $\vartheta'$  (relative to  $\mathbf{V}/\mathbf{V}'$ ) carries  $(x'', p'', q'', \lambda)$  to  $\vartheta'(x, p, q, \lambda) = (\lambda, \pi)$ . Using now 4.12(a), we see that there exists an isomorphism (necessarily unique)

(b)  $\iota : \mathbf{V}/\mathbf{V}' \xrightarrow{\sim} \tilde{\mathbf{V}}$  which carries  $(x'', p'', q'')$  to  $(\tilde{x}, \tilde{p}, \tilde{q})$ .

Thus, we have a map  $u$  from  $\Psi$  to the variety of all triples as in (a) such that (b) holds. Let  $\Psi'$  be the variety consisting of all  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}$  (without stability condition) such that the triple (a) attached to  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}$  satisfies (b).

Note that  $\Psi$  is an open subset of  $\Psi'$ . On the other hand, by Lemma 5.3,  $\Psi'$  is a vector bundle of dimension

(c)  $\sum_i \dim \mathbf{V}'_i (\dim \mathbf{D}_i - \dim \tilde{\mathbf{V}}_i + \sum_{h; h'=i} \dim \mathbf{V}'_{h''})$

over the variety of triples (a) satisfying (b). This variety of triples is itself a locally trivial fibration over the space of all surjective maps  $\mathbf{V} \rightarrow \tilde{\mathbf{V}}$  (in  $\mathcal{C}^0$ ) with fibre isomorphic to  $\Lambda_{0, \mathbf{V}', \lambda}$  (where  $\dim \mathbf{V}'_i = \dim \mathbf{V}_i - \dim \tilde{\mathbf{V}}_i$  for all  $i$ ). Using now 4.5(a), we see that this variety of triples has pure dimension equal to

(d)  $\dim \mathbf{V}_i \dim \tilde{\mathbf{V}}_i + d_{\mathbf{V}'}$

where  $\mathbf{V}'$  is as above. It follows that  $\Psi'$  (and hence also  $\Psi$ ) has pure dimension equal to the sum of (c) and (d). This is equal to the expression in the proposition, by 5.1(b). The proposition is proved.

**COROLLARY 5.6.** — *The fibre of the map  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \rightarrow Z_{\mathbf{D}}$  induced by  $\vartheta'$  at  $(\lambda, \pi) \in {}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}}$  has pure dimension  $(1/2)(\dim \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} - \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}})$ .*

## 6. Small maps.

6.1. Let  $E_1$  be the set of all  $\lambda \in E$  such that  $\check{R}_{\lambda} = \emptyset$ . Let  $Z_{\mathbf{D}}^{E_1}$ ,  $Z_{\mathbf{D}}^{E-E_1}$ ,  $Z_{\mathbf{D}}^0$  be the inverse images of  $E_1, E - E_1, \{0\}$  under the canonical map  $Z_{\mathbf{D}} \rightarrow E$ .

**LEMMA 6.2.** — *If  $\lambda \in E_1$ , then  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda} = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*s}$ .*

Let  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ . We associate to  $(x, p, q)$

$$\mathbf{V}', (x', 0, 0) \in \Lambda_{0, \mathbf{V}', \lambda}, \quad (x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{*s}$$

as in 5.2. We see that  $\Lambda_{0, \mathbf{V}', \lambda} \neq \emptyset$  hence, by 4.5(c), we have  $\mathbf{V}' = 0$ . Hence  $(x, p, q) = (x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*s}$ . Thus,  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda} = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{*s}$ . Passing to dual spaces we obtain  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda} = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^s$ . The lemma is proved.



LEMMA 6.3.

- (a)  $Z_{\mathbf{D}, \mathbf{V}}$  is open dense in  ${}_{\mathbf{V}}Z_{\mathbf{D}}$ .  
 (b) The canonical map  $\pi_{\mathbf{V}} : \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \rightarrow {}_{\mathbf{V}}Z_{\mathbf{D}}$  induced by  $\vartheta'$  restricts to a homeomorphism  $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D}, \mathbf{V}}) \xrightarrow{\sim} Z_{\mathbf{D}, \mathbf{V}}$ .

