ANNALES DE L'INSTITUT FOURIER

LÁSZLÓ LEMPERT

Approximation of holomorphic functions of infinitely many variables II

Annales de l'institut Fourier, tome 50, nº 2 (2000), p. 423-442

<http://www.numdam.org/item?id=AIF_2000__50_2_423_0>

Ann. Inst. Fourier, Grenoble
50, 2 (spécial Cinquantenaire) (2000), 423–442

APPROXIMATION OF HOLOMORPHIC FUNCTIONS OF INFINITELY MANY VARIABLES II

by László LEMPERT

Introduction.

In [L1] we formulated the following approximation problem:

Let X be a complex Banach space, $B(R) \subset X$ the ball of radius $R > 0$ about the origin, and f a holomorphic function on $B(R)$. Given $r \in (0, R)$ and $\epsilon > 0$, is there a function h, holomorphic on X, such that $|f - h| < \epsilon$ on $B(r)$?

We speculated that such approximation results would be important for future developments in complex analysis of infinite dimensions, and in [L1] we could settle the problem in the affirmative for the space $X = l^1(\Gamma)$, Γ any set. Later in [P] Patyi gave an important extension of this to so called l^1 sums of finite dimensional Banach spaces. Subsequent work indeed showed that whenever approximation as in the problem is possible, it follows almost automatically that the sheaf cohomology groups $H^q(\Omega, \mathcal{O})$ vanish for $\Omega \subset X$ pseudoconvex and $q \geqslant 1$, see [L3], Theorem 0.3.

In this paper we shall solve the approximation problem when $X = l^p(\Gamma)$, $1 \leqslant p < \infty$, or X has a countable unconditional basis $e_1, e_2, \ldots \in X$ (for definitions see Section 1). It is known that in the latter case the

Research partially supported by an NSF Grant.
Keywords: Holomorphic functions – Banach spaces.
Math. classification: 46G20 – 32A05.

topology of X can be induced by a norm $\| \ \|$ that satisfies for $x = \sum_1^\infty x(n)e_n \in X$ and $\xi_n \in \mathbb{C}$

$$(0.1) \quad \left\|\sum \xi_n x(n)e_n\right\| \leqslant \left\|\sum x(n)e_n\right\|, \quad \text{provided } |\xi_n| \leqslant 1, \quad n = 1, 2, \ldots$$

Most classical Banach spaces admit unconditional bases, see e.g. [S], $L^1(0,1)$ and $C[0,1]$ being stubborn exceptions.

Let V be a sequentially complete locally convex topological vector space over $\mathbb{C}$, whose topology is defined by a family Ψ of seminorms.

THEOREM 0.1. — *Suppose $X = l^p(\Gamma)$, $1 \leqslant p < \infty$, with the usual norm, or X has a countable unconditional basis $\{e_n\}$, and (0.1) is satisfied. Given $f : B(R) \to V$ holomorphic, $r \in (0, R)$, $\psi \in \Psi$, and $\epsilon > 0$, there is a holomorphic $h : X \to V$ such that $\psi(f - h) < \epsilon$ on $B(r)$.*

As stated above, this implies cohomology vanishing, *cf.* [L3], Theorem 0.1:

THEOREM 0.2. — *Let X be a Banach space with countable unconditional basis, V a Fréchet space, and $\mathcal{V}$ the sheaf of germs of V valued holomorphic functions on an open $\Omega \subset X$. If Ω is pseudoconvex then $H^q(\Omega, \mathcal{V}) = 0$, $q \geqslant 1$.*

In particular, sheaf cohomology groups vanish in separable Hilbert spaces. In fact, Banach spaces more general than those in Theorem 0.1 will be treated in this paper, see Theorem 4.2 and Definition 3.5. However, we cannot prove either of Theorems 0.1 or 0.2 for all separable Banach spaces. On the other hand, no counterexamples are known in Banach spaces either for the approximation problem or for cohomology vanishing. For cohomology vanishing and non-vanishing in general locally convex spaces, see [D1], [D2], [M], [MV], and the introduction of [L3].

The proof of Theorem 0.1 borrows ideas from [L1]. As there, one starts by expanding the function f in a monomial series

$$(0.2) \qquad f(x_1, x_2, \ldots) \sim \sum a_{k_1 k_2 \ldots} x_1^{k_1} x_2^{k_2} \ldots, \qquad a_{k_1 k_2 \ldots} \in V,$$

say, in the space $X = l^p(\mathbb{N})$. The first stumbling block is that, unless $p = 1$, the series in (0.2) does not necessarily converge to f. However, there is a class of functions for which the monomial series do converge. Fix an integer $m \geqslant p$, let $C_m \subset \mathbb{C}$ denote the group of m'th roots of unity, and G the multiplicative group of sequences $\theta_1, \theta_2, \ldots \in C_m$. This group acts on l^p by coordinatewise multiplication, and leaves the ball $B(R)$ invariant. It turns out that the estimates of [L1] can be used to conclude that the series in (0.2) converges to f whenever f is G invariant. For invariant functions the rest of the arguments of [L1] also works, and gives that by omitting certain "negligible" terms from the monomial series of f the remaining series represents an entire function h as required.

The question that we must still address is whether the case of an invariant function is of any relevance for general functions f. Indeed it is. In Section 2 we shall prove that for any holomorphic $f : B(R) \to V$ there is a holomorphic $F : B(R) \times X \to V$ such that $F(x,x) = f(x)$, $x \in B(R)$, and $F(\cdot, y)$ is G invariant for any $y \in X$. Thus $F(x,y)$ is invariant in x and entire in y. It turns out that this property suffices to produce a holomorphic $H : X \times X \to V$ such that $\psi(F - H) < \epsilon$ on $B(r) \times B(r)$, by a modification of the approach of [L1] alluded to above, see Theorem 4.1. Obviously, $h(x) = H(x,x)$ will then do.

While this synopsis dealt with the space $l^p(\mathbb{N})$ only, the same approach works for more general spaces X as long as an appropriate substitute is found for the group $G = (C_m)^{\mathbb{N}}$ above. When $X = l^p(\Gamma)$, one takes $G = (C_m)^{\Gamma}$; the relevant property of this group will be given in Theorem 3.4. For general spaces with countable unconditional bases a suitable G will be constructed in Section 5.

Acknowledgement. I am grateful to F. Bracci and C. Kiselman for helpful comments.

1. Basics.

In this paper V will always denote a sequentially complete, locally convex topological vector space over $\mathbb{C}$, whose topology is given by a family Ψ of seminorms. Suppose v, $v_j \in V$, for j belonging to some index set J. We write $\sum v_j = v$ to mean that for any $\psi \in \Psi$ and $\epsilon > 0$ there is a finite $J_0 \supset J$ such that $\psi(v - \sum_{J_1} v_j) < \epsilon$ whenever $J_1 \supset J_0$ is finite. If V happens to be a Fréchet space, all but countably many terms of a convergent series $\sum v_j$ must be zero. We say that a series $\sum v_j$ is normally convergent if $\sum \psi(v_j) < \infty$ for all $\psi \in \Psi$ (even though "seminormally" would be more appropriate). If only countably many v_j differ from zero then normal convergence of $\sum v_j$ implies $\sum v_j = v$ for some $v \in V$. Suppose S is an arbitrary set and $f_j : S \to V$, $j \in J$. We say that $\sum f_j$ converges normally on S if $\sum \sup_S \psi(f_j) < \infty$ for all $\psi \in \Psi$, and that $\sum f_j = f : S \to V$

uniformly if for every $\psi \in \Psi$ and $\epsilon > 0$ there is a finite $J_0 \subset J$ such that $\sup_S \psi(f - \sum_{J_1} f_j) < \epsilon$ whenever $J_1 \supset J_0$ is finite. Pointwise and normal convergence on S together imply uniform convergence on S. If S is a topological space, the f_j are continuous, and $f = \sum f_j$ converges uniformly on S then f is also continuous. Similarly, if S is an open subset of a locally convex space, the f_j are holomorphic, and $f = \sum f_j$ converges uniformly on S then f is holomorphic.

