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HYPERBOLICITY PROPERTIES OF QUOTIENT SURFACES BY FREELY OPERATING ARITHMETIC LATTICES

by E. OELJEKLAUS and C. SCHMERLING

0. Introduction.

During the last decades arithmetic quotients of bounded symmetric domains have been intensively studied. However, only few results on the hyperbolicity of these quotients are known. A complex space Y is called *hyperbolic*, if the Kobayashi pseudo metric d_Y on Y is a metric. Brody [Br] has proved that a compact complex space is hyperbolic if and only if every holomorphic map from $\mathbb{C}$ into Y is constant. In this paper we shall prove that a special class of twodimensional arithmetic quotients of bounded symmetric domains is hyperbolic.

THEOREM. — *Let D be a bounded symmetric domain in $\mathbb{C}^2$ and $\Gamma \subset \mathrm{Aut}^0 D$ an irreducible arithmetic lattice which operates freely on D. Then the cusp–compactification $\overline{X}$ of $X = D/\Gamma$ is hyperbolic.*

Note that Y is called *hyperbolic modulo* $S \subset Y$ if $d_Y(x,y) = 0$ implies that $x = y$ or that $x \in S$ and $y \in S$. Let $\pi : \tilde{X} \to \overline{X}$ be the minimal resolution of the singularities of $\overline{X}$ and $R := \pi^{-1}(\overline{X} \backslash X)$.

COROLLARY. — *$\tilde{X}$ is a minimal surface of general type and hyperbolic modulo R.*

Keywords: Bounded symmetric domain – Arithmetic lattice – Surface of general type.
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In the first section we briefly recall the relevant aspects of the theory of arithmetic quotients of bounded symmetric domains. Then we deal with the logarithmic canonical bundle. Our method of proving Kobayashi-hyperbolicity is to construct a so-called hyperbolic pseudo metric on $\overline{X}$. This is a hermitian pseudo metric with distance-decreasing property, which degenerates only on a proper subvariety. In our situation the push-down of the Bergman metric yields a Hermitian metric with negative holomorphic sectional curvature on $X = D/\Gamma$. We modify this metric by multiplying it with a suitable non–negative real function to get a pseudo metric on $\overline{X}$ which still has the distance-decreasing property for holomorphic mappings $\mathbb{H} \to \overline{X}$ and degenerates exactly in the cusp points. For this purpose the vanishing order of the function has to be carefully adjusted. We apply a method which was used by Schumacher and Takegoshi in the compact case [ST]. Of main importance for us were the results of Mumford in [Mu], in particular, the estimates for the degeneration of the induced metric on $X \subset \tilde{X}$ along the boundary divisor R, see also [Ko]. Our considerations are also based on the explicit description of the minimal resolution $\tilde{X}$ for the cusp singularities given by Hemperly [He] for arithmetic ball quotients and by Hirzebruch [Hi] for the Hilbert modular surfaces. In the situation $D = \mathbb{H}^2$ we apply a theorem of Tai [AMRT], also proved with different methods by Mumford [Mu], which guarantees for sufficiently small Γ many pluricanonical holomorphic sections over $\tilde{X}$ which vanish of high order along the boundary divisor R.

The above theorem is a generalization of results of the doctorate thesis [S] of the second-named author, who would like to thank Siegmund Kosarew for encouragement and helpful discussions.

1. Neat lattices and cusp resolution.

Let D be a bounded symmetric domain in $\mathbb{C}^2$ equipped with the Bergman metric and $\operatorname{Aut} D$ the real Lie group of biholomorphic automorphisms of D. Recall that D is either biholomorphic to the two dimensional ball $\mathbb{B}^2 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 < 1\}$ or to the twofold product $\mathbb{H}^2$ of the upper half plane. The groups $\operatorname{Aut} \mathbb{B}^2 = SU(2,1,\mathbb{C})/\mathbb{Z}_3$ and $\operatorname{Aut}^0 \mathbb{H}^2 = PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$ are simple and semisimple respectively.

Throughout this paper Γ denotes a lattice in $G := \operatorname{Aut}^0 D$, i.e. Γ is a discrete subgroup of G and the quotient G/Γ has finite volume with respect to the invariant Haar measure on G. The group Γ operates properly discontinuously on D. In particular, the groups $\Gamma_z = \{\gamma \in \Gamma | \gamma z = z\}$ are finite for all $z \in D$. The quotient D/Γ carries a complex structure such that the natural surjection $D \to D/\Gamma$ is a holomorphic covering, ramified in $z \in D$ if and only if $\Gamma_z \neq \{e\}$. A lattice Γ in $PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$ is called reducible, if Γ is *commensurable* with the direct product $\Lambda = \Gamma_1 \times \Gamma_2$ of discrete subgroups $\Gamma_1, \Gamma_2 \subset PGL(2,\mathbb{R})$, i.e. the intersection $\Gamma \cap \Lambda$ is a cofinite subgroup (a subgroup of finite index) of both Γ and Λ. Otherwise Γ is called *irreducible*.

At first we consider the case $D = \mathbb{H}^2$. Let $K = \mathbb{Q}(\sqrt{d})$ be a totally real quadratic number field, $d \in \mathbb{N}$ square-free, and o_K the ring of integers of K.

The embedding

$$a = r + s\sqrt{d} \mapsto r - s\sqrt{d} =: \overline{a},$$

$r, s \in \mathbb{Q}$, of K into $\mathbb{R}$ induces an embeddding

$$SL(2,K) \to SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), \quad A \mapsto (A, \overline{A}).$$

Now $SL(2,K)$ can be viewed as a subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and we define Γ_K to be the image of $SL(2,o_K)$ under the canonical surjection $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$.

