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ON HARDY SPACES IN COMPLEX ELLIPSOIDS

by Thomas HANSSON

1. Introduction.

A fundamental tool in the study of holomorphic functions of one complex variable is the Cauchy integral formula. Hence when studying holomorphic functions in a domain $\Omega \subset \mathbb{C}^n$ one wants a suitable generalisation of the Cauchy integral. One possible choice is the Szegő projection S, i.e. the $L^2(\partial\Omega)$ orthogonal projection on the space of (boundary values of) holomorphic functions. However, except for a few special domains, the kernel of S have no explicit expression. Thus to understand the mapping properties of the Szegő operator one has to estimate its kernel. For strictly pseudoconvex domains such estimates were obtained by Fefferman, [F].

Another, less canonical choice, is to use integral operators generated by Cauchy-Fantappié forms. In contrast to the Szegő operator, these integral operators have rather explicit kernels. More precisely, if Ω is a bounded domain in $\mathbb{C}^n$ with C^2-boundary and $q : \partial\Omega \times \Omega$ is a C^1-map that satisfies $\langle q(\zeta, z), \zeta - z\rangle \neq 0$, for every $\zeta \in \partial\Omega$ and $z \in \Omega$, then the operator H defined by

$$Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\Omega} \frac{f(\zeta) q \wedge (\bar\partial_\zeta q)^{n-1}}{\langle q, \zeta - z\rangle^n}, \quad z \in \Omega,$$

reproduces holomorphic functions. For example, if Ω is strictly geometrically convex with defining function ρ, we can take $q(\zeta, z) = \partial\rho(\zeta)$. To understand the mapping properties of H, the main difficulty is to prove

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that H is a bounded operator on $L^2(\partial\Omega)$. This was proved for strictly pseudoconvex domains (with some natural choice of q) in [KS].

Recently a lot of work has been done in this area on domains of finite type. For instance in [NRSW], estimates of the kernel of S have been obtained for pseudoconvex domains of finite type in $\mathbb{C}^2$. Such estimates have later also been achieved for convex domains of finite type in $\mathbb{C}^n$ ([McN], [McNS], [KL]). In these papers the Szegő projection has been defined with respect to the Euclidean surface measure on $\partial\Omega$. However there is another natural measure on the boundary to consider, as suggested by the integral defining H, namely the measure represented by the form $(2\pi i)^{-n}\partial\rho\wedge(\bar{\partial}\partial\rho)^{n-1}$. Denote this measure by dS. In strictly pseudoconvex domains, dS is equivalent to the ordinary surface measure $d\sigma$, but for domains of finite type it is essentially smaller near Levi-flat points. This can be compared with the situation in the polydisc, where one usually consider Hardy spaces with respect to the uniform measure on the torus. It is the object of this paper to illustrate the advantages of using the weighted measure dS, when studying for example boundedness properties of H. As far as we know, this measure has not been considered before in that context.

Now set $d(\zeta,z) = |\langle\partial\rho(\zeta),\zeta-z\rangle|+|\langle\partial\rho(z),z-\zeta\rangle|$. Then, if Ω is strictly pseudoconvex, it is known that d is a quasimetric on the boundary of the domain and that $dS(\sim d\sigma)$ satisfies the doubling condition with respect to d. Thus $(\partial\Omega, dS, d)$ is a space of homogeneous type (see [C]). Moreover, in this setting, the operator H can be viewed as a singular integral operator, and by the $T1$-theorem for spaces of homogeneous type it can be proved to be bounded on $L^2(\partial\Omega)$. (See also [Ha] or [KS] for an elementary proof of this fact.)

The object of this paper is to give a thorough investigation of these matters for the complex ellipsoids B^p, $p = (p_1, ..., p_n)$, $p_j \in \mathbb{Z}^+$, with defining function $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$. B^p can be viewed as a model case for domains of finite type and has previously been studied in for example [R1] and [BC]. We will see that $(\partial\Omega, dS, d)$ is a space of homogeneous type. Furthermore, in this setting we will be able to prove some important representation theorems for the Hardy space $H^1(B^p, dS)$, following classical lines.

Let $L^q(\partial B^p)$ and $H^q(B^p)$, $q \geq 1$, be defined with respect to the measure dS. Hence an analytic function f in B^p belongs to $H^q(B^p)$ if

$$||f||_{H^q} = \sup_{0<r<1} \left(\int_{\partial B_r^p} |f|^q dS_r \right)^{1/q} < \infty,$$

where $B_r^p = \{z \in \mathbb{C}^n, \rho(z) < -r\}$ and dS_r is the measure on ∂B_r^p, represented by the form $(2\pi i)^{-n}\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}$. Then our results are as follows.

THEOREM 1. — *The operator H maps $L^2(\partial B^p)$ boundedly into $H^2(B^p)$.*

Thus defining $Hf(z_0)$ for $z_0 \in \partial B^p$ as the limit of $Hf(z)$ when $z \in B^p$ approaches z_0 in the normal direction, then H is bounded, considered as an operator on $L^2(\partial B^p)$. Hence in particular H has an adjoint operator, denoted by H^*, on $L^2(\partial B^p)$. Moreover, defining $BMO(\partial B^p)$ in the usual way (see § 3) in terms of the quasimetric d and surface measure dS, we also have

THEOREM 2. — *The operators H and H^* are bounded on $BMO(\partial B^p)$.*

In particular we can regard $H^*(BMO)$ as a subspace of BMO.

In the unit ball B it is known that all bounded linear functionals on H^1 can be represented by functions that are holomorphic in B and with boundary function in BMO (see [FS]). This has also been proved to hold for all strictly pseudoconvex domains in $\mathbb{C}^n$ (see for example [AC]). In B^p though, we will be content with the following analogous statement.

THEOREM 3. — *The space $H^*(BMO)$ operates on $H^1(B^p)$ in the sense that the pairing*

$$\langle f, H^*b\rangle = \int_{\partial B^p} f\overline{H^*b}dS, \quad b \in BMO(\partial B^p),$$

has a continuous extension from $C^1(\overline{B^p}) \cap H^1(B^p)$ to $H^1(B^p)$. Conversely, any bounded linear functional on $H^1(B^p)$ occurs in this way, and the operator norm is comparable to the BMO norm of H^*b.

Hence Theorem 3 states that $(H^1)^* \cong H^*(BMO)$. The proof actually gives that $(H^1)^* \cong H^*(L^\infty)$, and thus in particular $H^*(BMO) = H^*(L^\infty)$.

Next we consider the atomic space $\mathcal{H}^1_{at}$ on the boundary, where atoms are defined with respect to the measure dS and quasimetric d.

More precisely, an atom is a measurable function a on ∂B^p that is either identically one or else there is some ball $B_\epsilon(z_0) = \{z \in \partial B^p, d(z, z_0) < \epsilon\}$ centred at $z_0 \in \partial B^p$ such that

$$\operatorname{supp} a \subset B_\epsilon(z_0), \tag{1.1}$$

$$||a||_{L^\infty} \leq \frac{1}{|B_\epsilon(z_0)|} \tag{1.2}$$

and

$$\int_{\partial B^p} a(z) dS(z) = 0, \tag{1.3}$$

where $|B_\epsilon(z_0)| = \displaystyle\int_{B_\epsilon(z_0)} dS$. The atomic space $\mathcal{H}^1_{at}(\partial B^p)$ then consists of functions f such that $f = \sum\limits_{j=1}^{\infty} \lambda_j a_j$ (in L^1), where $\lambda_j \in \mathbb{C}$ satisfies $\sum\limits_{j=1}^{\infty} |\lambda_j| < \infty$ and a_j are atoms. The norm $||f||_{at}$ is defined as the infimum of $\sum\limits_{j=1}^{\infty} |\lambda_j|$ over all representations $\sum\limits_{j=1}^{\infty} \lambda_j a_j$ for f. The "real variable" spaces $\mathcal{H}^1_{at}$ and BMO are related by (see [CW])

$$(\mathcal{H}^1_{at})^* \cong BMO, \quad \text{via the pairing } \langle f, b\rangle = \int_{\partial B^p} f\bar{b} dS \text{ for } b \in BMO. \tag{1.4}$$

Note that, by (1.4) and Theorem 2, H can be extended to a bounded operator on $\mathcal{H}^1_{at}$. Moreover, we define the holomorphic atomic space $H^1_{\mathrm{at}}(B^p)$ to consist of all functions F such that $F = Hf$ for some $f \in \mathcal{H}^1_{at}(\partial B^p)$, with norm $||F||_{\mathrm{at}} = \inf\{||f||_{at}\,;\ F = Hf,\ f \in \mathcal{H}^1_{at}(\partial B^p)\}$. Then it is easy to see that every $F \in H^1_{\mathrm{at}}$ is a H^1-function, but the converse is also true.

