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CHRISTIAN DUVAL

PIERRE LECOMTE

VALENTIN OVSIENKO

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CONFORMALLY EQUIVARIANT QUANTIZATION: EXISTENCE AND UNIQUENESS

by C. DUVAL, P. LECOMTE, V. OVSIENKO

In Memory of Moshe Flato and André Lichnerowicz

1. Introduction.

The general problem of quantization is often understood as the quest for a correspondence between the smooth functions of a given symplectic manifold, i.e., the classical observables, and (symmetric) operators on a certain associated Hilbert space, which are called the quantum observables. This correspondence must satisfy a number of additional properties that heavily depend upon the standpoint of the authors and are, by no means, universal.

One guiding principle for the search of a quantization procedure is to impose further coherence with some natural symmetry of phase space. This constitutes the foundations of the “orbit method” [14], geometric quantization [17], [26], [15] in the presence of symmetries, Moyal-Weyl quantization (see, e.g., [10]) defined by requiring invariance with respect to the linear symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ of $\mathbb{R}^{2n}$.

1.1. Equivariant quantization problem.

In all previous examples, the symmetry group was a Lie subgroup of the group of all symplectomorphisms of the symplectic manifold. Here, we

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will confine considerations to the case of cotangent bundles, T^*M , with their canonical (vertical) polarization. This polarization, together with the Liouville 1-form on T^*M , should be preserved by the symmetry group which naturally arises as the cotangent lift of a Lie group, G , acting on M .

We will thus look for an identification, as G -modules, between the space, $\mathcal{S}(M)$, of smooth functions on T^*M that are polynomial on the fibers and the space, $\mathcal{D}(M)$, of linear differential operators on M .

Note that the Moyal-Weyl quantization does not fit into this general framework in which the symmetries of configuration space, M , play a central rôle. Indeed, the action of $\mathrm{Sp}(2n, \mathbb{R})$ on $T^*\mathbb{R}^n$ does not descend to $\mathbb{R}^n$.

Now, G -equivariance between $\mathcal{S}(M)$ and $\mathcal{D}(M)$ is clearly too strong a requirement if G is the group, $\mathrm{Diff}(M)$, of all diffeomorphisms of M . We will therefore impose such an equivariance in the weaker case where G is a finite-dimensional Lie group whose action on M is to be only local.

The main tool we will be using is provided by the notion of (flat) G -structure. Let us recall that a G -structure on M is defined by a local action of G on M , compatible with a local identification of M with some homogeneous space G/H . More precisely, it is defined by an atlas of charts $(\varphi_\alpha, V_\alpha)$ with $\varphi_\alpha : V_\alpha \rightarrow G/H$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is given by an element of G . This new approach significantly differs from the more usual one which makes use of connections to intrinsically define quantization procedures and symbol calculus.

The G -structure we will consider in this article is the conformal structure with $G = \mathrm{SO}(p+1, q+1)$ modeled on the pseudo-Riemannian manifold $S^p \times S^q$.

Our purpose is to show that there exists, actually, a canonical isomorphism of $\mathrm{SO}(p+1, q+1)$ -modules between the space of symbols, $\mathcal{S}(M)$, and the space of differential operators, $\mathcal{D}(M)$.

Experience of other approaches to the quantization problem and of the geometrical study of differential equations prompts us to rather consider the space, $\mathcal{D}_{\lambda, \mu}(M)$, of differential operators with arguments and values in the space of tensor densities of weights λ and μ respectively. So, we will naturally need to study this space of differential operators as a $\mathrm{SO}(p+1, q+1)$ -module. As a consequence, the $\mathrm{SO}(p+1, q+1)$ -module of symbols will be twisted by the weight $\delta = \mu - \lambda$, and denoted by $\mathcal{S}_\delta(M)$.

Let us emphasize that equivariant quantization of G -structures has already been carried out in the case of projective structures, i.e. $\mathrm{SL}(n+1, \mathbb{R})$ -structures, in the recent papers [22], [20]. As for conformal structures, a first step towards their equivariant quantization was taken in [9] in the case of second order operators.

In the particular case $n = 1$, both conformal and projective structures coincide. We refer to [6] for a thorough study of $\mathrm{sl}(2, \mathbb{R})$ -equivariant quantization and of the corresponding invariant star-product. See also [28] for a classic monography on the structures of $\mathrm{sl}(2, \mathbb{R})$ -module on the space of differential operators on the real line.

There exist various approaches to the quantization problem, however, our viewpoint put emphasis on the equivariance condition with respect to a (maximal) group, G , of symmetry in the context of deformation quantization. Now, G -equivariance is the root of geometric quantization [26], [17], [15], Berezin quantization [2], [3], etc., but it seems to constitute a fairly new approach in the framework of symbol calculus, deformation theory and semi-classical approximations dealt with in this work.

1.2. Quantizing equivariantly conformal structures.

We outline here the main results we have obtained, and describe the general framework adopted in this article to answer the question raised in the preceding section.

We review in Section 2 the structures of the spaces of symbols $S_\delta(M)$ and of differential operators $\mathcal{D}_{\lambda, \mu}(M)$ as $\mathrm{Vect}(M)$ -modules. The Lie algebra, $\mathfrak{o}(p+1, q+1)$, of conformal Killing vectors of a conformally flat manifold (M, g) is then described. The restriction of the preceding $\mathrm{Vect}(M)$ -actions is also explicitly calculated.

Section 3 presents the main theorems which establish the existence and the uniqueness of a $\mathfrak{o}(p+1, q+1)$ -equivariant quantization in the special and fundamental case $\lambda = \mu$. It turns out that the value $\lambda = \mu = \frac{1}{2}$ guarantees that our quantization actually defines a star-product on T^*M . This is precisely the value of the weights used in geometric quantization. In the general case, we again obtain a canonical isomorphism of $\mathfrak{o}(p+1, q+1)$ -modules, except for an infinite series of values of $\delta = \mu - \lambda$, which we call resonances.

Section 4 is devoted to the algebra of invariants. One considers the action of $\mathfrak{o}(p+1, q+1)$ and the Euclidean subalgebra $\mathfrak{e}(p, q)$ on the space

of polynomials on $T^*\mathbb{R}^n$. Resorting to Weyl's theory of invariants, we characterize the commutant of $\mathfrak{o}(p+1, q+1)$ within the algebra of operators on the latter space of polynomials: it is a commutative associative algebra with two generators. The commutant of $e(p, q)$ has already been determined in [9]. These algebras of conformal and Euclidean invariants play a crucial rôle in our work and enable us to compute the Casimir operators C_δ and $C_{\lambda, \mu}$ of the $\mathfrak{o}(p+1, q+1)$ -actions on $\mathcal{S}_\delta(M)$ and $\mathcal{D}_{\lambda, \mu}(M)$ respectively.

In Section 5 we provide the proofs of the main theorems. It should be stressed that these proofs rely on the diagonalization of the preceding Casimir operators in an essential way. The same idea has already been exploited in [6] in the one-dimensional case, and in [20] in the case of projectively flat manifolds of dimension $n > 1$.

Section 6 is concerned with the explicit expression of the quantization map restricted to second order polynomials on T^*M . It is worth noticing that our conformally equivariant quantization on $T^*\mathbb{R}^n$ differs from the standard Weyl quantization to which it constitutes a new alternative.

This article is also related to various different subjects, namely representation theory, the theory of invariant differential operators and the cohomology of Lie algebras. A number of very concrete problems could be tackled in this framework. For example, the quantization of the geodesic flow has been achieved in a purely conformally invariant manner [9]. Also the Yamabe-Laplace operator (or conformal Laplacian), see [4], arose from the quantization of the same geodesic flow in a resonant case (recall that this operator is of special importance in field theory in a curved space-time, see, e.g., [24]).

Let us finally mention that this work opens up a number of original questions under current investigation, viz the determination of the $\mathfrak{o}(p+1, q+1)$ -invariant star-product and multi-dimensional Schwarzian derivative.

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2. Basic definitions and tools.

2.1. Differential operators on tensor densities.

Let us recall that a tensor density of degree λ on manifold M is a smooth section of the line bundle $\Delta_\lambda(M) = |\Lambda^n T^*M|^{\otimes \lambda}$ over M . The space of tensor densities of degree λ is naturally a $\text{Diff}(M)$ - and $\text{Vect}(M)$ -module. In this paper, we will consider the space $\mathcal{F}_\lambda(M)$ (or $\mathcal{F}_\lambda$ in short) of complex-valued smooth tensor densities, i.e., of the sections of $\Delta_\lambda(M) \otimes \mathbb{C}$.

The space $\mathcal{D}_{\lambda,\mu}$ of linear differential operators

$$(2.1) \quad A : \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$$

from λ -densities to μ -densities on M is naturally a $\text{Diff}(M)$ - and $\text{Vect}(M)$ -module. These modules have been studied and classified in [8], [21], [22], [12], [11], [23], [9], [19].

There is a filtration $\mathcal{D}_{\lambda,\mu}^0 \subset \mathcal{D}_{\lambda,\mu}^1 \subset \cdots \subset \mathcal{D}_{\lambda,\mu}^k \subset \cdots$, where the module of zero-order operators $\mathcal{D}_{\lambda,\mu}^0 \cong \mathcal{F}_{\mu-\lambda}$ consists of multiplication by $(\mu - \lambda)$ -densities. The higher-order modules are defined by induction: $A \in \mathcal{D}_{\lambda,\mu}^k$ if $[A, f] \in \mathcal{D}_{\lambda,\mu}^{k-1}$ for every $f \in C^\infty(M)$.