By a theorem of Selberg every irreducible non cocompact lattice Γ in $PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$ is commensurable to some Γ_K as above, up to an inner automorphism of $PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})$. Therefore these lattices are *arithmetic* in the sense of Borel [Bo].

Every lattice in $\operatorname{Aut}\mathbb{B}^2$ is irreducible. Note that there are non-arithmetic lattices in $\operatorname{Aut}\mathbb{B}^2$. However, if $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$ square-free, then every lattice Γ in $G := \operatorname{Aut}\mathbb{B}^2$ which is commensurable to $G \cap SL(3,o_K)/\mathbb{Z}_3$, is arithmetic.

If the quotient $X = D/\Gamma$ of D by an irreducible arithmetic lattice Γ is not compact, then it can be compactified by finitely many points (cusps) to a normal projective algebraic variety $\overline{X}$, the so-called cusp-compactification of X.

Next we roughly describe the cusp-compactification of quotients of bounded symmetric domains in $\mathbb{C}^2$ by irreducible arithmetic lattices. This construction is a special case of a general construction in arbitrary dimensions given by Baily and Borel [BB]. For details we refer to [He], [Ho] in the case $D = \mathbb{B}^2$ and to [Fr2] for $D = \mathbb{H}^2$.

The holomorphic action of $G = \mathrm{Aut}^0 D$ on D extends to a topological action on the topological closure $\overline{D}$ of D in $\mathbb{C}^2$, and for $x \in \partial D$ the isotropy group $P_x = \{g \in G \,|\, gx = x\}$ is a minimal parabolic subgroup of G with unipotent radical U_x. Let Γ be an irreducible arithmetic lattice in G. A point $x \in \partial D$ is called a rational boundary point (with respect to Γ) if $\Gamma \cap U_x$ is a cocompact lattice in U_x. The set of rational boundary points is invariant under the Γ–action on $\overline{D}$, and the Γ–orbit $\Gamma x = \{\gamma x \,|\, \gamma \in \Gamma\}$ of a rational boundary point is called a *cusp*. The assumptions on Γ imply that the set A_Γ of cusps is finite. The complex structure on $X := D/\Gamma$ can be extended to a (uniquely determined) complex structure on $\overline{X} := X \cup A_\Gamma$ such that $\overline{X}$ is a normal complex space, called the *cusp–compactification* of X. Every cusp is a singular point of $\overline{X}$. If Γ does not operate freely on D, then there are also finitely many singular points in X corresponding to $z \in D$ with $\Gamma_z \neq \{e\}$. In every cusp $\kappa \in A_\Gamma$ we fix a point x_κ and define $\Gamma_\kappa := \Gamma \cap P_{x_\kappa}$. Let Γ' be a cofinite normal subgroup of Γ. The cusp κ decomposes into finitely many Γ'–cusps $\kappa = \kappa_1' \cup \ldots \cup \kappa_{r_\kappa}'$, $r_\kappa = [\Gamma : \Gamma_\kappa \Gamma']$, and the holomorphic surjection $X' = D/\Gamma' \to X$ extends to a holomorphic map $\sigma : \overline{X'} \to \overline{X}$ with $\sigma^{-1}(\kappa) = \{\kappa' \in A_{\Gamma'} \,|\, \kappa' \subset \kappa\}$ and $\#\sigma^{-1}(\kappa) = r_\kappa$.

The vector space V_n of holomorphic Γ-automorphic forms of weight n on D is finite dimensional, and each basis of V_n provides a holomorphic embedding of $\overline{X}$ into some $\mathbb{P}^N$ for n$\in \mathbb{N}$ sufficiently large. In particular, $\overline{X}$ is a projective algebraic variety, and there is a very ample holomorphic line bundle $\mathcal{O}(n)$ on $\overline{X}$ such that the global holomorphic sections of $\mathcal{O}(n)$ are in 1-1-correspondence with the elements of V_n.

Hereafter we assume that Γ operates freely on D. Then $X = D/\Gamma$ is a complex manifold, and the natural surjection $D \to D/\Gamma$ is an unramified holomorphic covering. For the minimal resolution $\pi : \tilde{X} \to \overline{X}$ of the singularities of $\overline{X}$ the divisors $\pi^{-1}(\kappa)$, $\kappa \in A_\Gamma$, have normal crossings. Their structures have been described in detail by [He] and [Ho] for $D = \mathbb{B}^2$, and by [Hi] for $D = \mathbb{H}^2$:

a) $D = \mathbb{B}^2$: $E_\kappa := \pi^{-1}(\kappa)$ is a smooth elliptic curve for every $\kappa \in A_\Gamma$.

b) $D = \mathbb{H}^2$: $R_\kappa := \pi^{-1}(\kappa)$ is either a rational curve B_0 with an ordinary double point and self intersection number $B_0^2 \leq -1$ or a cycle of $r \geq 2$ smooth rational curves $B_0, \ldots, B_{r-1}$ with the following structure:
   - α) $r = 2$: B_1, B_2 intersect transversally in two points, and $B_1^2 \leq -2$, $B_2^2 \leq -3$.
   - β) $r \geq 3$: $B_i B_{i+1} = B_0 B_{r-1} = 1$, $B_i^2 \leq -2$, $B_{r-1}^2 \leq -3, 0 \leq i \leq r-2$.

