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SIMPLICITY OF NERETIN'S GROUP OF SPHEROMORPHISMS

by Christophe KAPOUDJIAN

Introduction.

Answering a question of I.M. Gelfand on the existence of analogues of heighest-weight representations of the diffeomorphism group of the circle in the case of p-adic transformation groups, Yu.A. Neretin constructed a group of transformations of the boundary $\partial\mathcal{T}_p$ of the regular tree $\mathcal{T}_p$ (*cf.* [12] and [13]): the group N_p of spheromorphisms (§1). When p is a prime integer, the boundary $\partial\mathcal{T}_p$ is naturally homeomorphic to the projective line on the field of p-adic numbers, and in any case, to a Cantor set.

Roughly speaking, a spheromorphism is a transformation induced in the boundary by a "piecewise" tree automorphism. The spheromorphism group is generated by two groups: on the one hand a Higman-Thompson group (§2), which is countable and almost-acts on the tree, respecting a local orientation of the edges, and on the other hand, the tree automorphism group (§3).

Exploiting simplicity theorems known for the generating two groups, and adapting some arguments of a simplicity theorem of Epstein, we finally prove the simplicity of N_p (the analogue of M.R. Herman's theorem on the simplicity of the orientation-preserving diffeomorphism group of the circle, *cf.* [7]), and of some of its subgroups (§4):

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THEOREM. — *For each integer $p \geq 2$, the spheromorphism group N_p is simple.*

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1. The Neretin group of spheromorphisms.

1.1. Let $\mathcal{T}_n$ be the regular tree whose vertices have valence $n+1$, with $n \geq 2$, and $\partial\mathcal{T}_n$ its boundary, or set of "ends", see e.g. [14] or [6].

We may describe the boundary $\partial\mathcal{T}_n$ as a compact ultrametric space: choose a vertex o of the tree $\mathcal{T}_n$. Each end is defined by a unique chain (i.e. a sequence of consecutive vertices $(o = x_0, x_1, \ldots)$ with $x_{i+2} \neq x_i$) starting from the origin o. The metric on $\partial\mathcal{T}_n$ is defined in the following way: Let ω, $\omega' \in \partial\mathcal{T}_n$ be respectively represented by the chains $(o = x_0, x_1, \ldots)$ and $(o = x'_0, x'_1, \ldots)$.

• If the intersection of the supports of the chains is reduced to $\{o\}$, then declare the distance between ω and ω' to be equal to 1: $d(\omega, \omega') = 1$.

• If $x_i = x'_i$ for $i = 0, \ldots, k$ and $x_{k+1} \neq x'_{k+1}$, then define $d(\omega, \omega') = \dfrac{n}{n+1} n^{-k}$.

It follows that a closed ball of radius $\dfrac{n}{n+1} n^{-k}$ is the set of all points of $\partial\mathcal{T}_n$ represented by chains containing a fixed finite chain $(o = x_0, x_1, \ldots, x_k)$, and that it is an open set. In fact, $\partial\mathcal{T}_n$ endowed with the metric d is a compact ultrametric space, homeomorphic to a Cantor set.

When p is prime, $\mathcal{T}_p$ is the Bruhat-Tits building of the p-adic Lie group $SL_2(\mathbb{Q}_p)$, just as the Poincaré disk D is the symmetric space of the real group $SL_2(\mathbb{R})$. The boundary $\partial\mathcal{T}_p$, which can be identified with $\mathbb{Q}_p P^1$, the projective line on $\mathbb{Q}_p$, may thus be viewed as the p-adic analogue of the circle.

1.2. Let $\partial\mathcal{T}_n$ still denote the boundary of the tree $\mathcal{T}_n$, $n \geq 2$. The group of spheromorphisms N_n can be defined as the group of transformations of $\partial\mathcal{T}_n$ induced by "piecewise" tree automorphisms:

Take a finite subtree of $\mathcal{T}_n$. Its complementary has finitely many connected components $L_1, \ldots, L_k$, called *branches*, all isomorphic to an infinite n-ary complete rooted tree. A subset ∂L of the boundary is naturally associated to each branch L: it consists of all the ends represented by the chains running over this branch. The k disjoint sets ∂L_j, $j = 1, \dots, k$ cover the boundary. We call $(L_1, \dots, L_k)$ a *broom*.