Since the canonical map  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \rightarrow E$  is a submersion and  $E_1$  is open dense in  $E$ , the inverse image of  $E_1$  under this map is open dense in  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}}$ . This inverse image is contained in  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s} / G_{\mathbf{V}}$  by 6.2. It follows that the open set  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s} / G_{\mathbf{V}}$  of  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}}$  is also dense. Applying the continuous surjective map  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \rightarrow {}_{\mathbf{V}}Z_{\mathbf{D}}$ , we deduce that the image of  $\Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s} / G_{\mathbf{V}}$ , that is  $Z_{\mathbf{D}, \mathbf{V}}$ , is dense in  ${}_{\mathbf{V}}Z_{\mathbf{D}}$ . It is open by 4.12. This proves (a).

We prove (b). It suffices to show that  $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D}, \mathbf{V}}) = \Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s} / G_{\mathbf{V}}$ . Let  $(x, p, q, \lambda) \in \pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D}, \mathbf{V}})$ . We associate to  $(x, p, q)$

$$\mathbf{V}', (x', 0, 0) \in \Lambda_{0, \mathbf{V}', \lambda}, \quad (x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{s, *s}$$

as in 5.2. We have automatically  $(x'', p'', q'') \in \Lambda_{\mathbf{D}, \mathbf{V}/\mathbf{V}', \lambda}^{s, *s}$  and, as in the proof of 5.5, there exists an isomorphism  $\iota : \mathbf{V}/\mathbf{V}' \xrightarrow{\sim} \mathbf{V}$  which carries  $(x'', p'', q'')$  to a triple in  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{s, *s}$ . In particular, we must have  $\mathbf{V}' = 0$  and  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{s, *s}$ . Thus,  $\pi_{\mathbf{V}}^{-1} \subset (Z_{\mathbf{D}, \mathbf{V}}) \subset \Lambda_{\mathbf{D}, \mathbf{V}}^{s, *s} / G_{\mathbf{V}}$ . The reverse inclusion is obvious. The lemma is proved.

LEMMA 6.4. — Let  $\mathbf{D}, \mathbf{V}, \tilde{\mathbf{V}} \in \mathcal{C}^0$ . If  $\mathbf{V}, \tilde{\mathbf{V}}$  are not isomorphic in  $\mathcal{C}^0$ , then  $\dim({}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}}) < \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}}$ .

If  $\lambda \in E_1$ , we have (by 6.2)  ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}}^{\lambda} = Z_{\mathbf{D}, \mathbf{V}} \cap Z_{\mathbf{D}}^{\lambda}$ , hence

$${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}} \cap Z_{\mathbf{D}}^{\lambda} = Z_{\mathbf{D}, \mathbf{V}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}} \cap Z_{\mathbf{D}}^{\lambda} = \emptyset.$$

(We use 4.12(d) and our hypothesis.) Thus,

$${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}} \subset Z_{\mathbf{D}, \tilde{\mathbf{V}}} \cap Z_{\mathbf{D}}^{E-E_1}.$$

It is therefore enough to prove that

$$(a) \quad \dim(Z_{\mathbf{D}, \tilde{\mathbf{V}}} \cap Z_{\mathbf{D}}^{E-E_1}) < \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}}.$$

By 4.12, the space in the left hand side of (a) is homeomorphic to the inverse image of  $E - E_1$  under the canonical map  $\Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^{s, *s} / G_{\tilde{\mathbf{V}}} \rightarrow E$ . Since this map is a submersion and  $E - E_1$  is a proper closed subset of  $E$ , the dimension of the inverse image of  $E - E_1$  is  $< \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}}$ . The lemma is proved.

