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ALCIDES LINS NETO

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A NOTE ON PROJECTIVE LEVI FLATS AND MINIMAL SETS OF ALGEBRAIC FOLIATIONS

by Alcides LINS NETO

1. Introduction.

A non trivial minimal set of a singular foliation $\mathcal{F}$ on a complex compact manifold M, is a closed set $\mathcal{M} \subset M$ with the following properties:

a) $\mathcal{M}$ is invariant for $\mathcal{F}$.

b) $\mathcal{M} \neq \varnothing$.

c) $\mathcal{M}$ does not contain singular points of $\mathcal{F}$.

d) $\mathcal{M}$ is minimal with respect to properties a), b) and c).

We shall use the abbreviation n.t.m.s. to denote "non trivial minimal set".

In [CLS1] the problem of the existence or not of n.t.m.s. for codimension one foliations of $\mathbb{CP}^2$ was studied. In particular it was proved that for any $k \geqslant 2$, the space of foliations of degree k contains an open non empty set, say A_k, such that any foliation $\mathcal{F} \in A_k$ has no n.t.m.s. It was proved also that if a foliation has an algebraic leaf then it has no n.t.m.s. Since foliations of degree 0 or 1 have always algebraic leaves, they cannot have n.t.m.s. In general, the question of the existence of foliations in $\mathbb{CP}^2$ having n.t.m.s. remains open. Concerning this problem we will prove in §2.1 the following result:

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THEOREM 1. — *Codimension 1 foliations on $\mathbb{CP}^n$, $n \geqslant 3$, have no n.t.m.s..*

In fact Theorem 1 will be a consequence of the following result:

THEOREM 1′. — *Any closed invariant set of a holomorphic codimension one foliation of $\mathbb{CP}^n, n \geqslant 3$, contains a singularity of the foliation.*

Another problem that we will consider is the existence of Levi flats in projective spaces. Let M be a complex manifold of complex dimension n and L be a C^1 submanifold of real codimension 1. Given $p \in L$, the tangent space $T_p(L)$ contains an unique complex subspace of complex dimension $n-1$ that we shall denote C_p. This defines a distribution $\mathcal{C}$ on L.

We say that L is a Levi flat if the distribution $\mathcal{C}$ is integrable. The integrability of $\mathcal{C}$ implies that L has a C^{k-1} foliation $\mathcal{F}$, whose leaves are tangent to the subspaces $C_p, p \in L$. Since the subspaces C_p are complex the leaves of $\mathcal{F}$ are holomorphic immersed submanifolds of complex dimension $n-1$.

In §2.2 we shall prove the following result:

THEOREM 2. — *For $n \geqslant 3$ there are no real analytic Levi flats on $\mathbb{CP}^n$.*

In §3 we will generalize Theorem 2. In order to state the main result that will be used, we consider the following situation:

Let M be a holomorphic manifold of complex dimension $n \geqslant 2$. Let V be an open set of M satisfying the following properties:

a) All connected components of V are Stein.

b) The closure $\overline{V}$ of V, is compact and connected.

We will denote by K the boundary $\overline{V} \setminus V$ of V. Given a neighborhood U of K, $0 \leqslant j \leqslant 2n$, and $p \in K$, let

$$(*) \qquad h_j : H_j(U,\mathbb{Z}) \longrightarrow H_j(\overline{V},\mathbb{Z}) \text{ and } i_j : \Pi_j(U,p) \longrightarrow \Pi_j(\overline{V},p)$$

be the homomorphisms induced by the inclusion $i : U \longrightarrow \overline{V}$ in the homology and homotopy groups respectively. The following result is a kind of generalization of Lefschetz Theorem on hyperplane sections (cf. [M]):

THEOREM 3. — *For any neighborhood A of K in $\overline{V}$ there is a neighborhood U of K such that $U \subset A$ and h_j and i_j as in (*) are isomorphisms for $j \leqslant n-2$ and are onto for $j = n-1$. In particular we have the following:*

(i) *K is connected.*

(ii) *If K is a C^1 real submanifold of M then the homomorphisms below are isomorphisms for $j \leqslant n-2$ and are onto for $j = n-1$:*

$$h_j : H_j(K,\mathbb{Z}) \longrightarrow H_j(\tilde{V},\mathbb{Z}) \text{ and } i_j : \Pi_j(K,p) \longrightarrow \Pi_j(\tilde{V},p).$$

(iii) *If K is a C^1 real submanifold of M and $n \geqslant 3$ then $\Pi_1(K,p)$ and $\Pi_1(\overline{V},p)$ are isomorphic.*

It is not difficult to see that Lefschetz Theorem on hyperplane sections is a consequence of Theorem 3. Another consequence is the following:

COROLLARY — *Let M be a compact complex manifold of complex dimension $n \geqslant 3$, with finite fundamental group. Then M cannot contain a real analytic Levi flat L such that all connected components of $M \backslash L$ are Stein.*

The above corollary follows from (iii) of Theorem 3 and Haefliger's Theorem, which says that a real analytic manifold with finite fundamental group admits no real analytic foliations (cf. [Ha]).

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2. Non trivial minimal sets and Levi flats.

In this section we prove Theorems 1′ and 2. In the proof we will use the following:

THEOREM [T]. — *Let U be an open connected subset of $\mathbb{CP}^n, n \geqslant 2$, satisfying the following properties:*

(i) *The boundary ∂U of U is not empty.*

(ii) *For any $p \in \partial U$ there exists a holomorphic embedding $f_p : B^{n-1} \longrightarrow \mathbb{CP}^n$, where B^{n-1} is the unit ball in $\mathbb{C}^{n-1}$, such that $f_p(0) = p$ and $f_p(B^{n-1}) \cap U = \varnothing$.*

Then U is Stein.