Remark. — Each ball for the metric d is of the form ∂L, and each ∂L is a finite union of balls. The family $\{\partial L : L \text{ branch}\}$ is a basis of closed-open sets for the topology defined by d.

Let $(L_1, \dots, L_k)$ and $(L'_1, \dots, L'_k)$ be two brooms of $\mathcal{T}_n$, σ a permutation of $\{1, \dots, k\}$. Let $\phi_j : L_j \to L'_{\sigma(j)}$ be a rooted tree isomorphism, $j = 1, \dots, k$. These k mappings induce a bijection $\phi = (\partial\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\dots,k}$ of the boundary. Such a broom appearing in the definition of ϕ is called *ϕ-adapted*, and is obviously not uniquely associated to ϕ. It is clear that the set of all the ϕ's defined by this procedure is a group of homeomorphisms of the boundary.

DEFINITION 1.1 (Spheromorphism group, [13]). — *For each $n \geq 2$, the set of all bijections $\phi = (\partial\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\dots,k}$ of the boundary $\partial\mathcal{T}_n$ is the spheromorphism group of Neretin, and is denoted N_n.*

Remarks. — 1) In view of this description, the automorphism group $\operatorname{Aut}\mathcal{T}_n$ of the tree embeds as a subgroup of N_n. The image of $\operatorname{Aut}\mathcal{T}_n$ in N_n is the set of spheromorphisms which possess an adapted broom with two branches.

2) When p is a prime integer, $\partial\mathcal{T}_p$ is homeomorphic to $\mathbb{Q}_pP^1$, and N_p contains the group An_p of locally analytic bijections of $\mathbb{Q}_pP^1$ (see [13]).

2. Higman-Thompson groups.

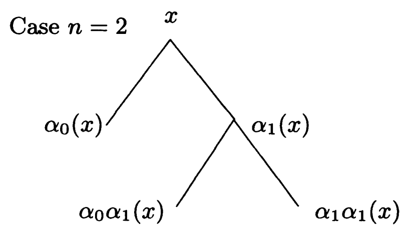
2.1. Definition of Higman-Thompson groups. In 1965, R.J. Thompson, interested in finitely presented groups with non-solvable word problem, introduced a group (denoted $G_{2,1}$ in the following) which happened to be the first known example of finitely generated infinite simple group [11]. Thompson's group was later generalized by G. Higman ([8]). For the description of the Higman-Thompson groups, we refer to [2]. See also [4].

Recall that a finite *n-ary rooted planar tree* is a finite tree T with root x realized in the oriented plane such that

– If T is not reduced to x, the valence of x is equal to n.

– The valence of a vertex $v \neq x$ is equal to 1 or $n+1$: if the valence of v is 1, we call v a *leaf* of the tree; if it is equal to $n+1$, v has n adjacent edges not contained in the geodesic joining the root x to v. We realize them by drawing them down from the vertex v. We order them from the left to the right and label their terminal vertices (opposite to v) $\alpha_0(v), \ldots, \alpha_{n-1}(v)$.

The set of leaves of a finite n-ary rooted tree T is called a *basis* and is denoted B_T.



DEFINITION 2.1. — *A simple expansion of a finite n-ary rooted tree T is any finite n-ary rooted tree T' obtained by the following procedure:*

- *Choose a vertex v in the base B_T.*
- *Make an expansion of v by drawing n edges down from it.*

We get a new tree T' whose basis $B_{T'}$ is deduced from B_T by replacing v by $\alpha_0(v), \ldots, \alpha_{n-1}(v)$.

An expansion T' of T is a tree obtained from T by making finitely many successive simple expansions. Any two trees T_1 and T_2 always possess a common expansion.