Our main concern in this paper will be function theory in the Banach space

$$l^p(\Gamma) = \Big\{x : \Gamma \to \mathbb{C} \mid \|x\| = \Big(\sum |x(\gamma)|^p\Big)^{1/p} < \infty\Big\},$$

with Γ an arbitrary set, $1 \leqslant p < \infty$; and in spaces with countable unconditional basis. However, what we have to say holds for more general complex Banach spaces $(X, \|\ \|)$ as well. All through the paper we shall assume that X has a not necessarily countable, unconditional basis—or basis, for short—, i.e., a collection $\{e_\gamma : \gamma \in \Gamma\} \subset X$, Γ some set, with the property that any $x \in X$ can be uniquely represented as

$$x = \sum_{\gamma\in\Gamma} x(\gamma) e_\gamma, \qquad x(\gamma) \in \mathbb{C}, \quad \gamma \in \Gamma. \tag{1.1}$$

We shall also assume that whenever (1.1) holds and $\xi_\gamma \in \mathbb{C}$ then

$$\Big\|\sum \xi_\gamma x(\gamma) e_\gamma\Big\| \leqslant \Big\|\sum x(\gamma) e_\gamma\Big\|, \qquad \text{provided } |\xi_\gamma| \leqslant 1, \quad \gamma \in \Gamma; \tag{1.2}$$

it is implied that the series on the left converges. If X has an unconditional basis $\{e_\gamma\}$ then (1.2) can always be achieved by replacing the original norm by an equivalent one, see e.g. [D2], Lemma 4.35 for the case when X is separable and [S], Theorem 17.5 in general. The spaces $l^p(\Gamma)$ have unconditional bases, and if e_γ is taken to be the characteristic function of $\{\gamma\} \subset \Gamma$ then (1.2) is also satisfied.

Later on we shall need a characterization of compactness in X.

PROPOSITION 1.1. — *A closed set $K \subset X$ is compact if and only if it is bounded, and for any $\epsilon > 0$ there is a finite $\Gamma_0 \subset \Gamma$ such that*

$$\Big\|\sum_{\Gamma\setminus\Gamma_0} x(\gamma) e_\gamma\Big\| < \epsilon, \ \textit{for all } x \in K. \tag{1.3}$$

This is a slight generalization of [DS] IV.5.5, and follows from [DS] IV.5.4.

Now let S_1 denote the multiplicative semigroup of those $\sigma : \Gamma \to [0, 1)$ for which the set $\{\gamma : \sigma(\gamma) \geqslant \epsilon\}$ is finite for all $\epsilon > 0$. This semigroup acts on X by the rule $(\sigma x)(\gamma) = \sigma(\gamma)x(\gamma)$. Denote the image of a set $L \subset X$ under σ by σL.

PROPOSITION 1.2. — *For any $\sigma \in S_1$ and $R < \infty$ the set $\sigma B(R)$ is relatively compact in $B(R)$. Conversely, for any compact $K \subset B(R)$ there are a $\sigma \in S_1$ and $L \subset B(R)$ compact such that $\sigma L = K$.*

Proof. — The closure of $\sigma B(R)$ is contained in $\overline{B(\max_\gamma \sigma(\gamma))} \subset B(R)$, and is compact by Proposition 1.1, since (1.3) will hold with $\Gamma_0 = \{\gamma : \sigma(\gamma) \geqslant \epsilon/R\}$. Conversely, if $K \subset B(R)$ is compact, choose $\theta \in (\sup_K(\|x\|/R)^{1/2}, 1)$, and construct a sequence $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma$, each Γ_n finite, such that

$$\Big\| \sum_{\Gamma\setminus\Gamma_n} x(\gamma)e_\gamma \Big\| \leqslant 4^{-n}(1-\theta)^n\theta^2 R, \quad x \in K, \quad n \geqslant 0.$$

This implies $x(\gamma) = 0$ if $x \in K$, $\gamma \notin \cup_n \Gamma_n = \Gamma^*$. Define

$$\sigma(\gamma) = \begin{cases} 2^{-n}\theta, & \text{for } \gamma \in \Gamma_{n+1} \setminus \Gamma_n \\ 0, & \text{for } \gamma \notin \Gamma^* \end{cases}$$

and a closed set $L = \{y \in X : \sigma y \in K,\ y(\gamma) = 0 \text{ if } \gamma \notin \Gamma^*\}$. Obviously $\sigma L = K$. Since for any $y \in L$, $N = 0, 1, \ldots$, and $x = \sigma y$

$$\Big\| \sum_{\Gamma\setminus\Gamma_N} y(\gamma)e_\gamma \Big\| \leqslant \sum_{n\geqslant N} \Big\| \sum_{\Gamma_{n+1}\setminus\Gamma_n} 2^n\theta^{-1}x(\gamma)e_\gamma \Big\| \leqslant \sum_{n\geqslant N} 2^{-n}(1-\theta)^n\theta R < 2^{-N}R,$$

$L \subset B(R)$ is compact by Proposition 1.1.

In this paper we shall freely use basic facts of infinite dimensional complex analysis; on such matters the reader is advised to consult [D2], [D3], [N]. We shall denote by $\mathcal{O}(M; M')$ the set of holomorphic mappings $M \to M'$.

2. Extensions.

Our goal in this section is to relate arbitrary holomorphic functions on a ball $B(R)$ in a Banach space $(X, \|\ \|)$ to those that are invariant under a group. We shall assume that X has a basis $\{e_\gamma\}$ as in Section 1, and (1.2) holds.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle group (with group operation written additively) and $T = T(\Gamma) = \{t : \Gamma \to S^1\}$ the typically infinite dimensional torus, endowed with the product topology. T acts continuously on X by

$$\rho_t\Big(\sum x(\gamma)e_\gamma\Big) = \sum e^{2\pi i t(\gamma)}x(\gamma)e_\gamma, \qquad t \in T, \quad x \in X.$$

Each ρ_t is an isometry because of (1.2), and so ρ leaves $B(R)$ invariant. Denote by dt the Haar measure on T normalized to have total mass 1. We define a multiindex to be a mapping $k : \Gamma \to \mathbb{N} \cup \{0\}$ such that $k(\gamma) = 0$ save for finitely many $\gamma \in \Gamma$. *In this paper k, κ will always denote multiindices.* If $t \in T$, we shall write $k \cdot t$ for $\sum_\gamma k(\gamma)t(\gamma) \in S^1$.

Any $f \in \mathcal{O}(B(R); V)$ can be expanded in a monomial series

$$f \sim \sum_k f_k, \qquad f_k = \int_T e^{-2\pi i k \cdot t} \rho_t^* f \, dt, \tag{2.1}$$

with the terms f_k holomorphic on $B(R)$. By considering the restrictions of f, f_k to subspaces spanned by finitely many basis vectors e_γ one establishes that the f_k are indeed monomials of the form

$$f_k(x) = a_k x^k = a_k \prod x(\gamma)^{k(\gamma)}, \qquad a_k \in V;$$

we use the convention $0^0 = 1$.