For the following considerations we have to put an even stronger assumption on Γ, namely that Γ is a *neat* subgroup of G. Recall that an element $g \in GL(n,\mathbb{C})$ is called neat, if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion-free. A subgroup $H \subset GL(n,\mathbb{C})$ is called neat if every element in H is neat. In our situation we call Γ a neat subgroup of G if every element of Γ has a neat representative in $SU(2,1,\mathbb{C})$ (resp. in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$). A neat arithmetic lattice Γ in G operates freely on D because every finite subgroup of Γ is trivial.

For every finitely generated number field with ring of integers o and every $\gamma \in SL(n,o)$, $\gamma \neq e$, the group $SL(n,o)$ contains a cofinite neat normal subgroup Λ_1 with $\gamma \notin \Lambda_1$, see [Bo]. It follows that every subgroup $\Lambda \subset SL(n,\mathbb{C})$, which is commensurable with $SL(n,o)$, has a similar property: If $\gamma \in \Lambda$, $\gamma \neq e$, and Λ_1 is a cofinite neat subgroup of $SL(n,o)$ with $\gamma \notin \Lambda_1$, then $\Lambda \cap \bigcap_{g\in\Lambda} g\Lambda_1 g^{-1}$ is a cofinite neat normal subgroup of Λ. This implies

LEMMA 1. — *Let D be a bounded symmetric domain in $\mathbb{C}^2$ and Γ an arithmetic subgroup of $G = \mathrm{Aut}^0 D$. For every finite set $M \subset \Gamma$ with $e \notin M$ the group Γ contains a cofinite neat normal subgroup Γ' with $M \cap \Gamma' = \emptyset$.* □

2. The logarithmical canonical bundle.

Beside the minimal resolution $\tilde{X} \to \overline{X}$ we take more generally into account all resolutions $\pi : \hat{X} \to \overline{X}$ of the singularities of $\overline{X}$ such that $R := \pi^{-1}(\overline{X}\backslash X)$ has normal crossings, and we identify X with $\pi^{-1}(X) \subset \hat{X}$. The canonical line bundle of X can be extended to a holomorphic line bundle $KX(\log R)$ over $\hat{X}$, which is characterized by the following property: For every open polycylinder $\Delta \subset \hat{X}$ satisfying

$$\Delta \cap R = \{(z,w) \in \Delta \mid z = 0\} \text{ (resp. } \Delta \cap R = \{(z,w) \in \Delta \mid zw = 0\})$$

a holomorphic section of $KX(\log R)$ over Δ has the form

$$a(z,w)\Big(\frac{dz}{z} \wedge dw\Big), \text{ resp. } a(z,w)\Big(\frac{dz \wedge dw}{zw}\Big),$$

where $a(z,w)$ is holomorphic in Δ. The bundle $KX(\log R)$ is called *logarithmical canonical bundle* of $\hat{X}$ (with respect to R). From the above property it follows immediately that $KX(\log R)$ is isomorphic to the holomorphic line bundle $\hat{K} \otimes L$, where $\hat{K}$ is the canonical line bundle of $\hat{X}$

and L is the holomorphic line bundle associated to the boundary divisor R. Note that $\dim H^0(\hat{X}, L^n) = 1$ for every $n \in \mathbb{N}$. For each irreducible neat arithmetic lattice Γ in G and any $n \in \mathbb{N}^+$ the pull-back $\pi^*(\mathcal{O}(n))$ of the ample line bundle $\mathcal{O}(n)$ on $\overline{X}$ is isomorphic to the line bundle $(KX(\log R))^{\otimes n}$ on $\hat{X}$, see [He] and [Mu], Prop. 3.3 and 3.4, and therefore isomorphic to $\hat{K}^n \otimes L^n$. We note for later use:

LEMMA 2. — *For every $x_0, x_1 \in X \subset \hat{X}$, $x_0 \neq x_1$, and $n \in \mathbb{N}$ sufficiently large, there exists some $s \in H^0(\hat{X}, \hat{K}^n \otimes L^{n-1})$ with $s(x_0) \neq 0$ and $s(x_1) = 0$.*

Proof. — The bundle $\mathcal{O}(n)$ is very ample for $n \gg 0$. In particular, there exists some $t \in H^0(\overline{X}, \mathcal{O}(n))$ with $t(\pi(x_0)) \neq 0$ and $t(\pi(x_1)) = t(y) = 0$ for every cusp point $y \in \overline{X} \backslash X$. Then $\pi^*(t) \in H^0(\hat{X}, \hat{K}^n \otimes L^n)$ vanishes on $R \cup \{x_1\}$ and $\pi^*(t)(x_0) \neq 0$. Define $s := \frac{\pi^*(t)}{d}$ for some $d \neq 0$ in $H^0(\hat{X}, L)$. □

For a cofinite normal subgroup Γ' of Γ let $\pi' : \hat{X}' \to \overline{X}'$ be a resolution of the singularities of the cusp–compactification $\overline{X}'$ of $X' = D/\Gamma'$ such that the divisor $R' := \hat{X}' \backslash X'$ has normal crossings. We denote by $\hat{K}'$ the canonical line bundle of $\hat{X}'$ and by L' the holomorphic line bundle on $\hat{X}'$ associated to the divisor R'. Let $\tilde{X}'$ be the minimal resolution with canonical bundle $\tilde{K}'$.