The elements of the Higman-Thompson groups will be represented by "symbols":

DEFINITION 2.2 (symbols). — *Consider a pair (T_1, T_2) of finite n-ary rooted trees with basis having the same cardinality. Let $\sigma : B_{T_1} \to B_{T_2}$ be a bijection from the basis of the first tree to the basis of the second one. We call the triple (T_1, T_2, σ) a symbol.*

A simple expansion of a symbol (T_1, T_2, σ) is any symbol (T_1', T_2', σ') thus obtained:

- *T_1' is a simple expansion of T_1, deduced from T_1 by expanding a vertex $v \in B_{T_1}$.*
- *Then T_2' is the expansion of T_2 realized from the vertex $\sigma(v)$.*

- $\sigma' : B_{T_1'} \to B_{T_2'}$ *is defined by*

$$\sigma'_{|B_{T_1}\setminus\{v\}} = \sigma_{|B_{T_1}\setminus\{v\}},$$
$$\sigma'(\alpha_i(v)) = \alpha_i(\sigma(v)) \ , i = 0, \ldots, n-1.$$

An expansion (T_1', T_2', σ') of the symbol (T_1, T_2, σ) is obtained from the latter by making finitely many simple expansions.

Declare now that (T_1, T_2, σ) and (T_1', T_2', σ') are equivalent if they possess a common expansion.

All the necessary vocabulary has been introduced to set the following:

DEFINITION 2.3 (Higman-Thompson groups). — *The set of equivalence classes of symbols $[(T_1, T_2, \sigma)]$ form a set G_n endowed with the following group structure:*

Two elements $[(T_1, T, \sigma)]$ and $[(T', T_2, \sigma')]$ being given, at the price of making expansions of their representing symbols, it may be supposed that $T = T'$. Then $\sigma'\sigma : B_{T_1} \to B_{T_2}$ can be defined, and we set

$$[(T_1, T, \sigma)][(T, T_2, \sigma')] = [(T_1, T_2, \sigma'\sigma)],$$

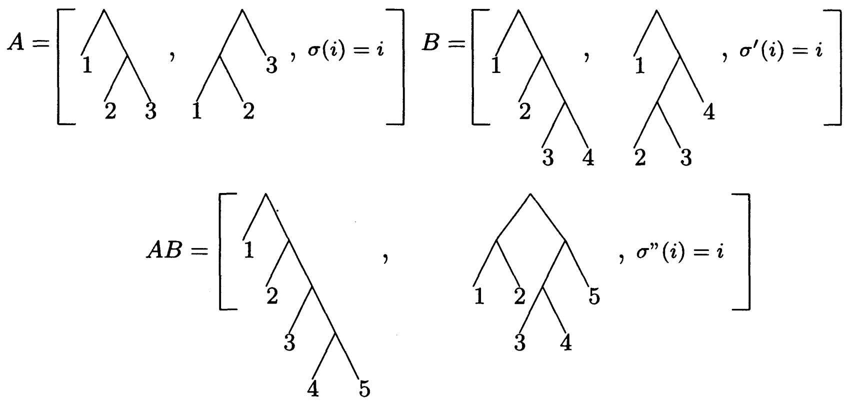
since it is easy to check that this definition is independent of the chosen symbols.

The neutral element is $[(T, T, \sigma = \mathrm{id})]$ represented by any symbol $(T, T, \sigma = \mathrm{id})$.

The inverse of $[(T_1, T_2, \sigma)]$ is $[(T_2, T_1, \sigma^{-1})]$.

The group G_n belongs to the family of Higman-Thompson groups.

Example ($n = 2$).



Recall that the leaves of a tree T (i.e. the vertices in B_T) are always labelled from the left to the right. Let (T, T', σ) be a symbol, and $\sigma : B_T = \{v_1, \ldots, v_k\} \to B_{T'} = \{v'_1, \ldots, v'_k\}$. There exists a unique permutation $\tau \in \mathcal{S}_k$ such that

$$\sigma(v_i) = v'_{\tau(i)} \ \forall i = 1, \ldots, k.$$

Then define $\theta(\sigma) = \epsilon(\tau)$ the signature of τ. An easy calculation shows that if $(\widetilde{T}, \widetilde{T'}, \tilde{\sigma})$ is a simple expansion of the symbol (T, T', σ), then

$$\theta(\tilde{\sigma}) = \theta(\sigma)(-1)^{n-1},$$

so that when n is an odd integer, $\theta(\sigma)$ is independent of the chosen symbol, and we get the group epimorphism

$$\theta : G_n \to \mathbb{Z}/2\mathbb{Z}$$
$$\theta([(T, T', \sigma)]) = \epsilon(\tau).$$

Generalization. Let $r \geq 1$ be a fixed integer. First consider pairs of r-uplets of finite n-ary rooted trees $((T_1, \ldots, T_r), (T'_1, \ldots, T'_r))$, and bijections σ from $B_{T_1} \cup \ldots \cup B_{T_r}$ to $B_{T'_1} \cup \ldots \cup B_{T'_r}$ (We do not ask σ to map B_{T_i} onto $B_{T'_i}$). We always suppose the r-uplet of trees to be ordered from the left (T_1) to the right (T_r). Any triple $((T_1, \ldots, T_r), (T'_1, \ldots, T'_r), \sigma)$ is called an r-symbol. Similarly to the case $r = 1$, we define the group $G_{n,r}$ where the elements are represented by r-symbols. Of course, $G_{n,1} = G_n$.