Note that condition (ii) implies that U is locally pseudo-convex, that is, if $p \in \partial U$, then there exists a neighborhood V of p such that $V \cap U$ is

Stein. Therefore Theorem [T] is a consequence of a Theorem of Takeuchi (cf. [T] and [E]).

As a consequence we have the following:

COROLLARY — *Let $K \subset \mathbb{CP}^n, n \geqslant 2$, be either a Levi flat or a closed invariant set of a holomorphic singular foliation of codimension 1 of $\mathbb{CP}^n$, which does not contain singular points of the foliation. Then every connected component of $U = \mathbb{CP}^n \setminus K$ is Stein.*

2.1 Proof of Theorem 1′.

The idea is to prove that the singular set of a holomorphic foliation of codimension 1 on $\mathbb{CP}^n$ must have at least one irreducible component of complex codimension 2. This fact together with the corollary of Theorem [T] implies Theorem 1′. In fact, if a foliation $\mathcal{F}$ on $\mathbb{CP}^n, n \geqslant 3$, had a closed invariant set K such that $K \cap \operatorname{sing}(\mathcal{F}) = \varnothing$, then, by the corollary of Theorem [T], the connected components of the open set $U = \mathbb{CP}^n \setminus K$ would be Stein. On the other hand the singular set of $\mathcal{F}$ contains some irreducible component, say S, of dimension $\geqslant 1$. Since a Stein open set cannot contain a compact analytic subset of dimension greater than zero, this would imply that $S \cap K \neq \varnothing$, which is a contradiction.

We will consider the following situation:

Let $\mathcal{F}$ be a foliation of degree k on $\mathbb{CP}^2$, with finite singular set $\operatorname{sing}(\mathcal{F})$. Given $p \in \operatorname{sing}(\mathcal{F})$ let X be a holomorphic vector field in a neighborhood V of p which is tangent to $\mathcal{F}$. Suppose that the linear part $A = DX(p)$ of X at p is non singular. The Baum-Bott index of $\mathcal{F}$ at p is defined in this case by

$$BB(\mathcal{F}, p) = (\operatorname{trace}(A))^2/\det(A).$$

The Baum-Bott index can be defined for any isolated singularity (cf. [BB] and [MB]), but we will use it only in the non degenerate case.

We will use the following result, wich is a consequence of a theorem of Baum and Bott (cf. [AL-N]):

PROPOSITION 1. — *Let $\mathcal{F}$ be a foliation of degree k on $\mathbb{CP}^2$, with singular set $\operatorname{sing}(\mathcal{F})$ of complex codimension 2. Then*

$$\sum_{p \in \operatorname{sing}(\mathcal{F})} BB(\mathcal{F}, p) = (k+2)^2.$$

In particular $\sum_{p \in \mathrm{sing}(\mathcal{F})} BB(\mathcal{F}, p)$ *is positive for any foliation* $\mathcal{F}$ *on* $\mathbb{CP}^2$.

Now let $\mathcal{G}$ be a codimension one holomorphic foliation on $\mathbb{CP}^n$, $n \geqslant 3$, and suppose by contradiction that all irreducible components of $\mathrm{sing}(\mathcal{G})$ have complex codimension $\geqslant 3$. Let $E \subset \mathbb{CP}^n$ be a 2-plane in general position with respect to $\mathcal{G}$. The 2-plane E is a linear embedding of $i : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^n$, where $E = i(\mathbb{CP}^2)$, with the following properties:

a) E is not containned in any leaf of $\mathcal{G}$.

b) E intersects transversally all smooth strata of $\mathrm{sing}(\mathcal{G})$.

c) Outside $E \cap \mathrm{sing}(\mathcal{G})$, the set of tangencies of $\mathcal{G}$ with E has codimension at most 2 in E.

In Proposition 1 of [CLS2] it is proved that the set of all 2-planes satisfying (a), (b) and (c) is open and dense in the Grassmanian of 2-planes in $\mathbb{CP}^n$. It follows from (a) that the $i^*(\mathcal{G}) = \mathcal{F}$ is a codimension one foliation on $\mathbb{CP}^2$. Condition (b) and the fact that all irreducible components of $\mathrm{sing}(\mathcal{G})$ have codimension $\geqslant 3$, imply that $E \cap \mathrm{sing}(\mathcal{G}) = \varnothing$, so that the singularities of $\mathcal{F}$ correspond to the tangencies of E with some leaves of $\mathcal{G}$, and so (c) implies that $\mathrm{sing}(\mathcal{F})$ is finite.

Let $p \in \mathrm{sing}(\mathcal{F})$. Since $i(p) = p'$ is not a singular point of $\mathcal{G}$, this foliation has a holomorphic first integral, say g, defined in a neighborhood U of p', where $dg(p') \neq 0$. Let $f = g \circ i$. Observe that f is a local first integral of $\mathcal{F}$. Since p' is a tangency of $\mathcal{G}$ with E, we must have $df(p) = 0$, so that p is an isolated singularity of f and of $\mathcal{F}$. We say that p' is a tangency of Morse type if p is a Morse singularity for f, that is in some coordinate system (x, y) around p such that $x(p) = y(p) = 0$ we have $f(x, y) = f(p) + xy$. If this is the case, then the foliation $\mathcal{F}$ is defined in a neighborhood of $p = (0, 0)$ by the vector field $x\partial/\partial x - y\partial/\partial y$, so that $BB(\mathcal{F}, p) = 0$.