2.2. Symbols with values in tensor densities.

Consider the space $\mathcal{S} = \Gamma(S(TM))$ of contravariant symmetric tensor fields on M which is naturally a $\text{Diff}(M)$ - and $\text{Vect}(M)$ -module. We will define the space of δ -weighted symbols on T^*M as the space of sections

$$(2.2) \quad \mathcal{S}_\delta = \Gamma(S(TM) \otimes \Delta_\delta(M)).$$

The space $\mathcal{S}_\delta$ is also, naturally, a $\text{Diff}(M)$ - and $\text{Vect}(M)$ -module.

Again, there is a filtration $\mathcal{S}_\delta^0 \subset \mathcal{S}_\delta^1 \subset \cdots \subset \mathcal{S}_\delta^k \subset \cdots$, where $\mathcal{S}_\delta^k$ denotes the space of symbols of degree less or equal to k . In contrast to the filtration on the space $\mathcal{D}_{\lambda,\mu}$ of differential operators, the above filtration on the space (2.2) of symbols actually leads to a $\text{Diff}(M)$ -invariant graduation

$$(2.3) \quad \mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{S}_{k,\delta}$$

where $\mathcal{S}_{k,\delta}$ denotes the space of homogeneous polynomials (isomorphic to $\mathcal{S}_\delta^k / \mathcal{S}_\delta^{k-1}$).

2.3. $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{S}_\delta$ as $\text{Vect}(M)$ -modules.

We will always assume M orientable and identify $\mathcal{F}_\lambda$ to $C^\infty(M) \otimes \mathbb{C}$ by the choice of a volume form, Vol , on M .

It is clear from the definition of $\mathcal{D}_{\lambda,\mu}$ that the corresponding $\text{Vect}(M)$ -action, $\mathcal{L}^{\lambda,\mu}$, is given by

$$(2.4) \quad \mathcal{L}_X^{\lambda,\mu}(A) = L_X^\mu A - A L_X^\lambda$$

where $X \in \text{Vect}(M)$, and L_X^λ is the standard Lie derivative of λ -densities $\mathcal{F}_\lambda$. Now, any λ -density being represented by $f |\text{Vol}|^\lambda$ for some $f \in C^\infty(M) \otimes \mathbb{C}$, the Lie derivative L_X^λ is thus given by

$$(2.5) \quad L_X^\lambda(f) = X(f) + \lambda \text{Div}(X) f$$

where $\text{Div}(X) = L_X(\text{Vol})/\text{Vol}$.

As to the $\text{Vect}(M)$ -action, L^δ , on $\mathcal{S}_\delta$, it reads

$$(2.6) \quad L_X^\delta(P) = L_X(P) + \delta \text{Div}(X) P$$

where L_X denotes here the Lie derivative of contravariant tensors given by the cotangent lift of $X \in \text{Vect}(M)$.

2.4. The modules $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$ and $\mathcal{S}_\delta(\mathbb{R}^n)$.

In a given coordinate system $(x^1, \dots, x^n)$ on $\mathbb{R}^n$, the expression of a differential operator $A \in \mathcal{D}_{\lambda,\mu}^k$ (see (2.1)) reads

$$(2.7) \quad A = A_k^{i_1 \dots i_k} \partial_{i_1} \dots \partial_{i_k} + \dots + A_1^i \partial_i + A_0$$

where $\partial_i = \partial/\partial x^i$, the coefficient $A_\ell^{i_1 \dots i_\ell} \in C^\infty(\mathbb{R}^n)$ being symmetric in $i_1, \dots, i_\ell$ for $\ell = 0, 1, \dots, k$. From now on we suppose a summation over repeated indices.

The local expression of a symbol $P \in \mathcal{S}_\delta^k$ (see (2.2)), in the canonical coordinate system $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ on $T^*\mathbb{R}^n$ is then

$$(2.8) \quad P = P_k^{i_1 \dots i_k} \xi_{i_1} \dots \xi_{i_k} + \dots + P_1^i \xi_i + P_0$$

where $P_\ell^{i_1 \dots i_\ell} \in C^\infty(\mathbb{R}^n)$ represent the components of symmetric contravariant tensor fields on $\mathbb{R}^n$ (for $\ell = 0, 1, \dots, k$).

As vector spaces, $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{S}_\delta$ are clearly isomorphic, though not in a canonical way. For example, the normal ordering map

$$(2.9) \quad \sigma : A_k^{i_1 \dots i_k} \partial_{i_1} \dots \partial_{i_k} \mapsto A_k^{i_1 \dots i_k} \xi_{i_1} \dots \xi_{i_k}$$

defines such an isomorphism.

The $\text{Vect}(\mathbb{R}^n)$ -action (2.4) on differential operators is, of course, different from the standard $\text{Vect}(\mathbb{R}^n)$ -action (2.6) on polynomials. We will, therefore, distinguish the two $\text{Vect}(\mathbb{R}^n)$ -modules

$$(2.10) \quad \mathcal{D}_{\lambda,\mu} = (\text{Pol}(T^*\mathbb{R}^n), \mathcal{L}^{\lambda,\mu}),$$

$$(2.11) \quad \mathcal{S}_\delta = (\text{Pol}(T^*\mathbb{R}^n), L^\delta).$$

In particular, a vector field X corresponds to a first-order polynomial, $X = X^i \xi_i$. The operator of Lie derivative is then given by the Hamiltonian vector field

$$(2.12) \quad L_X^\delta = \partial_{\xi_i} X \partial_i - \partial_i X \partial_{\xi_i} + \delta DX,$$

where $D = \partial_{\xi_i} \partial_i$ is the divergence operator (see Section 4.1). This local expression precisely corresponds to the previous expression (2.6).

One easily proves the

PROPOSITION 2.1. — *The $\text{Vect}(\mathbb{R}^n)$ -action on $\mathcal{D}_{\lambda,\mu}$ has the following form:*

$$(2.13) \quad \begin{aligned} \mathcal{L}_X^{\lambda,\mu} = & L_X^\delta - \frac{1}{2} \partial_i \partial_j X \partial_{\xi_i} \partial_{\xi_j} - \lambda (\partial_i \circ D) X \partial_{\xi_i} \\ & + (\text{higher order derivatives } \partial_{i_1} \dots \partial_{i_\ell} X) \end{aligned}$$

where $\delta = \mu - \lambda$.

2.5. Conformally flat manifolds.

Throughout this paper we will deal with conformally flat manifolds. Let us recall that a smooth pseudo-Riemannian manifold (M, g) is conformally flat if, for every $x \in M$, there exists a neighborhood V_x of x and $F \in C^\infty(V_x, \mathbb{R}_+^*)$ such that (V_x, g) is flat with the new metric $g = Fg$.

The basic example of a Riemannian n -dimensional conformally flat manifold is the sphere S^n with its canonical metric, and $S^p \times S^q$ in the case of signature $p - q$. A conformally flat manifold is locally identified with such

a homogeneous space and thus admits a local action of $\mathrm{SO}(p+1, q+1)$. The associated, locally defined, action of the Lie algebra $\mathfrak{o}(p+1, q+1)$ corresponds (if $n = p+q \geq 3$) to that of the subalgebra of the vector fields X solutions of

$$L_X g = f g$$

for some $f \in C^\infty(M)$ depending upon X .

It is well known that a conformally flat manifold admits an atlas in which $\mathfrak{o}(p+1, q+1)$ is generated by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i} \\ X_{ij} &= x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i} \\ X_0 &= x^i \frac{\partial}{\partial x^i} \\ \bar{X}_i &= x_j x^j \frac{\partial}{\partial x^i} - 2x_i x^j \frac{\partial}{\partial x^j} \end{aligned} \quad (2.14)$$

where $i, j = 1, \dots, n$ and $x_i = g_{ij} x^j$. In the sequel, indices will be raised and lowered by means of the (flat) metric g .

Let us introduce the following nested Lie subalgebras that will be considered below, namely

$$(2.15) \quad \mathfrak{o}(p, q) \subset \mathfrak{e}(p, q) \subset \mathfrak{ce}(p, q) \subset \mathfrak{o}(p+1, q+1)$$

where $\mathfrak{o}(p, q)$ is generated by the X_{ij} , the Euclidean subalgebra $\mathfrak{e}(p, q)$ by X_{ij} and X_i and the Lie algebra $\mathfrak{ce}(p, q) = \mathfrak{e}(p, q) \rtimes \mathbb{R}$ by X_{ij} , X_i and X_0 .

Remark 2.2. — It is worth noticing that the conformal Lie algebra $\mathfrak{o}(p+1, q+1)$ is maximal in the Lie algebra $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}^n)$ of polynomial vector fields in the following sense: any bigger subalgebra of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}^n)$ necessarily coincides with $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}^n)$. See [5] for a simple proof. The uniqueness and the canonical character of our quantization procedure definitely originates from this maximality property of $\mathfrak{o}(p+1, q+1)$. See also [25] for a classification of a class of maximal Lie subalgebras of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}^n)$.

Remark 2.3. — From now on, we will use local coordinate systems adapted to the flat conformal structure on M in which the generators of $\mathfrak{o}(p+1, q+1)$ retain the form (2.14). This flat conformal structure precisely corresponds to a $\mathrm{SO}(p+1, q+1)$ -structure on M (cf. Introduction) defined by the atlas of these adapted coordinate systems. Clearly, our formulæ will

prove to be independent of the particular choice of an adapted coordinate system and to be globally defined.

2.6. Explicit formulæ for the $\mathfrak{o}(p+1, q+1)$ -actions.

As a first application of the preceding results, let us compute the action of the conformal algebra on $\mathcal{S}_\delta$ and on $\mathcal{D}_{\lambda,\mu}$ which is given by the following two propositions.