When $\dim X < \infty$, it follows from simple Cauchy–estimates that the series in (2.1) converges to f normally on compact subsets $K \subset B(R)$. This implies convergence of (2.1) on a dense subset of $B(R)$ for general X: if $X_0 \subset X$ is the linear span of the basis vectors e_γ, then $f = \sum f_k$ on $B(R) \cap X_0$. However, $\sum f_k$ will rarely converge on all of $B(R)$.

There is one more player to be introduced in this section, a subgroup $G \subset T$. Fix a function $d : \Gamma \to \mathbb{N}$, and let G consist of those $t \in T$ for which $d(\gamma)t(\gamma) = 0 \in S^1$ for all γ. Thus $G = \prod_\Gamma(\mathbb{Z}[1/d(\gamma)]/\mathbb{Z}) \subset \prod_\Gamma(\mathbb{R}/\mathbb{Z}) = T$. Below we shall write dk for the multiindex $\gamma \mapsto d(\gamma)k(\gamma)$, and $\kappa < d$ to mean $\kappa(\gamma) < d(\gamma)$ for all γ.

THEOREM 2.1. — *Suppose $\sum f_k(x) = \sum a_k x^k$ is the monomial expansion of a function $f \in \mathcal{O}(B(R); V)$. If $x \in B(R) \cap X_0$ and $y \in X_0$, the series*

$$\sum_k \sum_{\kappa < d} a_{dk+\kappa} x^{dk} y^\kappa \tag{2.2}$$

converges normally. Furthermore, the function that (2.2) represents extends from $(B(R) \cap X_0) \times X_0$ to a holomorphic function $F : B(R) \times X \to V$ that satisfies

$$F(x, x) = f(x), \qquad x \in B(R), \tag{2.3}$$

$$F(\rho_t x, y) = F(x, y), \quad t \in G, \quad (x, y) \in B(R) \times X. \tag{2.4}$$

This will be proved through a sequence of propositions.

PROPOSITION 2.2. — *Let Z_0 be a dense subspace of a Banach space Z, $\Omega \subset Z$ open. A function $h : \Omega \cap Z_0 \to V$ extends to an $H \in \mathcal{O}(\Omega; V)$ if and only if (i) for any one dimensional affine subspace $L \subset Z_0$ the function $h|L$ is holomorphic, and (ii) for each seminorm $\psi \in \Psi$ and $K \subset \Omega$ compact $\sup_{K \cap Z_0} \psi(h) < \infty$.*

Proof. — One direction being trivial, we shall only verify that (i) and (ii) imply h extends holomorphically to Ω. For fixed $\psi \in \Psi$, any $z \in \Omega$ has a neighborhood $\omega \subset \Omega$ on which $\psi(h)$ is bounded. Indeed, otherwise there would exist a sequence $z_n \in \Omega \cap Z_0$, $z_n \to z$, such that $\sup_n \psi(h(z_n)) = \infty$, contradicting (ii) with $K = \{z, z_1, z_2, \ldots\}$. From the Cauchy representation formula applied to various one dimensional restrictions $h|L$ one can read off that z has a neighborhood ω' such that $\psi(h(z_1) - h(z_2))/\|z_1 - z_2\|$ is bounded for $z_1 \neq z_2 \in \omega' \cap Z_0$. It follows that $h(z_n)$ is a Cauchy sequence whenever $Z_0 \ni z_n \to z$: let $H(z)$ denote its limit, which is clearly independent of the choice of z_n. It is immediate that the function $H : \Omega \to V$ is continuous. All we need to show now is that $H|L$ is holomorphic for an arbitrary one dimensional affine subspace $L \subset Z$. Given such an L, construct a sequence of affine maps $u_n : L \to Z_0$ that converge to id_L. Since $H \circ u_n = h \circ u_n$ are holomorphic, so is their locally uniform limit $H|L$. Hence H is indeed holomorphic.

PROPOSITION 2.3. — *Let W be an arbitrary Banach space, $U \subset W$ open, and let furthermore X be a Banach space with basis $\{e_\gamma\}_{\gamma \in \Gamma}$ as above. Suppose H is a V valued holomorphic function on some connected neighborhood Ω of $U \times \{0\} \subset W \oplus X$ such that for all $(w, y) \in \Omega$, $\gamma \in \Gamma$ the function $\eta \mapsto H(w, y + \eta e_\gamma)$ is the restriction of a polynomial of $\eta \in \mathbb{C}$. Then H continues analytically to $U \times X$.*

Proof. — Suppose first $\Omega = U \times B(\epsilon)$, $\epsilon > 0$. For any $(w, y) \in U \times B(\epsilon)$ and finite $\Gamma_0 \subset \Gamma$ the function $H(w, y + \sum_{\gamma \in \Gamma_0} \eta_\gamma e_\gamma)$ is the restriction of a polynomial in η_γ, $\gamma \in \Gamma_0$, with coefficients depending holomorphically on w, y. This polynomial then provides an analytic continuation of H to the open set

$$D(\Gamma_0) = \Big\{(w, x) \in U \times X : \Big\| \sum_{\gamma \notin \Gamma_0} x(\gamma) e_\gamma \Big\| < \epsilon \Big\}.$$

Since $D(\Gamma_0 \cup \Gamma_1) \supset D(\Gamma_0) \cup D(\Gamma_1)$, it follows that H continues analytically to $\cup_{\Gamma_0} D(\Gamma_0) = U \times X$.

A general Ω as in the proposition will contain a neighborhood of

$U\times\{0\}$ of the form $\cup_{j\in J}U_j\times B(\epsilon_j)$, with J some index set and $U_j\subset U$ open. By virtue of what we have already proved H continues to $\bigcup U_j\times X=U\times X$, q.e.d.

PROPOSITION 2.4. — *In the situation of Theorem 2.1 assume Γ is finite. Then the series (2.2) converges normally for each $(x,y)\in B(R)\times X$ and its sum is a holomorphic function.*

Proof. — Notice that the series (2.2) multiplied by x^d is the Taylor series of the function

$$(1/|G|)\sum_{\kappa<d}\sum_{t\in G}e^{-2\pi i\kappa\cdot t}f(\rho_t x)x^{d-\kappa}y^{\kappa},$$

which is holomorphic on $B(R)\times X$. It follows that (2.2) indeed converges to a holomorphic function, normally on compact subsets of $B(R)\times X$.

Proof of Theorem 2.1. — The normal convergence of (2.2) follows from Proposition 2.4; its sum $h:(B(R)\cap X_0)\times X_0\to V$ is holomorphic when restricted to finite dimensional subspaces of $X_0\times X_0$ by the same proposition. Our next concern is to show that h extends holomorphically to

$$\Omega=\{(x,y)\in X\oplus X:\|x\|+3\|y\|<R\};$$

this will involve estimating h on $\Omega\cap(X_0\times X_0)$.