Now observe that the 2-plane E can be deformed a little bit to a 2-plane E' in such a way that all tangencies of E' with $\mathcal{G}$ are of Morse type. This assertion is an easy consequence of the following facts:

(i) Morse type singularities are stable by small perturbations.

(ii) Let $g : B^n(0, 2) \longrightarrow \mathbb{C}$ be a holomorphic function, where $g(0) = 0$ and $dg(0) \neq 0$, say $\partial g/\partial x_n(0) \neq 0$. Let $f(x, y) = g(x, y, 0, \ldots, 0)$. Assume that $df(0, 0) = 0$ and that $df(x, y) \neq 0$ for $(x, y) \in B^2(0, 2) \setminus \{0\}$. Then, given $\epsilon > 0$, there are $a, b, c \in \mathbb{C}$, with $|a|$, $|b|$, $|c| < \epsilon$ and such that all singularities of $h(x,y) = g(x,y,\ldots,ax+by+c)$ in $B^2(0,1)$ are of Morse type.

We leave the details of the proof for the reader.

Finally, observe that we have obtained a foliation $\mathcal{F}' = \mathcal{G}|_{E'}$ such that for any $p \in \text{sing}(\mathcal{F}')$ we have $BB(\mathcal{F}',p) = 0$. This contradicts Proposition 1, so that $\text{sing}(\mathcal{G})$ must have some component of codimension 2.

2.2. Proof of Theorem 2.

The idea is to use Theorem 1′, and the following result:

THEOREM 4 . — *Let L be a real analytic Levi flat in $\mathbb{CP}^n, n \geqslant 2$. Then there exists a holomorphic codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ such that L is $\mathcal{F}$-invariant.*

Clearly Theorem 4 follows from the corollary of Theorem [T] and of the following lemmas:

LEMMA 1. — *Let M be a complex manifold and $L \subset M$ be a real analytic Levi flat. Then, there exist a neighborhood U of L and a holomorphic codimension one foliation $\mathcal{G}$ on U such that L is $\mathcal{G}$-invariant.*

Observe that, if $\mathcal{F}$ is the foliation on L defined by the integrable distribution $\mathcal{C}$ of complex hyperplanes in L, then $\mathcal{G}|_L = \mathcal{F}$.

LEMMA 2 . — *Let V be a Stein manifold and $K \subset V$ be a compact set such that $U = V \setminus K$ is connected. Then any holomorphic codimension one foliation $\mathcal{G}$ on U, such that* $\text{cod}(\text{sing}(\mathcal{G})) \geqslant 2$, *can be extended to a holomorphic foliation on V.*

2.2.1 - Proof of Lemma 1.

We will use the following fact in the proof:

ASSERTION. — For any $p \in L$ there exists a holomorphic function H, defined in a neighborhood U of p, such that $dH(p) \neq 0$ and $L \cap U = v^{-1}(0)$, where $v = \Im(H)$.

The above assertion is well known and can be proved by using Frobenius Theorem and the results of [To]. Since its proof is not long we will give it at the end of §2.2.1. Let us prove Lemma 1 from the assertion.

Observe first that the assertion implies that for any $p \in L$ there exists a holomorphic coordinate system

$$\phi = (x, y) : U \longrightarrow \mathbb{C}^n, \text{ where } x : U \longrightarrow \mathbb{C}^{n-1} \text{ and } y : U \longrightarrow \mathbb{C}$$

such that $L \cap U = y_2^{-1}(0)$, where $y_2 = \Im(y)$.

It follows from the above observation that it is possible to find a holomorphic atlas of a neighborhood V of L

$$\mathcal{U} = \{U_j, \phi_j = (x_j, y_j)\}_{j \in J} \text{ where } (x_j, y_j) : U_j \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$$

such that

(i) If $i, j \in J$, then $U_{i,j} = U_i \cap U_j$ is connected and homeomorphic to a ball in $\mathbb{C}^n$, if not empty.

(ii) For any $j \in J$ we have $U_j \cap L \neq \varnothing$ and is homeomorphic to a ball in L.

(iii) For any $i, j \in J$ such that $U_{i,j} \neq \varnothing$ then $L \cap U_{i,j} \neq \varnothing$ and is homeomorphic to a ball in L.

(iv) For any $j \in J$ we have $U_j \cap L = \{\Im(y_j) = 0\}$.

Now let $i, j \in J$ be such that $U_i \cap U_j \neq \varnothing$ and consider the change of chart

$$\phi = \phi_{i,j} = \phi_i \circ (\phi_j)^{-1} : \phi_j(U_{i,j}) \longrightarrow \phi_i(U_{i,j}).$$

In order to simplify the notations let us call $x_i = x$, $y_i = y$, $x_j = z$ and $y_j = w$, so that $\phi(x, y) = (z(x, y), w(x, y))$. It follows from (i), (ii) and (iii) above, that the domain of w, $\phi_j(U_{i,j})$, is homeomorphic to a ball and contains a ball $B \subset \mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Moreover, condition (iv) implies that if $(x, t) \in B$ $(t \in \mathbb{R})$, then $w(x, t) \in \mathbb{R}$, so that the holomorphic function $x \mapsto w(x, t)$ (t fixed) must be constant. This implies that the map $(x, t) \in B \mapsto w(x, t)$ does not depend on x. Since w is holomorphic it follows that $(x, y) \mapsto w(x, y)$ does not depends on x.