THEOREM 4. — *Every $F \in H^1(B^p)$ is in $H^1_{\mathrm{at}}(B^p)$, and conversely, with $||F||_{\mathrm{at}} \sim ||F||_{H^1}$.*

A classical result for the unit disk in $\mathbb{C}$ is that every H^1-function can be factorized as a product of two H^2-functions. This can be shown by first factorizing out the zeros by means of a Blaschke product and then simply taking square roots. In higher dimensions there are no analogue of the Blaschke products available. Nevertheless, using the atomic decomposition for H^1, Coifman, Rochberg and Weiss, [CRW], proved a substitute result for the ball B in $\mathbb{C}^n$. This is generalized in our setting to B^p as follows

THEOREM 5. — *For every $F \in H^1(B^p)$ there are two sequences $\{G_j\}$ and $\{H_j\}$ in $H^2(B^p)$ such that*

$$F(z) = \sum_{j=1}^{\infty} G_j(z) H_j(z), \quad z \in B^p,$$

with

$$\sum_{j=1}^{\infty} ||G_j||_{H^2}||H_j||_{H^2} \sim ||F||_{H^1}.$$

Note that, by Theorem 4, it is enough to prove that every holomorphic atom Ha admits such a decomposition. In fact we will prove that $Ha = G_jH_j$ where $||G_j||_{H^2}||H_j||_{H^2} \leq C$.

Remarks and notations. — In this paper $\lesssim$ means $\leq C$, where C is a nonzero constant at most depending on the dimension n and the index p in B^p, and $\sim$ means both $\lesssim$ and $\gtrsim$. Moreover, when f is a function defined in B^p and its values over the boundary are to be considered, then f is understood to be replaced by its limit function at the boundary (i.e. $f(z_0)$, $z_0 \in \partial B^p$ is the limit of $f(z)$ when $z \in B^p$ approaches z_0 in the normal direction). In this way, H^1 can be considered as a closed subspace of L^1, and $||f||_{H^1} \sim ||f||_{L^1}$. Also a function in H^1 can be approximated by holomorphic functions that are continuous up to the boundary (i.e. if $f \in H^1$ then $f_r \to f$ in L^1, where f_r , $0 < r < 1$, is defined by $f_r(z) = f(rz)$, $z \in \overline{B^p}$.)

The paper is organized as follows. — In Section 2 the appropriate geometry of B^p is presented in more detail. In Section 3 we prove the L^2-boundedness of the operator H (Theorem 1) by applying the $T1$-theorem for spaces of homogeneous type. The boundedness on BMO (Theorem 2) is also considered. In Section 4 the duality in Theorem 3 is proved. The principal idea in the proof is due to Fefferman and Stein. Also the atomic decomposition of H^1 (Theorem 4) is proved in this section. It formalizes the idea that H^1 and H^1_{at} are equal since they have the same dual space. Finally in Section 5 we prove the factorization theorem (Theorem 5).

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2. The boundary as a space of homogeneous type.

In this section the appropriate geometry of B^p is presented in more detail. In the integral operator H defined in the introduction, we chose

$q = \partial\rho$ where $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$. Since the real part of $\langle \partial\rho(\zeta), \zeta - z\rangle$ is, apart from a nonzero smooth factor, the Euclidean distance from z to the real tangent plane at $\zeta \in \partial B^p$ and since B^p is convex this distance is positive if $z \in B^p$.

Moreover, with this choice of q, the form $(2\pi i)^{-n}\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}$ represents a positive measure dS on the boundary. Recall that a $(2n-1)$-form α represents the Euclidean surface measure on ∂B^p if $\dfrac{d\rho}{|d\rho|} \wedge \alpha = \bigwedge_{j=1}^{n} dx_j \wedge dy_j$. As we have

$$\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1} = (n-1)! \sum_{j=1}^{n} \left(p_j |\zeta_j|^{2(p_j-1)} \bar\zeta_j \prod_{k\neq j} p_k^2 |\zeta_k|^{2(p_k-1)} d\zeta_j \wedge \bigwedge_{k\neq j} d\zeta_k \wedge d\bar\zeta_k \right)$$

and

$$\frac{d\rho(\zeta)}{|d\rho(\zeta)|} \wedge \partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1} = \frac{(n-1)!}{|d\rho(\zeta)|} \sum_{j=1}^{n} |\zeta_j|^{2p_j} \prod_{j=1}^{n} p_j^2 |\zeta_j|^{2(p_j-1)} \bigwedge_{j=1}^{n} d\zeta_j \wedge d\bar\zeta_j,$$

we see that $dS \sim w\, d\sigma$ where

$$w(\zeta) = \prod_{j=1}^{n} p_j^2 |\zeta_j|^{2(p_j-1)}.$$

Later on we will consider the volume measure dV represented by the form $(2\pi i)^{-n}(\bar\partial\partial\rho)^n$. It corresponds naturally to dS, as they (essentially) are induced by the same Hermitian metric corresponding to the $(1,1)$-form $i\partial\bar\partial\rho$ (see §4). Note that $(2\pi i)^{-n}(\bar\partial\partial\rho)^n = d((2\pi i)^{-n}\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1})$ and that $dV \sim w\, d\lambda$, where $d\lambda$ is Lebesgue measure. For further use we write $|U| = \int_U dS$ when $U \subset \partial B^p$.

Now set $d(\zeta, z) = |v(\zeta, z)| + |v(z, \zeta)|$, where $v(\zeta, z) = \langle \partial\rho(\zeta), \zeta - z\rangle = \sum_{j=1}^{n} p_j |\zeta_j|^{2(p_j-1)} \bar\zeta_j (\zeta_j - z_j)$, and let $B_\epsilon(z) = \{\zeta \in \partial B^p, d(\zeta, z) < \epsilon\}$ be the corresponding balls. Then, generalizing the idea from the strictly pseudoconvex case, we will see that $(\partial B^p, dS, d)$ is a space of homogeneous type. The distance d have been considered before in for example [BL] and is in fact equivalent to the quasimetric used by McNeal for convex domains (see for instance [McN]). Hence for the proofs of Lemmas 1, 2 and 4 below, which concern estimates on v and d, we refer to the paper [BL]. Detailed proofs can also be found in [Ha].

Since clearly $d(\zeta, z)$ is symmetric and equals zero if and only if $\zeta = z$, the following lemma shows that d is a quasimetric on the boundary.

LEMMA 1. — *For $z, \zeta, w \in \partial B^p$ we have*

i) $|v(\zeta, z)| \sim |v(z, \zeta)|$

and

ii) $d(\zeta, z) \lesssim d(\zeta, w) + d(w, z)$.

Remark. — Note that if $z \in B_\epsilon(z_0)$ and $\zeta \in B^c_{C\epsilon}(z_0) = \partial B^p \backslash B_{C\epsilon}(z_0)$, for C sufficiently large, then by Lemma 1 we have $d(\zeta, z) \sim d(\zeta, z_0)$.

For later purposes we also need estimates of $v(\zeta, z)$ near the boundary.

LEMMA 2. — *For $z \in \mathbb{C}^n$ let $\tilde{z}$ denote the projection onto ∂B^p determined by $z_j = (1 + \rho(z))^{\frac{1}{2p_j}} \tilde{z}_j$. Then for $\zeta \in \overline{B^p}$ we have*

i) $|v(z, \zeta) - \rho(z)| \sim |v(\tilde{z}, \tilde{\zeta})| - \rho(z) - \rho(\zeta)$, *if* $z \in \overline{B^p}$

and

ii) $|v(z, \zeta)| \sim |v(\tilde{z}, \tilde{\zeta})| + \rho(z) - \rho(\zeta)$, *if* $z \in (B^p)^c$.

To see that $(\partial B^p, d, dS)$ is a space of homogeneous type it remains to check the doubling condition $|B_{2\epsilon}(z_0)| \lesssim |B_\epsilon(z_0)|$. This clearly follows from the stronger statement $|B_\epsilon(z_0)| \sim \epsilon^n$, that is contained in the following lemma.

LEMMA 3. — *If $z \in \partial B^p$ then*

i) $\int_{|v(\zeta,z)|<\epsilon} dS(\zeta) \sim \epsilon^n$,

ii) $\int_{|v(z,\zeta)|<\epsilon, \zeta \in B^p} dV(\zeta) \sim \epsilon^{n+1}$

and

iii) $\int_{|v(\zeta,z)|<\epsilon, \zeta \in (B^p)^c} dV(\zeta) \sim \epsilon^{n+1}$.

Proof. — The estimate i) is known for the ball. The general case can be proved by a change of variables $\zeta_j \to \zeta_j^{p_j}$. To do this we need a simple connection between v and the ball related $\tilde{v}$ defined by $\tilde{v}(\zeta, z) = \langle \bar{\zeta}^p, \zeta^p - z^p \rangle$ where $z^p = (z_1^{p_1}, ..., z_n^{p_n})$, i.e. $\tilde{v}(\zeta, z) = v_B(\zeta^p, z^p)$ where v_B is the corresponding v for the ball. They are related by

$$v(\zeta,z)=\tilde{v}(\zeta,z)+\sum_{j=1}^{n}\sum_{k=2}^{p_j}\binom{p_j}{k}\bar{\zeta}_j^{p_j}\zeta_j^{p_j-k}(z_j-\zeta_j)^k.$$

Now one can prove that $|\tilde{v}(\zeta,z)|\lesssim|v(\zeta,z)|$ (see [BL]) and hence

$$\int_{|v(\zeta,z)|<\epsilon}dS(\zeta)\le\int_{|\tilde{v}(\zeta,z)|\lesssim\epsilon}dS(\zeta)\sim\int_{|v_B(\zeta,z^p)|\lesssim\epsilon}d\sigma_B(\zeta)\sim\epsilon^n.$$

Likewise we get the converse inequality by observing that if we restrict the integration to $\zeta\in E_z$, where $E_z=\{\zeta\in\partial B^p,\ |\arg\zeta_j-\arg z_j|<\pi/p_j\}$, then $|v(\zeta,z)|\lesssim|\tilde{v}(\zeta,z)|$ (see [BL]).