LEMMA 3. — *Let Γ be an irreducible arithmetic lattice in* $\mathrm{Aut}^0 \mathbb{H}^2$. *Then Γ contains a cofinite neat normal subgroup Γ' with the following properties:*

1) *$\tilde{X}'$ is a minimal surface of general type.*
2) *For every finite set $B \subset X' \subset \tilde{X}'$ there exist $n_0 \in \mathbb{N}$ and $s \in H^0(\tilde{X}', \tilde{K}'^{n_0} \otimes L'^{-1})$ with $s(x) \neq 0$ for every $x \in B$.*

Proof. — We may (and will) assume that $\mathbb{H}^2/\Gamma$ is not compact and, by Lemma 1, that Γ is neat. Let Γ be commensurable with $\Gamma_K = SL(2, o_K)$ for some totally real quadratic number field K. For $n \in \mathbb{N}$ let $\Gamma_K(n)$ be the principal congruence subgroup of Γ_K with respect to n. The cusp–compactification of $\mathbb{H}^2/\Gamma_K(n)$ does not contain any rational or elliptic curve if n is sufficiently large [Fr1]. Choose $\Gamma' \subset \Gamma_K(n)$ to be a cofinite normal subgroup of Γ. Then $\tilde{X}'$ is a minimal projective algebraic surface with only finitely many rational (and no elliptic) curves, all of them contained in R'. Therefore $\tilde{X}'$ is either a minimal surface of general type or a minimal $K3$–surface, but the latter case would contradict Lemma 2, since $\dim H^0(\tilde{X}', L'^n) = 1$ for every $n \in \mathbb{N}$.

To prove 2) note that the pluricanonical map f_k maps $\tilde{X}'$ holomorphically and birationally onto $f_k(\tilde{X}') \subset \mathbb{P}^N$ for $k \geq 5$. It suffices to study the case $B = \{x_0\}$. Since x_0 is not contained in any (-2)–curve, $f_k(x_0) \notin f_k(R')$. For $m \gg 0$ there is an irreducible hypersurface $S \subset \mathbb{P}^N$ of degree m with $f_k(x_0) \notin S$, $f_k(R') \subset S$. This implies the existence of a holomorphic section $t \in H^0(\tilde{X}', \tilde{K}'^{mk})$ which vanishes exactly on $f_k^{-1}(S)$. Define $n_0 := mk$ and $s := \frac{t}{d'}$ for some $d' \neq 0$ in $H^0(\tilde{X}', L')$. □

Finally we will study the case $D = \mathbb{B}^2$ in more detail, assuming that Γ is a neat arithmetic lattice in $\operatorname{Aut}\mathbb{B}^2$. Let Γ' be a cofinite normal subgroup of Γ. Since $E' := \tilde{X}'\backslash X'$ is a disjoint union of finitely many smooth elliptic curves $E'_{\kappa'}$, $\kappa' \in A_{\Gamma'}$, the unramified holomorphic covering map $X' \to X$ of degree $[\Gamma : \Gamma']$ extends to a holomorphic map $p : \tilde{X}' \to \tilde{X}$, ramified along E'. We can easily calculate the ramification order of p over E_κ, $\kappa \in A_\Gamma$. Let $E'_{\kappa,1}, \ldots, E'_{\kappa,r_\kappa}$ be the irreducible components of $p^{-1}(E_\kappa)$ with $r_\kappa = [\Gamma : \Gamma_\kappa\Gamma']$ and $\Gamma'_\kappa := \Gamma' \cap \Gamma_\kappa$. The map p is ramified of order $[\Gamma : \Gamma'] \cdot [\Gamma : \Gamma_\kappa\Gamma']^{-1} = [\Gamma_\kappa : \Gamma'_\kappa]$ along these curves and the ramification divisor of p equals

$$D = \sum_{\kappa \in A_\Gamma} \sum_{\nu_\kappa = 1}^{r_\kappa} ([\Gamma_\kappa : \Gamma'_\kappa] - 1) E'_{\kappa,\nu_\kappa}.$$

Let $E'_\kappa := \sum_{\nu_\kappa=1}^{r_\kappa} E_{\kappa,\nu_\kappa}$ for $\kappa \in A_\Gamma$. Since $\tilde{K}' = p^*(\tilde{K}) \otimes [D]$ (cf. [BPV], p. 41) and since $p^*(\tilde{L}) = \bigotimes_{\kappa \in A_\Gamma} [E'_\kappa]^{[\Gamma_\kappa : \Gamma'_\kappa]}$, we obtain

LEMMA 4. — *Let Γ be a neat arithmetic lattice in* Aut $\mathbb{B}^2$ *and let Γ' be a cofinite normal subgroup of Γ. Then*

$$p^*(\tilde{K}^n \otimes L^{n-1}) = \tilde{K}'^n \otimes \bigotimes_{\kappa \in A_\Gamma} [E'_\kappa]^{n - [\Gamma_\kappa : \Gamma'_\kappa]}$$

for all $n \in \mathbb{N}$. □

3. Degeneration of the Bergman metric along the boundary divisor.

In 1977 Mumford generalized the Proportionality Theorem of Hirzebruch to the case of non-compact quotients of bounded symmetric domains by neat arithmetic lattices. For our setting we extract the following result from the Main Theorem and Proposition 3.4 in [Mu], which yields a description of the invariant Bergman metric h on X along the boundary divisor $R = \hat{X} \backslash X$ on $\hat{X}$.