The above argument implies the coordinate changes $\phi_{i,j}$, are of the form

$$\phi_{i,j}(x_j, y_j) = (x_i(x_j, y_j), y_i(y_j)).$$

Therefore the atlas $\mathcal{U}$ defines a codimension one foliation on the neighborhood $V = \cup_j U_j$ of L.

Proof of the assertion. — Let $\mathcal{F}$ be the foliation on L defined by the distribution of complex hyperplanes $\mathcal{C}$. Since L is real analytic $\mathcal{F}$ is also real analytic. Fix a point $p \in L$ and a holomorphic coordinate system $(\phi = (x,y), U)$ with the following properties:

a) $x = (x_1, \ldots, x_{n-1}) : U \longrightarrow \mathbb{C}^{n-1}$ and $y : U \longrightarrow \mathbb{C}$.

b) $p \in U$ and $\phi(p) = 0$.

c) The surface $\{x = 0\}$ is transversal to the leaves of $\mathcal{F}$.

Condition (c) implies that $L \cap \{x = 0\}$ is a real analytic curve $\gamma(t) = (0, y(t))$. After a holomorphic change of variables in the coordinate y we can suppose that $y(t) = t$, which means that $L \cap \{x = 0\}$ is the real axis in the y-plane. Let F_t be the leaf of $\mathcal{F}$ through $\gamma(t)$. Since F_t is holomorphic it can be written locally as the graph of a function, say $y = \varphi(x,t)$, where φ is real analytic, holomorphic with respect to x and $\varphi(0,t) = t$. If we set $\varphi = u + iv$, where $u = \mathbb{R}(\varphi)$ and $v = \Im(\varphi)$, then the hypersurface L can be defined locally around p by eliminating t in the equation $y_1 - u(x,t) = 0$ $(y = y_1 + iy_2)$, say $t = h(x, y_1)$, and substituting h in $y_2 - v(x,t)$, so that in a neighborhood of p, L is given by $y_2 = v(x, h(x, y_1))$.

Now, let $\varphi(x,t) = \sum_{n=1}^{\infty} a_n(x).t^n$ be the Taylor series in t of φ around (0,0). Extend φ to a neighborhood of (0,0) in $\mathbb{C}^{n-1} \times \mathbb{C}$ by setting $\varphi(x,z) = \sum_{n=1}^{\infty} a_n(x).z^n$, $z \in \mathbb{C}$. Let $F(x,y,z) = y - \varphi(x,z)$. Since $F(0,0,0) = 0$ and $\partial F/\partial z(0,0,0) \neq 0$, by the implicit function theorem, there exists a holomorphic function $H(x,y)$, defined in a neighborhood of (0,0), such that $\varphi(x, H(x,y)) = y$. Finally, observe that L can be defined in a neighborhood of (0,0) by $\Im(H(x,y)) = 0$. We leave the proof of this fact for the reader. This ends the proof of Lemma 1.

2.2.2. Proof of Lemma 2.

Let $f : V \longrightarrow \mathbb{R}$ be a strictly-pluri-subharmonic C^∞ exhaustion of V. Since $\lim_{p \to \infty} f(p) = +\infty$, the sets $M_t = \{p \in V;\ f(p) \leqslant t\}$ are compact and $K \subset M_t$ for $t \geqslant t_0$, so that the foliation $\mathcal{G}$ is defined on $V_t = V \setminus M_t$, for $t \geqslant t_0$. The idea is to prove the following:

(*) Suppose that $\mathcal{G}_t$ is a codimension one holomorphic foliation defined on V_t, such that $\operatorname{cod}(\operatorname{sing}(\mathcal{G}_t)) \geqslant 2$. Then there exists $\epsilon > 0$ such that $\mathcal{G}_t$ can be extended to a foliation on $V_{t-\epsilon}$.

Since $m = \inf\{f\} > -\infty$, it is clear that (*) implies Lemma 2. On the other hand, since $f^{-1}(t)$ is compact, it is not difficult to see that (*) is a consequence of the following:

(**) Let $\mathcal{G}_t$ be as in (*) and $p \in f^{-1}(t)$. Then $\mathcal{G}_t$ can be extended to $V_t \cup W$, where W is a neighborhood of p.

In order to prove (**) we use the following results:

LEMMA A (cf. [ST]). — *Given $p \in f^{-1}(t)$ there exists a biholomorphism $\phi : W \longrightarrow W' \subset \mathbb{C}^n$, where W is a neighborhood of p, and a Hartog's domain $H \subset W'$ such that $\phi^{-1}(H) \subset V_t$ and $p \in \phi^{-1}(\widehat{H})$.*

A Hartog's domain is an open set H of $\mathbb{C}^n, n \geqslant 2$ of the form $H = (U' \times \Delta(r)) \cup (U \times (\Delta(r) \setminus \overline{\Delta}(r')))$, where U and U' are open sets of $\mathbb{C}^{n-1}$, U connected, $U \supset U' \neq \varnothing$, $\Delta(r)$ is the disk of radius r in $\mathbb{C}$ and $0 < r' < r$. The set $\widehat{H}$ is, by definition $U \times \Delta(r)$. We observe that in Lemma A, we can suppose that U and U' are polydisks.