PROPOSITION 2.4. — *The action of $\mathfrak{o}(p+1, q+1)$ on $\mathcal{S}_\delta$ reads*

$$\begin{aligned}
 L_{X_i}^\delta &= \partial_i \\
 L_{X_{ij}}^\delta &= x_i \partial_j - x_j \partial_i + \xi_i \partial_{\xi_j} - \xi_j \partial_{\xi_i} \\
 (2.16) \quad L_{X_0}^\delta &= x^i \partial_i - \xi_i \partial_{\xi_i} + n\delta \\
 L_{\bar{X}_i}^\delta &= x_j x^j \partial_i - 2x_i x^j \partial_j - 2(\xi_i x_j - \xi_j x_i) \partial_{\xi_j} + 2\xi_j x^j \partial_{\xi_i} - 2n\delta x_i.
 \end{aligned}$$

Proof. — These expressions follow from the explicit form (2.14) of the $\mathfrak{o}(p+1, q+1)$ -generators, and from (2.12). $\square$

PROPOSITION 2.5. — *The action of $\mathfrak{o}(p+1, q+1)$ on $\mathcal{D}_{\lambda,\mu}$ reads*

$$(2.17) \quad \mathcal{L}_X^{\lambda,\mu} = L_X^\delta$$

for all $X \in \mathfrak{ce}(p, q)$, where $\delta = \mu - \lambda$; one furthermore has

$$(2.18) \quad \mathcal{L}_{\bar{X}_i}^{\lambda,\mu} = L_{\bar{X}_i}^\delta - \xi_i T + 2(\mathcal{E} + n\lambda) \partial_{\xi_i}$$

for all infinitesimal inversions $\bar{X}_i$ (with $i = 1, \dots, n$) where $T = \partial_{\xi_j} \partial_{\xi_j}$ is the trace and $\mathcal{E} = \xi_j \partial_{\xi_j}$ the Euler operator. (See Section 4.1.)

Proof. — This is a direct consequence of Proposition 2.1, and of the formulæ (2.14). $\square$

Remark 2.6. — The formula (2.18) captures the difference between the $\mathfrak{o}(p+1, q+1)$ -modules $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{S}_\delta$. Note that the operator $\mathcal{L}_X^{\lambda,\mu} - L_X^\delta$ is nilpotent since it maps $\mathcal{S}_\delta^k$ to $\mathcal{S}_\delta^{k-1}$.

3. Main results.

In this section we formulate the main results of this article. All the proofs will be given in Section 5.

3.1. Quantization in the case $\delta = 0$.

Let us consider first the special case $\delta = \mu - \lambda = 0$ and use the shorthand notation $\mathcal{S} \equiv \mathcal{S}_0$ and $\mathcal{D}_\lambda \equiv \mathcal{D}_{\lambda,\lambda}$. Although this is definitely not the most general case to start with, this zero value of the shift is of central importance to relate our conformally equivariant quantization to the more traditional procedures such as geometric or deformation quantization.

THEOREM 3.1. — (i) *There exists an isomorphism of $\mathfrak{o}(p+1, q+1)$ -modules*

$$(3.1) \quad \tilde{\mathcal{Q}}_\lambda : \mathcal{S} \rightarrow \mathcal{D}_\lambda.$$

(ii) *This isomorphism is unique provided the principal symbol be preserved at each order, i.e., provided it reads $\tilde{\mathcal{Q}}_\lambda = \text{Id} + \mathcal{N}_\lambda$ with nilpotent part $\mathcal{N}_\lambda : \mathcal{S}^k \rightarrow \mathcal{S}^{k-1}$.*

Let us introduce a new operator on symbols that will eventually insure the symmetry of the corresponding differential operators. Define $\mathcal{I}_\hbar : \mathcal{S}_\delta \rightarrow \mathcal{S}_\delta[i\hbar]$ by

$$(3.2) \quad \mathcal{I}_\hbar(P)(\xi) = P(i\hbar\xi).$$

Note that we will understand $\hbar$ either as a formal parameter or as a fixed real number, depending upon the context.

Remark 3.2. — It is evident that $\mathcal{I}_\hbar$ is an invariant operator, i.e., $[L_X^\delta, \mathcal{I}_\hbar] = 0$ for all $X \in \text{Vect}(\mathbb{R}^n)$.

DEFINITION 3.3. — *We will call conformally equivariant quantization the $\mathfrak{o}(p+1, q+1)$ -equivariant map $\mathcal{Q}_{\lambda;\hbar} : \mathcal{S} \rightarrow \mathcal{D}_\lambda[i\hbar]$ defined by*

$$(3.3) \quad \mathcal{Q}_{\lambda;\hbar} = \tilde{\mathcal{Q}}_\lambda \circ \mathcal{I}_\hbar$$

where $\hbar$ is a formal parameter and $\mathcal{I}_\hbar$ is given by (3.2).

Theorem 3.1 and the preceding definition enable us to look for a conformally invariant star-product on the space of symbols $\mathcal{S}$ over T^*M . In fact, as soon as one gets an isomorphism such as (3.3), one can readily define an associative bilinear operation (depending on λ)

$$(3.4) \quad *_{\lambda;\hbar} : \mathcal{S} \otimes \mathcal{S} \rightarrow \mathcal{S}[[i\hbar]]$$

such that

$$(3.5) \quad \mathcal{Q}_{\lambda;\hbar}(P *_{\lambda;\hbar} Q) = \mathcal{Q}_{\lambda;\hbar}(P) \circ \mathcal{Q}_{\lambda;\hbar}(Q).$$

Recall that an associative operation $*_{\hbar} : \mathcal{S} \otimes \mathcal{S} \rightarrow \mathcal{S}[[\hbar]]$ is called a star-product [1], [7], [10] if it is of the form

$$(3.6) \quad P *_{\hbar} Q = PQ + \frac{i\hbar}{2} \{P, Q\} + O(\hbar^2)$$

where $\{\cdot, \cdot\}$ stands for the Poisson bracket on T^*M , and is given by bi-differential operators at each order in $\hbar$.

THEOREM 3.4. — *The associative, conformally invariant, operation $*_{\lambda, \hbar}$ defined by (3.5) is a star-product if and only if $\lambda = \frac{1}{2}$.*

Let us emphasize that this theorem provides us precisely with the value of λ used in geometric quantization and, in some sense, links the latter to deformation quantization.

3.2. General formulation. Resonant values of δ .

In this section we formulate our result about the isomorphism of the $\mathfrak{o}(p+1, q+1)$ -modules $\mathcal{S}_{\delta}$ and $\mathcal{D}_{\lambda, \mu}$ in the general situation.

The discussion below mainly relies on the structure of the spectrum of the Casimir operator $\mathcal{C}_{\lambda, \mu}$ of (the $\mathfrak{o}(p+1, q+1)$ -module) $\mathcal{D}_{\lambda, \mu}$. Indeed, the Casimir operator C_{δ} of $\mathcal{S}_{\delta}$ turns out to be diagonalizable. Therefore, a necessary condition for the $\mathfrak{o}(p+1, q+1)$ -modules $\mathcal{S}_{\delta}$ and $\mathcal{D}_{\lambda, \mu}$ to be isomorphic is that $\mathcal{C}_{\lambda, \mu}$ be diagonalizable. This is of course the case if its eigenvalues are “simple”, while some problems could arise otherwise. The latter case occurs only if the shift $\delta = \mu - \lambda$ belongs to the set

$$(3.7) \quad \Sigma = \{\delta_{k, \ell; s, t} \mid k, \ell, s, t \in \mathbb{N}; k > \ell; 2s \leq k; 2t \leq \ell\}$$

where

$$(3.8) \quad \delta_{k, \ell; s, t} = \frac{1}{n(k - \ell)} ((k - \ell + t - s)(k + \ell - 2(s + t) + n - 1) + (s - t)(k + \ell + 1) + 2(kt - \ell s)).$$

The elements of Σ will be called *resonances*.

THEOREM 3.5 (Generic case). — *If $n = p + q \geq 2$ and $\delta \notin \Sigma$, then there exists an isomorphism of $\mathfrak{o}(p+1, q+1)$ -modules*

$$(3.9) \quad \tilde{Q}_{\lambda, \mu} : \mathcal{S}_{\delta} \rightarrow \mathcal{D}_{\lambda, \mu}$$

which is unique provided the principal symbol be preserved at each order.

If $\delta \in \Sigma$, the Casimir operator $\mathcal{C}_{\lambda,\mu}$ has “multiple” eigenvalues and, in some cases, is even not diagonalizable. The corresponding *critical* values of δ are difficult to determine; however, they belong to

$$(3.10) \quad \Sigma_0 = \{\delta_{k,\ell;s,t} \in \Sigma \mid 0 \leq s - t \leq k - \ell\}.$$

THEOREM 3.6 (Resonant case). — *If $n = p + q \geq 2$ and $\delta \in \Sigma \setminus \Sigma_0$, then there exists an isomorphism (3.9) of $\mathfrak{o}(p + 1, q + 1)$ -modules which is unique provided the principal symbol be preserved at each order.*

The proofs of Theorems 3.5 and 3.6 both consist mainly in showing that $\mathcal{C}_{\lambda,\mu}$ is diagonalizable. In the case of Theorem 3.5, this is quite immediate to prove, whereas the resonant case is much more involved.

Remark 3.7. — The isomorphisms of $\mathfrak{o}(p + 1, q + 1)$ -modules obtained in the above results are in fact described in terms of local operators: they are expressed by differential formulae in the coordinates (x^i) and (ξ_i) of the bundle T^*M . It is worth noticing that this is not an assumption but just a fact.