If $(x,y)\in\Omega$ and $q\geqslant 1$ is sufficiently close to 1, define $z=z_{x,y,q}\in B(R)$ by $z(\gamma)=q|x(\gamma)|+3|y(\gamma)|$ and $\Gamma_{xy}=\{\gamma:z(\gamma)\neq 0\}$. When $(x,y)\in\Omega\cap(X_0\times X_0)$, in (2.2) only those multiindices k,κ will contribute that are supported in Γ_{xy}. Hence, assuming now $q>1$

(2.5)

$$\begin{aligned}h(x,y)&=\int_T\prod_{\gamma\in\Gamma_{xy}}\sum_{n=0}^{\infty}\sum_{\nu=0}^{d(\gamma)-1}x(\gamma)^{d(\gamma)n}y(\gamma)^{\nu}(z(\gamma)e^{2\pi it(\gamma)})^{-d(\gamma)n-\nu}f(\rho_t z)dt\\&=\int_T\prod_{\gamma\in\Gamma_{xy}}\Lambda(d(\gamma),x(\gamma)e^{-2\pi it(\gamma)},y(\gamma)e^{-2\pi it(\gamma)})f(\rho_t z)dt,\end{aligned}$$

where for $\delta\in\mathbb{N}$, ξ, $\eta\in\mathbb{C}$ we set $\zeta=q|\xi|+3|\eta|$ and, provided $\zeta\neq 0$,

$$\Lambda(\delta,\xi,\eta)=-1+2\,\mathrm{Re}\sum_{n=0}^{\infty}\sum_{\nu=0}^{\delta-1}\xi^{\delta n}\eta^{\nu}\zeta^{-\delta n-\nu}.$$

Note that while multiplying out the products in (2.5) gives more terms in the second line than in the first, the excessive terms integrate to zero; hence the two integrals in (2.5) are indeed equal. The terms in Λ corresponding to $\nu = 0$ plus the constant term add up to

$$\operatorname{Re}\frac{\zeta^\delta+\xi^\delta}{\zeta^\delta-\xi^\delta} = \frac{\zeta^{2\delta}-|\xi|^{2\delta}}{|\zeta^\delta-\xi^\delta|^2} = \frac{\zeta^\delta(\zeta^\delta+|\xi|^\delta)}{|\zeta^\delta-\xi^\delta|^2}\left(1-\frac{|\xi|^\delta}{\zeta^\delta}\right),$$

while the sum of the remaining terms is

$$2\operatorname{Re}\frac{\zeta^\delta}{\zeta^\delta-\xi^\delta}\sum_{\nu=1}^{\delta-1}\left(\frac{\eta}{\zeta}\right)^\nu \geqslant -2\frac{\zeta^\delta(\zeta^\delta+|\xi|^\delta)}{|\zeta^\delta-\xi^\delta|^2}\frac{|\eta|}{\zeta}\sum_{\nu=0}^{\infty}3^{-\nu} = -3\frac{\zeta^\delta(\zeta^\delta+|\xi|^\delta)}{|\zeta^\delta-\xi^\delta|^2}\frac{|\eta|}{\zeta}.$$

Hence

$$\Lambda(\delta,\xi,\eta) \geqslant \frac{\zeta^\delta(\zeta^\delta+|\xi|^\delta)}{|\zeta^\delta-\xi^\delta|^2}\left(1-\frac{|\xi|}{\zeta}-\frac{3|\eta|}{\zeta}\right) \geqslant 0,$$

so that (2.5) implies with any $\psi\in\Psi$

$$\psi(h(x,y)) \leqslant \sup_{t\in T}\psi(f(\rho_t z))\int_T\prod_{\gamma\in\Gamma_{xy}}\Lambda(d(\gamma),x(\gamma)e^{-2\pi i t(\gamma)},y(\gamma)e^{-2\pi i t(\gamma)})dt.$$

Since the integral above is 1, we obtain

$$\psi(h(x,y)) \leqslant \sup_{t\in T}\psi(f(\rho_t z_{x,y,q})) \tag{2.6}$$

first for $q>1$, and then, by continuity, also for $q=1$. If $K\subset\Omega$ is compact then so is the set $L=\{\rho_t z_{x,y,1} : t\in T, (x,y)\in K\}\subset B(R)$, whence (2.6) implies

$$\sup_{K\cap(X_0\times X_0)}\psi(h)\leqslant\sup_L\psi(f)<\infty.$$

Therefore h extends to $H\in\mathcal{O}(\Omega;V)$ by Proposition 2.2.

Now (2.2) makes it clear that (wherever defined) the function $\eta\mapsto h(x,y+\eta e_\gamma)$ is a polynomial of degree less than $d(\gamma)$; the same must hold for H as well. Proposition 2.3 therefore implies that H continues analytically to $F\in\mathcal{O}(B(R)\times X)$. By uniqueness of analytic continuation, $F|_{X_0\times X_0}=h$. One reads off from (2.2) that (2.3), (2.4) are satisfied when $x\in B(R)\cap X_0$, $y\in X_0$, whence (2.3), (2.4) follow for all x,y because X_0 is dense in X.

Remark 2.5. — With F of Theorem 2.1 the function $F(x,y+\xi e_\gamma)$ is a polynomial in $\xi\in\mathbb{C}$ of degree less than $d(\gamma)$, for every x, y, γ. It is easy to check that this property along with (2.3) and (2.4) uniquely determines $F\in\mathcal{O}(B(R)\times X;V)$.

3. Expansions.

Our approach to approximations will be through expansions; the terms in the expansions below will be indexed by pairs (k, n), with $k : \Gamma \to \mathbb{N} \cup \{0\}$ a multiindex and $n \in \mathbb{N} \cup \{0\}$. To measure the size of the terms asymptotically it will be convenient to introduce the following notation.

Recall that S_1 denotes the multiplicative semigroup of those $\sigma : \Gamma \to [0, 1)$ for which the set $\{\gamma \in \Gamma : \sigma(\gamma) > \epsilon\}$ is finite for all $\epsilon > 0$. Suppose for each $n \in \mathbb{N} \cup \{0\}$ we are given a set $\mathcal{K}_n$ of multiindices, $\mathcal{K} = \cup_n \mathcal{K}_n \times \{n\}$; and for each $(k, n) \in \mathcal{K}$ we are given a number $c_{kn} \geqslant 0$.

DEFINITION 3.1. — *If $0 \leqslant \lambda \leqslant \infty$, we shall write $\lambda \rhd (c_{kn})_{\mathcal{K}}$ resp. $\lambda \trianglerighteq (c_{kn})_{\mathcal{K}}$ to mean $\sup_{k \in \mathcal{K}_n} c_{kn}\sigma^k < \infty$ for all n, $\sigma \in S_1$, and*

$$\limsup_{n} \sup_{k \in \mathcal{K}_n} (c_{kn}\sigma^k)^{1/n} < \lambda \quad \textit{resp.} \leqslant \lambda, \quad \textit{for all } \sigma \in S_1.$$

If some $\mathcal{K}_n = \emptyset$, the corresponding sup is understood to be $-\infty$. Below we shall drop reference to $\mathcal{K}$ and just write $\lambda \rhd$ resp. $\lambda \trianglerighteq c_{kn}$ if $\mathcal{K}$ consists of all pairs (k, n) for which c_{kn} is defined. This notion is natural in the study of power series. For example, when Γ consists of a single element, so that multiindices can be identified with nonnegative integers, one can check that the power series $\sum a_{kn}x^k y^n$ converges in the bidisc $|x| < 1/\lambda$, $|y| < 1$ precisely when $\lambda \trianglerighteq |a_{kn}|$. It is also possible to describe convergence in other Reinhardt domains in terms of a relation $\lambda \trianglerighteq c_{kn}$, where c_{kn} is $|a_{kn}|$ times an appropriate weight. Theorem 3.7 below will generalize this observation, cf. also Proposition 5.1.

Returning to an arbitrary Γ, the following rules should be remembered. Here and later $\#k$ stands for the cardinality of the set $\{\gamma : k(\gamma) \neq 0\}$. Thus $0 \leqslant \#k < \infty$.