THEOREM 6.5. — *The canonical map  $\pi_{\mathbf{V}} : \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \rightarrow {}_{\mathbf{V}}Z_{\mathbf{D}}$  is small.*

By 5.1,  $\Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}}$  is smooth of pure dimension; by 4.13,  $\pi_{\mathbf{V}}$  is proper. From 4.12 we see that the sets  ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}}$  (for various  $\tilde{\mathbf{V}} \in \mathcal{C}^0$ ) form a partition of  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  into locally closed subvarieties. Only finitely many of these pieces are non-empty. One of them,  $Z_{\mathbf{D}, \mathbf{V}}$  is open dense in  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  and  $\pi_{\mathbf{V}}$  is a homeomorphism over this open set. It is then enough to show that for any other piece, that is  ${}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}, \tilde{\mathbf{V}}}$  with  $\tilde{\mathbf{V}}, \mathbf{V}$  not isomorphic, twice the dimension of any fibre over a point in the piece plus the dimension of the piece is strictly less than  $\dim {}_{\mathbf{V}}Z_{\mathbf{D}}$ . Using 6.4 and 5.6 we see that this sum is strictly less than

$$\begin{aligned} \dim \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} - \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}} + \dim \Lambda_{\mathbf{D}, \tilde{\mathbf{V}}}^s / G_{\tilde{\mathbf{V}}} &= \dim \Lambda_{\mathbf{D}, \mathbf{V}}^s / G_{\mathbf{V}} \\ &= \dim {}_{\mathbf{V}}Z_{\mathbf{D}}. \end{aligned}$$

The theorem is proved.

6.6. The previous result should be compared with [N2, 10.11] which can be reformulated to say that  $\Lambda_{\mathbf{D}, \mathbf{V}, 0}^s / G_{\mathbf{V}} \rightarrow {}_{\mathbf{V}}Z_{\mathbf{D}} \cap Z_{\mathbf{D}}^0$  is semismall. That result is essentially equivalent to [N2, 7.2], which in turn is proved using the special case of Corollary 5.6 with  $\lambda = 0$ . The proof of this special case given in [N2, 7.2] is based on the method of proof of [L3, 12.3]. This proof does not generalize to the case  $\lambda \neq 0$ , where the arguments in Section 5 are needed.

6.7. If  $Y'$  is an irreducible complex algebraic variety, the intersection cohomology complex  $IC(Y')$  is well defined. (We normalize it so that its restriction to an open dense subset of  $Y'$  is  $\mathbf{C}$ .) If  $Y$  is an arbitrary complex algebraic variety, the intersection cohomology complex of  $Y$  is defined as  $IC(Y) = \oplus_{Y'} IC(Y')$  where  $Y'$  runs over the set of irreducible components of  $Y$  and  $IC(Y')$  is extended to the whole of  $Y$  by 0 outside  $Y'$ . If  $Y$  is equidimensional and  $\pi_1 : \tilde{Y} \rightarrow Y$  is a small map then, from the definitions,  $IC(Y) = (\pi_1)_*(\mathbf{C})$ .

COROLLARY 6.8. — *We have canonically  $IC({}_{\mathbf{V}}Z_{\mathbf{D}}) = (\pi_{\mathbf{V}})_*(\mathbf{C})$  (as complexes on  ${}_{\mathbf{V}}Z_{\mathbf{D}}$ ).*

Note that  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  is equidimensional.

LEMMA 6.9.

(a) *For any  $\mathbf{V} \in \mathcal{C}^0$ ,  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  is a union of irreducible components of  $Z_{\mathbf{D}}$ .*

(b) Any irreducible component of  $Z_{\mathbf{D}}$  is contained in some  $\mathbf{v}Z_{\mathbf{D}}$  and the isomorphism class of such  $\mathbf{V}$  in  $\mathcal{C}^0$  is uniquely determined.

By 5.1, any irreducible component of  $\Lambda_{\mathbf{D},\mathbf{V}}^s$  meets  $\Lambda_{\mathbf{D},\mathbf{V},\lambda}^s$  for some  $\lambda \in E_1$ . Hence any irreducible component of  $\mathbf{v}Z_{\mathbf{D}}$  meets  $Z_{\mathbf{D}}^{E_1}$ . Since the closed subsets  $\mathbf{v}Z_{\mathbf{D}}$  cover  $Z_{\mathbf{D}}$ , it follows that any irreducible component of  $Z_{\mathbf{D}}$  meets  $Z_{\mathbf{D}}^{E_1}$ . By the proof of 6.3(a),  $Z_{\mathbf{D}}^{E_1}$  is open dense in  $Z_{\mathbf{D}}$ . Hence the irreducible components of  $Z_{\mathbf{D}}$  are exactly the closures of the irreducible components of  $Z_{\mathbf{D}}^{E_1}$ .