LEVI'S THEOREM (cf. [S]). — *Let H be a Hartog's domain and f be a meromorphic function on H. Then f can be extended to a meromorphic function on $\widehat{H}$.*

Let us finish the proof of Lemma 2. Consider the biholomorphism ϕ as in Lemma A and let $\mathcal{G}'$ be the restriction of $\phi_\star(\mathcal{G}_t)$ to $H \subset \mathbb{C}^n$. We will prove that there exists an integrable holomorphic 1-form ω on $\widehat{H}$ such that $\mathcal{G}'$ is defined by the differential equation $\omega \mid_H = 0$. This will prove the lemma.

The foliation $\mathcal{G}'$ is defined locally by integrable 1-forms, so that there exist a covering of H by open sets $\mathcal{U} = (U_j)_{j \in J}$ and collections $(\omega_j)_{j \in J}$, $(h_{i,j})_{U_{i,j} \neq \varnothing}$ ($U_{i,j} = U_i \cap U_j$), such that

a) ω_j is an integrable 1-form on U_j such that $\operatorname{cod}(\operatorname{sing}(\omega_j)) \geqslant 2$ and $\mathcal{G}' \mid_{U_j}$ is defined by $\omega_j = 0$.

b) If $U_{i,j} \neq \varnothing$ then $h_{i,j} \in \mathcal{O}^\star(U_{i,j})$ and on $U_{i,j}$ we have $\omega_i = h_{i,j} \,.\, \omega_j$.

Since $\widehat{H} \subset \mathbb{C}^n$ we can write

$$\omega_j = \sum_{i=1}^{n} g_i^j . dx_i \ , \ \text{where} \ \ g_i^j \in \mathcal{O}(U_j).$$

Observe that condition (b) implies that if $U_{j,\ell} \neq \varnothing$ then

$$(*) \qquad g_i^j = h_{j,\ell} . g_i^\ell \ \text{ for all } \ i = 1, \ldots, n.$$

Since $\widehat{H}$ is connected, it follows that for some $i \in \{1, \ldots, n\}$ we must have $g_i^j \not\equiv 0$ for all $j \in J$. We suppose $i = n$, so that g_i^j / g_n^j defines a meromorphic function f_i^j on U_j for all $i = 1, \ldots, n-1$. Now, $(*)$ implies that if $U_{j,\ell} \neq \varnothing$, then $f_i^j = f_i^\ell$ on $U_{j,\ell}$, so that, for all $i = 1, \ldots, n-1$, there exists a meromorphic function f_i on H, such that $f_i\, |_{U_j} = f_i^j$. It follows from Levi's Theorem that f_i can be extended to a meromorphic function on $\widehat{H}$, which we call still f_i.

Consider the meromorphic 1-form η defined on $\widehat{H}$ by

$$\eta = dx_n + \sum_{i=1}^{n-1} f_i \,.\, dx_i$$

Since $\widehat{H}$ is a polydisk, it follows that there exists $h \in \mathcal{O}(\widehat{H})$ and a holomorphic 1-form ω on $\widehat{H}$ such that $\mathrm{cod}(\mathrm{sing}(\omega) \geqslant 2$ and $\eta = \dfrac{1}{h}.\omega$. It is not difficult to see that for all $j \in J$ we have $\omega\, |_{U_j} = g_j.\omega_j$ for some $g_j \in \mathcal{O}^*(U_j)$, so that ω is integrable and the foliation defined by $\omega = 0$ on $\widehat{H}$ extends $\mathcal{G}'$. This ends the proof of Lemma 2.

3. Theorem 3 and its corollary.

In this section we prove Theorem 3 and its corollary.

Let V, $\overline{V}$ and K$=\overline{V} \setminus V$, be as in Theorem 3. Fix a neighborhood A of K in $\overline{V}$. We need a lemma.

LEMMA 3. — *There exist neighborhoods U_1 and U of K in $\overline{V}$ and a flow $\varphi : \mathbb{R} \times \overline{V} \longrightarrow \overline{V}$ such that*

(a) $U_1 \subset \overline{U}_1 \subset U \subset A$.

(b) $\varphi_t(p) = p, \ \forall\ p \in \overline{U_1}$, *where* $\varphi_t(p) = \varphi(t,p)$.

(c) $\varphi\, |_{\mathbb{R}\times V}$ *is* C^∞. *Let X be the C^∞ vector field on V which generates φ.*

(d) *All singularities of X in $V \setminus \overline{U}_1$ are hyperbolic.*

(e) *If $p \in V \setminus \overline{U_1}$ is a singularity of X, then its stable manifold, $W^S(p)$, has (real) dimension at most n.*

(f) *If $p \in V \setminus U$ is a singularity of X, then $W^S(p)$ is contained in $V \setminus U$.*

(g) *If $F \subset V$ is a compact subset of $\overline{V}$ such that $F \cap W^S(q) = \varnothing$ for any singularity q of X in $V \setminus U$, then there exists $s_0 > 0$ such that $\varphi_t(F) \subset U$ for $t \geqslant s_0$.*

3.1. Proof of Lemma 3.

Let N be a connected component of V. Since N is Stein, there exists on N a C^∞ strictly-pluri-subharmonic exhaustion, say f. We will use the following facts:

(1) The set of exhaustions of N, is open in the Whitney C^0 topology of $C^0(N,\mathbb{R})$.

(2) The set of C^2 Morse functions is open and dense in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

(3) The set of C^∞ functions is dense in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

(4) The set of strictly-pluri-subharmonic functions of class C^2 on N, say $\mathrm{SPSH}^2(N,\mathbb{R})$, is open in the Whitney C^2 topology of $C^2(N,\mathbb{R})$.

The definition of the Whitney topology and the proofs of (1), (2) and (3) can be found in [H]. Let us prove (4).