Remark 3.8. — If $\delta \in \Sigma_0$, then there are values of the weights λ and μ for which the sought isomorphism does exist (being, however, not necessarily unique). We have no precise statement for this degenerate case, but in the example of second order symbols, the table (6.6) provides special values of λ and μ leading to an isomorphism (3.9).

Remark 3.9. — One easily finds values of n for which $0 \in \Sigma$ (for instance, $n = 2$, for which $\delta_{4,3;2,0} = 0$). However, we will show that if $\delta = 0$ is resonant, it is not critical (Lemma 5.12).

Remark 3.10. — In the one-dimensional case, $n = 1$, the above theorems still hold true but the resonances are simply $\delta = 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$ and appear in [6], [11]. (The conformal structure is then replaced by the natural $\mathfrak{sl}(2, \mathbb{R})$ infinitesimal action.)

Again, we will introduce the quantization map as the $\mathfrak{o}(p + 1, q + 1)$ -equivariant map $\mathcal{Q}_{\lambda,\mu;\hbar} : \mathcal{S}_\delta \rightarrow \mathcal{D}_\lambda[\hbar]$ defined by

$$(3.11) \quad \mathcal{Q}_{\lambda,\mu;\hbar} = \tilde{\mathcal{Q}}_{\lambda,\mu} \circ \mathcal{I}_\hbar$$

as a natural generalization of (3.3).

Let us recall that if $\lambda + \mu = 1$, there exists, for compactly-supported densities, a $\text{Vect}(M)$ -invariant pairing $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathbb{C}$ defined by

$$(3.12) \quad \varphi \otimes \psi \mapsto \int_M \overline{\varphi} \psi.$$

We can then formulate the important

COROLLARY 3.11. — Assume $\delta \notin \Sigma_0$ and $\lambda + \mu = 1$. The quantization

$$(3.13) \quad \check{P} = \mathcal{Q}_{\frac{1-\delta}{2}, \frac{1+\delta}{2}; \hbar}(P)$$

of any symbol $P \in \mathcal{S}_\delta$ is a symmetric (formally self-adjoint) operator.

Proof. — Let us denote by A^* the adjoint of $A \in \mathcal{D}_{\lambda, 1-\lambda}$ with respect to the pairing (3.12). Consider the symmetric operator

$$\text{Symm}(\mathcal{Q}_{\lambda, \mu; \hbar}(P)) = \frac{1}{2} (\mathcal{Q}_{\lambda, \mu; \hbar}(P) + (\mathcal{Q}_{\lambda, \mu; \hbar}(P))^*)$$

which exists whenever $\lambda + \mu = 1$. Notice that it has the same principal symbol as $\mathcal{Q}_{\lambda, \mu; \hbar}(P)$. Now, the map $\text{Symm}(\mathcal{Q}_{\lambda, \mu; \hbar})$ is obviously $\mathfrak{o}(p+1, q+1)$ -equivariant. Theorems 3.5 and 3.6 just apply and yield $\text{Symm}(\mathcal{Q}_{\lambda, \mu; \hbar}(P)) = \mathcal{Q}_{\lambda, \mu; \hbar}(P)$. $\square$

The particular case $\delta = 0$ is of special importance and related to Theorem 3.4 since $\lambda = \mu = \frac{1}{2}$.

3.3. Quantum Hamiltonians.

To recover the traditional Schrödinger picture of quantum mechanics, one needs to associate to the operator $\check{P}$ resulting from our quantization map (3.13) an operator

$$(3.14) \quad \hat{P} : \mathcal{F}_0 \rightarrow \mathcal{F}_0$$

on the space of complex-valued functions on a conformally flat manifold (M, rg) .

Using the natural identification $\mathcal{F}_0 \rightarrow \mathcal{F}_\lambda$ between tensor densities and smooth functions given (see Section 2.3) by

$$(3.15) \quad f \mapsto f |\text{Vol}_g|^\lambda,$$

one can introduce the differential operator, $\hat{P}$, defined by the commutative diagram

$$(3.16) \quad \begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\hat{P}} & \mathcal{F}_0 \\ |\text{Vol}_g|^\lambda \downarrow & & \downarrow |\text{Vol}_g|^\mu \\ \mathcal{F}_\lambda & \xrightarrow{\check{P}} & \mathcal{F}_\mu \end{array}$$

where $\check{P}$ is given by (3.13) in the case $\lambda + \mu = 1$.

Remark 3.12. — So far, we only needed a conformal class of metrics to define a conformally equivariant quantization map. But, in the current construction, we definitely make a particular choice of metric, g , in the latter class to express the operator $\hat{P}$.

In the case $\delta = 0$, which is relevant for quantum mechanics, the operator $\check{P}$ admits a prolongation as a (formally) self-adjoint operator on the Hilbert space $\overline{\mathcal{F}_{\frac{1}{2}}}$ (the completion of the space of compactly supported half-densities with Hermitian inner product (3.12)). It will be therefore legitimate to call $\hat{P}$ the quantum Hamiltonian associated with the Hamiltonian $P \in \mathcal{S} \cong \text{Pol}(T^*\mathbb{R}^n)$. This quantum Hamiltonian is then a (formally) self-adjoint operator on the space $L^2(M, |\text{Vol}_g|)$.

4. Conformally invariant operators.

The space $\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]$ of polynomials on $T^*\mathbb{R}^n$ is naturally a module over the Lie algebra, $\text{Vect}_{\text{Pol}}(\mathbb{R}^n)$, of polynomial vector fields on $\mathbb{R}^n$. This module structure is induced by the $\text{Vect}(\mathbb{R}^n)$ action on $T^*\mathbb{R}^n$. But, we will rather consider, as in Section 2.4, the deformed action (2.12) depending on a parameter δ ; we will henceforth denote this module by $\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]_\delta$.

DEFINITION 4.1. — We denote by $\text{End}_{\text{diff}}(\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n])$ the subspace

$$\mathbb{C}\left[x^1, \dots, x^n, \xi_1, \dots, \xi_n, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right]$$

of polynomial differential operators on $\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]$.

DEFINITION 4.2. — To any Lie algebra $\mathfrak{g} \subset \text{Vect}_{\text{Pol}}(\mathbb{R}^n)$ we associate its commutant, $\mathfrak{g}^\dagger$, as the Lie subalgebra of $\text{End}_{\text{diff}}(\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n])$ of those operators that commute with $\mathfrak{g}$.

This classical notion of commutant has first been considered in the context of differential operators by Kirillov [16].

4.1. Algebra of Euclidean invariants.

To work out a conformally equivariant quantization map, we need to study first equivariance with respect to the Euclidean subalgebra $\mathfrak{e}(p, q)$. To this end, we will introduce the commutant $\mathfrak{e}(p, q)^\dagger$.

Let us recall the structure of $e(p, q)^!$ which has been shown [9] to be the associative algebra generated by the operators

$$(4.1) \quad R = \xi^i \xi_i, \quad E = \xi_i \frac{\partial}{\partial \xi_i} + \frac{n}{2}, \quad T = \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi_i}$$

whose commutation relations are those of $\mathfrak{sl}(2, \mathbb{R})$ together with

$$(4.2) \quad G = \xi^i \frac{\partial}{\partial x^i}, \quad D = \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x^i}, \quad \Delta = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x_i}$$

which generate the Heisenberg Lie algebra $\mathfrak{h}_1$.

We will find it useful to deal with the Euler operator

$$(4.3) \quad \mathcal{E} = E - \frac{n}{2}.$$

An example of Howe dual pairs of (non semi-simple) Lie algebras is given by

THEOREM 4.3 [9]. — *The commutant $e(p, q)^!$ in $\text{End}_{\text{diff}}(\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n])$ is isomorphic to $U(\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}_1)/\mathcal{Z}$ where the ideal, $\mathcal{Z}$, is as follows:*

(i) if $n = 2$, the ideal $\mathcal{Z}$ is generated by

$$(4.4) \quad Z = \left(C + \frac{3}{2}\right) \Delta + \frac{1}{4} \left([D, [G, C]]_+ - [G, [D, C]]_+ \right),$$

where $[\cdot, \cdot]_+$ stands for the anticommutator, and

$$(4.5) \quad C = E^2 - \frac{1}{2} [R, T]_+$$

is the Casimir element of $\mathfrak{sl}(2, \mathbb{R})$,

(ii) if $n \geq 3$, one has

$$(4.6) \quad \mathcal{Z} = \{0\}.$$

This theorem is a generalization of the celebrated Brauer-Weyl Theorem [27] (see also [16], [9]).

Let us mention that we will, actually, need considering invariant operators with respect to homotheties generated by X_0 (see (2.14) and (2.15)) inside $e(p, q)^!$. We readily have the

COROLLARY 4.4. — *The commutant $\text{ce}(p, q)^!$ in $\text{End}_{\text{diff}}(\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]_\delta)$ is the associative algebra generated (see (4.1) and (4.2)) by*

$$(4.7) \quad E, \quad R_0 = R T, \quad D, \quad G_0 = G T, \quad \Delta_0 = \Delta T.$$

4.2. Algebra of conformal invariants.

The commutant $\text{o}(p+1, q+1)^!$, is given by the following corollary of Theorem 4.3.

COROLLARY 4.5. — *The commutant $\text{o}(p+1, q+1)^!$ in*

$$\text{End}_{\text{diff}}(\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]_\delta)$$

is, for $n \geq 3$, the commutative associative algebra generated by E and R_0 .

Proof. — In view of the preceding corollary, we need only the commutation relations of the operators (4.7) with the generators $L_{X_i}^\delta$ of inversions given in (2.16) in order to determine $\text{o}(p+1, q+1)^!$.