PROPOSITION 3.2. — *Suppose $\lambda < \infty$.*

(a) *$\lambda \trianglerighteq (c_{kn})_{\mathcal{K}}$ holds precisely when $\mu \rhd (c_{kn})_{\mathcal{K}}$ for all $\mu > \lambda$.*

(b) *If $\lambda \rhd (c_{kn})_{\mathcal{K}}$ then for every $\sigma \in S_1$ there is a number C such that $c_{kn}\sigma^k < C\lambda^n$.*

(c) *If $\lambda \trianglerighteq (c_{kn})_{\mathcal{K}}$, $\mu \rhd (d_{kn})_{\mathcal{K}}$ and α, β are positive numbers then*

$$(c_{kn}^{\alpha} d_{kn}^{\beta})_{\mathcal{K}} \begin{cases} \lhd \lambda^{\alpha}\mu^{\beta}, & \textit{if } \lambda > 0 \\ \trianglelefteq 0, & \textit{if } \lambda = 0. \end{cases}$$

Here $\lambda^{\alpha}\infty^{\beta}$ is understood to mean ∞ if $\lambda > 0$.

(d) $q \trianglerighteq Q^{\#k}q^n$ *if* $q, Q > 0$.

Proof. — (a), (b), and (c) are straightforward consequences of Definition 3.1. It will suffice to verify (d) when $Q \geqslant 1$. Choose $\sigma \in S_1$, let $\Gamma_0 = \{\gamma \in \Gamma : \sigma(\gamma) \geqslant 1/Q\}$, and define $\tau \in S_1$ by

$$\tau(\gamma) = \begin{cases} \sigma(\gamma), & \text{for } \gamma \in \Gamma_0 \\ Q\sigma(\gamma), & \text{for } \gamma \notin \Gamma_0. \end{cases}$$

Then $Q^{\#k}\sigma^k \leqslant Q^{|\Gamma_0|}\tau^k \leqslant Q^{|\Gamma_0|}$ and so $\limsup_n \sup_k (Q^{\#k}q^n\sigma^k)^{1/n} \leqslant q$.

Now consider a Banach space X with basis $\{e_\gamma : \gamma \in \Gamma\}$ satisfying (1.2); the action ρ of the torus $T = \{t : \Gamma \to S^1\}$ as in Section 2; and a second, this time arbitrary Banach space Y with norm $\| \quad \|_Y$ and balls $B_Y(P) = \{y \in Y : \|y\|_Y < P\}$, $0 < P < \infty$. The action ρ induces an action $\overline{\rho}$ of $T \times S^1$ on $X \times Y$:

$$\overline{\rho}_{t,s}(x, y) = (\rho_t x, e^{2\pi i s}y), \quad t \in T,\ s \in S^1,\ x \in X,\ y \in Y.$$

Any $f \in \mathcal{O}(B(R) \times B_Y(P); V)$ can be expanded in a series

$$f \sim \sum_{n=0}^{\infty}\sum_k f_{kn}, \qquad f_{kn} = \int_{T\times S^1} e^{-2\pi i(k\cdot t+ns)}\overline{\rho}^*_{t,s} f\, dtds. \tag{3.1}$$

Upon inspecting restrictions of f, f_{kn} to subspaces spanned by finitely many $e_\gamma \in X$ and a single $y \in Y$, one finds that $f_{kn}(x, y) = a_{kn}(y)x^k$, with $a_{kn} \in \mathcal{O}(Y; V)$ homogeneous of degree n; and also that the series $\sum\sum f_{kn}$ converges to f on $(B(R) \cap X_0) \times B_Y(P)$, X_0 denoting the linear span of the basis vectors e_γ. (3.1) will be called the homonomial expansion of f, f_{kn} the homonomial components. In general, a series $\sum h_{kn}(x, y) = \sum b_{kn}(y)x^k$ with $b_{kn} \in \mathcal{O}(Y; V)$ homogeneous of degree n will be called a homonomial series. Our goal in this section is to estimate the terms of homonomial expansions. We start with a crude estimate:

THEOREM 3.3. — *If* $0 < R, \delta < \infty$, *the homonomial components* f_{kn} *of* $f \in \mathcal{O}(B(R) \times B_Y(\delta); V)$ *satisfy*

$$\infty \rhd \sup_{B(R)\times B_Y(P)} \psi(f_{kn}), \qquad \psi \in \Psi,\ P < \infty. \tag{3.2}$$

Proof. — Recall from Section 1 that the semigroup S_1 acts on X. Fix $\sigma \in S_1$. The set $\sigma B(R) \subset B(R)$ being relatively compact by Proposition 1.2, for any $\psi \in \Psi$ there is an $\epsilon \in (0, \delta)$ such that

$$\sup_{\sigma B(R)\times B_Y(\epsilon)} \psi(f) = A < \infty.$$

This implies the homonomial components in (3.1) satisfy

$$A \geqslant \sup_{\sigma B(R)\times B_Y(\epsilon)} \psi(f_{kn}) = \sigma^k \sup_{B(R)\times B_Y(\epsilon)} \psi(f_{kn}).$$

Hence $\sigma^k \sup_{B(R)\times B_Y(P)} \psi(f_{kn}) \leqslant A(P/\epsilon)^n$ by homogeneity, and (3.2) indeed holds.

The estimate in Theorem 3.3 is not sharp. To describe the size of homonomial components more accurately we fix a subgroup $G \subset T$, and consider only G invariant functions. Expansions of such functions can be understood by comparing them with a certain series Δ that generalizes the geometric series from finite dimensional analysis. In the context of the space $X = l^1(\Gamma)$ and trivial G this function was first introduced in [L1], [L2]. Let

$$M_k = \sup_{B(1)} |x^k| \qquad \text{and} \tag{3.3}$$

$$\Delta_G(q,x) = \sum_k{}^G |q|^{\#k}|x^k|/M_k, \quad q \in \mathbb{C},\ x \in B(1), \tag{3.4}$$

where $\sum^G$ indicates summation over those monomials x^k that are G invariant. In plain English, k should satisfy $k \cdot t = 0 \in S^1$ if $t \in G$. Approximation in the spaces $X = l^p(\Gamma)$, with the canonical basis, depends on the following

THEOREM 3.4. — *If $X = l^p(\Gamma)$, $m \geqslant p$ is an integer, and $G = \{t \in T : mt = 0\}$ then*

(a) *(3.4) converges uniformly on compact subsets of $\mathbb{C} \times B(1)$, and Δ_G is continuous.*

(b) *For every $r < 1$ there is a $q > 0$ such that $\Delta_G(q,\cdot)$ is bounded on $B(r)$.*

Proof. — We start by computing M_k. Denote $\sum_\gamma k(\gamma)$ by $|k|$, and assuming $k \neq 0$ let $z_k = k^{1/p}|k|^{-1/p}$, a point in the closed unit ball. Then $M_k \geqslant z_k^k = k^{k/p}|k|^{-|k|/p}$. In fact, the inequality between the arithmetic and geometric means implies that $M_k = k^{k/p}|k|^{-|k|/p}$; this holds also when $k = 0$. Thus in the special case $p = m = 1$ the series (3.4) agrees with the series studied in [L1], [L2], and this case of the theorem is the content of [L1], Théorème 2.1 and [L2], Lemma 4.1. Let us write Δ^1 for the function Δ_G corresponding to that case, and $B^1(1)$ for the unit ball in $l^1(\Gamma)$: thus $\Delta^1(q,x) = \sum |q|^{\#k}|x^k||k|^{|k|}k^{-k}$ is continuous on $\mathbb{C} \times B^1(1)$.