As in the case $r = 1$, the morphism $\theta : G_{n,r} \to \mathbb{Z}/2\mathbb{Z}$ can be defined provided n is odd. We set $G'_{n,r} = \operatorname{Ker}\theta$. If n is even, we agree that $G'_{n,r} = G_{n,r}$. We are now ready to cite the simplicity theorem:

THEOREM 2.1 ([2]). — *The group $G'_{n,r}$ is the commutator subgroup of $G_{n,r}$, and every non-trivial subgroup normalized by $G'_{n,r}$ contains it. In particular, $G_{n,r}$ is simple if n is even, and if n is odd, $G_{n,r}$ contains a simple group of index 2, namely $G'_{n,r} = [G_{n,r}, G_{n,r}]$.*

2.2. Embedding of $G_{n,1} = G_n$ and $G_{n,2}$ into the Neretin group N_n. The finite n-ary rooted trees we used in the definition of the Higman-Thompson groups may be canonically embedded in a chosen branch L of the regular tree $\mathcal{T}_n$, by simply completing the finite tree to an infinite n-ary rooted tree and then, identifying it to the branch L. Denote by L' the branch opposite to L in $\mathcal{T}_n$ (linked to L by an edge). Each $g \in G_{n,1}$, defined by a symbol (T_1, T_2, σ), induces a spheromorphism $\tilde{g}$ in an obious way: if (v_i^1) (resp. (v_i^2)) are the leaves of T_1 (resp. T_2), denote by L_i^1 (resp. L_i^2) the subbranch of L whose root is v_i^1 (resp. v_i^2). Then $\tilde{g}$ is induced on ∂L by the collection $(L_i^1 \xrightarrow{\cong} L_{\sigma_i}^2)_i$, the isomorphisms respecting the left-to-right order of the edges of the branches. On $\partial L'$, one imposes $\tilde{g}_{|\partial L'} = \mathrm{id}_{|\partial L'}$. The embedding

$$G_{n,1} \hookrightarrow N_n$$

is now obtained.

On the other hand, we need the two branches L and L' like above to realize $G_{n,2}$ in N_n. Each $g \in G_{n,2}$ will induce a spheromorphism by a procedure analogous to the previous one. It will appear in the following that, as far as we are concerned with the Neretin group N_n, $G_{n,2}$ is more relevant than the group $G_{n,1} = G_n$ itself.

3. The group $\mathrm{Aut}\,\mathcal{T}_n$ of automorphisms of the tree $\mathcal{T}_n$, $n \geq 2$.

3.1. Simplicity theorem. In [15], the author gave a theorem of simplicity of a class of groups of automorphisms of a tree:

DEFINITION 3.1. — *Let A be a tree, G be a group of automorphisms of A, C be a (finite or infinite) chain of A, and F the fixator of C in G. For each vertex x of A, let $\pi(x)$ be the nearest vertex from x in C. For each vertex s of C, the set $\pi^{-1}(s)$ (which constitutes a subtree of A) is invariant under the action of F; denote by F_s the group of permutations of this set induced by F. There is a natural homomorphism*

$$F \longrightarrow \prod_{s \in \mathrm{Vert}(C)} F_s, \qquad (1)$$

where Vert(C) *denotes the set of vertices of* C.

We say that the group G *possesses the property (P) if the homomorphism* (1) *is an isomorphism for all chains* C *(i.e. the actions of* F *on the sets* $\pi^{-1}(s)$ *are independent from each other).*

For example the group of all automorphisms of A possesses the property (P).