The closed subsets  $\mathbf{v}Z_{\mathbf{D}} \cap Z_{\mathbf{D}}^{E_1}$  of  $Z_{\mathbf{D}}^{E_1}$  coincide with the subsets  $Z_{\mathbf{D},\mathbf{V}} \cap Z_{\mathbf{D}}^{E_1}$  of  $Z_{\mathbf{D}}^{E_1}$  and these form a partition of  $Z_{\mathbf{D}}^{E_1}$  by 4.12(d). Hence  $\mathbf{v}Z_{\mathbf{D}} \cap Z_{\mathbf{D}}^{E_1}$  are both open and closed in  $Z_{\mathbf{D}}^{E_1}$  hence are unions of irreducible components of  $Z_{\mathbf{D}}^{E_1}$ . The lemma follows.

6.10. One expects that  $\Lambda_{\mathbf{D},\mathbf{V}}^s$  is connected (if non-empty). This is equivalent to the property that  $\Lambda_{\mathbf{D},\mathbf{V},0}^s$  is connected (if non-empty) which is stated in [N2, 6.2] but, as Nakajima informed me, the proof given there is incorrect. If we assume that this property holds, then 6.9 would have a simpler form, namely that *the  $\mathbf{v}Z_{\mathbf{D}}$  which are non-empty are precisely the irreducible components of  $Z_{\mathbf{D}}$ .*

6.11. Let  $\tilde{Z}_{\mathbf{D}}$  be the disjoint union  $\sqcup_{\mathbf{V}} \Lambda_{\mathbf{D},\mathbf{V}}^s / G_{\mathbf{V}}$  where  $\mathbf{V}$  runs over a set of representatives for the isomorphism classes of objects of  $\mathcal{C}^0$ . This is a finite union since  $\Lambda_{\mathbf{D},\mathbf{V}}^s$  is empty for all but finitely many  $\mathbf{V}$  (see [L4, 5.14]). Moreover,  $\tilde{Z}_{\mathbf{D}}$  is canonically defined (independent of the choice of representatives) due to the fact that we factor by  $G_{\mathbf{V}}$ . Let  $\pi : \tilde{Z}_{\mathbf{D}} \rightarrow Z_{\mathbf{D}}$  be the morphism whose restriction to  $\Lambda_{\mathbf{D},\mathbf{V}}^s / G_{\mathbf{V}}$  is  $\pi_{\mathbf{V}}$  for any  $\mathbf{V}$ . From 6.8 and 6.9 we deduce the following result.

COROLLARY 6.12. — *We have canonically  $IC(Z_{\mathbf{D}}) = \pi_*(\mathbf{C})$  (as complexes on  $Z_{\mathbf{D}}$ ).*

6.13. The action of  $W$  on  $Z_{\mathbf{D}}$  given by 1.5, 2.2 is denoted by  $w : z \mapsto w(z)$ . From definitions one checks that this action is through morphisms of algebraic varieties. Since  $IC(Z_{\mathbf{D}})$  is canonically attached to  $Z_{\mathbf{D}}$ , for any  $w \in W$  we have a canonical isomorphism  $\gamma_w : w^*IC(Z_{\mathbf{D}}) \xrightarrow{\sim} IC(Z_{\mathbf{D}})$ . Moreover, for  $w, w' \in W$ ,  $\gamma_{ww'}$  is equal to the composition

$$w'^*w^*IC(Z_{\mathbf{D}}) \xrightarrow{w'^*\gamma_w} w'^*IC(Z_{\mathbf{D}}) \xrightarrow{\gamma_{w'}} IC(Z_{\mathbf{D}}).$$