Straightforward calculation leads to

$$(4.8) \quad \begin{aligned} [E, L_{X_i}^\delta] &= 0 \\ [R_0, L_{X_i}^\delta] &= 0 \\ [G_0, L_{X_i}^\delta] &= 2(R_0 \partial_{\xi^i} + (2 - n\delta)\xi_i T) \\ [D, L_{X_i}^\delta] &= 2(-\xi_i T + 2\mathcal{E} \partial_{\xi^i} + n(1 - \delta)\partial_{\xi^i}) \\ [\Delta_0, L_{X_i}^\delta] &= 4(\mathcal{E} \partial_i T + G_0 \partial_{\xi^i} - \xi_i D T) + 2(2 + n(1 - 2\delta))\partial_i T \end{aligned}$$

for $i = 1, \dots, n$.

Now, Theorem 4.3 guarantees that the monomials $W_{e,r,d,g,\ell} = E^e R_0^r D^d G_0^g \Delta_0^\ell$ are independent; they are of degree

$$(4.9) \quad \deg(W_{e,r,d,g,\ell}) = e + 2r + d + 2g + 2\ell$$

as differential operators in ξ_i . We thus have to look for the commutant of inversions in $\text{ce}(p, q)^!$, i.e., to determine the linear combinations of the previous monomials that commute with $L_{X_i}^\delta$. The commutator $[W_{e,r,d,g,\ell}, L_{X_i}^\delta]$ is a differential operator (in ξ_i) of degree $\deg(W_{e,r,d,g,\ell}) + 1$. We are therefore led to study the principal symbol of this operator, which can be easily computed with the help of (4.8). In order to make our calculations

more tractable, let us rather deal with the principal symbol of the operator

$\sum_{i=1}^n \xi^i [W_{e,r,d,g,\ell}, L_{\bar{X}_i}^\delta]$; it is of the form

$$(4.10) \quad \begin{aligned} & 2g E^{e+1} R_0^{r+1} D^d G_0^{g-1} \Delta_0^\ell \\ & - 2d(E^e R_0^{r+1} D^{d-1} G_0^g \Delta_0^\ell - 2E^{e+2} R_0^r D^{d-1} G_0^g \Delta_0^\ell) \\ & + 4\ell(2E^{e+1} R_0^r D^d G_0^{g+1} \Delta_0^{\ell-1} - E^e R_0^{r+1} D^{d+1} G_0^g \Delta_0^{\ell-1}). \end{aligned}$$

We then seek the linear combinations of the monomials $W_{e,r,d,g,\ell}$, of fixed degree (4.9), for which the previous expression is identically zero. Resorting to Theorem 4.3, we immediately get $g = 0$ since the first term in (4.10) is clearly independent of the others. The same is true for the next two terms, yielding $d = 0$ and $\ell = 0$. $\square$

4.3. Casimir operator C_δ .

We have computed in Section 4.2 the commutant $\mathfrak{o}(p+1, q+1)^1$ of the conformal Lie algebra. Now, representation theory tells us that there exists a distinguished invariant within this commutant, namely the Casimir operator.

Recall that the Casimir operator of a given representation $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ of a semi-simple Lie algebra $\mathfrak{g}$ is

$$(4.11) \quad C = B^{\alpha\beta} \rho(X_\alpha) \rho(X_\beta)$$

where B is the Killing metric and (X_α) any basis of $\mathfrak{g}$ with $B_{\alpha\beta} = B(X_\alpha, X_\beta)$, the components of the associated Gram matrix. It is well known that the Casimir operator is invariant, i.e. $[C, \rho(X)] = 0$ for all $X \in \mathfrak{g}$.

In this section, we will provide the explicit calculation of the Casimir operator of the $\mathfrak{o}(p+1, q+1)$ -action, L^δ , on $\mathcal{S}_\delta$ given by Proposition 2.4. We choose the Killing form as

$$(4.12) \quad B(X, Y) = -\frac{1}{2} \text{Tr}(XY)$$

where $X, Y \in \mathfrak{o}(p+1, q+1)$ in their $(n+2) \times (n+2)$ matrix realization. We can then give the explicit formula for this Casimir operator, denoted by C_δ , in terms of the invariant operators (4.7).

PROPOSITION 4.6. — *One has*

$$(4.13) \quad C_\delta = R_0 + 2(1 + n(\delta - 1) - \mathcal{E})\mathcal{E} - n^2\delta(\delta - 1)$$

where $n = p + q$.

Proof. — The matrix realization of the $\mathfrak{o}(p+1, q+1)$ -generators (2.14) is given by

$$(4.14) \quad X_i = \begin{pmatrix} 0 & -\sqrt{2}e_i & 0 \\ 0 & 0 & 0 \\ \sqrt{2}e_i^b & 0 & 0 \end{pmatrix},$$

$$(4.15) \quad X_{ij} = \begin{pmatrix} e_j e_i^b - e_i e_j^b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(4.16) \quad \bar{X}_i = \begin{pmatrix} 0 & 0 & \sqrt{2}e_i \\ -\sqrt{2}e_i^b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where (e_i) is the canonical basis of $\mathbb{R}^n$ and $(e_i^b = g(e_i))$ its dual basis associated with the metric g .

A simple calculation yields the basis $(X^\alpha = B^{\alpha\beta} X_\beta)$ of $\mathfrak{o}(p+1, q+1)$ dual to (X_α) with respect to the Killing metric (4.12). One gets

$$(4.17) \quad \begin{aligned} X^i &= -\frac{1}{2} g^{ij} \bar{X}_j \\ X^{ij} &= g^{ik} g^{j\ell} X_{k\ell} \\ X^0 &= -X_0 \\ \bar{X}^i &= -\frac{1}{2} g^{ij} X_j. \end{aligned}$$

Using the $\mathfrak{o}(p+1, q+1)$ -action, L^δ , on $\mathcal{S}_\delta$ given in (2.16), one shows immediately that the Casimir operator

$$(4.18) \quad C_\delta = \frac{1}{2} g^{ik} g^{j\ell} L_{X_{ij}}^\delta L_{X_{k\ell}}^\delta - (L_{X_0}^\delta)^2 - \frac{1}{2} g^{ij} L_{X_i}^\delta L_{\bar{X}_j}^\delta - \frac{1}{2} g^{ij} L_{\bar{X}_i}^\delta L_{X_j}^\delta$$

actually retains the form (4.13). $\square$

Remark 4.7. — It is worth noticing that the Casimir operator (4.13) can be alternatively expressed in terms of the Casimir operator, C , (see (4.5)) and the Cartan generator, E , of $\mathfrak{sl}(2, \mathbb{R})$. One finds

$$(4.19) \quad C_\delta = -C - (E - n\delta)^2 - n \left(1 - \frac{n}{2}\right).$$

4.4. Casimir operator $\mathcal{C}_{\lambda, \mu}$.

The Casimir operator of the $\mathfrak{o}(p+1, q+1)$ -action on $\mathcal{D}_{\lambda, \mu}$ is defined, accordingly, by

$$(4.20) \quad \mathcal{C}_{\lambda, \mu} = B^{\alpha\beta} \mathcal{L}_{X_\alpha}^{\lambda, \mu} \mathcal{L}_{X_\beta}^{\lambda, \mu}.$$

PROPOSITION 4.8. — *The Casimir operator of the $\mathfrak{o}(p+1, q+1)$ -action on $\mathcal{D}_{\lambda, \mu}$ is of the form*

$$(4.21) \quad \mathcal{C}_{\lambda, \mu} = C_\delta + G_0 - 2(n\lambda + \mathcal{E})D.$$

Proof. — The explicit formula for $\mathcal{C}_{\lambda, \mu}$ is obviously obtained by replacing L^δ by $\mathcal{L}^{\lambda, \mu}$ in (4.18). Applying then (2.17) and (2.18) to that expression immediately leads to the result (4.21).

5. Proofs of the main theorems.

Throughout this section we use, for convenience, the local identification (2.9) of $\mathcal{D}_{\lambda, \mu}$ and $\mathcal{S}_\delta$.

5.1. Diagonalization of the Casimir operator C_δ .

We have already mentioned that we will study the diagonalization of the Casimir operators C_δ and $\mathcal{C}_{\lambda, \mu}$. Here, we understand that an endomorphism of an infinite-dimensional space is diagonalizable if any element of the latter is a (finite) sum of eigenvectors of the former.

Let us recall that $[\mathcal{E}, R_0] = 0$, so that $[C_\delta, \mathcal{E}] = 0$ and $[C_\delta, R_0] = 0$. We can thus simultaneously diagonalize the three operators $\mathcal{E}$, R_0 and C_δ .

LEMMA 5.1. — *The eigenvectors of the operator R_0 restricted to the space $\mathcal{S}_{k, \delta}$ of homogeneous polynomials (see (2.3)) are of the form*

$$(5.1) \quad P_{k, s} = R^s Q,$$

where R is given by (4.1) and $Q \in \mathcal{S}_{k-2s, \delta}$ is tracefree (harmonic), viz $TQ = 0$, and

$$(5.2) \quad s \in \{0, 1, 2, \dots, [k/2]\}.$$

The associated eigenvalues are

$$(5.3) \quad \varrho_{k, s} = 2s(n + 2(k - s - 1)).$$

For a proof, see, e.g., [27].

We readily have the following

COROLLARY 5.2. — *The spectrum of the Casimir operator C_δ is given by*

$$(5.4) \quad \gamma_{k,s} = 2s(n + 2(k - s - 1)) + 2k(1 + n(\delta - 1) - k) - n^2\delta(\delta - 1).$$

Proof. — This follows immediately from Lemma 5.1 and the expression (4.13) of the Casimir operator C_δ . $\square$

We have thus the following useful decomposition: every $P \in \mathcal{S}_\delta$ can be decomposed as a locally finite sum

$$(5.5) \quad P = \sum_{k=0}^{\infty} \sum_{s=0}^{[k/2]} P_{k,s}$$

where $[k/2]$ is the integer part of $k/2$. In other words, we have a direct sum decomposition

$$(5.6) \quad \mathcal{S}_\delta = \bigoplus_{\substack{k=0 \\ s \leq [k/2]}}^{\infty} \mathcal{S}_{(k,s),\delta}$$

into eigenspaces of $\mathcal{E}$ and R_0 . (See Theorem (5.6.A) in [27].)