By using i) and Lemma 2, the statements ii) and iii) now easily follow. If we change coordinates $\zeta\to(t,\tilde{\zeta})$, where $t=-\rho(\zeta)$ and $\tilde{\zeta}$ is the projection of ζ onto ∂B^p (in the meaning of Lemma 2), then $dV(\zeta)\sim dt\,dS(\tilde{\zeta})$ and hence

$$\begin{aligned}\int_{|v(z,\zeta)|<\epsilon,\zeta\in B^p}dV(\zeta)&\sim\int\int_{|v(z,\tilde{\zeta})|+t<\epsilon}dtdS(\tilde{\zeta})\\&=\int_0^\epsilon\left(\int_{|v(z,\tilde{\zeta})|<\epsilon-t}dS(\tilde{\zeta})\right)dt\sim\int_0^\epsilon(\epsilon-t)^n dt\sim\epsilon^{n+1}.\end{aligned}$$

Similarly iii) is proved. □

Note that i) in Lemma 3 implies that for $z\in\partial B^p$ and $\alpha>0$ we have

$$\int_{d(\zeta,z)>\epsilon}\frac{dS(\zeta)}{d(\zeta,z)^{n+\alpha}}\lesssim\frac{1}{\epsilon^\alpha},$$

where the constant in the inequality depend on α. This is most easily seen by splitting the integral in dyadic parts. In the same way one can see that ii) in Lemma 3 implies that for $z\in\partial B^p$ and $\alpha,\beta>0$ such that $\beta-\alpha>-1$, we have

$$\int_{B^p}\frac{(-\rho(\zeta))^{\beta-\alpha}dV(\zeta)}{|v(z,\zeta)|^{n+1-\alpha}}<\infty. \tag{2.2}$$

For further use we also need the following smoothness result on $v(\zeta,z)$.

LEMMA 4. — *With $\delta=(2\max\{p_j\})^{-1}$ we have for $z,\zeta,w\in\partial B^p$*

i) $|v(w,\zeta)-v(z,\zeta)+v(z,w)|\lesssim d(w,\zeta)^\delta d(z,w)^{1-\delta}+d(w,\zeta)^{1-\delta}d(z,w)^\delta$

and

ii) *there exists a constant C such that if $z,w\in B_\epsilon(z_0)$ and $\zeta\in B^c_{C\epsilon}(z_0)$ then*

$$|v(w,\zeta)-v(z,\zeta)|+|v(\zeta,w)-v(\zeta,z)|\lesssim\epsilon^\delta d(\zeta,z_0)^{1-\delta}.$$

Moreover, with δ as in Lemma 4, we also have the estimate $|z-\zeta| \lesssim |v(z,\zeta)-\rho(z)|^{\delta}$, for $z \in B^p$ and $\zeta \in \partial B^p$ (see [BL]).

3. Boundedness properties of H.

With $v(\zeta,z)$ and dS as in Section 2 the operator H defined for $f \in L^1(\partial B^p)$ by

$$Hf(z) = \int_{\partial B^p} \frac{f(\zeta)\,dS(\zeta)}{v(\zeta,z)^n}, \; z \in B^p$$

reproduces holomorphic functions that is sufficiently smooth up to the boundary. In fact, since $C^1(\overline{B^p}) \cap H^1(B^p)$ is dense in $H^1(B^p)$, we have $Hf = f$ when $f \in H^1(B^p)$.

In this section we prove that the operator H is bounded from L^2 to H^2. This is done by applying Coifman's generalization of the $T1$-theorem to spaces of homogeneous type. Then we use this L^2-boundedness to prove that H and its adjoint H^* are bounded operators on BMO. The proof is an adaption of the classical argument in [FS] to the non-isotropic geometry of ∂B^p.

We begin by recalling the $T1$-theorem. Before we state it, we need some basic definitions concerning singular integral operators. A nice presentation of the $T1$-theorem, for a general space of homogeneous type, can be found in [C, Ch. 6]. Here we just consider the specific space $(\partial B^p, d, dS)$, where (by the C^∞-manifold structure of ∂B^p) the presentation can be slightly simplified. A kernel $k : \partial B^p \times \partial B^p \backslash \{\zeta = z\} \to \mathbb{C}^n$ is said to be standard if there exist constants c and $\delta > 0$ such that for all $\zeta, z, w \in \partial B^p$, with $\epsilon = d(z,\zeta) > 0$ and $d(z,w) < c\epsilon$, we have

$$|k(z,\zeta)| \lesssim \frac{1}{|B_\epsilon(z)|} \tag{3.2}$$

and

$$|k(w,\zeta)-k(z,\zeta)| + |k(\zeta,w)-k(\zeta,z)| \lesssim \left(\frac{d(z,w)}{d(z,\zeta)}\right)^{\delta} \frac{1}{|B_\epsilon(z)|}. \tag{3.3}$$

Then we say that T is a singular integral operator if it is a continuous linear operator from $C^\infty(\partial B^p)$ to the space of distributions on ∂B^p that is associated to a standard kernel k in the sense that

$$\langle Tf, g\rangle = \int_{\partial B^p}\int_{\partial B^p} k(\zeta,z) f(z) g(\zeta) dS(z) dS(\zeta),$$

for all $f, g \in C^\infty$ with disjoint supports. The transpose T^t of a singular integral operator T is defined by the relation

$$\langle Tf, g\rangle = \langle f, T^t g\rangle \quad \text{for all } f, g \in C^\infty.$$

To formulate the $T1$-theorem we also need the notion of weak boundedness. For $w \in \partial B^p$ and $\delta, \epsilon > 0$ define $A(\delta, w, \epsilon)$ to be the set of all $\phi \in C^\infty$ supported in $B_\epsilon(w)$ satisfying $||\phi||_\infty \leq 1$ and for all $\zeta, z \in \partial B^p$

$$|\phi(\zeta) - \phi(z)| \leq \frac{d(\zeta, z)^\delta}{\epsilon^\delta}.$$

Then we say that a singular integral operator T is weakly bounded if there exist δ small enough (see [C] for enough requirements) such that for all $w \in \partial B^p, \epsilon > 0$ and $\phi, \tau \in A(\delta, w, \epsilon)$

$$|\langle T\phi, \tau\rangle| \lesssim |B_\epsilon(w)|.$$

The $T1$-theorem can then be stated as follows

PROPOSITION 1. — *A singular integral operator T on a space of homogeneous type is bounded on L^2 if and only if it is weakly bounded and $T1, T^t 1 \in BMO$.*

Recall also that any singular integral operator that is bounded on L^2 is bounded on L^q for $q > 1$. For the definition of BMO see (3.7) below. However, in the proof of Theorem 1 we show that $H^t 1 \in L^\infty$ and thus (as $H1 = 1$) it is enough to know that $L^\infty \subset BMO$.

Proof of Theorem 1. — To apply the $T1$-theorem to H we first note that $1/v(\zeta, z)^n$ is a standard kernel. By Lemmas 1 and 3 the estimate in (3.2) is clearly fulfilled. Also making use of Lemma 4, the estimate in (3.3) follows. For example

$$\left|\frac{1}{v(w,\zeta)^n} - \frac{1}{v(z,\zeta)^n}\right| = \left|\frac{(v(z,\zeta) - v(w,\zeta))\sum_{j=0}^{n-1} v(z,\zeta)^j v(w,\zeta)^{n-1-j}}{v(z,\zeta)^n v(w,\zeta)^n}\right|$$
$$\lesssim \frac{d(z,w)^\delta \epsilon^{1-\delta}}{\epsilon^{n+1}} \sim \left(\frac{d(z,w)}{d(z,\zeta)}\right)^\delta \frac{1}{|B_\epsilon(z)|},$$

since if c is sufficiently small then $d(z,\zeta) \sim d(w,\zeta)$ when $d(z,w) < cd(z,\zeta)$.

Next we need to define H as an operator on ∂B^p. Let $Hf(z_0)$, $z_0 \in \partial B^p$, be the limit of $Hf(z)$ when $z \in B^p$ approaches z_0 in the normal direction. When $f \in L^2$ it is not immediately clear that this limit exist. However, since $H1 = 1$, we have

$$Hf(z) = \int_{\partial B^p} \frac{f(\zeta) - f(z_0)}{v(\zeta, z)^n} \, dS(\zeta) + f(z_0), \quad z \in B^p,$$

and if f is say C^1, the integral on the right is easily seen to converge when $z \to z_0 \in \partial B^p$, so

$$Hf(z_0) = \int_{\partial B^p} \frac{f(\zeta) - f(z_0)}{v(\zeta, z_0)^n} \, dS(\zeta) + f(z_0). \tag{3.4}$$

This can equally well be taken as definition of Hf when f is a C^1-function. Moreover, it is easy to check that H is a singular integral operator associated to $1/v(\zeta, z)^n$.