In other words, the action of  $W$  on  $Z_{\mathbf{D}}$  lifts canonically to an action of  $W$  on  $IC(Z_{\mathbf{D}})$  hence (by 6.12) to an action of  $W$  on the complex

$\pi_*(\mathbf{C})$ . In particular, by passage to stalks, we see that for any  $W$ -orbit  $\mathcal{O}$  on  $Z_{\mathbf{D}}$  we have a natural action of  $W$  on the cohomology spaces of  $\sqcup_{z \in \mathcal{O}} \pi^{-1} \subset (z)$ . Also, we have an induced  $W$ -action on the cohomology spaces of  $\tilde{Z}_{\mathbf{D}}^0 = \pi^{-1} \subset (Z_{\mathbf{D}}^0) = \sqcup_{\mathbf{V}} \Lambda_{\mathbf{D}, \mathbf{V}, 0} / G_{\mathbf{V}}$ .

6.14. By 2.3, 2.4 we have an action of  $G_{\mathbf{D}} \times \mathbf{C}^*$  on  $Z_{\mathbf{D}}$ . This is an algebraic group action. Moreover,  $G_{\mathbf{D}} \times \mathbf{C}^*$  acts naturally on  $\tilde{Z}_{\mathbf{D}}$  so that  $\pi$  is  $(G_{\mathbf{D}} \times \mathbf{C}^*)$ -equivariant. The construction of the  $W$ -action in 6.13 extends automatically to the  $(G_{\mathbf{D}} \times \mathbf{C}^*)$ -equivariant setting in the same way as the construction [L1] of the Springer representation was extended to the equivariant setting in [L2]. This gives for example a natural  $W$ -action on

$$H_*^{G_{\mathbf{D}} \times \mathbf{C}^*}(\tilde{Z}_{\mathbf{D}}^0) = \oplus_{\mathbf{V}} H_*^{G_{\mathbf{D}} \times \mathbf{C}^*}(\Lambda_{\mathbf{D}, \mathbf{V}, 0} / G_{\mathbf{V}})$$

(equivariant homology) where  $\tilde{Z}_{\mathbf{D}}^0 = \pi^{-1} \subset (Z_{\mathbf{D}}^0)$ .

6.15. Consider the fibre product  $\tilde{Z}_{\mathbf{D}}^0 \times_{Z_{\mathbf{D}}^0} \tilde{Z}_{\mathbf{D}}^0$ . (This is homeomorphic to a variety in [N2, Sec.7].) Just as in [L1], from the  $W$ -action on  $\pi_*(\mathbf{C})$  in 6.13, we obtain a  $W \times W$ -action on  $H_*^{G_{\mathbf{D}} \times \mathbf{C}^*}(\tilde{Z}_{\mathbf{D}}^0 \times_{Z_{\mathbf{D}}^0} \tilde{Z}_{\mathbf{D}}^0)$ .

## 7. Weight spaces.

7.1. From the definition of  $\pi : \tilde{Z}_{\mathbf{D}} \rightarrow Z_{\mathbf{D}}$ , we have a canonical “weight” decomposition

$$\pi_*(\mathbf{C}) = \oplus_{\mathbf{V}} (\pi_{\mathbf{V}})_*(\mathbf{C})$$

where  $\mathbf{V}$  runs over a set of representatives for the isomorphism classes of objects of  $\mathcal{C}^0$  such that  $\Lambda_{\mathbf{D}, \mathbf{V}}^s \neq \emptyset$  and  $(\pi_{\mathbf{V}})_*(\mathbf{C})$  is extended to the whole of  $Z_{\mathbf{D}}$  by 0 outside  $\mathbf{V}Z_{\mathbf{D}}$ . In this section we describe the relationship between the  $W$ -action on  $\pi_*(\mathbf{C})$  (see 6.13) and this “weight decomposition”.