LEMMA 5 (Mumford). — *Let D be a bounded symmetric domain in $\mathbb{C}^2$ and Γ a neat irreducible arithmetic lattice in $G = \mathrm{Aut}^0 D$. The holomorphic tangent bundle T and the canonical line bundle of $X = D/\Gamma$ can be extended to holomorphic vector bundles $\hat{T}$ and $KX(\log R)$ on $\hat{X}$ such that h and $(\det h)^{-1}$ define singular metrics on $\hat{T}$ and $KX(\log R)$ respectively, in the following sense:*

For every open polycylinder $\Delta \subset \hat{X}$ satisfying

$$\Delta \cap R = \{(z,w) \in \Delta \mid z = 0\} \ (\text{resp.}\ \Delta \cap R = \{(z,w) \in \Delta \mid zw = 0\})$$

and for every local basis of $\hat{T}$ and of $KX(\log R)$ over Δ the following is true:

1) *There are constants $C > 0$ and $m, n \in \mathbb{N}$ such that :*

$$|h_{ij}(z,w)| \le C \cdot (\log|z|)^{2m},$$
$$(\det h)^{-1}(z,w) \le C \cdot (\log|z|)^{2n}, (\text{resp.}\ |h_{ij}(z,w)| \le C \cdot (\log|z| + \log|w|)^{2m},$$
$$(\det h)^{-1}(z,w) \le C \cdot (\log|z| + \log|w|)^{2n}$$

for every $(z,w) \in \Delta \cap X$.

2) *For every holomorphic section $s \in H^0(\Delta \cap X, T)$ the following statements are equivalent:*

i) $s = \hat{s}_{|\Delta \cap X}$ *for some* $\hat{s} \in H^0(\Delta, \hat{T})$

ii)

$$h_{(z,w)}(s,s) \le C \cdot (\log|z|)^{2m} \ (\text{resp.}\ h_{(z,w)}(s,s) \le C \cdot (\log|z| + \log|w|)^{2m}$$

for some $C > 0$ and some $m \in \mathbb{N}$.

iii)

$$s(z,w) = a(z,w) \cdot \left(z\frac{\partial}{\partial z}\right) + b(z,w)\frac{\partial}{\partial w} (\text{resp.}\ s(z,w) = a(z,w) \cdot \left(z\frac{\partial}{\partial z}\right) + b(z,w) \cdot \left(w\frac{\partial}{\partial w}\right) \ \text{with}\ a, b \in \mathcal{O}(\Delta).$$

Proof. — (See [Mu] Main Theorem) For the equivalence i)⇔ ii) see [Mu], Prop.1.3. For the equivalence ii)⇔ iii) we refer to [Mu] Prop. 3.4. □

From Lemma 5 we immediately deduce the following estimates for the Bergman metric h in a neighbourhood of the boundary divisor R:

COROLLARY. — *Let Δ be an open polycylinder in $\tilde{X}$ with $\Delta \cap R = \{(z,w) \in \Delta \mid z = 0\}$ (resp. $\Delta \cap R = \{(z,w) \in \Delta \mid z \cdot w = 0\}$).*

For every local basis of T over Δ there exist constants $C > 0$ and $m \in \mathbb{N}$ such that

$$|h_{ij}(z,w)| \le C \cdot (\log|z|)^{2m} \cdot |z|^{-2} (\text{resp. } |h_{ij}(z,w)| \le C \cdot (\log|z| + \log|w|)^{2m} \cdot |z|^{-2} \cdot |w|^{-2}$$

for every $(z,w) \in \Delta \cap X$. □

In the situation $D = \mathbb{H}^2$ we shall apply a general theorem of Tai [AMRT], Theorem 2, later also proved with different methods by Mumford [Mu], Theorem 4.1 and Proposition 4.2, which, in this situation, reads as follows:

If Γ is an irreducible neat arithmetic lattice in $\mathrm{Aut}^0 D$, then for some cofinite normal subgroup Γ' of Γ every desingularization of $\overline{X'} = \overline{D/\Gamma'}$ is of general type.

The unramified covering $X' \to X$ can be extended to a meromorphic map $p : \tilde{X}' \to \tilde{X}$. A finite sequence of suitable blow ups of $\tilde{X}'$ leads to a desingularization $\hat{X}'$ of $\overline{X'}$ and a birational holomorphic map $\tau : \hat{X}' \to \tilde{X}'$ such that $\hat{p} := p \circ \tau$ is holomorphic. Note that $\hat{X}' \backslash X'$ has normal crossings. A central step in the above cited proofs of Tai's Theorem is a verification of the following statement, see [AMRT], p. 301 and [Mu], p. 270–272.

LEMMA 6. — *Let $m \in \mathbb{N}$ and $s \in H^0(\tilde{X}, \tilde{K}^m \otimes L^m)$ with $s(x) = 0$ for every $x \in \tilde{X} \backslash X$. There exists a cofinite normal subgroup Γ' in Γ such that $\hat{p}^*(s) \in H^0(\hat{X}', \hat{K}'^m)$.* □

4. Construction of a hyperbolic pseudo metric.

Now we construct a continuous Hermitian pseudo metric on the cusp-compactification $\overline{X}$ of $X = D/\Gamma$ for a neat irreducible arithmetic lattice Γ in $\mathrm{Aut}^0 D$. More precisely, we construct such a pseudo metric on a suitable desingularization $\hat{X}$ of $\overline{X}$ and push it down to $\tilde{X}$ and $\overline{X}$. This pseudo metric will be distance-decreasing for holomorphic mappings from the unit disk $\mathbb{E}$ into $\overline{X}$ and vanishes exactly in the cusp points. By an averaging process such a pseudo metric can be constructed for every freely operating irreducible arithmetic lattice.