To see that H is bounded on L^2 it remains to verify the conditions in Proposition 2. As H reproduces holomorphic functions we have $H1 = 1 \in BMO$. Thus we turn our attention directly to $H^t 1$. Since $v(\zeta, z)$ is holomorphic in z we have by Stokes' theorem

$$\int_{\partial B^p_r} \frac{\partial \rho(z) \wedge (\bar{\partial}\partial \rho(z))^{n-1}}{v(\zeta, z)^n} = \int_{B^p_r} \frac{(\bar{\partial}\partial \rho(z))^n}{v(\zeta, z)^n},$$

where $B^p_r = \{z \in \mathbb{C}^n, \, \rho(z) < -r\}$. Note here that, by (2.2), the right hand side converges (uniformly in $\zeta \in \partial B^p$) when $r \to 1$. Then by Fubini's theorem

$$\int_{\partial B^p_r} Hf dS = \int_{\partial B^p} f(\zeta) \left(\int_{B^p_r} \frac{dV(z)}{v(\zeta, z)^n} \right) dS(\zeta). \tag{3.5}$$

If $f \in C^\infty$ then Hf is continuous on $\overline{B^p}$, so letting $r \to 1$ in (3.5) we get

$$\int_{\partial B^p} f \overline{H^t 1} dS = \int_{\partial B^p} Hf dS = \int_{\partial B^p} f(\zeta) \left(\int_{B^p} \frac{dV(z)}{v(\zeta, z)^n} \right) dS(\zeta).$$

Hence we must have

$$H^t 1(z) = \int_{B^p} \frac{dV(\zeta)}{\overline{v(z, \zeta)}^n}, \; z \in \partial B^p. \tag{3.6}$$

In particular, by (2.2), we have $H^t 1 \in L^\infty \subset BMO$.

Thus it remains to check that H is weakly bounded. This follows from

LEMMA 5. — *If $z \in \partial B^p$ then*

$$\left| \int_{d(\zeta, z) > \epsilon} \frac{dS(\zeta)}{v(\zeta, z)^n} \right| \lesssim 1.$$

Proof. — Since $d(\zeta,z) \lesssim |v(\zeta,z)|$ we have
$$\left|\int_{d(\zeta,z)>\epsilon} \frac{dS(\zeta)}{v(\zeta,z)^n}\right| \leq \left|\int_{|v(\zeta,z)|>\epsilon} \frac{dS(\zeta)}{v(\zeta,z)^n}\right| + \int_{\epsilon \gtrsim |v(\zeta,z)|>\epsilon} \frac{dS(\zeta)}{|v(\zeta,z)|^n}.$$
Here the second integral on the right is clearly bounded. To estimate the first one we use Stokes' theorem on the domain $\{\zeta \in (B^p)^c, |v(\zeta,z)| > \epsilon, \rho(\zeta) < 1\}$. Since $\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}/v(\zeta,z)^n$ is a closed form where $\zeta \neq z$, and as
$$\int_{\rho(\zeta)=1} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}}{v(\zeta,z)^n} = 1$$
(by Proposition 1 applied to the domain $\{\rho < 1\}$), we deduce that
$$\int_{|v(\zeta,z)|>\epsilon} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}}{v(\zeta,z)^n} = 1 - \int_{|v(\zeta,z)|=\epsilon, \zeta\in(B^p)^c} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}}{v(\zeta,z)^n}.$$
When integrating with respect to ζ such that $|v(\zeta,z)| = \epsilon$, we can write $1/v(\zeta,z)^n = \overline{v(\zeta,z)}^n/\epsilon^{2n}$. Thus if we use Stokes' theorem again, now on the domain $\{\zeta \in (B^p)^c, |v(\zeta,z)| < \epsilon\}$, we get
$$\begin{aligned}
&\int_{|v(\zeta,z)|=\epsilon, \zeta\in(B^p)^c} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}}{v(\zeta,z)^n} \\
&= \frac{1}{\epsilon^{2n}} \int_{|v(\zeta,z)|=\epsilon, \zeta\in(B^p)^c} \overline{v(\zeta,z)}^n \partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1} \\
&= \frac{1}{\epsilon^{2n}} \int_{|v(\zeta,z)|<\epsilon} \overline{v(\zeta,z)}^n \partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1} \\
&\quad + \frac{1}{\epsilon^{2n}} \int_{|v(\zeta,z)|<\epsilon, \zeta\in(B^p)^c} d_\zeta \overline{v(\zeta,z)}^n \wedge \partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1} \\
&\quad + \frac{1}{\epsilon^{2n}} \int_{|v(\zeta,z)|<\epsilon, \zeta\in(B^p)^c} \overline{v(\zeta,z)}^n (\bar\partial\partial\rho(\zeta))^n.
\end{aligned}$$
Since $|d_\zeta \overline{v(\zeta,z)}^n \wedge \partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}| \lesssim |v(\zeta,z)|^{n-1} w(\zeta)$ we conclude that
$$\begin{aligned}
\left|\int_{|v(\zeta,z)|>\epsilon} \frac{dS(\zeta)}{v(\zeta,z)^n}\right| &\lesssim \frac{1}{\epsilon^n} \int_{|v(\zeta,z)|<\epsilon} dS(\zeta) \\
&\quad + \frac{1}{\epsilon^{n+1}} \int_{|v(\zeta,z)|<\epsilon, \zeta\in(B^p)^c} dV(\zeta) + \frac{1}{\epsilon^n} \int_{|v(\zeta,z)|<\epsilon, \zeta\in(B^p)^c} dV(\zeta).
\end{aligned}$$
Hence, by i) and iii) in Lemma 3, we are done. □

Remark. — In a similar way one can prove that
$$\int_{d(\zeta,z)>\epsilon} \frac{dS(\zeta)}{v(z,\zeta)^n} \lesssim 1.$$

To see that H is weakly bounded is now easy. If $\phi \in A(\delta, w, \epsilon)$ and C is sufficiently large then

$$H\phi(z) = \int_{d(\zeta,w)\leq C\epsilon} \frac{\phi(\zeta) - \phi(z)}{v(\zeta,z)^n} dS(\zeta) + \phi(z)\left(1 - \int_{d(\zeta,w)>C\epsilon} \frac{1}{v(\zeta,z)^n} dS(\zeta)\right).$$

By Lemma 5 the second term on the right is uniformly bounded, and for the first one we have

$$\left|\int_{d(\zeta,w)\leq C\epsilon} \frac{\phi(\zeta) - \phi(z)}{v(\zeta,z)^n} dS(\zeta)\right| \lesssim \frac{1}{\epsilon^\delta}\int_{d(\zeta,w)\leq C\epsilon} \frac{1}{d(\zeta,z)^{n-\delta}} dS(\zeta) \lesssim \frac{\epsilon^\delta}{\epsilon^\delta} = 1,$$

so $||H\phi(z)||_\infty \lesssim 1$ for all $\phi \in A(\delta, w, \epsilon)$. Hence if also $\tau \in A(\delta, w, \epsilon)$ then

$$|\langle H\phi, \tau\rangle| \lesssim \int_{\partial B^p} |\tau| dS = \int_{d(\zeta,w)<\epsilon} |\tau| dS \leq |(B_\epsilon(w))|.$$

Thus by the $T1$-theorem it follows that H, defined on the boundary by (3.4) for C^1-functions, can be extended to a bounded operator on L^2. Then to see that H (as defined in B^p by (3.1)) takes L^2 boundedly into H^2 is easy. For $f \in L^2$ take a sequence of functions $f_n \in C^1(\overline{B^p})$ such that $f_n \to f$ in L^2. Then

$$\int_{\partial B^p_r} |Hf_n|^2 dS \leq \int_{\partial B^p} |Hf_n|^2 dS \lesssim \int_{\partial B^p} |f_n|^2 dS,$$

and hence letting $n \to \infty$ we get $\int_{\partial B^p_r} |Hf|^2 dS \lesssim \int_{\partial B^p} |f|^2 dS$ for all $r > 0$. Thus $||Hf||_{H^2} \lesssim ||f||_{L^2}$ and Theorem 1 is proved. □

Now for $f \in C^1$ consider $Hf(z)$ as defined on the boundary by taking limits from the interior. For functions F in H^2 this limit exist almost everywhere and we have $||F||_{H^2} \sim ||F||_{L^2}$. Hence we conclude by Theorem 1 that $Hf(z)$, $z \in \partial B^p$, can be defined in this way for all L^2-functions. Moreover, this limit operator is bounded on L^2. In fact it is the same operator as the extended operator given by the $T1$-theorem (since they coincide on smooth functions). Then, as an operator on L^2, it has an adjoint operator which we denote by H^*. Note that H^* is a singular integral operator associated to the standard kernel $1/\overline{v(z,\zeta)}^n$.