We are now able to explain the origin of the resonant values (3.8) of $\delta = \mu - \lambda$.

LEMMA 5.3. — *One has $\delta = \delta_{k,\ell;s,t}$ if and only if $\gamma_{k,s} = \gamma_{\ell,t}$.*

Proof. — This is straightforward from (5.4). $\square$

5.2. Diagonalization of the Casimir operator $\mathcal{C}_{\lambda,\mu}$.

We establish, in this section, the main technical statement that helps us to prove the existence of an equivariant quantization map for almost all values of the shift δ .

The expression (4.21) of the Casimir operator is of the form $\mathcal{C}_{\lambda,\mu} = C_\delta + N_\lambda$ with nilpotent part $N_\lambda : \mathcal{S}_\delta^k \rightarrow \mathcal{S}_\delta^{k-1}$ (for $k = 0, 1, 2, \dots$), see Section 4.4. This implies that any solution $P \in \mathcal{D}_{\lambda,\mu}^k$ of the equation

$$(5.7) \quad \mathcal{C}_{\lambda,\mu} P = \gamma P$$

is of the form $P = P_{k,s} + P'$ where $P_{k,s}$ is as in (5.1) and $P' \in \mathcal{D}_{\lambda,\mu}^{k-1}$; the eigenvalue γ clearly coincides with $\gamma_{k,s}$ given by (5.4).

PROPOSITION 5.4 (Generic case). — *If $\delta \notin \Sigma$, then the eigenvalue problem (5.7) for the Casimir operator $C_{\lambda,\mu}$ has a solution if and only if $\gamma = \gamma_{k,s}$ for some k, s (as given by (5.4)). The corresponding eigenvectors are uniquely determined by their principal symbols, arbitrarily taken in $\mathcal{S}_{(k,s),\delta}$.*

Proof. — The highest degree component of the eigenvalue equation (5.7) is just the eigenvalue equation for C_δ . Hence, a solution of (5.7) is necessarily of the form $\gamma = \gamma_{k,s}$ and

$$(5.8) \quad P = P_{k,s} + \sum_{\substack{\ell < k \\ t \leq [\ell/2]}} P_{\ell,t}$$

according to the decomposition (5.6). The remainder of equation (5.7) reads now

$$(5.9) \quad \sum_{\substack{\ell < k \\ t \leq [\ell/2]}} (\gamma_{k,s} - \gamma_{\ell,t}) P_{\ell,t} = N_\lambda P.$$

Since $\delta \notin \Sigma$, by Lemma 5.3, the coefficients $\gamma_{k,s} - \gamma_{\ell,t}$ do not vanish. In view of the nilpotency of N_λ , the result follows immediately. $\square$

In order to handle the case of non-critical resonant values of δ , we need the following

LEMMA 5.5. — *If $P \in \mathcal{S}_{(k,s),\delta}$, then*

- (i) *the polynomials $D(P)$ and $G_0(P)$ belong to $\mathcal{S}_{(k-1,s-1),\delta} \oplus \mathcal{S}_{(k-1,s),\delta}$,*
- (ii) *the polynomial $\Delta_0(P)$ belongs to $\mathcal{S}_{(k-2,s-2),\delta} \oplus \mathcal{S}_{(k-2,s-1),\delta} \oplus \mathcal{S}_{(k-2,s),\delta}$.*

Proof. — Any polynomial $P \in \mathcal{S}_{(k,s),\delta}$ is of the form $P = R^s Q$ where $Q \in \mathcal{S}_{(k-2s,0),\delta}$ is harmonic, see (5.1). One has

$$\begin{aligned} D(P) &= [D, R^s](Q) + R^s D(Q) \\ &= \sum_{r=0}^{s-1} R^r [D, R] R^{s-r-1}(Q) + R^s D(Q) \\ &= 2s R^{s-1} G(Q) + R^s D(Q) \end{aligned}$$

since $[D, R] = 2G$ and $[R, G] = 0$ in the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}_1$ (see Section 4.1).

At last, $D(Q)$ is harmonic because $[T, D] = 0$ and one furthermore easily checks that $G(Q) \in \mathcal{S}_{(k-2s+1,0),\delta} \oplus \mathcal{S}_{(k-2s+1,1),\delta}$. Hence, $D(P) \in \mathcal{S}_{(k-1,s-1),\delta} \oplus \mathcal{S}_{(k-1,s),\delta}$.

The proof for $G_0(P)$ and $\Delta_0(P)$ is analogous and will be omitted. $\square$

Let us then introduce the space

$$(5.10) \quad \tilde{\mathcal{S}}_{(k,s),\delta} = \bigoplus_{0 \leq s-t \leq k-\ell} \mathcal{S}_{(\ell,t),\delta}$$

which is “generated” by the tree

$$(5.11) \quad \begin{array}{ccccc} & & \mathcal{S}_{(k,s),\delta} & & \\ & \swarrow & & \searrow & \\ \mathcal{S}_{(k-1,s-1),\delta} & & & & \mathcal{S}_{(k-1,s),\delta} \\ \swarrow & & & & \searrow \\ \dots & & \mathcal{S}_{(k-2,s-1),\delta} & & \dots \\ & & \dots & & \end{array}$$

In view of the preceding lemma, $\tilde{\mathcal{S}}_{(k,s),\delta}$ is stabilized by the Casimir operator $\mathcal{C}_{\lambda,\mu}$. Moreover, if $\delta \notin \Sigma_0$ (see (3.10)), $\gamma_{\ell,t} \neq \gamma_{k,s}$ whenever $\mathcal{S}_{(\ell,t),\delta} \subset \tilde{\mathcal{S}}_{(k,s),\delta}$.

PROPOSITION 5.6 (Resonant case). — *If $\delta \in \Sigma \setminus \Sigma_0$, then the eigenvalue problem (5.7) for the Casimir operator $\mathcal{C}_{\lambda,\mu}$ has a solution if and only if $\gamma = \gamma_{k,s}$ for some k, s (as given by (5.4)). Every $P_{k,s} \in \mathcal{S}_{(k,s),\delta}$ is the principal symbol of an eigenvector of $\mathcal{C}_{\lambda,\mu}$. This eigenvector is uniquely determined provided it belongs to $\tilde{\mathcal{S}}_{(k,s),\delta}$.*

Proof. — One proceeds just as in the proof of Proposition 5.4. The fact that one restricts ℓ, t in the decomposition (5.8) to those values for which $\mathcal{S}_{(\ell,t),\delta} \subset \tilde{\mathcal{S}}_{(k,s),\delta}$ again insures that the coefficients in equation (5.9) do not vanish. $\square$

Remark 5.7. — In Proposition 5.6, if $\delta \neq \delta_{k,\ell;s,t}$ for some ℓ, t , then any eigenvector with principal symbol in $\mathcal{S}_{(k,s),\delta}$ necessarily belongs to $\tilde{\mathcal{S}}_{(k,s),\delta}$.

If δ is not critical, it then follows from Propositions 5.4 and 5.6 that every $P_{k,s} \in \mathcal{S}_{(k,s),\delta}$ is the principal symbol of an eigenvector $\tilde{P}_{k,s} \in \tilde{\mathcal{S}}_{(k,s),\delta}$ of the Casimir operator $\mathcal{C}_{\lambda,\mu}$. Hence the

COROLLARY 5.8. — *If $\delta \notin \Sigma_0$ the Casimir operator $\mathcal{C}_{\lambda,\mu}$ is diagonalizable.*

5.3. Proof of Theorems 3.5 and 3.6.

Let us show that the diagonalization of the Casimir operator actually leads to the determination of a unique isomorphism of $\mathfrak{o}(p+1, q+1)$ -modules $\tilde{\mathcal{Q}}_{\lambda, \mu} : \mathcal{S}_\delta \rightarrow \mathcal{D}_{\lambda, \mu}$. Looking for a map $\tilde{\mathcal{Q}}_{\lambda, \mu}$ such that

$$(5.12) \quad \begin{array}{ccc} \mathcal{D}_{\lambda, \mu} & \xrightarrow{C_{\lambda, \mu}} & \mathcal{D}_{\lambda, \mu} \\ \tilde{\mathcal{Q}}_{\lambda, \mu} \uparrow & & \tilde{\mathcal{Q}}_{\lambda, \mu} \uparrow \\ \mathcal{S}_\delta & \xrightarrow{C_\delta} & \mathcal{S}_\delta \end{array}$$

be a commutative diagram, we are led to the

DEFINITION 5.9. — *The linear map $\tilde{\mathcal{Q}}_{\lambda, \mu} : \mathcal{S}_\delta \rightarrow \mathcal{D}_{\lambda, \mu}$ is defined by*

$$(5.13) \quad \tilde{\mathcal{Q}}_{\lambda, \mu}(P_{k, s}) = \tilde{P}_{k, s}$$

using the decomposition (5.5).

This map has, obviously, the following properties:

1. $C_{\lambda, \mu} \tilde{\mathcal{Q}}_{\lambda, \mu} = \tilde{\mathcal{Q}}_{\lambda, \mu} C_\delta$,
2. $\tilde{\mathcal{Q}}_{\lambda, \mu} = \text{Id} + \mathcal{N}_{\lambda, \mu}$ with nilpotent part $\mathcal{N}_{\lambda, \mu} : \mathcal{S}_\delta^k \rightarrow \mathcal{S}_\delta^{k-1}$.