The standard Bergman metric on D is a Kähler-Einstein metric with Kähler-Einstein coefficient $\lambda = -1$. The ball $\mathbb{B}^2$ has constant negative holomorphic sectional curvature $K_{\mathbb{B}^2} := -\frac{2}{3}$, and the holomorphic sectional curvature of $\mathbb{H}^2$ is bounded from above by $K_{\mathbb{H}^2} := -\frac{1}{2}$.

PROPOSITION. — *Let D be a bounded symmetric domain in $\mathbb{C}^2$ and Γ an irreducible arithmetic lattice in* $\mathrm{Aut}^0 D$ *which operates freely on D. For every $x_0 \in X \subset \tilde{X}$ there exists a continuous function $\phi : \tilde{X} \to \mathbb{R}_{\geq 0}$ with $\phi(x_0) > 0$ and the following properties:*

1) *The curve $R = \tilde{X} \backslash X$ is contained in $N_\phi := \{x \in \tilde{X} \mid \phi(x) = 0\}$, and ϕ is smooth on $\tilde{X} \backslash N_\phi$.*

2) *The product $\phi \cdot h$ defines a continuous pseudo metric on $\tilde{X}$ which is identically zero on R.*

3) *There exists $a \in \mathbb{R}$, $0 < a < -K_D$, such that for every holomorphic map $f : \mathbb{E} \to \tilde{X}$ and every $\zeta \in f^{-1}(\tilde{X} \backslash N_\phi)$ the equation*

$$\frac{\partial^2 \log(\phi \circ f)}{\partial \zeta \partial \overline{\zeta}}(\zeta) = (-a) \frac{\partial^2 \log \det(f^* h)}{\partial \zeta \partial \overline{\zeta}}(\zeta) \tag{1}$$

holds.

Proof. — It suffices to prove the proposition for some cofinite neat normal subgroup $\Gamma' \subset \Gamma$, replacing $\{x_0\}$ by some finite set $B \subset X' = D/\Gamma' \subset \tilde{X}'$. In fact, if $p : \tilde{X}' \to \tilde{X}$ is the meromorphic extension of the natural holomorphic covering map $X' \to X$ and if ϕ satisfies the above conditions on $\tilde{X}'$ with $B := p^{-1}(x_0)$, then

$$\psi : \tilde{X} \to \mathbb{R}_{\geq 0},\ \psi(x) := \sqrt[n]{\prod_{y \in p^{-1}(x)} \phi(y)},$$

$n = [\Gamma : \Gamma']$, has the desired properties for $\tilde{X}$ and x_0. Therefore we assume that Γ is an irreducible neat arithmetic lattice, see Lemma 1.

We study the cases $D = \mathbb{B}^2$ and $D = \mathbb{H}^2$ separately.

I. $D = \mathbb{B}^2$

We fix $n_0 \in \mathbb{N}$ such $t(x_0) \neq 0$ for some $t \in H^0(\tilde{X}, \tilde{K}^{n_0} \otimes L^{n_0 - 1}))$ and choose a cofinite normal subgroup Γ' of Γ with $[\Gamma_\kappa : \Gamma'_\kappa] \geq 2n_0$ for all cusps $\kappa \in A_\Gamma$, see Lemma 2 and Lemma 1. We define

$$s := p^*(t) \in H^0(\tilde{X}', p^*(\tilde{K}^{n_0} \otimes L^{n_0 - 1})).$$

Note that $p : \tilde{X}' \to \tilde{X}$ is holomorphic. In the terminology of Lemma 4 let $d \in H^0(\tilde{X}', \bigotimes_{\kappa\in A_\Gamma}[E'_\kappa]^{[\Gamma_\kappa:\Gamma'_\kappa]})$ be a holomorphic section with zero-divisor $\sum_{\kappa\in A_\Gamma}[\Gamma_\kappa : \Gamma'_\kappa]E'_\kappa$. Applying Lemma 4 we note that

$$\phi(x) := (|s(x)|^2 \cdot |d(x)|^2 \cdot (\det h(x))^{-n_0})^{\frac{4}{7n_0}} \quad \text{for } x \in X',$$

$$\phi(x) := 0 \quad \text{for} \quad x \in R' = \tilde{X}'\backslash X',$$

is a non-negative real function on $\tilde{X}'$.

Let $\Delta \subset \tilde{X}'$ be an open polycylinder with $\Delta \cap E' = \{(z,w) \in \Delta | z = 0\}$. Lemma 5 yields for all $(z,w) \in \Delta \cap X'$:

$$|\phi(z,w)| \le C(\log|z|)^{2n}|z|^{\frac{4\cdot 4n_0}{7n_0}}$$

with suitable constants $C > 0$ and $n \in \mathbb{N}$. In particular, ϕ satisfies condition 1) on $\tilde{X}'$. For the pseudo metric $\phi \cdot h$ the corollary to Lemma 5 yields

$$|\phi(z,w) \cdot h_{ij}(z,w)| \le C \cdot (\log|z|)^{2m} \cdot |z|^{\frac{2}{7}}$$

for all $(z,w) \in \Delta \cap X'$ and suitable constants $C > 0$ and $m \in \mathbb{N}$. Therefore the pseudo metric $\phi \cdot h$ satisfies condition 2) on $\tilde{X}'$.