Before we turn to the proof of Theorem 2 we recall the definition of BMO. The space $BMO(\partial B^p)$ consists of those $b \in L^1(\partial B^p)$ such that the norm

(3.7)

$$||b||_{BMO} = \operatorname{supp}_{z_0,\epsilon} \frac{1}{|B_\epsilon(z_0)|}\int_{B_\epsilon(z_0)} |b(z) - b_{B_\epsilon(z_0)}| dS(z) + \int_{\partial B^p} |b(z)| dS(z)$$

is finite, where $b_{B_\epsilon(z_0)}$ is the mean value of b on $B_\epsilon(z_0)$. Recall also that by the John-Nirenberg inequality for BMO-functions we have

$$\frac{1}{|B_\epsilon(z_0)|}\int_{B_\epsilon(z_0)} |b(z) - b_{B_\epsilon(z_0)}|^q dS(z) \lesssim ||b||^q_{BMO}, \quad q > 1. \tag{3.8}$$

Proof of Theorem 2. — The idea in the proof of Theorem 2 is standard and in fact the same as in the original paper by Fefferman and Stein on Hardy Spaces ([FS]). Hence some details below are left for the reader. To prove that H is bounded on BMO let $b \in BMO$ and fix a ball B_ϵ with center in $z_0 \in \partial B^p$. Then write $b = b_1 + b_2 + b_3$ where $b_1 = b_{B_{C\epsilon}(z_0)}, b_2 = (b - b_1)\chi_{B_{C\epsilon}(z_0)}$ and $b_3 = (b - b_1)(1 - \chi_{B_{C\epsilon}(z_0)})$. Here $\chi_{B_{C\epsilon}(z_0)}$ is the characteristic function for $B_{C\epsilon}(z_0)$ and C is a sufficiently large constant (depending on the constant in the triangle inequality for d). As $Hb_1 = b_1$ is constant we have

$$\frac{1}{|B_\epsilon|}\int_{B_\epsilon} |Hb_1(z) - (Hb_1)_{B_\epsilon}|dS(z) = 0.$$

Moreover, since H is bounded on L^2, we have by Jensen's inequality and (3.8)

$$\begin{aligned}
&\left(\frac{1}{|B_\epsilon|}\int_{B_\epsilon} |Hb_2 - (Hb_2)_{B_\epsilon}|dS\right)^2 \lesssim \left(\frac{1}{|B_\epsilon|}\int_{B_\epsilon} |Hb_2|dS\right)^2\\
&\leq \frac{1}{|B_\epsilon|}\int_{\partial B^p} |Hb_2|^2 dS \lesssim \frac{1}{|B_\epsilon|}\int_{\partial B^p} |b_2|^2 dS\\
&= \frac{1}{|B_\epsilon|}\int_{B_{C\epsilon}} |b - b_{B_{C\epsilon}}|^2 dS \lesssim ||b||^2_{BMO}.
\end{aligned}$$

Finally by Lemma 4 we have

$$\begin{aligned}
&\frac{1}{|B_\epsilon|}\int_{B_\epsilon} |Hb_3(z) - (Hb_3)_{B_\epsilon}|dS(z)\\
&\lesssim \frac{1}{|B_\epsilon|^2}\int_{B_\epsilon}\left(\int_{B_\epsilon}\left(\int_{\partial B^p}\left|\frac{1}{v(\zeta,z)^n} - \frac{1}{v(\zeta,w)^n}\right| |b_3(\zeta)|dS(\zeta)\right) dS(w)\right) dS(z)\\
&\lesssim \int_{\partial B^p} \epsilon^\delta d(\zeta,z_0)^{1-\delta} d(\zeta,z_0)^{n+1}|b_3(\zeta)|dS(\zeta) \lesssim \epsilon^\delta \int_{B^c_{C\epsilon}} \frac{|b(\zeta) - b_{B_{C\epsilon}}|}{d(\zeta,z_0)^{n+\delta}} dS(\zeta)\\
&\lesssim ||b||_{BMO}
\end{aligned}$$

where $\delta = (2\max\{p_j\})^{-1}$. The last inequality can be seen by splitting the integral in dyadic parts (compare for instance [FS]).

To see that $||Hb||_{L^1} \lesssim ||b||_{BMO}$ write instead $b = b_{\partial B^p} + (b - b_{\partial B^p})$. Then $||Hb_{\partial B^p}||_{L^1} = ||b_{\partial B^p}||_{L^1} \lesssim ||b||_{L^1} \leq ||b||_{BMO}$ and the term $H(b -$

$b_{\partial B^p}$) can be estimated as Hb_2 above. This proves that H is bounded on BMO.

In the same way we can prove that H^* is bounded on BMO, with some extra effort on the term H^*b_1. Although H^*1 is not identically 1, one can easily see by Lemma 4 that it is Hölder continuous of order δ (recall the explicit expression (3.6) for H^*1). Moreover one can show that $|b_1| \lesssim \log \frac{1}{\epsilon}||b||_{BMO}$ (see for example [FS]) and hence

$$\begin{aligned}
&\frac{1}{|B_\epsilon|}\int_{B_\epsilon} |H^*b_1(z) - (H^*b_1)_{B_\epsilon}|dS(z) \\
&\leq \frac{|b_1|}{|B_\epsilon|^2}\int_{B_\epsilon}\int_{B_\epsilon} |H^*1(z) - H^*1(\zeta)|dS(\zeta)dS(z) \\
&\lesssim |b_1|\cdot \epsilon^\delta \lesssim ||b||_{BMO}\epsilon^\delta \log\frac{1}{\epsilon} \lesssim ||b||_{BMO}.
\end{aligned}$$

The terms H^*b_2 and H^*b_3 are treated as the corresponding ones for H, and it follows that also H^* is bounded on BMO. □

4. Duality and atomic decomposition of H^1.

In this section we prove Theorem 3 and Theorem 4. In the proof of Theorem 3 we need a version of Green's identity with respect to the non-isotropic structure of B^p. For this reason we begin by considering the Hermitian metric corresponding to the $(1,1)-$form $\omega = i\partial\bar\partial\rho$, where $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$ (A general presentation of Hermitian metrics can be found in [R2, Ch.3]). That is, for $(1,0)$-forms u and v, we define the inner product by

$$\langle u, v\rangle_\omega \frac{\omega^n}{n!} = iu \wedge \bar v \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

This is extended to all 1-forms by setting $\langle \bar u, \bar v\rangle_\omega = \overline{\langle u, v\rangle_\omega}$, $\langle u, \bar v\rangle_\omega = 0$ and $\langle \bar u, v\rangle_\omega = 0$. It should be noted that $\langle\cdot,\cdot\rangle_\omega$ degenerates at points $z \in \mathbb{C}^n$ where $z_j = 0$ for some j such that $p_j \neq 1$. Since

$$\omega^n/n! = i^n \prod_{k=1}^{n} p_k^2|z_k|^{2(p_k-1)} \bigwedge_{k=1}^{n} dz_k \wedge d\bar z_k$$

and

$$\omega^{n-1}/(n-1)! = i^{n-1}\sum_{j=1}^{n}\left(\prod_{k\neq j} p_k^2|z_k|^{2(p_k-1)} \bigwedge_{k\neq j} dz_k \wedge d\bar z_k\right),$$

we have

$$\langle u,v\rangle_\omega \bigwedge_{k=1}^{n} dz_k\wedge d\bar z_k = u\wedge\bar v\wedge\sum_{j=1}^{n}\left(\frac{1}{p_j^2|z_j|^{2(p_j-1)}}\bigwedge_{k\neq j} dz_k\wedge d\bar z_k\right),$$

for $(1,0)-$forms u and v. In particular

$$|\partial f(z)|_\omega^2=\sum_{j=1}^{n}\frac{1}{p_j^2|z_j|^{2(p_j-1)}}\left|\frac{\partial f}{\partial z_j}\right|^2.$$

Thus, if we set $e_j=p_j|z_j|^{p_j-1}dz_j$, then $\{e_j,\bar e_j\}_{j=1}^n$ is an orthonormal basis for the set of all $1-$forms, in the metric $\langle\cdot,\cdot\rangle_\omega$. The metric $\langle\cdot,\cdot\rangle_\omega$ induces a volume measure, denoted by dV_ω, and a surface measure, denoted by dS_ω. The volume measure dV_ω is represented by $\bigwedge_{j=1}^{n} ie_j\wedge\bar e_j=\omega^n/n!$, and a $(2n-1)$-form α represents dS_ω if $(d\rho/|d\rho|_\omega)\wedge\alpha=\omega^n/n!$. Since on ∂B^p

$$|d\rho|_\omega^2=2|\partial\rho|_\omega^2=2\sum_{j=1}^{n}\frac{p_j^2|z_j|^{2(2p_j-1)}}{p_j^2|z_j|^{2(p_j-1)}}=2\sum_{j=1}^{n}|z_j|^{2p_j}=2,$$

we see that, up to a constant, dS_ω coincides with the measure dS, represented by the form $(2\pi i)^{-n}\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}$ (see § 2). Note also that dV_ω only differs by a constant from the measure dV, represented by the form $(2\pi i)^{-n}(\bar\partial\partial\rho(\zeta))^n$ (see § 2). Next we calculate the Laplacian $\Delta_\omega=d^*d$, where d^* is the adjoint of d with respect to the metric $\langle\cdot,\cdot\rangle_\omega$. On every test function φ, with compact support in B^p, we have by Stokes' theorem

$$\int_{B^p} d^*df\cdot\overline{\varphi}\,dV_\omega=\int_{B^p}\langle df,d\varphi\rangle_\omega dV_\omega=\int_{B^p} i\,(\partial-\bar\partial)f\wedge\overline{d\varphi}\wedge\frac{\omega^{n-1}}{(n-1)!}$$
$$=\int_{B^p} i\,d(\partial-\bar\partial)f\wedge\overline{\varphi}\frac{\omega^{n-1}}{(n-1)!}=\int_{B^p}\left(\sum_{j=1}^{n}\frac{-2}{p_j^2|z_j|^{2(p_j-1)}}\frac{\partial^2 f}{\partial z_j\partial\bar z_j}\right)\overline{\varphi}\,dV_\omega.$$

Hence

$$\Delta_\omega f=\sum_{j=i}^{n}\frac{-2}{p_j^2|z_j|^{2(p_j-1)}}\frac{\partial^2 f}{\partial z_j\partial\bar z_j},$$

and Green's identity with respect to the metric $\langle\cdot,\cdot\rangle_\omega$ is valid, i.e.

$$\int_{B^p}(f\Delta_\omega g-g\Delta_\omega f)dV_\omega=\int_{\partial B^p}\left(g\frac{\partial f}{\partial n}-f\frac{\partial g}{\partial n}\right)dS_\omega, \tag{4.1}$$

where $\partial f/\partial n=\langle df,d\rho/|d\rho|_\omega\rangle_\omega$. As the metric degenerates, and the Laplacian has singularities, at points $z\in\mathbb{C}^n$ where $z_j=0$ for some j such that $p_j\neq 1$, one could be worried about the validity of (4.1). However there is no singularity in $\Delta_\omega h dV_\omega$ and $(\partial h/\partial n) dS_\omega$, if h is sufficiently smooth. Hence (4.1) can be obtained, for instance by first deriving Green's identity with respect to the metric which corresponds to $i\partial\bar{\partial}(\rho + \epsilon\rho_B)$, where ρ_B is the defining function for the ball, and taking limits as $\epsilon \to 0$.