For general p, m consider the continuous map $g : B(1) \to B^1(1)$ given by $g(x)(\gamma) = |x(\gamma)|^p$, $x \in B(1)$, $\gamma \in \Gamma$. With $x \in B(1)$ and $y = g(x) \in B^1(1)$

$$\Delta_G(q,x) = \sum_k |q|^{\# k}|x^k|^m |k|^{m|k|/p} k^{-mk/p} = \sum_k \{|Q|^{\# k}|y^k||k|^{|k|}k^{-k}\}^{m/p},$$

where $Q = q^{p/m}$; and the latter series is termwise dominated by the series

$$\Delta^1(Q,y)^{(m/p)-1} \sum |Q|^{\# k}|y^k||k|^{|k|}k^{-k}.$$

Thus the theorem follows from the special case of Δ^1.

Approximations not only in $l^p(\Gamma)$ but in other spaces also depend on the existence of a subgroup $G \subset T$ that satisfies (a), (b) of Theorem 3.4. It turns out that (b) implies (a), which leads to the following

DEFINITION 3.5. — *We shall say that a subgroup $G \subset T$ bridles the space X if for every $r < 1$ there is a $q > 0$ such that $\Delta_G(q, \cdot)$ is bounded on $B(r)$.*

PROPOSITION 3.6. — *If G bridles X then (3.4) converges uniformly on compact subsets of $\mathbb{C} \times \mathbb{B}(\mathbb{K})$, and Δ_G is continuous.*

Proof. — Fix $r \in (0,1)$ and choose $q_0 > 0$ so that $\Delta_G(q_0, \cdot)$ is bounded on $B(r)$. For a finite $\Gamma_0 \subset \Gamma$ and $Q \geqslant q_0$ consider the open set

$$\Omega(\Gamma_0,Q) = \Big\{(q,x) \in \mathbb{C}\times X : |q| < Q, \Big\| \sum_{\Gamma_0} x(\gamma)e_\gamma + (Q/q_0) \sum_{\Gamma\setminus\Gamma_0} x(\gamma)e_\gamma \Big\| < r\Big\}.$$

Notice that $\cup_{\Gamma_0,Q}\Omega(\Gamma_0,Q) = \mathbb{C} \times B(r)$. If $(q,x) \in \Omega(\Gamma_0,Q)$, define

$$y = y_x = \sum_{\Gamma_0} x(\gamma)e_\gamma + (Q/q_0)\sum_{\Gamma\setminus\Gamma_0} x(\gamma)e_\gamma \in B(r).$$

Then $(|q|/q_0)^{\# k}|x^k| \leqslant (Q/q_0)^{|\Gamma_0|}|y^k|$, whence

$$\Delta_G(q,x) \leqslant (Q/q_0)^{|\Gamma_0|}\Delta_G(q_0,y)$$

shows Δ_G is bounded on $\Omega(\Gamma_0,Q)$. Proposition 3.6 now follows from [L2], Propositions 2.1 and 4.2.

Returning to a general X, assume $f \in \mathcal{O}(B(R)\times B(P);V)$ is invariant under a subgroup $G \subset T$:

(3.5) $\qquad f(\rho_t x, y) = f(x,y), \qquad t \in G,\ x \in B(R),\ y \in B_Y(P).$

In this case the homonomial components f_{kn} are also G invariant, whence (3.1) becomes a G invariant expansion

$$f(x,y) \sim \sum_{n=0}^{\infty} \sum_{k} {}^{G} f_{kn}(x,y) = \sum_{n=0}^{\infty} \sum_{k} {}^{G} a_{kn}(y) x^k. \tag{3.6}$$

Assuming G bridles X, a precise description of the series thus gotten is available. The theorem below generalizes the Cauchy–Hadamard formula, to which it reduces when $X = (0)$, $Y = \mathbb{C}$.

THEOREM 3.7. — *Suppose $G \subset T$ bridles X.*

(a) *The homonomial components f_{kn} of a G invariant $f \in \mathcal{O}(B(R) \times B_Y(P); V)$ satisfy*

$$1 \trianglerighteq \sup_{B(R)\times K} \psi(f_{kn}), \qquad \psi \in \Psi,\ K \subset B_Y(P) \text{ compact}, \tag{3.7}$$

and the series

$$\sum_{n=0}^{\infty} \sum_{k} {}^{G} f_{kn} \tag{3.8}$$

converges to f, uniformly on compact subsets of $B(R) \times B_Y(P)$.

(b) *Conversely, if $a_{kn} \in \mathcal{O}(Y; V)$ are homogeneous of degree n, and $f_{kn}(x,y) = a_{kn}(y)x^k$ satisfy (3.7) then (3.8) converges to some G invariant $h \in \mathcal{O}(B(R) \times B_Y(P); V)$, uniformly on compacts. The homonomial expansion of h is (3.8).*

Special cases of this theorem were proved in [R], [L1], [L2]. In our argument below we shall estimate homonomial terms f_{kn} using

$$\psi(f_{kn}(z,y)) \leqslant |z^k| R^{-|k|} \sup_{B(R)} \psi(f_{kn}(\cdot, y))/M_k, \qquad \psi \in \Psi; \tag{3.9}$$

this follows easily from (3.3).

Proof. — We shall assume $R = P = 1$, since the substitution $(x,y) \mapsto (Rx, Py)$ will reduce the general case to this one.

(b) To prove uniform convergence on compacts it suffices to treat compact sets of form $L \times K$, $L \subset B(1), K \subset B_Y(1)$. Let $\sigma \in S_1$ and $L_1 \subset B(1)$ compact such that $\sigma L_1 = L$, *cf.* Proposition 1.2. Fix $q \in (\max_K \|y\|_Y, 1)$ and $\psi \in \Psi$. Then $K/q \subset B_Y(1)$, and

$$1 > q \trianglerighteq q^n \sup_{B(1)\times K/q} \psi(f_{kn}) = \sup_{B(1)\times K} \psi(f_{kn}),$$

by Proposition 3.2(c), (d), (3.7), and homogeneity. Hence by (a), (b) of the same proposition there are C and $\lambda < 1$ such that

$$\sigma^k \sup_{B(1)\times K} \psi(f_{kn}) < C\lambda^n.$$

With an arbitrary $x \in L$, $y \in K$, and $z \in L_1$ such that $x = \sigma z$ we have therefore

$$\psi(f_{kn}(x,y)) = \sigma^k \psi(f_{kn}(z,y)) \leqslant C\lambda^n |z^k|/M_k,$$

by (3.9). Since by Proposition 3.6 the series $\sum_n \sum_k^G \lambda^n |z^k|/M_k$ uniformly converges for $z \in L_1$, (3.8) is also uniformly convergent on $L \times K$. Its sum h is easily seen to be holomorphic (*cf.* Propositions 2.1, 2.2 of [L2] that deal with the case $V = \mathbb{C}$), and G invariant. Upon computing the integrals in (3.1) with $f = h$ one obtains that (3.8) is indeed the homonomial expansion of h.

(a) To verify (3.7) it can be assumed K is invariant under the S^1 action $y \mapsto e^{2\pi i s} y$. With $\sigma \in S_1$ fixed, $A = \sup_{\sigma B(1)\times K} \psi(f) < \infty$ since $\sigma B(1) \times K$ is relatively compact in $B(1) \times B_Y(1)$ by Proposition 1.2. (3.1) implies

$$A \geqslant \sup_{\sigma B(1)\times K} \psi(f_{kn}) = \sigma^k \sup_{B(1)\times K} \psi(f_{kn}),$$

whence (3.7) follows. Furthermore, by part (b) (3.8) converges to a holomorphic function h, uniformly on compact subsets of $B(1) \times B_Y(1)$. As h and f agree on the dense set $(B(1) \cap X_0) \times B_Y(1)$, they must agree everywhere.