Finally,

$$\frac{\partial^2 \log(\phi \circ f)}{\partial\overline{\zeta}\partial\zeta}(\zeta) = -\frac{4}{7}\,\frac{\partial^2 \log(\det(f^*h))}{\partial\zeta\partial\overline{\zeta}}(\zeta)$$

for every $\zeta \in f^{-1}(X'\backslash N_\phi)$. Hence for $a := \frac{4}{7} < \frac{2}{3} = -K_{\mathbb{B}^2}$ equation (1) holds.

II. $D = \mathbb{H}^2$

According to Lemma 3 and the above remarks we assume that there exist $n_0 \in \mathbb{N}$ and $s \in H^0(\tilde{X}, \tilde{K}^{n_0} \otimes L^{-1})$ with $s(x_0) \neq 0$. Let $d \in H^0(\tilde{X}, L^{n_0+1})$ with zero-divisor $(n_0+1)R$. Then $sd \in H^0(\tilde{X}, \tilde{K}^{n_0} \otimes L^{n_0})$, and Lemma 6 yields a cofinite normal subgroup $\Gamma' \subset \Gamma$ such that $t := \hat{p}^*(sd) \in H^0(\hat{X}', \hat{K}'^{n_0})$, where $\hat{p} : \hat{X}' \to \tilde{X}$ is the holomorphic map used in Lemma 6. Let $d' \in H^0(\hat{X}', L'^{n_0})$ with zero-divisor n_0R'. We define a non-negative real function $\hat{\phi}$ on $\hat{X}'$ as follows:

$$\hat{\phi}(x) := (|td'(x)|^2 \cdot (\det h(x))^{-n_0})^{\frac{2}{4n_0+1}} \quad \text{for} \quad x \in X',$$

$$\hat{\phi}(x) := 0 \quad \text{for} \quad x \in R' = \hat{X}'\backslash X'.$$

Applying Lemma 5 in both cases $\Delta \cap R' = \{(z,w) \in \Delta \,|\, z = 0\}$ and $\Delta \cap R' = \{(z,w) \in \Delta \,|\, z \cdot w = 0\}$ we see that $\hat{\phi}$ satisfies condition 1) on $\hat{X}'$. It follows from the corollary to Lemma 5 that $\hat{\phi} \cdot h$ also satisfies condition 2) on $\hat{X}'$ since $\frac{2(4n_0+2)}{4n_0+1} > 2$ and

$$|\hat{\phi}(z,w) \cdot h_{ij}(z,w)| \le C \cdot (\log|z|)^{2m} \cdot |z|^{(2(n_0+1)+2n_0)\frac{2}{4n_0+1}-2},$$

$$\text{resp.} \quad |\hat{\phi} \cdot h_{ij}(z,w)| \le C \cdot (\log|z| + \log|w|)^{2m} \cdot |zw|^{\frac{2(4n_0+2)}{4n_0+1}-2}$$

for every $(z,w) \in \Delta \cap X'$ and for suitable constants $C > 0$ and $m \in \mathbb{N}$.

Finally

$$\frac{\partial^2 \log(\hat{\phi} \circ f)}{\partial\zeta\partial\overline{\zeta}}(\zeta) = -\frac{2n_0}{4n_0+1}\frac{\partial^2 \log(\det(f^*h))}{\partial\zeta\partial\overline{\zeta}}(\zeta)$$

for every $\zeta \in f^{-1}(X'\backslash N_{\hat{\phi}})$. Hence for $a := \frac{2n_0}{4n_0+1} < \frac{1}{2} = -K_{\mathbb{H}^2}$ equation (1) holds for $\hat{\phi}$ instead of ϕ. Let $\tau : \hat{X}' \to \tilde{X}'$ be the birational holomorphic map introduced at the end of Section 3. Then $\phi := \hat{\phi} \circ \tau^{-1}$ obviously satisfies the statement of the proposition for $\tilde{X}'$. □

Recall that the Kobayashi pseudo metric d_Y on a connected complex space Y dominates every continuous pseudo metric ρ on Y which is distance–decreasing with respect to holomorphic mappings from the unit disk $\mathbb{E}$ into Y:

$$d_Y(y,z) \geq \rho(y,z) \text{ for all } y,z \in Y.$$

In particular, if $y_0 \in Y$ and $\rho(y_0,z) > 0$ for all $z \neq y_0$ in some neighborhood of y_0, then $d_Y(y_0,z) > 0$ for all $z \neq y_0$ in Y.

THEOREM. — *Let D be a bounded symmetric domain in $\mathbb{C}^2$ and $\Gamma \subset \mathrm{Aut}^0 D$ an irreducible arithmetic lattice operating freely on D. Then the cusp–compactification $\overline{X}$ of D/Γ is hyperbolic.*

Proof. — Let $x_0 \in D/\Gamma \subset \tilde{X}$. By the above remark it suffices to show the existence of a continuous pseudo metric ρ_{x_0} on $\tilde{X}$ which is distance–decreasing with respect to holomorphic mappings $\mathbb{E} \to \tilde{X}$ and satisfies $\rho_{x_0}(x_0,y) > 0$ for all $y \neq x_0$ in some neighborhood U of x_0. In fact, this implies that $d_{\tilde{X}}(x,y) = 0$ if and only if $x = y$ or $x,y \in R$. In particular, the image of every holomorphic map $\mathbb{C} \to \tilde{X}$ is contained in R, and every holomorphic map $\mathbb{C} \to \overline{X}$ has to be constant. Let K_D and a be defined as above and $\phi : \tilde{X} \to \mathbb{R}_{\geq 0}$ a function which satisfies the statement of the proposition. It is sufficient to show that for some $r > 0$ the pseudo metric $r \cdot \phi \cdot h$ on $\tilde{X}$ is distance–decreasing with respect to holomorphic mappings $f : \mathbb{E} \to \tilde{X}$.