THEOREM 3.1 (J. Tits). — *Let* A *be a tree,* G *be a group of automorphisms of* A*, and* G^+ *be the subgroup generated by the stabilizers of the edges of* A *in* G*. Suppose that* G *possesses the property* (P)*, conserves no proper non-empty subtree of* A *and fixes no end of* A*. Then each subgroup of* G *normalized by* G^+ *and not reduced to the identity contains* G^+*. In particular,* G^+ *is a simple group or is reduced to the identity.*

Example 1. — $A = \mathcal{T}_n$, $n \geq 2$, $G = \operatorname{Aut}\mathcal{T}_n$. It happens that $G^+ = \operatorname{Aut}^+ \mathcal{T}_n$ coincides with the group of type-preserving automorphisms of the tree. So $\operatorname{Aut}^+ \mathcal{T}_n$ is a simple group, of index 2 in $\operatorname{Aut}\mathcal{T}_n$.

Example 2. — Equipped Bruhat-Tits trees.

Let $p \geq 2$ be a prime integer. In [13], the author defines an *equipment* on the tree $\mathcal{T}_p$ as the specification, for each vertex v, of a labelling of its adjacent edges $({l^v}_0, \ldots, {l^v}_{p-1}, {l^v}_\infty)$ by the points of $\mathbb{F}_pP^1$. If v and v' are linked by an edge $l = {l^v}_i = {l^{v'}}_j$, there is no reason that $i = j$.

We denote by $\widetilde{\mathcal{T}_p}$ such an equipped tree, and define the subgroup $\operatorname{Aut}\widetilde{\mathcal{T}_p}$ of $\operatorname{Aut}\mathcal{T}_p$ as the set of tree automorphisms such that their restrictions to the adjacent edges of a vertex belong to $PSL_2(\mathbb{F}_p)$. Since $\operatorname{Aut}\widetilde{\mathcal{T}_p}$ obviously satisfies property (P), conserves no proper non-empty subtree of $\mathcal{T}_p$ and fixes no end, the group $(\operatorname{Aut}\widetilde{\mathcal{T}_p})^+$ is simple.

Two equipped trees $\widetilde{\mathcal{T}_p}^1$ and $\widetilde{\mathcal{T}_p}^2$ being given, we use the transitivity of $SL_2(\mathbb{F}_p)$ on $\mathbb{F}_pP^1$ to construct a tree isomorphism $\widetilde{\mathcal{T}_p}^1 \to \widetilde{\mathcal{T}_p}^2$ respecting the equipments. Such an isomorphism conjugates $\operatorname{Aut}\widetilde{\mathcal{T}_p}^1$ and $\operatorname{Aut}\widetilde{\mathcal{T}_p}^2$.

3.2. A family of subgroups of N_n.

DEFINITION 3.2. — *If* G *is a subgroup of* $\operatorname{Aut}\mathcal{T}_n$ *we define*

$$(N_n)_G := < G_{n,2}, G^+ >$$

the subgroup of N_n *generated by* $G_{n,2}$ *and* G^+.

Example 1. — If $G = \mathrm{Aut}\,\mathcal{T}_n$, $(N_n)_G = N_n$. In this case, we can even show:

PROPOSITION 3.1. — *The subgroups* $[G_n, G_n]$ *and* $\mathrm{Aut}^+\,\mathcal{T}_n$ *of the group* N_n, $n \geq 2$, *generate the group* N_n.

Proof. — Let us denote by L the chosen branch of the tree $\mathcal{T}_n$ where we realized the Higman-Thompson group G_n. If L' is the branch opposite to L (i.e., linked with L by an edge), then the boundaries of L and L' partition the whole boundary of the tree: $\partial L \cup \partial L' = \partial \mathcal{T}_n$.

First case. — Suppose that $\phi \in N_n$ possesses a broom $(L_i)_{i=1,\dots,I}$ such that $\phi_{|\partial L_1} = \mathrm{id}_{|\partial L_1}$. At the price of making an expansion of L_1, one can suppose that L_1 and L' have the same type (i.e. their roots have the same type). Then there exists $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k(L') = L_1$. So $k^{-1}\phi k_{|\partial L'} = \mathrm{id}_{|\partial L'}$. Let us now consider $k^{-1}\phi k_{|\partial L}$. It may be seen as the composite

$$\partial L \xrightarrow{\tau} \partial L \xrightarrow{\sigma} \partial L$$

with $\tau \in G_n$ and $\sigma \in \mathrm{Aut}^+\,\mathcal{T}_n$, $\sigma_{|L'} = \mathrm{id}_{|L'}$. Then on the whole boundary $\partial\mathcal{T}_n$, $k^{-1}\phi k = \sigma\tau$.