COROLLARY 3.8. — *A G invariant homonomial series (3.8) is the homonomial expansion of a function $f \in \mathcal{O}(B(R) \times Y; V)$ if and only if*

$$0 \trianglerighteq \sup_{B(R)\times K} \psi(f_{kn}) \tag{3.10}$$

for all $\psi \in \Psi$ and $K \subset Y$ compact.

Proof. — If (3.10) holds then (3.7) holds as well for all $P < \infty$, whence Theorem 3.7(b) implies that (3.8) is indeed the homonomial expansion of some $f \in \mathcal{O}(B(R)\times Y; V)$. Conversely, if (3.8) is the expansion of such an f then Theorem 3.7(a) implies (3.7) holds for all $P < \infty$. Fix $q > 0$ and replace K by K/q in (3.7) to obtain

$$1 \trianglerighteq \sup_{B(R)\times K/q} \psi(f_{kn}) = q^{-n} \sup_{B(R)\times K} \psi(f_{kn})$$

by virtue of homogeneity. Hence $q \trianglerighteq \sup_{B(R)\times K} \psi(f_{kn})$ for all $q > 0$ and (3.10) follows, *cf.* Proposition 3.2 (c,d,a).

4. Approximations.

Again, let X be a Banach space with basis satisfying (1.2); ρ the action of the torus T, as in Section 2; and $(Y, \| \ \|_Y)$ another Banach space.

THEOREM 4.1. — *If a subgroup $G \subset T$ bridles X, and $f \in \mathcal{O}(B(R) \times Y; V)$ is G invariant, then for any $\varphi \in \Psi$, $r < R$, $\Pi < \infty$, and $\epsilon > 0$ there is an $h \in \mathcal{O}(X \times Y; V)$ such that $\varphi(f-h) < \epsilon$ on $B(r) \times B_Y(\Pi)$.*

Proof. — Let

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{k} {}^G f_{kn}(x,y), \qquad x \in B(R),\ y \in Y \tag{4.1}$$

be the homonomial expansion of f. With $\varphi, \dots, \epsilon$ as in the theorem, $\theta \in (1, R/r)$ fixed, and $Q > 1$, $\epsilon_1 > 0$ to be determined later, for each $n = 0, 1, \dots$ define

$$\begin{aligned} \mathcal{K}_n &= \{k : \sup_{B(R)\times B_Y(\Pi)} \varphi(f_{kn}) \geqslant 2^{-n} Q^{-\#k} \theta^{|k|} \epsilon_1\}, \\ \mathcal{K} &= \bigcup_n \mathcal{K}_n \times \{n\}, \qquad h = \sum_{(k,n)\in\mathcal{K}} f_{kn}. \end{aligned} \tag{4.2}$$

To show that h is holomorphic on $X \times Y$, by Corollary 3.8 we need to check

$$0 \trianglerighteq \Big(\sup_{B(P)\times K} \psi(f_{kn}) \Big)_{\mathcal{K}}, \tag{4.3}$$

for all $P < \infty$, $\psi \in \Psi$, and $K \subset Y$ compact. In fact, by homogeneity, it will suffice to show (4.3) for $K \subset B_Y(\Pi)$. If need be, ψ can be replaced by $\psi + \varphi$ to arrange that $\psi \geqslant \varphi$. In addition to these, we shall also assume, as we may, $P > R$. (4.3) will be derived from the following three estimates:

$$0 \trianglerighteq \max_{B(R)\times K} \psi(f_{kn}), \tag{4.4}$$

$$2 \trianglerighteq \Big(Q^{-\#k} \theta^{|k|} / \sup_{B(R)\times B_Y(\Pi)} \psi(f_{kn}) \Big)_{\mathcal{K}} = (c_{kn})_{\mathcal{K}}, \tag{4.5}$$

$$\infty = 1 \cdot \infty \triangleright Q^{\#k} \sup_{B(R)\times B_Y(\Pi)} \psi(f_{kn}) = d_{kn}, \tag{4.6}$$

that follow respectively from Corollary 3.8, from (4.2), and from Theorem 3.3 and Proposition 3.2(c), (d). Choose $\alpha > 0$ and $\beta = \alpha + 1$ so that $\theta^{\alpha/2} = P/R$, then (4.5) and (4.6) imply

$$\infty = 2^\alpha \infty^\beta \rhd (c^\alpha_{kn} d^\beta_{kn})_{\mathcal{K}} = \Big(Q^{\#k}(P/R)^{2|k|} \sup_{B(R)\times B_Y(\Pi)} \psi(f_{kn})\Big)_{\mathcal{K}},$$

and so

$$0 = 0^{1/2}\infty^{1/2} \trianglerighteq \Big(\sqrt{Q}^{\#k}(P/R)^{|k|} \sup_{B(R)\times K} \psi(f_{kn})\Big)_{\mathcal{K}}$$

by (4.4) and Proposition 3.2(c). This latter implies (4.3); therefore h is indeed entire.

Using (3.9) and (4.2) to estimate $\varphi(f - h)$ we obtain for $(x, y) \in B(r) \times B_Y(\Pi)$

$$\begin{aligned}\varphi(f(x,y) - h(x,y)) &\leqslant \sum_{(k,n)\notin\mathcal{K}} \varphi(f_{kn}(x,y)) \\ &\leqslant \epsilon_1 \sum_n 2^{-n} \sum_k {}^G Q^{-\#k}|x^k|\theta^{|k|}R^{-|k|}/M_k = 2\epsilon_1 \Delta_G(1/Q, \theta x/R).\end{aligned}$$

Fix Q so large that $\Delta_G(1/Q, \cdot)$ is bounded on $B(\theta r/R)$. Then

$$\varphi(f - h) \leqslant 2\epsilon_1 \sup_{B(\theta r/R)} \Delta_G(1/Q, \cdot) < \epsilon$$

on $B(r) \times B_Y(\Pi)$ as required, provided $\epsilon_1 > 0$ is small enough.

Recall that in Section 2 we associated with a function $d : \Gamma \to \mathbb{N}$ the group

$$G = \prod_\Gamma (\mathbb{Z}[1/d(\gamma)]/\mathbb{Z}) \subset T. \tag{4.7}$$

THEOREM 4.2. — *If for some $d : \Gamma \to \mathbb{N}$ the group (4.7) bridles X, then for any $f \in \mathcal{O}(B(R); V)$, $\varphi \in \Psi$, $r < R$, and $\epsilon > 0$ there is an $h \in \mathcal{O}(X; V)$ such that $\varphi(f - h) < \epsilon$ on $B(r)$.*

Proof. — By Theorem 2.1 there is an $F \in \mathcal{O}(B(R) \times X; V)$, G invariant in the first variable, such that $F(x, x) = f(x)$ for $x \in B(R)$, and by Theorem 4.1 there is an $H \in \mathcal{O}(X \times X; V)$ such that $\varphi(F - H) < \epsilon$ on $B(r) \times B(r)$. Hence $h(x) = H(x, x)$ will do.

5. Consummation.

Our goal here is to construct for any Banach space X with a countable unconditional basis a group that bridles X. To do so we shall have to compare the functions Δ_G corresponding to different Banach spaces; when necessary, we shall write $\Delta_{G,X}$ to indicate the underlying space.