5.3.1. Proof of Theorem 3.5.

Let us first prove that

$$(5.14) \quad \mathcal{L}_X^{\lambda, \mu} \tilde{\mathcal{Q}}_{\lambda, \mu} = \tilde{\mathcal{Q}}_{\lambda, \mu} L_X^\delta$$

for all $X \in \mathfrak{o}(p+1, q+1)$. From Property 1, we see that if $P_{k, s} \in \mathcal{S}_{(k, s), \delta}$ then $\mathcal{L}_X^{\lambda, \mu} \tilde{\mathcal{Q}}_{\lambda, \mu} P_{k, s}$ and $\tilde{\mathcal{Q}}_{\lambda, \mu} L_X^\delta P_{k, s}$ are both eigenvectors of the Casimir operator $C_{\lambda, \mu}$ associated with the same eigenvalue $\gamma_{k, s}$. Moreover, Property 2 and (2.13) entail that $\mathcal{L}_X^{\lambda, \mu} \tilde{\mathcal{Q}}_{\lambda, \mu} P_{k, s}$ and $\tilde{\mathcal{Q}}_{\lambda, \mu} L_X^\delta P_{k, s}$ have the same principal symbol, namely $L_X^\delta P_{k, s}$. It follows from Proposition 5.4 that these eigenvectors actually coincide. Hence, the existence of the sought quantization map.

Now, to prove the uniqueness, it suffices to note that an isomorphism of the $\mathfrak{o}(p+1, q+1)$ -modules $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda, \mu}$ necessarily intertwines the corresponding Casimir operators. If it moreover preserves the principal symbol, then Proposition 5.4 shows that it is, indeed, $\tilde{\mathcal{Q}}_{\lambda, \mu}$.

The proof of Theorem 3.5 is complete.

5.3.2. Proof of Theorem 3.6.

This proof is built on the same pattern as the previous one. But, since δ has resonant values, we must resort to Proposition 5.6 instead of Proposition 5.4. This is done at the expense of some preparation due to the fact that the uniqueness of the eigenvector $\tilde{P}_{k,s}$ is guaranteed only within $\tilde{S}_{(k,s),\delta}$.

The above proof of the equivariance property (5.14) should now be completed with the help of

LEMMA 5.10. — *For every $X \in \mathfrak{o}(p+1, q+1)$, one has $L_X^\delta \mathcal{S}_{(k,s),\delta} \subset \mathcal{S}_{(k,s),\delta}$ and $\mathcal{L}_X^{\lambda,\mu} \tilde{\mathcal{S}}_{(k,s),\delta} \subset \tilde{\mathcal{S}}_{(k,s),\delta}$.*

Proof. — The first inclusion easily follows from Corollary 4.5. As to the second one, we then proceed as in the proof of Lemma 5.5, using (2.18). □

The existence of the isomorphism $\tilde{\mathcal{Q}}_{\lambda,\mu}$ is thus proven.

In the same way, the uniqueness of the sought isomorphism is established as in the proof of Theorem 3.5, provided we apply the following

LEMMA 5.11. — *Any linear map $\tilde{\mathcal{Q}} : \mathcal{S}_\delta \rightarrow \mathcal{S}_\delta$ that intertwines the $\mathfrak{ce}(p, q)$ -action and do not increase the degree stabilizes each space $\tilde{\mathcal{S}}_{(k,s),\delta}$.*

Proof. — It has been shown in Lemma 7.1 of [19] (see also Theorem 5.1 of [22]) that such a map $\tilde{\mathcal{Q}}$ is necessarily a differential operator with constant coefficients. We can thus apply Corollary 4.4: $\tilde{\mathcal{Q}}$ is a polynomial in the operators (4.7). We conclude by using Lemma 5.5. □

This ends the proof of Theorem 3.6.

5.4. Proof of Theorem 3.1.

In order to prove Theorem 3.1, it is enough to show that $\delta = 0$ is not a critical value, i.e., $0 \notin \Sigma_0$. This follows from the stronger

LEMMA 5.12. — *If the following inequalities hold*

$$(5.15) \quad 0 \leq s - t \leq k - \ell,$$

one has $\delta_{k,\ell;s,t} > 0$.

Proof. — In the expression (3.8), both factors in the first term, $k - \ell + t - s$ and $k + \ell - 2(s + t) + n - 1$, are non-negative in view of (5.15) and (5.2). Also (5.15) yields $kt - \ell s \geq -\ell(s - t)$ so that the second term is bounded from below by $(s - t)(k - \ell + 1)$, which is non-negative. We have just shown that $\delta \geq 0$.

Now, if $s = t$, one has $n\delta = k + \ell + n - 2t - 1 \geq k$ because of (5.2). The result follows since $k > 0$ in (3.8). $\square$

5.5. Proof of Theorem 3.4: Conformally invariant star-product.

Throughout this section we will only consider the case $\lambda = \mu$. Let us give an explicit expression for the quantization map (3.3) up to the second order in $\hbar$.

PROPOSITION 5.13. — *If $P_{k,s} \in \mathcal{S}_{(k,s)}$ is a homogeneous polynomial (see (5.6)) with $k > 2$, then*

(i) *if $s > 0$, the quantization map is of the form*
(5.16)

$$\mathcal{Q}_{\lambda;\hbar}(P_{k,s}) = P_{k,s} + \frac{i\hbar}{2} \left(D(P_{k,s}) + \frac{(1 - 2\lambda)n}{s(2s - 2k - n + 2)} G_0(P_{k,s}) \right) + O(\hbar^2),$$

(ii) *if $s = 0$ (i.e., the harmonic case), one has*

$$(5.17) \quad \mathcal{Q}_{\lambda;\hbar}(P_{k,0}) = P_{k,0} + i\hbar \left(\frac{n\lambda + k - 1}{n + 2(k - 1)} D(P_{k,0}) \right) + O(\hbar^2).$$

Proof. — (i) An eigenvector of the Casimir operator $\mathcal{C}_{\lambda,\lambda}$, with principal symbol $P_{k,s}$, is of the form

$$P = P_{k,s} + P_{k-1,s} + P_{k-1,s-1} + \text{terms of degree} \leq k - 2.$$

(See the formula (4.21) and Lemma 5.5.)

The eigenvalue problem (5.7) therefore leads to $\gamma_{k,s}P_{k,s} = \gamma P_{k,s}$, where $\gamma_{k,s}$ is the eigenvalue of the Casimir operator C_0 given by (5.4), and to

$$(5.18) \quad \begin{aligned} \gamma_{k-1,s}P_{k-1,s} + \gamma_{k-1,s-1}P_{k-1,s-1} &= \gamma(P_{k-1,s} + P_{k-1,s-1}) \\ &+ 2(n\lambda + k - 1)D(P_{k,s}) - G_0(P_{k,s}). \end{aligned}$$

In order to solve this equation for $P_{k-1,s}$ and $P_{k-1,s-1}$, one needs to introduce the projectors

$$\Pi_1^{k-1,s} = \frac{R_0 - \varrho_{k-1,s-1}}{\varrho_{k-1,s} - \varrho_{k-1,s-1}} \quad \text{and} \quad \Pi_2^{k-1,s} = \frac{R_0 - \varrho_{k-1,s}}{\varrho_{k-1,s-1} - \varrho_{k-1,s}}$$

from the space $\mathcal{S}_{(k-1,s)} \oplus \mathcal{S}_{(k-1,s-1)}$ to the first and second summand respectively, where $\varrho_{k,s}$ is the eigenvalue (5.3) of R_0 .

From equation (5.18) one gets

$$\begin{aligned} P_{k-1,s} &= - \frac{\Pi_1^{k-1,s}}{\gamma_{k,s} - \gamma_{k-1,s}} (2(n\lambda + k - 1)D(P_{k,s}) - G_0(P_{k,s})) \\ P_{k-1,s-1} &= - \frac{\Pi_2^{k-1,s}}{\gamma_{k,s} - \gamma_{k-1,s-1}} (2(n\lambda + k - 1)D(P_{k,s}) - G_0(P_{k,s})). \end{aligned}$$

To rewrite the previous expression in terms of $D(P_{k,s})$ and $G_0(P_{k,s})$, one resorts to the following formulæ:

$$\begin{aligned} R_0 D(P_{k,s}) &= \varrho_{k,s} D(P_{k,s}) - 2G_0(P_{k,s}) \\ R_0 G_0(P_{k,s}) &= 2D(P_{k,s}) + (\varrho_{k,s} - 2(n + 2k - 2)) G_0(P_{k,s}) \end{aligned}$$

obtained with the help of the commutation relations of the generators (4.1) and (4.2) of $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}_1$. A lengthy but straightforward calculation gives

$$P_{k-1,s} + P_{k-1,s-1} = \frac{1}{2} D(P_{k,s}) + \frac{(1 - 2\lambda)n}{2s(2s - 2k - n + 2)} G_0(P_{k,s}).$$

Then, the definition (3.3) of the quantization map yields the formula (5.16).

(ii) In the harmonic case, $s = 0$, the equation (5.18) reduces to

$$(5.19) \quad \gamma_{k-1,0} P_{k-1,0} = \gamma_{k,0} P_{k-1,0} + 2(n\lambda + k - 1)D(P_{k,0})$$

since $G_0(P_{k,0}) = 0$. With the help of (5.4), one gets the formula (5.17). $\square$

Remark 5.14. — In the lower-order cases $k \leq 2$, there exists an explicit formula for the quantization map; it is given by the two formulæ (6.2), (6.3) and (6.4) below.