LEMMA 7.2. — *Let  $i \in I$  and let  $\mathbf{V} \in \mathcal{C}^0$  be such that  $Z_{\mathbf{D}, \mathbf{V}} \cap Z_{\mathbf{D}}^{E_1} \neq \emptyset$ . Then*

- (a) *there exists  $\mathbf{V}' \in \mathcal{C}^0$  such that  $\mathbf{V}_j = \mathbf{V}'_j$  for  $j \in I - \{i\}$  and  $\dim \mathbf{V}_i + \dim \mathbf{V}'_i = \dim \mathbf{D}_i + \sum_{h; h'=i} \dim \mathbf{V}_{h''}$ ;*
- (b)  *$s_i(Z_{\mathbf{D}, \mathbf{V}} \cap Z_{\mathbf{D}}^{E_1}) = Z_{\mathbf{D}, \mathbf{V}'} \cap Z_{\mathbf{D}}^{E_1}$ .*

We can find  $(x, p, q, \lambda) \in \Lambda_{\mathbf{D}, \mathbf{V}}^{s, s}$  with  $\lambda \in E_1$ . Let  $U, b : U \rightarrow \mathbf{V}_i$  be as in 3.2(a). Since  $\lambda \in E_1$ , we have  $\lambda_i \neq 0$ . By the argument in 3.4,  $b$  is surjective. Hence  $\dim \mathbf{V}_i \geq \dim U = \dim \mathbf{D}_i + \sum_{h; h'=i} \dim \mathbf{V}_{h''}$  and (a) follows.

We prove (b). Since for  $\lambda \in E_1$  we have  $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}^{s, *s} = \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  and  $\Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}^{s, *s} = \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$  where  $\lambda' = s_i(\lambda) \in E_1$ , (see 6.1), it suffices to show that the following diagram of sets is commutative:

$$\begin{array}{ccc} \Lambda_{\mathbf{D}, \mathbf{V}, \lambda} & \xleftarrow{r} F \xrightarrow{r'} & \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'} \\ \downarrow & & \downarrow \\ Z_{\mathbf{D}} & \xrightarrow{s_i} & Z_{\mathbf{D}} \end{array}$$

for  $\lambda \in E_1$ . Here the left vertical map is induced by  $\vartheta'$ , the right vertical map is the analogous map for  $\mathbf{V}', \lambda'$ , and  $r, r'$  are as in 3.3, 3.5.

Let  $((x, p, q); (x', p', q')) \in F$ . Let  $\pi \in \tilde{\mathcal{F}}_{\mathbf{D}}$  (resp.  $\pi' \in \tilde{\mathcal{F}}_{\mathbf{D}}$ ) be defined in terms of  $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$  (resp.  $(x', p', q') \in \Lambda_{\mathbf{D}, \mathbf{V}', \lambda'}$ ) as in 4.6. We must show that  $\pi' = s_i^\lambda(\pi)$ . It is enough to show that

$$(c) \quad \pi'_{[f]} = \pi_{s_i^\lambda[f]}$$

for any  $f \in \mathbf{I}$  of form  $i_1, i_2, \dots, i_s$ . By definition,

$$(d) \quad \pi'_{[f]} = q'_{i_1} x'_{i_1, i_2} x'_{i_2, i_3} \cdots x'_{i_{s-1}, i_s} p'_{i_s}$$

where the product of the  $x'$  is taken to be 1 if  $s = 1$ . We wish to convert the right hand side of (d) into an expression involving only  $q_j, x_{kl}, p_j$  (rather than  $q'_j, x'_{kl}, p'_j$ ). We will achieve this by a repeated use of the identities 3.2(b2), (c). Assume first that  $s = 1$  so that  $f$  is  $j$  and  $\pi'_{[f]} = q'_j p'_j$  for some  $j$ . If  $j \neq i$ , then by 3.2(c) we have  $q'_j p'_j = q_j p_j = \pi_{[f]} = \pi_{s_i^\lambda[f]}$ . If  $j = i$ , then by 3.2(b2) we have

$$q'_i p'_i = q_i p_i + \lambda_i = \pi_{[f] + \lambda_i u_i} = \pi_{s_i^\lambda[f]}.$$