Let $h \in C^2(N,\mathbb{R})$ and let $\mathcal{L}_h$ be the Levi form of h, which is defined in a holomorphic coordinate system $(x = (x_1,\dots,x_n), U)$ of N by

$$\mathcal{L}_h(p).v \;=\; \sum_{i,j} \partial^2 h/\partial x_i \partial \overline{x_j}(x)\ .v_i.\overline{v_j}\ ,$$
$$\text{where } (p,v) \in TU \text{ and } (x,(v_1,\dots,v_n)) = Tx(p,v).$$

By definition, h is strictly-pluri-subharmonic if, and only if, $\mathcal{L}_h(p).v > 0$ for all $(p,v) \in TN$, with $v \neq 0$. Let us fix a riemannian metric g on N with norm $|\,.\,|$. Given $h \in \mathrm{SPSH}^2(N,\mathbb{R})$ define $k_h : N \longrightarrow \mathbb{R}$ by

$$k_h(p) = \inf\{\mathcal{L}_h(p).v\ ;\ v \in T_pN \text{ and } \mid v\mid_p = 1\}$$

so that $k_h > 0$ on N. Now, for a fixed $h_0 \in \mathrm{SPSH}^2(N,\mathbb{R})$ let

$$\mathcal{U} \;=\; \{h \in C^2(N,\mathbb{R});\ \mid \mathcal{L}_h(p).v - \mathcal{L}_{h_0}(p).v \mid < 1/2\ .k_{h_0}(p)$$
$$\forall\ (p,v) \in N \text{ with } \mid v\mid_p = 1\}.$$

Since $\mathcal{L}_h$ and k_h depend only on the second jet of h, it follows from the definition of the Whitney topology that $\mathcal{U}$ is a neighborhood of h_0 in $C^2(N,\mathbb{R})$. Moreover if $h \in \mathcal{U}$, then $k_h > 0$, so that $h \in \mathrm{SPSH}^2(N,\mathbb{R})$, which proves (4).

It follows from (1), (2),(3) and (4) that we can suppose that f is a Morse function. Let g be the riemannian metric on N fixed before. Let

$Y = \mathrm{grad}_{\mathrm{g}}(f)$, which is defined by $df_p.v = \mathrm{g}_p(v, Y(p))$, for any $(p,v) \in TN$. Let Y_t be the flow of Y.

The following facts are well known (cf. [M] and [Sm]):

(5) f is strictly increasing along non singular orbits of Y.

(6) The singularities of Y are the points p of N for which $df_p = 0$.

(7) Given p such that $df_p = 0$, there exists a C^∞ coordinate system $x = (x_1, \dots, x_{2n})$ around p such that

$$(*) \qquad f(x) \ = \ f(\mathrm{p}) + \sum_{j=1}^{2n} b_j.(x_j)^2 \ + \ \text{h.o.t.} \ , \quad \text{where} \ \ b_j \in \mathbb{R} \setminus \{0\}.$$

The number $i(p)$ of negative $b_{j's}$ in $(*)$, is an invariant of f and p and is called the Morse index of f at p.

(8) If p and $i(p)$ are as in (7), then p is a hyperbolic singularity of Y and its stable manifold, $W^s(p)$, has (real) dimension $i(p)$.

We need the following:

Assertion. — For any singularity p of Y, we have $i(p) \leqslant n$.

Proof. — It follows from Theorem 1.4.15, pg. 29 of [HL], that there exists a holomorphic coordinate system $(z = (z_1, \dots, z_n), W)$ around p, such that $z(p) = 0$ and the expression of f in W is of the form

$$(*) \qquad f(z) \ = f(p) + \sum_{j=1}^{n} [(1 + a_j).x_j^2 + (1 - a_j).y_j^2] \ + \text{h.o.t.},$$

where $z_j = x_j + i\ y_j$ and $1 \neq a_j \geqslant 0$ (because f is a Morse function). This implies the assertion.

Let us consider now the open set $A' = A \cap N$. Since $\overline{V}$ is compact, it follows that $N \setminus A'$ is compact, which implies that there exists $t_0 \in \mathbb{R}$ such that $f^{-1}((-\infty, t]) \supset N \setminus A'$, for $t \geqslant t_0$. Fix $t_2 > t_1 > t_0$ and let $N_j = f^{-1}(t_j, +\infty)$, $j = 1, 2$, so that $A' \supset \overline{N_1} \supset N_1 \supset \overline{N_2}$. We choose t_1 in such a way that $f^{-1}(t_1)$ does not contain singularities of Y, which implies that Y is transverse to $f^{-1}(t_1)$ and points inward N_1. Observe also that Y has finitely many singularities on $N \setminus \overline{N_2}$.

Now, let $p \in N \setminus \overline{N_j}$, $j = 1, 2$ be a singularity of Y. It follows from (5), from the definition of N_j and from the fact that

$$W^s(p) \ = \ \{q \ ; \lim_{t \to +\infty} Y_t(q) = p\}$$

that

(9) $W^s(p) \subset N \setminus \overline{N_j}$, for $j = 1, 2$.

Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞ function such that $\phi(t) = 0$ for $t \geqslant t_2$, $\phi(t) = 1$ for $t \leqslant t_1$ and $\phi(t) > 0$ for $t \in (t_1, t_2)$.

Let X^N be the C^∞ vector field on N defined by $X^N(p) = \phi(f(p)).Y(p)$, and denote by X_t^N its flow. It is not difficult to verify the following facts:

(10) $X_t^N(p) = p$, $\forall\ p \in \overline{N_2}$.