PROPOSITION 5.1. — *Suppose that X is a finite dimensional Banach space with basis $\{e_\gamma : \gamma \in \Gamma\}$, and (1.2) is satisfied. Then $\Delta_G(q,\cdot)$ is bounded on $B(r)$ for any $r<1$, $q\in\mathbb{C}$, and subgroup $G\subset T$.*

Proof. — We may assume $|q|\geqslant 1$. By homogeneity $|x^k|\leqslant M_k r^{|k|}$ if $x\in B(r)$, so that

$$\Delta_G(q,x)\leqslant |q|^{|\Gamma|}\sum_k r^{|k|}=|q|^{|\Gamma|}(1-r)^{-|\Gamma|}.$$

Now suppose X has a countable unconditional basis $\{e_\gamma : \gamma\in\mathbb{N}\}$, and (0.1) holds. If $\Gamma_1\subset\mathbb{N}$ and $d:\Gamma_1\to\mathbb{N}$, set

$$G(d)=\{t:\mathbb{N}\to S^1 \mid d(\gamma)t(\gamma)=0 \text{ for all } \gamma\in\Gamma_1\}\subset T.$$

Observe that a monomial x^k is $G(d)$ invariant precisely when $d(\gamma)$ divides $k(\gamma)$ for all $\gamma\in\Gamma_1$ and $k(\gamma)=0$ for $\gamma\notin\Gamma_1$. Clearly if $\Gamma_1\subset\Gamma_2$ and $d':\Gamma_2\to\mathbb{N}$ is a continuation of d then $G(d')\subset G(d)$, and $\Delta_{G(d')}\geqslant\Delta_{G(d)}$.

PROPOSITION 5.2. — *For any $\delta:\{1,\dots,N-1\}\to\mathbb{N}$, $r\in(0,1)$, and $q\in\mathbb{C}$*

(a) *$\Delta_{G(\delta)}(q,\cdot)$ is bounded on $B(r)$;*

(b) *given $\epsilon>0$ one can define a continuation $\delta':\{1,\dots,N\}\to\mathbb{N}$ of δ so that*

$$\Delta_{G(\delta')}(q,x)\leqslant\Delta_{G(\delta)}(q,x)+\epsilon,\quad x\in B(r). \tag{5.1}$$

Proof. — Let $\pi:X\to X$ denote the projection

$$\pi\Big(\sum_{\gamma\in\mathbb{N}}x(\gamma)e_\gamma\Big)=\sum_{\gamma\leqslant N}x(\gamma)e_\gamma.$$

If x^k is a $G(\delta)$ invariant monomial then $k(\gamma)=0$ for $\gamma\geqslant N$. Hence

$$\Delta_{G(\delta),X}(q,x)=\Delta_{G(\delta),X}(q,\pi x)\leqslant\Delta_{\{0\},\pi X}(q,\pi x),$$

and (a) follows from Proposition 5.1. Next, for any $m \in \mathbb{N}$ define a continuation $\delta_m : \{1, \dots, N\} \to \mathbb{N}$ of δ by $\delta_m(N) = m$. Since any $G(\delta_m)$ invariant monomial is of form $x^k = x^\kappa x(N)^{jm}$, with x^κ $G(\delta)$ invariant and $j \in \mathbb{N} \cup \{0\}$,

$$(5.2) \qquad \Delta_{G(\delta_m)}(q, x) - \Delta_{G(\delta)}(q, x) \leqslant \sum_{k(N) \geqslant m}{}^{G(\delta_1)} |q|^{\#k} |x^k| / M_k.$$

The right hand side here is dominated by tail sums of the series

$$(5.3) \qquad \sum_k{}^{G(\delta_1)} |q|^{\#k} |x^k| / M_k = \Delta_{G(\delta_1)}(q, x) = \Delta_{G(\delta_1)}(q, \pi x),$$

which converges when $x \in B(1)$ again by Proposition 5.1. Since finite dimensional power series converge locally uniformly within their domain of convergence, we can choose m so large that

$$\sum_{k(N) \geqslant m}{}^{G(\delta_1)} |q|^{\#k} |y^k| / M_k \leqslant \epsilon, \qquad y \in \pi B(r).$$

Set $\delta' = \delta_m$; then (5.2), (5.3) imply (5.1).

THEOREM 5.3. — *If X is a Banach space with countable unconditional basis $\{e_\gamma : \gamma \in \mathbb{N}\}$ and (0.1) is satisfied then there is a function $d : \mathbb{N} \to \mathbb{N}$ such that $G(d)$ bridles X.*

Proof. — Fix an increasing sequence $0 < r_N \to 1$, and using Proposition 5.2(b) inductively construct $d : \mathbb{N} \to \mathbb{N}$ so that $d_N = d|\{1, \dots, N\}$ satisfy

$$(5.4) \qquad \Delta_{G(d_{N+1})}(1, x) \leqslant \Delta_{G(d_N)}(1, x) + 2^{-N}, \qquad x \in B(r_N).$$

We claim that $G(d)$ bridles X. Indeed, let $r < 1$ and fix N so that $r \leqslant r_N$. Suppose $x \in B(r)$ has only finitely many nonzero coordinates, say $x(\gamma) = 0$ if $\gamma \geqslant M > N$. Then

(5.5)

$$\Delta_{G(d)}(1, x) = \Delta_{G(d_M)}(1, x) \leqslant \Delta_{G(d_N)}(1, x) + \sum_N^{M-1} 2^{-n} \leqslant 1 + \sup_{B(r)} \Delta_{G(d_N)}(1, \cdot)$$

by (5.4). The partial sums of $\Delta_{G(d)}$ being continuous functions, (5.5) implies

$$\sup_{B(r)} \Delta_{G(d)}(1, \cdot) \leqslant 1 + \sup_{B(r)} \Delta_{G(d_N)}(1, \cdot),$$

and this latter is finite according to Proposition 5.2(a), q.e.d.

Theorem 0.1 now follows from Theorem 4.2, because Theorems 3.4 resp. 5.3 provide the group G of form (4.7) that bridles X.

BIBLIOGRAPHY

[D1] S. DINEEN, Cousin's first problem on certain locally convex topological vector spaces, An. Acad. Brasil. Cienc., 48 (1976), 11–12.

[D2] S. DINEEN, Complex Analysis in Locally Convex Spaces, North Holland, Amsterdam, 1981.

[D3] S. DINEEN, Complex Analysis on Infinite Dimensional Spaces, Springer, Berlin, 1999.

[DS] N. DUNFORD, T. SCHWARTZ, Linear Operators I, John Wiley & Sons, New York, 1988.

[L1] L. LEMPERT, Approximation de fonctions holomorphes d'un nombre infini de variables, Ann. Inst. Fourier, 49–4 (1999), 1293–1304.

[L2] L. LEMPERT, The Dolbeault complex in infinite dimensions, II, J. Amer. Math. Soc., 12 (1999), 775–793.

[L3] L. LEMPERT, The Dolbeault complex in infinite dimensions III, manuscript.

[M] P. MAZET, Analytic Sets in Locally Convex Spaces, North Holland, Amsterdam, 1984.

[MV] R. MEISE and D. VOGT, Counterexamples in holomorphic functions on nuclear Fréchet spaces, Math. Z., 182 (1983), 167–177.

[N] P. NOVERRAZ, Pseudo-convexité, convexité polynomiale et domaines d'holomorphie en dimension infinie, North Holland, Amsterdam, 1973.

[P] I. PATYI, On the $\overline{\partial}$-equation in a Banach space, Bull. Soc. Math. France, to appear.

[R] R. A. RYAN, Holomorphic mappings in l^1, Trans. Amer. Soc., 302 (1987), 797–811.

[S] I. SINGER, Bases in Banach spaces I-II, Springer, Berlin, 1981.

Manuscrit reçu le 7 juillet 1999.

László LEMPERT,
Purdue University
Department of Mathematics
West Lafayette, IN 47907–1395 (USA).
lempert@math.purdue.edu