With this preliminary result, we are ready to prove the announced theorem.

PROPOSITION 5.15. — *Given a differential linear operator $\mathcal{Q} : \mathcal{S} \rightarrow \mathcal{D}_\lambda[[i\hbar]]$ of the form*

$$\mathcal{Q}(P) = P + i\hbar(\alpha D(P) + \beta G_0(P)) + O(\hbar^2),$$

the associative product $*$: $\mathcal{S} \otimes \mathcal{S} \rightarrow \mathcal{S}$ defined by $\mathcal{Q}(P * Q) = \mathcal{Q}(P) \circ \mathcal{Q}(Q)$ is a star-product if and only if $\alpha = \frac{1}{2}$ and $\beta = 0$.

Proof. — One can consider the inverse map $\mathcal{Q}^{-1} : \mathcal{D}_\lambda[[i\hbar]] \rightarrow \mathcal{S}[[i\hbar]]$, which is given by $\mathcal{Q}^{-1}(P) = P - i\hbar(\alpha D(P) + \beta G_0(P)) + O(\hbar^2)$. Using the well-known composition formula for differential operators

$$\begin{aligned} P \circ Q &= \sum_{\ell=0}^{\infty} \frac{(i\hbar)^\ell}{\ell!} \partial_{\xi_{i_1}} \cdots \partial_{\xi_{i_\ell}}(P) \partial_{i_1} \cdots \partial_{i_\ell}(Q), \\ (5.20) \quad &= PQ + i\hbar \partial_{\xi_j}(P) \partial_j(Q) + O(\hbar^2) \end{aligned}$$

one obtains

$$\begin{aligned} P * Q &= \mathcal{Q}^{-1}(\mathcal{Q}(P) \circ \mathcal{Q}(Q)) \\ &= PQ + i\hbar \partial_{\xi_j}(P) \partial_j(Q) + i\hbar \alpha \left(D(P)Q + PD(Q) - D(PQ) \right) \\ &\quad + i\hbar \beta \left(G_0(P)Q + PG_0(Q) - G_0(PQ) \right) \\ &\quad + O(\hbar^2) \\ &= PQ + \frac{i\hbar}{2} \{P, Q\} + i\hbar \left(\alpha - \frac{1}{2} \right) \left(D(P)Q + PD(Q) - D(PQ) \right) \\ &\quad + i\hbar \beta \left(G_0(P)Q + PG_0(Q) - G_0(PQ) \right) \\ &\quad + O(\hbar^2). \end{aligned}$$

Recall that the Hochschild boundary of a 1-cochain $A \in \text{End}(\mathcal{S}_\delta)$ is given by $(dA)(P, Q) = A(P)Q + PA(Q) - A(PQ)$ and observe that the preceding expression is therefore

$$(5.21) \quad P * Q = PQ + \frac{i\hbar}{2} \{P, Q\} + i\hbar d \left(\left(\alpha - \frac{1}{2} \right) D + \beta G_0 \right) (P, Q) + O(\hbar^2).$$

One sees that $P * Q$ satisfies the definition (3.6) of a star-product if and only if $\alpha = \frac{1}{2}$ and $\beta = 0$. $\square$

The operation (3.4) is, actually, given by bi-differential operators because the quantization map $\mathcal{Q}_{\lambda;\hbar}$ given by (3.3) and its inverse are differential operators at each order in $\hbar$. Indeed, we have $\tilde{\mathcal{Q}}_\lambda = \text{Id} + \mathcal{N}_\lambda$ as in Theorem 3.1, so that $(\mathcal{Q}_{\lambda;\hbar})^{-1}$ is a differential operator as is $\mathcal{Q}_{\lambda;\hbar}$.

Theorem 3.4 follows now from the preceding two propositions in the case $k > 2$ and from the explicit formula (6.4) in the case $k \leq 2$.

6. Quantizing second-order polynomials.

This problem has first been solved in [9]. It was proved that if $n = p + q \geq 2$, there exists an isomorphism of $\mathfrak{o}(p + 1, q + 1)$ -modules $\tilde{\mathcal{Q}}_{\lambda, \mu}^2 : \mathcal{S}_{\delta}^2 \xrightarrow{\cong} \mathcal{D}_{\lambda, \mu}^2$ where $\delta = \mu - \lambda$, provided

$$(6.1) \quad \delta \notin \left\{ \frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n} \right\}.$$

This result is clearly consistent with the general Theorem 3.5. Moreover, the latter guarantees the uniqueness of such an isomorphism under the further condition that the principal symbol be preserved at each order.

6.1. Explicit formulæ.

In the non-resonant case, the explicit formula for the unique isomorphism has also been computed in [9]. One has

$$(6.2) \quad \tilde{\mathcal{Q}}_{\lambda, \mu}^2 = \text{Id} + \gamma_1 G_0 + \gamma_2 D + \gamma_3 \mathcal{E}D + \gamma_4 \Delta_0 + \gamma_5 D^2$$

where the numerical coefficients are given by

$$(6.3) \quad \begin{aligned} \gamma_1 &= \frac{n(\lambda + \mu - 1)}{2(n\delta - 2)(n(\delta - 1) - 2)}, \\ \gamma_2 &= \frac{\lambda}{1 - \delta}, \\ \gamma_3 &= \frac{1 - \lambda - \mu}{(\delta - 1)(n(\delta - 1) - 2)}, \\ \gamma_4 &= \frac{n\lambda(2 + (4\lambda - 1)n + (2\lambda^2 - \lambda\mu - \mu^2 + 2\mu - 1)n^2)}{2(n(\delta - 1) - 1)(n(2\delta - 1) - 2)(n\delta - 2)(n(\delta - 1) - 2)}, \\ \gamma_5 &= \frac{n\lambda(n\lambda + 1)}{2(n(\delta - 1) - 1)(n(\delta - 1) - 2)}. \end{aligned}$$

In particular, the half-density quantization map (3.3) is given by

$$(6.4) \quad \mathcal{Q}_{\frac{1}{2}, \hbar}^2 = \text{Id} + \frac{i\hbar}{2} D - \frac{\hbar^2}{8} \left(\frac{n}{(n+1)(n+2)} \Delta_0 + \frac{n}{(n+1)} D^2 \right).$$

Remark 6.1. — At this stage, it is interesting to see how our conformally equivariant quantization compares with the Weyl quantization on

$T^*\mathbb{R}^n$. In our framework, the Weyl quantization map, $\mathcal{Q}_{\text{Weyl}}$, retains the very elegant form

$$(6.5) \quad \begin{aligned} \mathcal{Q}_{\text{Weyl}} &= \exp\left(\frac{i\hbar}{2}D\right) \\ &= \text{Id} + \frac{i\hbar}{2}D - \frac{\hbar^2}{8}D^2 + O(\hbar^3) \end{aligned}$$

where the divergence operator D is as in (4.2). (See, e.g., [10] p. 87.)

6.2. Study of the resonant modules.

For the sake of completeness, let us study in some more detail the particular modules of differential operators corresponding to the resonances (6.1). It has been shown [9] that, for each resonant value of δ , there exist pairs (λ, μ) of weights such that the $\mathfrak{o}(p+1, q+1)$ -modules $\mathcal{S}_\delta^2$ and $\mathcal{D}_{\lambda, \mu}^2$ are isomorphic, namely

$$(6.6) \quad \begin{array}{c|c|c|c|c|c} \delta & \frac{2}{n} & \frac{n+2}{2n} & 1 & \frac{n+1}{n} & \frac{n+2}{n} \\ \hline \lambda & \frac{n-2}{2n} & 0, \frac{n-2}{2n} & 0 & 0, -\frac{1}{n} & -\frac{1}{n} \\ \hline \mu & \frac{n+2}{2n} & \frac{n+2}{2n}, 1 & 1 & \frac{n+1}{n}, 1 & \frac{n+1}{n} \end{array}$$

However, in these cases, the isomorphism is not unique. For the particular values $\delta = 2/n, 1, (n+2)/n$, there is a unique choice of (λ, μ) which, furthermore, leads to symmetric quantized symbols; for example (see [9]) the so-called Yamabe operator (also known as the conformal Laplacian) shows up naturally in the first resonant case in (6.6).

6.3. Quantizing the geodesic flow.

Let us finally illustrate our quantization procedure with a specific and important example, namely the quantization of the geodesic flow on a conformally flat manifold (M, g) .

Consider, on T^*M , the quadratic Hamiltonian

$$H = g^{ij}\xi_i\xi_j.$$

whose flow projects onto the geodesics of (M, g) .

Let us put $\lambda = \mu = \frac{1}{2}$ and apply, using (6.4), the construction of the quantum Hamiltonian (3.14) spelled out in Section 3.3. In doing so, we recover a result obtained in [9], namely

$$(6.7) \quad \hat{H} = -\hbar^2 \left(\Delta_g - \frac{n^2}{4(n-1)(n+2)} R_g \right)$$

where R_g stands for the scalar curvature of (M, g) . The operator (6.7) is therefore a natural candidate for the quantum Hamiltonian of the geodesic flow on a pseudo-Riemannian manifold.

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C. DUVAL,
 Université de la Méditerranée et CPT-CNRS
 Luminy Case 907
 F-13228 Marseille cedex 9.
 duval@cpt.univ-mrs.fr
 &
 P. LECOMTE,
 Université de Liège
 Institut de Mathématiques
 Sart Tilman
 Gde Traverse, 12 (B 37)
 B-4000 Liège
 plecomte@ulg.ac.be
 &
 V. OVSIENKO
 CNRS, Centre de physique théorique
 CPT-CNRS
 Luminy Case 907
 F-13288 Marseille cedex 9.
 ovsienko@cpt.univ-mrs.fr