When n is odd, $\mathrm{Aut}^+\,\mathcal{T}_n \cap (G_n \setminus [G_n, G_n]) \neq \emptyset$, so that it can be supposed that $\tau \in [G_n, G_n]$.

Second case: general case. — (a) Suppose there exists L_i in the broom adapted to ϕ such that ∂L_i and $\phi(\partial L_i) = \partial L'_i$ have the same type. Then there exists $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k\phi(\partial L_i) = \partial L_i$ and $k \circ \phi_{|\partial L_i} = \mathrm{id}_{|\partial L_i}$. The first case enables to conclude.

(b) If not, for all i, the types of ∂L_i and $\phi(\partial L_i)$ are opposite. Then we use an element τ_0 of G_n (it is possible to find it of the form $[\tau_1, \tau_2]$) such that for some branch L_0, $\tau_0(L_0)$ and L_0 have opposite types. At the price of making an expansion of L_1 to make $\phi(\partial L_1)$ and ∂L_0 have the same type, there exists some $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k\phi(\partial L_1) = \partial L_0$. The types of L_1 and L_0 are still opposite. Then $\tau_0 k\phi(\partial L_1) = \tau_0(\partial L_0) = \partial L'_0$, and the types of L_1 and L'_0 coincide. Hence $\tau_0 k\phi$ satisfies the condition of case (a).

It follows that ϕ may be written as a product of elements of G_n and $\mathrm{Aut}^+\,\mathcal{T}_n$.

Example 2. — Now p is a prime integer. Let $\widetilde{\mathcal{T}}_p$ be any equipment on the tree $\mathcal{T}_p$ such that the elements of $G_{p,2}$ are induced by piecewise tree automorphisms of $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}$ (*cf.* §3.1, Example 2).

If $G = \mathrm{Aut}\,\widetilde{\mathcal{T}_p}$, then we claim that $(N_p)_G$ is the group denoted $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ in [13]:

$$\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) = \{\phi = (\phi_j : L_j \to L'_j)_j, \quad \phi_j = \text{ restriction of some element of } \mathrm{Aut}\,\widetilde{\mathcal{T}_p}\}.$$

Indeed, $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ contains G, and because of the condition on the equipment, it contains $G_{p,2}$. So, $< G, G_{p,2} > \subset \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$. On the other hand, every $\phi \in \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ can be written $\phi = \psi \circ \tau$, where $\tau = (L_j \to L'_j)_j$ belongs to $G_{p,2}$, and $\psi = (\psi_j = L'_j \to L'_j)_j$, with ψ_j induced by some element of G, which can be modified to be supported in the branch L'_j. It follows that $\psi_j \in G^+$, and $\psi = \prod_j \psi_j \in G^+$. Thus

$$< G, G_{p,2} > \subset \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) \subset < G^+, G_{p,2} >,$$

and the inclusions are equalities. Then $(N_p)_{\mathrm{Aut}\,\widetilde{\mathcal{T}_p}} = \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ as claimed.

Remarks. — 1) Any isomorphism of equipped trees $\widetilde{\mathcal{T}_p} \to \widetilde{\mathcal{T}_p}'$ conjugates $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ and $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}')$.

2) If $p = 2$, the group $PSL_2(\mathbb{F}_2)$ is the full symmetric group $\mathcal{S}_3$, so that $\mathrm{Diff}^+(\widetilde{\mathcal{T}_2}) = N_2$.

4. Simplicity of $(N_p)_G$.

We now give the main theorem of the article, valid for any integer $p \geq 2$:

THEOREM 4.1. — *Let G be a subgroup of $\mathrm{Aut}\,\mathcal{T}_p$ such that*

1. *G^+ is simple (e.g. G satisfies the conditions of Theorem 3.1),*
2. *If p is odd, $G^+ \cap (G_{p,2} \setminus [G_{p,2}, G_{p,2}])$ is non-empty,*
3. *G^+ possesses two non-commuting elements supported in a branch of the tree.*

Then the group $(N_p)_G$ is simple.