Before we proceed with the proof of Theorem 3 we also need a result on Carleson measures. We say that μ is a Carleson measure on B^p if $\mu(Q_\epsilon(z_0)) \lesssim |B_\epsilon(z_0)|$ for all $\epsilon > 0$ and $z_0 \in \partial B^p$, where $Q_\epsilon(z_0) = \{z \in B^p \ , \ |v(z_0, z)| < \epsilon\}$. Such measures satisfy the Carleson-Hörmander inequality.

LEMMA 6. — *If $q > 1$ then*

$$\int_{B^p} |f|^q d\mu \lesssim \int_{\partial B^p} |f|^q dS$$

for all plurisubharmonic functions f such that $\sup\limits_{0<r<1} \int_{\partial B^p_r} |f|^q dS_r < \infty.$

Proof of Lemma 6. — The Carleson-Hörmander inequality is known for the ball $B = B^{(1,\dots,1)}$ (see [H]), and the general case can be obtained by the change of variables $z \to z^p$ (i.e. $z_j \to z_j^{p_j}$, $j = 1, \dots, n$). In fact, if μ is a Carleson measure on B^p then the measure $\tilde{\mu}$, defined by $\tilde{\mu}(U) = \mu(\{z \in \mathbb{C}^n, \ z^p \in U\})$ is a Carleson measure on B (this follows from results obtained in § 2). Moreover, if f is a plurisubharmonic function on B^p, then $\tilde{f}(w) = \sum\limits_{z^p = w} |f(z)|$ defines a plurisubharmonic function on B. Hence, by the Carleson Hörmander inequality in the ball, we get

$$\begin{aligned}\int_{B^p} |f(z)|^q d\mu(z) &\leq \int_{B^p} |\tilde{f}(z^p)|^q d\mu(z) = \int_B |\tilde{f}(z)|^q d\tilde{\mu}(z) \\ &\lesssim \int_{\partial B} |\tilde{f}(z)|^q d\sigma(z) \sim \int_{\partial B^p} |\tilde{f}(z^p)|^q dS(z) \lesssim \int_{\partial B^p} |f(z)|^q dS(z).\end{aligned}$$

□

If we require f to be holomorphic, and hence $\log |f|$ is plurisubharmonic, the inequality in Lemma 6 is satisfied also for $q \leq 1$. In particular the inequality is satisfied with $q = 1$ and $f \in H^1$, which is needed in the following proof.

Proof of Theorem 3. — The main difficulty in proving the duality $(H^1)^* \cong H^*(BMO)$ is to obtain the estimate

$$\left| \int_{\partial B^p} f \bar{b} dS \right| \lesssim ||f||_{H^1} ||b||_{BMO}, \tag{4.2}$$

for $f \in C^1(\overline{B^p}) \cap H^1(B^p)$ and $b \in BMO$. From (4.2) it follows that every BMO-function, defined as a functional on $C^1(\overline{B^p}) \cap H^1(B^p)$ by $\langle f, b\rangle = \int_{\partial B^p} f\bar{b}dS$, can be extended to a continuous linear functional on H^1. Conversely, any functional on H^1 is given by some $u \in L^\infty$ and hence also by H^*u, since $\int_{\partial B^p} f\bar{u}dS = \int_{\partial B^p} Hf\bar{u}dS = \int_{\partial B^p} f\overline{H^*u}dS$ if $f \in H^1$. Moreover, if $\langle f, H^*u\rangle = 0$ for all $f \in H^1$, then we must have $H^*u = 0$, since taking $f = HH^*u$ we get $0 = \langle HH^*u, H^*u\rangle = \langle H^*u, H^*u\rangle = ||H^*u||_{L^2}^2$. Thus the representation is unique, and by the open mapping theorem it follows that $||H^*u||_{BMO} \sim ||H^*u||_{(H^1)^*}$.

To prove (4.2) we will use the representation $f(z) = Hf(z)$ which has no (simple) meaning when $z \in \partial B^p$. Therefore, instead we prove that

$$\left|\int_{\partial B_r^p} f\bar{b}dS\right| \lesssim ||f||_{H^1}||b||_{BMO},$$

for all $r > 0$, where $B_r^p = \{z \in \mathbb{C}^n,\ \rho(z) < -r\}$. Here b is extended from ∂B^p to a function defined on $\overline{B^p}$ by letting b be constant in the normal direction near the boundary. From this then (4.2) immediately follows.

By Lemma 2 we can define, for $z \in B^p$,

$$H_r^*b(z) = \int_{\partial B_r^p} \frac{b(\zeta)dS(\zeta)}{\overline{(v(z,\zeta) - \rho(z))^n}}.$$

Then $H_r^*b \in C^\infty(\overline{B^p})$, and by Fubini's theorem

$$\int_{\partial B_r^p} f\bar{b}dS = \int_{\partial B_r^p} Hf\bar{b}dS = \int_{\partial B^p} f\overline{H_r^*b}dS.$$

In what follows we write H^*b instead of H_r^*b and leave it for the reader to examine the independence of r in the estimates. By Stokes' theorem we get

$$\begin{aligned}\int_{\partial B^p} f\overline{H^*b}\,\bar{\partial}\rho \wedge (\bar{\partial}\partial\rho)^{n-1} &= \int_{B^p} \overline{H^*b}\,\partial f \wedge \bar{\partial}\rho \wedge (\bar{\partial}\partial\rho)^{n-1} \\ &\quad + \int_{B^p} f\ \partial\overline{H^*b} \wedge \bar{\partial}\rho \wedge (\bar{\partial}\partial\rho)^{n-1} \\ &\quad - \int_{B^p} f\overline{H^*b}\ (\bar{\partial}\partial\rho)^n = I_1 + I_2 + I_3.\end{aligned}$$

Here I_1 can be regarded as the main term. Another application of Stokes' theorem gives

$$I_1 = \int_{B^p} \rho\partial\overline{H^*b} \wedge \partial f \wedge (\bar{\partial}\partial\rho)^{n-1}.$$

To estimate this integral we need

LEMMA 7. — *With f and b as above we have*

i) $\int_{B^p} -\rho|\partial f|_\omega^2|f|^{q-2}dV \lesssim \int_{\partial B^p} |f|^q dS$

and

ii) $-\rho|\partial H^*b|_\omega^2$ *is a Carleson measure with norm less than* $||b||_{BMO}^2$.

Before we turn to the proof of Lemma 7 we complete the proof of Theorem 3. By Lemma 7, Schwarz' inequality and the Carleson-Hörmander inequality for functions in H^1 we have

$$\left|\int_{B^p} \rho\, \overline{\partial \overline{H^* b}} \wedge \partial f \wedge (\bar\partial\partial\rho)^{n-1}\right| \lesssim \int_{B^p} -\rho|\partial H^*b|_\omega|\partial f|_\omega dV$$

$$\lesssim \left(\int_{B^p} -\rho|\partial H^*b|_\omega^2|f|dV\right)^{1/2}\left(\int_{B^p} -\rho\frac{|\partial f|_\omega^2}{|f|}dV\right)^{1/2} \lesssim ||b||_{BMO}||f||_{H^1}.$$

The estimate of I_2 and I_3 is easier. By writing

$$v(z,\zeta)-\rho(z) = 1-\sum_{j=1}^{n}\zeta_j^{p_j}\bar z_j^{p_j} + \sum_{j=1}^{n}\sum_{k=2}^{p_j}\binom{p_j}{k}\bar z_j^{p_j}z_j^{p_j-k}(\zeta_j-z_j)^k$$

we see, by the comment after Lemma 4, that

$$\left|\frac{\partial}{\partial z_j}(v(z,\zeta)-\rho(z))\right| \lesssim |z_j|^{p_j}|z-\zeta| \lesssim |z_j|^{p_j}|v(z,\zeta)-\rho(z)|^\delta. \tag{4.3}$$