We apply a method which was used in the compact case in [ST] and verify the inequality

$$u_f(\zeta) := \frac{1}{2}\,\phi(f(\zeta)) \cdot (f^*h)(\zeta)(1-|\zeta|^2)^2 \leq \frac{\|\phi\|_\infty}{-(a+K_D)} =: \frac{1}{r}$$

for every $\zeta \in \mathbb{E}$. Recall that $a < -K_D$ and

$$\frac{\partial^2 \log(\phi \circ f)}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) = (-a)\frac{\partial^2 \log(\det(f^*h))}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) = -a(f^*h)(\zeta_0).$$

Following the standard proof of the Lemma of Ahlfors we note that there exists $\zeta_0 \in f^{-1}(X \backslash N_\phi)$ with

$$u_f(\zeta_0) = \max\{u_f(\zeta) \,|\, \zeta \in \mathbb{E}\}, \quad \zeta_0 = \zeta_0(f),$$

and therefore

$$\begin{aligned} 0 &\geq \frac{\partial^2 \log u_f}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) \\ &= \frac{\partial^2 \log(\phi \circ f)}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) + \frac{\partial^2 \log(f^* h)}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) + 2\,\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \log(1-|\zeta|^2)|_{\zeta=\zeta_0}. \end{aligned}$$

The sectional curvature condition for the metric h yields

$$\frac{\partial^2 \log(f^* h)}{\partial\zeta\partial\overline{\zeta}}(\zeta_0) \geq -K_D(f^* h)(\zeta_0)$$

and obviously $\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \log(1-|\zeta|^2)|_{\zeta=\zeta_0} = \frac{-1}{(1-|\zeta_0|^2)^2}$.

Finally we obtain

$$0 \geq -a((f^* h)(\zeta_0)) - K_D(f^* h)(\zeta_0) - \frac{2}{(1-|\zeta_0|^2)^2},$$

hence

$$-\frac{1}{2}(a+K_D)(f^* h)(\zeta_0) \cdot (1-|\zeta_0|^2)^2 \leq 1.$$

This proves

$$\begin{aligned} &u_f(\zeta) \leq u_f(\zeta_0) \\ &= \frac{\phi(f(\zeta_0))}{-(a+K_D)} \cdot \frac{1}{2}(-(a+K_D))(f^*(h))(\zeta_0)(1-|\zeta_0|^2)^2 \leq \frac{\|\phi\|_\infty}{-(a+K_D)} \end{aligned}$$

for every $\zeta \in \mathbb{E}$. □

COROLLARY. — *Let $\pi : \tilde{X} \to X$ be the minimal resolution of the singularities of $\overline{X}$. Then $\tilde{X}$ is a minimal surface of general type and hyperbolic modulo $R = \pi^{-1}(\overline{X} \backslash X)$.* □

BIBLIOGRAPHY

[AMRT] A. ASH, D. MUMFORD, M. RAPOPORT, Y. TAI Smooth compactification of locally symmetric varieties. Lie groups: History, Frontiers and application, vol. IV, Math. Sci. Press, 1975.

[BB] W. BAILY, A. BOREL Compactification of arithmetic quotients of bounded symmetric domains, Ann. of Math., 84 (1966), 422–528.

[Bo] A. Borel Introduction aux groupes arithmetiques, Hermann, 1969.

[BPV] W. Barth, C. Peters, A. van de Ven Compact complex surfaces, Erg. d. Mathematik, 3. Folge, Bd. 4, Springer (1984).

[Br] R. Brody Compact manifolds and hyperbolicity, Trans. AMS, 235 (1976), 213–219.

[Fr1] E. Freitag Eine Bemerkung zur Theorie der Hilbertschen Modulmannigfaltigkeiten hoher Stufe, Math. Z., 171 (1980), 27–35.

[Fr2] E. Freitag Hilbert modular forms, Springer, 1990.

[GR] H. Grauert, H. Reckziegel Hermitesche Metriken und normale Familien holomorpher Abbildungen, Math. Z., 89 (1965), 108–125.

[He] J. Hemperly The parabolic contribution to the number of linearly independent automorphic forms on a certain bounded domain, Amer. J. of Math., 94 (1972), 1078–1110.

[Hi] F. Hirzebruch Hilbert modular surfaces, Enseign. Math., (1973), 183–281.

[Ho] R.-P. Holzapfel Ball and Surface Arithmetics, Aspects of mathematics, Vol. 29, Vieweg, 1998.

[Ko] R. Kobayashi Einstein-Kähler metrics on open algebraic surfaces of general type, Tohoku Math. J., 37 (1985), 43–77.

[Mu] D. Mumford Hirzebruch's proportionality theorem in the non-compact case, Inv. Math., 42 (1977), 239–272.

[S] C. Schmerling Eine Hyperbolizitätsuntersuchung für reine arithmetische Quotientenflächen symmetrischer beschränkter Gebiete, Dissertation, Bremen, 1997.

[ST] G. Schumacher, K. Takegoshi Hyperbolicity and branched coverings, Math. Ann., 286 (1990), 537–548.

[vdG] G. van der Geer Hilbert modular surfaces, Erg. d. Math., 3, Folge, Vol. 16, Springer, 1988.

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