Condition 2. implies that $(N_p)_G$ is generated by G^+ and ${G_{p,2}}' = [G_{p,2}, G_{p,2}]$, since $G_{p,2}$ is generated by ${G_{p,2}}'$ together with any element in $G_{p,2} \setminus {G_{p,2}}'$.

COROLLARY 4.1. — *For each integer $p \geq 2$, the group N_p of all spheromorphisms is simple.*

For each prime number $p \geq 3$ and for any choice of equipment of the tree $\mathcal{T}_p$, the commutator subgroup $[\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})]$ is simple, and there is a short exact sequence

$$1 \to [\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})] \longrightarrow \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) \xrightarrow{\tilde{\theta}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

In other words, $H_1(\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof of Corollary 4.1. — $G = \mathrm{Aut}\,\mathcal{T}_p$ obviously satisfies all the conditions of the theorem above.

As for the statements about $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$, they can be proven by using a particular equipment, since for different equipments the groups are conjugated. So, remembering that $\mathcal{T}_p$ is obtained by gluing by an edge the two branches L and L' appearing in the definition of $G_{p,2}$, define the equipment $\widetilde{\mathcal{T}_p}^0$ in the following way: label the p edges drawn down from a vertex from 0 (on the left) to $p-1$ (on the right), whereas the edge pointing towards the root of the branch (L or L') is labelled ∞. Then setting $G = \mathrm{Aut}\,\widetilde{\mathcal{T}_p}^0$, we have $(N_p)_G = \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)$ (*cf.* §3.2 Example 2). But condition 2 of Theorem 4.1 fails for such G. We recalled in Section 2 that when p is odd, there is an epimorphism

$$\theta : G_{p,2} \to \mathbb{Z}/2\mathbb{Z}$$

whose kernel is the simple group $[G_{p,2}, G_{p,2}]$. It happens that θ may be extended to the group $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)$: if $\phi = (\phi_j : L_j \to L'_{\sigma(j)})_j$, where the indices of the branches label their roots from the left to the right (suppose the branches involved to be subbranches of L or L'), $\tilde{\theta}(\phi)$ will be the signature of σ. Indeed, if we refine some branch L_j into $L_{j_0} \cup L_{j_1} \cup \ldots \cup L_{j_{p-1}}$, then ϕ_j induces

$$\phi_{j_i} : L_{j_i} \to L'_{\sigma(j)_{k_i}} \quad i = 0, 1, \ldots, p-1,$$

with $i \in \mathbb{F}_p \to k_i \in \mathbb{F}_p$ in $B \subset PSL_2(\mathbb{F}_p)$, the stabilizer of ∞. Since B lies in the alternating group $\mathcal{A}_p$ on a set with p elements, the permutation deduced from σ has the same signature as in the case $k_i = i\ \forall i \in \mathbb{F}_p$. But then we saw (*cf.* §2) that, since p is odd, the signature remains unchanged. So

$$\tilde{\theta} : \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is a well-defined homomorphism.

It is clear that the kernel of $\tilde{\theta}$ is generated by $[G_{p,2}, G_{p,2}]$ and $(\mathrm{Aut}\, \widetilde{\mathcal{T}_p}^0)^+$, and the proof of the theorem will show that this group is simple. Now the kernel contains the commutator subgroup $[\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)]$, which is normal and non-trivial, consequently it coincides with the kernel.

Proof of Theorem 4.1. — Let $H \lhd (N_p)_G$ be a non-trivial normal subgroup of $(N_p)_G$. Then $H \cap G^+$ is normal in G^+ and $H \cap [G_{p,2}, G_{p,2}]$ is normal in $[G_{p,2}, G_{p,2}]$. Hence either $H \supset G^+$ or $H \cap G^+ = \{\mathrm{id}\}$, and either $H \supset [G_{p,2} G_{p,2}]$ or $H \cap [G_{p,2}, G_{p,2}] = \{\mathrm{id}\}$.