(11) If $p \in N \setminus \overline{N_2}$ and $o_X(p)$, $o_Y(p)$ are the orbits of X^N and Y through p, respectively, then $o_X(p) = o_Y(p) \cap (N \setminus \overline{N_2})$.

(12) f is strictly increasing along non singular orbits of X^N.

(13) The singularities of X^N on $N \setminus \overline{N_2}$ are hyperbolic and are singularities of Y.

(14) If $p \in N \setminus \overline{N_j}$, $j = 1, 2$, is a singularity of X^N , then its stable manifold coincides with $W^s(p)$, the stable manifold of p with respect to Y. In particular (13) is true for X^N.

(15) If $p \in N \setminus \overline{N_1}$ does not belongs to the stable manifold of some singularity of X^N in $N \setminus \overline{N_1}$, then there exists $s_0 > 0$ and a neighborhood W_p of p such that if $t > s_0$, then $X_t^N(W_p) \subset N_1$.

This last assertion follows from the continuity of the flow and from (12).

Let us finish the proof of Lemma 3. Consider the decomposition of V in connected components, $V = \bigcup_{j \in J} N^j$. For each $j \in J$, let f^j be a Morse strictly-pluri-subharmonic exhaustion of N^j. Let $A^j = A \cap N^j$ and $N_2^j \subset N_1^j$ be like N_1 and N_2 considered before. Let $X^{N^j} = X^j$ be a vector field on N^j which satisfies properties (10), ..., (15). Define the flow $\varphi : \mathbb{R} \times \overline{V} \longrightarrow \overline{V}$ by $\varphi(t, p) = p$ if $p \in K$ and $\varphi(t, p) = X_t^j(p)$ if $p \in N^j$. Observe that φ satisfies (c) of Lemma 3.

Since f^j is an exhaustion of N^j for any $j \in J$, it is not difficult to see that $U = \cup_j N_1^j \cup K$ and $U_1 = \bigcup_j N_2^j \cup K$ are neighborhoods of K and satisfy (a) of Lemma 3. Observe that this fact and (10) imply that φ is continuous and satisfies (b), (d) and (f) of Lemma 3. On the other hand property (g) of Lemma 3 can be easily checked from (15). We leave the details for the reader. This ends the proof of Lemma 3.

3.2. Proof of Theorem 3.

Let $V, \overline{V}, K$ and A be as in Theorem 3. Let U, U_1 and φ be as in Lemma 3. Fix $p \in K$ and consider the homomorphisms induced by the inclusion $i : U \longrightarrow \overline{V}$:

$$(*)\ h_q : H_q(U,\mathbb{Z}) \longrightarrow H_q(\overline{V},\mathbb{Z}) \text{ and } i_q : \Pi_q(U,p) \longrightarrow \Pi_q(\overline{V},p).$$

Let us consider first the homotopy case. Consider i_q as above and let us prove that it is onto for $1 \leqslant q \leqslant n-1$. We will consider a class $[g]$ in $\Pi_q(\overline{V},p)$ represented by a continuous map $g : S^q \longrightarrow \overline{V}$, where S^q is the unit sphere in $\mathbb{R}^{q+1}$ and $g(e) = p$ for some fixed $e \in S^q$. Let $V_1 = V \setminus \overline{U_1}$ and consider the open set $A = g^{-1}(V_1) \subset S^q$. It follows from standard arguments of differential topology that it is possible to find a continuous map $h : S^q \longrightarrow \overline{V}$ such that

(1) h coincides with g in $S^q \setminus A$.

(2) h is homotopic to g.

(3) h is C^∞ in A.

Let $p_1, \dots, p_m$ be the singularities of X in $V \setminus \overline{U}$, and $W = \bigcup_{j=1}^{m} W^s(p_j)$. Since for all $j = 1, \dots, m$ we have $q + \dim_{\mathbb{R}}(W^s(p_j)) \leqslant 2n-1 < 2n = \dim_{\mathbb{R}}(V)$, it follows from transversality theory that it is possible to find h in such a way that

(4) $h(A) \cap W = \varnothing$, so that $h(S^q) \cap W = \varnothing$ (by (f) of Lemma 3 and (1)).

It follows from (g) of Lemma 3 that there exists $t > 0$ such that $\varphi_t(h(S^q)) \subset U$. Since φ_t is homotopic to the identity of $\overline{V}$, this implies that i_q is onto.

Let us prove that i_q is injective if $1 \leqslant q \leqslant n-2$. Let $[g] \in \Pi_q(U,p)$ be such that $i_q[g] = 0$ and $g : S^q \longrightarrow U$ be a representative of $[g]$. Let $\overline{B^{q+1}}$ be the closed unit ball in $\mathbb{R}^{q+1}$ and $G : \overline{B^{q+1}} \longrightarrow \overline{V}$ be a continuous map such that $G\,|_{S^{q+1}} = g$. Let $A = G^{-1}(V_1)$. It follows from standard arguments of differential topology that it is possible to find a continuous map $H : \overline{B^{q+1}} \longrightarrow \overline{V}$ such that

(5) H coincides with G in $\overline{B^{q+1}} \setminus A$.

(6) H is homotopic to G.

(7) H is C^∞ in A.

Moreover, since for all $j = 1, \ldots, m$ we have $q+1+\dim_{\mathbb{R}}(W^{S}(p_j)) \leqslant 2n-1 < 2n = \dim_{\mathbb{R}}(V)$, it follows from transversality theory that it is possible to find H in such a way that

(8) $H(S^q) \cap W = \varnothing$.