Hence $b \to (-\rho(z))^{1-\delta/2}|\partial\overline{H^*b}(z)|$ is an operator with more than integrable kernel, and thus it can easily be seen to take BMO boundedly into L^∞. In particular $|\partial\overline{H^*b}(z)|dV$ is a Carleson measure with norm less than $||b||_{BMO}$ and by the Carleson-Hörmander inequality it follows that $|I_2| \lesssim ||f||_{H^1}||b||_{BMO}$. Similarly I_3 is estimated, by using that $(-\rho(z))^{\delta/2}|H^*b(z)| \lesssim ||b||_{BMO}$. □

Proof of Lemma 7. — Set $\phi = \frac{q}{2}\log|f|^2$. Green's identity (with respect to the metric $\langle\cdot,\cdot\rangle_\omega$) yields

$$\int_{B^p}\left(\rho\Delta_\omega e^\phi - e^\phi\Delta_\omega\rho\right)dV_\omega = \int_{\partial B^p} e^\phi\frac{\partial\rho}{\partial n}dS_\omega.$$

Here $\Delta_\omega e^\phi = e^\phi\Delta_\omega\phi - 2|\partial\phi|_\omega^2 e^\phi$ and $\partial\rho/\partial n = \sqrt{2}$. Thus, since $\Delta_\omega\rho$ and $\Delta_\omega\phi$ both are negative, we have

$$\int_{B^p} -\rho|\partial\phi|_\omega^2 e^\phi dV_\omega \lesssim \int_{\partial B^p} e^\phi dS_\omega.$$

Hence $i)$ in Lemma 7 follows, as $\partial\phi = q\partial f/2f$.

Finally, to prove ii) we have to show that

$$\int_{Q_\epsilon(z_0)} -\rho|\partial H^* b|^2_\omega dV \lesssim ||b||^2_{BMO}|B_\epsilon(z_0)|$$

for every $z_0 \in \partial B^p$ and $\epsilon > 0$. As in the proof of Theorem 2 write $b = b_1 + b_2 + b_3$, where $b_1 = b_{B_{C\epsilon}(z_0)}, b_2 = (b - b_1)\chi_{B_{C\epsilon}(z_0)}$ and $b_3 = (b - b_1)(1 - \chi_{B_{C\epsilon}(z_0)})$. Then we extend b_1, b_2 and b_3 to functions defined on $\overline{B^p}$ in the same way as b. Since dS also can be represented by the form $-(2\pi i)^{-n}\bar\partial\rho \wedge (\bar\partial\partial\rho)^{n-1}$, and since $\partial_\zeta(\overline{v(z,\zeta) - \rho(z)}) = 0$, we have by Stokes' theorem

$$\begin{aligned} H^*1(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{\partial B^p_r} \frac{-\bar\partial\rho(\zeta) \wedge (\bar\partial\partial\rho(\zeta))^{n-1}}{(\overline{v(z,\zeta) - \rho(z)})^n} \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{B^p_r} \frac{(\bar\partial\partial\rho(\zeta))^n}{(\overline{v(z,\zeta) - \rho(z)})^n}. \end{aligned}$$

Since $|(\partial/\partial z_j)(\overline{v(z,\zeta) - \rho(z)})| \lesssim |z_j|^{p_j - 1}$ and $-\rho(z) \le |v(z,\zeta) - \rho(z)|$ we have by (2.2)

$$\begin{aligned} -\rho(z)\left|\frac{\partial}{\partial z_j} H^*1(z)\right|^2 &\lesssim \left(\int_{B^p_r} \frac{(\partial/\partial z_j)(\overline{v(z,\zeta) - \rho(z)})dV(\zeta)}{|v(z,\zeta) - \rho(z)|^{n+1/2}}\right)^2 \\ &\lesssim |z_j|^{2(p_j - 1)}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{Q_\epsilon(z_0)} -\rho|\partial H^* b_1|^2_\omega dV &\lesssim |b_1|^2 \int_{Q_\epsilon(z_0)} dV \\ &\lesssim ||b||^2_{BMO}\epsilon^{n+1}\log^2(1/\epsilon) \lesssim ||b||^2_{BMO}|B_\epsilon(z_0)|. \end{aligned}$$

Furthermore, by repeated use of Stokes' theorem, we have

$$\begin{aligned} &\int_{Q_\epsilon(z_0)} -\rho|\partial H^* b_2|^2_\omega dV \\ &\le \int_{B^p} -\rho|\partial H^* b_2|^2_\omega dV = \int_{B^p} -\rho i\; \partial H^* b_2 \wedge \overline{\partial H^* b_2} \wedge (\bar\partial\partial\rho)^{n-1} \\ &= \int_{B^p} -iH^* b_2\; \overline{\partial H^* b_2} \wedge \partial\rho \wedge (\bar\partial\partial\rho)^{n-1} \\ &\quad + \int_{B^p} -\rho i H^* b_2\; \partial\overline{\partial H^* b_2} \wedge (\bar\partial\partial\rho)^{n-1} \\ &= \int_{B^p} i\overline{H^* b_2}\; \bar\partial H^* b_2 \wedge \partial\rho \wedge (\bar\partial\partial\rho)^{n-1} + \int_{B^p} i|H^* b_2|^2(\bar\partial\partial\rho)^n \\ &\quad + \int_{\partial B^p} i|H^* b_2|^2 \partial\rho \wedge (\bar\partial\partial\rho)^{n-1} + \int_{B^p} -\rho i\bar\partial H^* b_2 \wedge \partial\overline{H^* b_2} \wedge (\bar\partial\partial\rho)^{n-1} \\ &\quad + \int_{B^p} iH^* b_2 \partial\overline{H^* b_2} \wedge \bar\partial\rho \wedge (\bar\partial\partial\rho)^{n-1} = J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

By (3.8) and the boundedness of H^*, considered as an operator on $L^2(\partial B^p)$, we have

$$|J_3| \lesssim \int_{\partial B^p} |b_2|^2 dS \lesssim ||b||^2_{BMO}|B_\epsilon(z_0)|.$$

The integrals J_1, J_2, J_4 and J_5 are easier and can be estimated in a similar fashion. For example, by (4.3), we see that

$$(-\rho(z))^{1-\delta/2}|\frac{\partial}{\partial z_j}\overline{H^*\overline{b_2}}(z)| \lesssim |z_j|^{p_j} \int_{\partial B^p_r} \frac{|b_2(\zeta)|dS(\zeta)}{|v(z,\zeta)-\rho(z)|^{n-\delta/2}}$$

and hence by (2.2),

$$\begin{aligned} |J_4| &\lesssim \int_{B^p} \frac{1}{(-\rho(z))^{1-\delta}} \left(\int_{\partial B^p_r} \frac{|b_2(\zeta)|dS(\zeta)}{|v(z,\zeta)-\rho(z)|^{n-\delta/2}}\right)^2 dV(z) \\ &\lesssim \int_{\partial B^p_r} |b_2(\zeta)|^2 \left(\int_{B^p} \frac{dV(z)}{(-\rho(z))^{1-\delta}|v(z,\zeta)-\rho(z)|^{n-\delta/2}}\right) dS(\zeta) \\ &\lesssim \int_{\partial B^p_r} |b_2(\zeta)|^2 dS(\zeta) \lesssim ||b||^2_{BMO}|B_\epsilon(z_0)|. \end{aligned}$$

Finally to estimate the contribution from b_3 we observe that if $z \in Q_\epsilon(z_0)$ then, as in the proof of Theorem 2, we have

$$|\frac{\partial}{\partial z_j} H^* b_3(z)| \lesssim |z_j|^{p_j-1} \int_{B^c_{C_\epsilon}(z_0)} \frac{|b(\zeta)-b_{B_{C_\epsilon}}|dS(\zeta)}{|v(z,\zeta)-\rho(z)|^{n+1}} \lesssim |z_j|^{p_j-1}\frac{||b||_{BMO}}{\epsilon}.$$

Hence

$$\int_{Q_\epsilon(z_0)} -\rho|\partial H^* b_3|^2_\omega dV \lesssim \frac{||b||^2_{BMO}}{\epsilon}|Q_\epsilon(z_0)| \lesssim ||b||^2_{BMO}|B_\epsilon(z_0)|.$$

This concludes the proof of Lemma 7, and hence also of Theorem 3. □

Using (1.4) and (4.2) we can now easily prove Theorem 4.