Assume now that  $s \geq 2$ . In the right hand side of (d) we may

- replace any two consecutive factors  $x'_{i_{t-1}, i_t} x'_{i_t, i_{t+1}}$  such that  $i_{t-1} = i_{t+1}, i_t = i$  by  $x_{i_{t-1}, i_t} x_{i_t, i_{t+1}} - \varepsilon_{i_t, i_{t+1}} \lambda_i$  (using 3.2(b2));
- replace any two consecutive factors  $x'_{i_{t-1}, i_t} x'_{i_t, i_{t+1}}$  such that  $i_{t-1} \neq i_{t+1}, i_t = i$  by  $x_{i_{t-1}, i_t} x_{i_t, i_{t+1}}$  (using 3.2(b2)),
- if  $i_1 = i$  we replace  $q'_i x'_{i, i_2}$  by  $q_i x_{i, i_2}$  (using 3.2(b2)),
- if  $i_s = i$  we replace  $x'_{i_{s-1}, i} p'_i$  by  $x_{i_{s-1}, i} p_i$  (using 3.2(b2)),
- the remaining factors will be of the form  $x'_{kl}$  or  $q'_k$  or  $p'_l$  with  $k \neq i \neq l$  and can be replaced by  $x_{kl}$  or  $q_k$  or  $p_l$  (using 3.2(c)).

The resulting expression is clearly equal to  $\pi_{s_i^\lambda[f]}$ . This proves (c) hence also (b). The lemma is proved.

**PROPOSITION 7.3.** — *The subvarieties  ${}_{\mathbf{V}}Z_{\mathbf{D}}$  of  $Z_{\mathbf{D}}$  (for various  $\mathbf{V}$ ) are permuted among themselves by the  $W$ -action on  $Z_{\mathbf{D}}$ .*

It suffices to show that, given  $i \in I$  and  $\mathbf{V} \in \mathcal{C}^0$  such that  $\mathbf{v}Z_{\mathbf{D}} \neq \emptyset$ , we have  $s_i(\mathbf{v}Z_{\mathbf{D}}) = \mathbf{v}'Z_{\mathbf{D}}$  for some  $\mathbf{V}'$ . As in the proof of 6.3,  $Z_{\mathbf{D},\mathbf{v}} \cap Z_{\mathbf{D}}^{E_1}$  is open dense in  $\mathbf{v}Z_{\mathbf{D}}$ . Hence it suffices to show that, given  $i \in I$  and  $\mathbf{V} \in \mathcal{C}^0$  such that  $Z_{\mathbf{D},\mathbf{v}} \cap Z_{\mathbf{D}}^{E_1} \neq \emptyset$ , we have  $s_i(Z_{\mathbf{D},\mathbf{v}} \cap Z_{\mathbf{D}}^{E_1}) = Z_{\mathbf{D},\mathbf{v}'} \cap Z_{\mathbf{D}}^{E_1}$  for some  $\mathbf{V}'$ . But this follows from Lemma 7.2. The proposition is proved.

7.4. From the proof of 7.3 we see that if  $\mathbf{v}Z_{\mathbf{D}} \neq \emptyset$  and  $w \in W$ , then  $w(\mathbf{v}Z_{\mathbf{D}}) = \tilde{\mathbf{v}}Z_{\mathbf{D}}$  where  $\tilde{\mathbf{V}} \in \mathcal{C}^0$  is characterized by the equation

$$\sum_j \dim \mathbf{D}_j \varpi_j - \sum_j \dim \tilde{\mathbf{V}}_j \alpha_j = w \left( \sum_j \dim \mathbf{D}_j \varpi_j - \sum_j \dim \mathbf{V}_j \alpha_j \right)$$

in  $E$  at least if  $w = s_i$ ; but then this automatically holds for general  $w$ . It follows that  $w$  carries the summand  $(\pi_{\mathbf{V}})_*(\mathbf{C})$  of  $\pi_*(\mathbf{C})$  onto the summand  $(\pi_{\tilde{\mathbf{V}}})_*(\mathbf{C})$  where  $\tilde{\mathbf{V}}$  is as above.

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