It follows from (g) of Lemma 3 that there exists $t > 0$ such that $\varphi_t(H(\overline{B^{q+1}})) \subset U$. This implies that $[g] = 0$ in $\Pi_q(U,p)$, so that i_q is injective.

The proof in the homology case is similar. Let us sketch it.

We will work in the singular homology theory, with the notations of [G]. If $c = \sum_{j=1}^{k} \nu_j \sigma_j$ is a q-chain in $\overline{V}$ then each simplex σ_j is a continuous map from the standard simplex

$$\Delta_q = \left\{p \in \mathbb{R}^q;\ p = \sum_{i=0}^{q} t_i.E_i\ ,\ 0 \leqslant t_i \leqslant 1, \sum_i t_i \leqslant 1\right\}$$

into $\overline{V}$. We will use the notation $\operatorname{supp}(c) = \bigcup_j \sigma_j(\Delta_q)$. A sub-simplex σ_j^I, $I = (i_0 < i_1 < \ldots < i_r)$, $r \leqslant q$, is, by definition, the restriction of σ_j to Δ_q^I, where

$$\Delta_q^I = \left\{p \in \mathbb{R}^q;\ p = \sum_{j=0}^{r} t_j.E_{i_j}\ ,\ 0 \leqslant t_j \leqslant 1, \sum_j t_j \leqslant 1\right\}.$$

We will say that c is C^∞ if for all j and all I the restriction of σ_j^I to the open subsets $(\sigma_j^I)^{-1}(V)$ of Δ_q^I is C^∞.

Let us prove that h_q is onto if $0 \leqslant q \leqslant n-1$. Let $[c] \in H_q(\overline{V},\mathbb{Z})$ and $c = \sum_{j=1}^{k} \nu_j \sigma_j$ be a representative of $[c]$. It follows from standard arguments of homology theory that we can suppose that all σ_j^I are C^∞ and transversal to W. Since $0 \leqslant q \leqslant n-1$ this implies that $W \cap \operatorname{supp}(c) = \varnothing$. On the other hand, (g) of Lemma 3 implies that there exists $t > 0$ such that $\varphi_t(\operatorname{supp}(c)) \subset U$. Since φ_t is homotopic to the identity, it follows that h_q is onto.

Let us prove that h_q is injective if $0 \leqslant q \leqslant n-2$. Let $[c] \in H_q(U,\mathbb{Z})$ be such that $h_q[c] = 0$. Let c be a C^∞ representative of [c]. It follows from standard arguments of homology theory that there exists a C^∞ (q+1)-chain $c' = \sum_j \nu_j.\sigma_j$ on $\overline{V}$, such that $\partial c' = c$ and all σ_j^I are transversal to W. Since q+1$\leqslant n-1$ this implies that $W \cap \text{supp}(c') = \varnothing$. On the other hand, (g) of Lemma 3 implies that there exists $t > 0$ such that $\varphi_t(\text{supp}(c')) \subset U$. Since φ_t is homotopic to the identity, it follows that h_q is injective.

It remains to prove (i) and (ii) of Theorem 3 (since (iii) follows from (ii)).

Proof of (i). — Since $n \geqslant 2$, it follows from the theorem that there exists a a collection $\{U_n\}_{n=1}^{\infty}$ of open neighborhoods of K such that $\cap_n U_n = K$ and for all n

$$h_0 : H_0(U_n, \mathbb{Z}) \longrightarrow H_0(\overline{V}, \mathbb{Z})$$

is an isomorphism. On the other hand, this implies that $\overline{U_n}$ is compact and connected for all n. Therefore K is connected.

Proof of (ii). — Let us suppose now that K is a C^1 submanifold of M. Let B be a tubular neighborhood of K with projection $\pi : B \longrightarrow K$. We have two possibilities:

1st - $B \setminus K$ has one connected component.

2nd - $B \setminus K$ has two connected components, say B_1 and B_2 (if K has real codimension 1).

In the first case $\overline{V} \cap B = B$ and B is a neighborhood of K in $\overline{V}$. In the second case we have two possibilities: either $\overline{V} \cap B = B$, or $\overline{V} \cap B = B_j \cup K$ for $j = 1$ or 2 and $B_j \cup K$ is a neighborhood of K in $\overline{V}$. In any case we will set $A = \overline{V} \cap B$, so that A is a neighborhood of K in $\overline{V}$. It is well known that the homomorphisms induced by the inclusion $K \longrightarrow A$

$$h_q' : H_q(K, \mathbb{Z}) \longrightarrow H_q(A, \mathbb{Z}) \text{ and } i_q' : \Pi_q(K, p) \longrightarrow \Pi_q(A, p)$$

are isomorphisms for all q.

On the other hand there exists a neighborhood U of K in $\overline{V}$ such that $U \subset A$ and the homomorphisms induced by the inclusion $U \longrightarrow \overline{V}$

$$h_q'' : H_q(U, \mathbb{Z}) \longrightarrow H_q(\overline{V}, \mathbb{Z}) \text{ and } i_q'' : \Pi_q(U, p) \longrightarrow \Pi_q(\overline{V}, p)$$

are onto if $q \leqslant n-1$ and injectives if $q \leqslant n-2$. It is not difficult to see that this implies (ii). This finishes the proof of Theorem 3.

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Alcides LINS NETO,
Instituto de Matematica Pura e Aplicada
Estrada Dona Castorina, 110
Horto
Rio de Janeiro (Brasil).
alcides@impa.br.