Proof of Theorem 4. — We need to prove that $H^1_{\mathrm{at}} = H^1$ with equivalent norms. To see that H^1_{at} can be continuously embedded in H^1 amounts to prove that H takes $\mathcal{H}^1_{at}$ boundedly into H^1. Since

$$\int_{B^p_r} \left| H\left(\sum_j \lambda_j a_j\right)\right| dS \le \sum_j |\lambda_j| \int_{\partial B^p_r} |Ha_j| dS,$$

this follows from

LEMMA 8. — $||Ha||_{H^1} \lesssim 1$ *for all atoms a.*

Proof. — If $\operatorname{supp} a \subset B_\epsilon(z_0)$ and $z \notin B_{C\epsilon}(z_0)$ where C is sufficiently large, then by Lemma 4 and (1.1)–(1.3)

$$\begin{aligned}
|Ha(z)| &= \left|\int_{\partial B^p} \frac{a(\zeta)}{v(\zeta,z)^n} dS(\zeta)\right| \\
&= \left|\int_{\partial B^p} \left(\frac{1}{v(\zeta,z)^n} - \frac{1}{v(z_0,z)^n}\right) a(\zeta) dS(\zeta)\right| \\
&= \left|\int_{B_\epsilon(z_0)} \frac{v(z_0,z)^n - v(\zeta,z)^n}{v(\zeta,z)^n v(z_0,z)^n} a(\zeta) dS(\zeta)\right| \\
&\lesssim \int_{B_\epsilon(z_0)} \frac{|v(z_0,z) - v(\zeta,z)|}{d(\zeta,z)^{n+1}} |a(\zeta)| dS(\zeta) \lesssim \frac{\epsilon^\delta}{d(z_0,z)^{n+\delta}}.
\end{aligned}$$

Thus by (2.1)

$$\int_{B^c_{C\epsilon}(z_0)} |Ha(z)| dS(z) \lesssim \epsilon^\delta \int_{B^c_{C\epsilon}(z_0)} \frac{dS(z)}{d(z_0,z)^{n+\delta}} \lesssim 1.$$

Since H is bounded on L^2 we also have

$$\begin{aligned}
\int_{B_{C\epsilon}(z_0)} |Ha(z)| dS(z) &\lesssim \left(\int_{B_{C\epsilon}(z_0)} dS(z)\right)^{1/2} \left(\int_{B_{C\epsilon}(z_0)} |Ha(z)|^2 dS(z)\right)^{1/2} \\
&\lesssim (C\epsilon)^{n/2} \left(\int_{B_\epsilon(z_0)} |a(z)|^2 dS(z)\right)^{1/2} \lesssim 1.
\end{aligned}$$

□

The hard part in Theorem 4 is to see that $H^1 \subset H^1_{\mathrm{at}}$. Thus consider $f \in H^1$. Since $f = Hf$ it is enough to prove that f, as a function on the boundary, belongs to $\mathcal{H}^1_{at}$ and that $||f||_{at} \lesssim ||f||_{H^1}$. To see this, take an approximating sequence $f_n \in C^1(\overline{B^p}) \cap H^1(B^p)$ such that $||f - f_n||_{H^1} \to 0$. Since $f_n \in C^1(\partial B^p)$ we have $f_n \in \mathcal{H}^1_{at}$ and hence by (4.2) and the duality (1.4) we get

$$||f_n||_{at} = \sup_{||b||_{BMO} \le 1} \left|\int_{\partial B^p} f_n \bar{b} dS\right| \lesssim ||f_n||_{H^1}.$$

It follows (by applying this inequality to $f_n - f_m$) that f_n is a Cauchy sequence in $\mathcal{H}^1_{at}$, and since $\mathcal{H}^1_{at}$ is a Banach space, f_n converges to some $g \in \mathcal{H}^1_{at}$. As $f_n \to f$ in L^1 and $f_n \to g$ in L^1 ($\mathcal{H}^1_{at}$ is continuously embedded in L^1) we must have $f = g$ and by above $||f||_{at} \lesssim ||f||_{H^1}$. This completes the proof of Theorem 4. □

5. The factorization theorem.

In this section we prove Theorem 5. If $F \in H^1$, then by Theorem 4 we have $F = \sum_k \lambda_k H a_k$, where $\sum_k |\lambda_k| \sim \|F\|_{H^1}$ and a_k are atoms. Therefore it suffices to prove the factorization for holomorphic atoms. This in turn follows from the L^2-boundedness and the existence of a good support function.

Suppose that a is an atom with support in $B_\epsilon(z_0)$. Then set $G(z) = (v(z_0,z)+\epsilon)^\alpha Ha(z)$ and $H(z) = (v(z_0,z)+\epsilon)^{-\alpha}$, where α is to be specified below. Note that $(v(z_0,z)+\epsilon)^\alpha$ is analytic in $z \in B^p$ even if α is not an integer, because $\operatorname{Re} v(z_0,z) > 0$. We have $Ha = G \cdot H$, so we are done if we can prove that $||G||_{H^2}||H||_{H^2} \lesssim 1$. In fact we show that if α is between $n/2$ and $n/2+\delta$ (here δ is the same as in Lemma 4) then

$$(5.2) \qquad ||G||_{H^2} \lesssim \epsilon^{\alpha-\frac{n}{2}}$$

and

$$(5.3) \qquad ||H||_{H^2} \lesssim \epsilon^{-\alpha+\frac{n}{2}}.$$

The estimate (5.3) is easy. We have

$$\int_{B_\epsilon(z_0)} \frac{dS(z)}{|v(z_0,z)+\epsilon|^{2\alpha}} \lesssim \frac{1}{\epsilon^{2\alpha}}|B_\epsilon(z_0)| \sim \epsilon^{n-2\alpha},$$

and by (2.1)

$$\int_{B_\epsilon^c(z_0)} \frac{dS(z)}{|v(z_0,z)+\epsilon|^{2\alpha}} \lesssim \int_{B_\epsilon^c(z_0)} \frac{dS(z)}{d(z_0,z)^{2\alpha}} \lesssim \epsilon^{n-2\alpha},$$

since $\alpha > \frac{n}{2}$. To verify (5.2) recall that, from the proof of Lemma 8, we have

$$|Ha(z)| \lesssim \frac{\epsilon^\delta}{d(z_0,z)^{n+\delta}} \quad \text{if } z \in B_{C\epsilon}^c(z_0),$$

where C is a large constant depending on the constant in the triangle inequality for d (see remark after Lemma 1). Hence by (2.1)

$$\int_{B_{C\epsilon}^c(z_0)} |v(z_0,z)+\epsilon|^{2\alpha}|Ha(z)|^2 dS(z)$$

$$\lesssim \epsilon^{2\delta}\int_{B_{C\epsilon}^c(z_0)} \frac{dS(z)}{d(z_0,z)^{2(n+\delta-\alpha)}} \lesssim \epsilon^{2\alpha-n},$$

since $\alpha < n/2+\delta$. On the other hand, since H is bounded on L^2, we have

$$\int_{B_{C\epsilon}(z_0)} |v(z_0,z)+h|^{2\alpha}|Ha(z)|^2dS(z) \lesssim \epsilon^{2\alpha}\int_{B_{C\epsilon}(z_0)}|Ha(z)|^2dS(z)$$
$$\lesssim \epsilon^{2\alpha}\int_{B_\epsilon(z_0)}|a(\zeta)|^2dS(\zeta) \lesssim \epsilon^{2\alpha-n},$$

which proves (5.2). □

BIBLIOGRAPHY

[AC] M. ANDERSSON, H. CARLSSON, Wolff type estimates and the H^p Corona problem in strictly pseudoconvex domains, Ark. Mat., 32 (1994), 255–276.

[BC] A. BONAMI, Ph. CHARPENTIER, Solutions de l'équation $\bar\partial$ et zéros de la classe de Nevanlinna dans certains domaines faiblement pseudoconvexes, Ann. Inst. Fourier, 32–4 (1982), 53–89.

[BL] A. BONAMI, N. LOHOUÉ, Projecteurs de Bergman et Szegő pour une classe de domaines faiblement pseudo-convexes et estimations L^p, Composito Math., 46 (1982), 159–226.

[C] M. CHRIST, Lectures on Singular Integral Operators, Regional conference Series in Math. N° 77 (1989).

[CRW] R.R. COIFMAN, R. ROCHBERG, G. WEISS, Factorization theorems for Hardy spaces in several variables, Annals of Math., 103 (1976), 611–635.

[CW] R.R. COIFMAN, G. WEISS, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83 (1977), 569–643.

[F] C. FEFFERMAN, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Inventiones Math., 26 (1974), 1–66.

[FS] C. FEFFERMAN, E.M. STEIN, H^p spaces of Several Variables, Acta Mathematica, 129 (1972), 137–193.

[H] L. HÖRMANDER, L^p Estimates for (Pluri-) Subharmonic Functions, Math. Scand., 20 (1967), 65–78.

[Ha] T. HANSSON, On Representation Theorems for Hardy Spaces in Several Complex Variables, thesis (1997).

[KL] S.G. KRANTZ, S-Y. LI, Duality Theorems for Hardy and Bergman spaces on Convex Domains of Finite Type in $\mathbb{C}^n$, Ann. Inst. Fourier, 45–5 (1995), 1305–1327.

[KS] N. KERZMAN, E.M. STEIN, The Szegő kernel in terms of Cauchy-Fantappie kernels, Duke Math. Jour., 45, N° 2 (1978), 197–224.

[McN] J. MCNEAL, Estimates on the Bergman kernels of convex domains, Adv. Math., 109, N° 1 (1994), 108–139. Journal of functional analysis, 108 (1992), 361–373.

[McNS] J. MCNEAL, E.M. STEIN, Mapping properties of the Bergman projection on convex domains of finite type, Duke Math. Jour., 73, N° 1 (1994), 177–199.

[NRSW] A. NAGEL, J.P. ROSAY, E.M. STEIN, S. WAINGER, Estimates for the Bergman and Szegő kernels in $\mathbb{C}^2$, Ann. Math., 129 (1989), 113–149.

[R1] R.M. RANGE, On Hölder Estimates for $\bar{\partial}u = f$ on Weakly Pseudoconvex Domains, Several Complex Variables (Cortona, 1976/1977), 247–267.

[R2] R.M. RANGE, Holomorphic Functions and Integral Representations in Several Complex Variables, Springer-Verlag, 1986.

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