So we will prove that the cases $H \cap G^+ = \{\mathrm{id}\}$ and $H \cap [G_{p,2}, G_{p,2}] = \{\mathrm{id}\}$ do not occur. We will use some arguments of a theorem of Epstein ([5] and [1]):

THEOREM 4.2 (Epstein, 1970). — *Let X be a paracompact Hausdorff topological space, Γ a group of homeomorphisms of X, and $\mathcal{U}$ a basis of open sets for the topology of X. The Epstein axioms for the triple $(X, \Gamma, \mathcal{U})$ are:*

1. Axiom 1: *If $U \in \mathcal{U}$ and $g \in \Gamma$, then $gU \in \mathcal{U}$.*
2. Axiom 2: *Γ acts transitively on $\mathcal{U}$.*
3. Axiom 3: *Let $g \in \Gamma$, $U \in \mathcal{U}$ and $\mathcal{B}$ an open covering of X; then there exists an integer n and $g_1, \ldots, g_n \in \Gamma$ and $V_1, \ldots, V_n \in \mathcal{B}$ such that*

   (i) $g = g_n g_{n-1} \cdots g_1$,

   (ii) $\mathrm{supp}\,(g_i) \subset V_i$,

   (iii) $\mathrm{supp}\,(g_i) \cup (g_{i-1} \ldots g_1 \bar{U}) \neq X,\ 1 \leq i \leq n$.

Suppose the triple $(X, \Gamma, \mathcal{U})$ as above satisfies the Epstein axioms. Then if H is a non-trivial subgroup of Γ that is normalized by $[\Gamma, \Gamma]$, then $[\Gamma, \Gamma] \subset H$. In particular, the group $[\Gamma, \Gamma]$ is simple.

The simplicity of $[\mathrm{Diff}^+(S^1), \mathrm{Diff}^+(S^1)]$ was an easy corollary of this theorem. M.R. Herman finally proved $\mathrm{Diff}^+(S^1)$ was perfect, hence simple ([7]). For more details, we suggest the reader to refer to the very interesting book [1].

In the case of a non-connected topological space and a non trivial group Γ, axiom 3 can never be satisfied (see [5]). Consequently, we will not be able to use the preceding theorem directly to prove the simplicity of $(N_p)_G$. However, setting $X = \partial \mathcal{T}_p$, $\mathcal{U} = \{\partial L : L \text{ branch of } \mathcal{T}_p\}$ and

$\Gamma = (N_p)_G$, it is easy to see that the triple $(\partial\mathcal{T}_p, (N_p)_G, \mathcal{U})$ satisfies axiom 2 and a

"modified axiom 1": If $U \in \mathcal{U}$ and $g \in \Gamma$, then there exists $U' \in \mathcal{U}$, $U' \subset U$, such that $gU' \in \mathcal{U}$.

Then we can show that two lemmas, which are steps in the proof of the Epstein theorem, still hold in our case:

LEMMA 4.1 (from 1.4.2 in [5], or Lemma 2.2.5 in [1]). — *Let $(X, \Gamma, \mathcal{U})$ be a triple satisfying the modified axiom 1 and axiom 2. Let $V_0 \in \mathcal{U}$ and $h \in \Gamma$ with $\operatorname{supp} h \subset V_0$, and suppose that $H \lhd \Gamma$ is a non-trivial normal subgroup of Γ. Then there exists some $\rho \in H$ such that $\rho_{|V_0} = h_{|V_0}$.*

Proof. — Choose any $\alpha \in H$ with $\alpha \neq \mathrm{id}$, and find $x \in X$ such that $\alpha(x) \neq x$. Choose a small neighborhood $U \in \mathcal{U}$ of x such that $U \cap \alpha^{-1}(U) = \emptyset$. Next, take V, $W \in \mathcal{U}$ such that $V \cap W = \emptyset$, $\bar{V} \cup \bar{W} \subset U$, $x \in V$. Suppose first that $V_0 = V$. By axiom 2, there exists $g \in \Gamma$ with $gW = V$. Define

$$\rho = [\alpha, [g, h]] = \alpha^{-1}[g, h]^{-1}\alpha[g, h].$$

Then $\rho \in \Gamma$ since $H \lhd \Gamma$. We can verify that

$$\rho = \begin{cases} h \text{ on } V, \\ g^{-1}h^{-1}g \text{ on } W, \\ \alpha^{-1}h\alpha \text{ on } \alpha^{-1}V, \\ \alpha^{-1}g^{-1}h^{-1}g\alpha \text{ on } \alpha^{-1}W, \\ \mathrm{id} \text{ elsewhere.} \end{cases}$$

Now if $V_0 \neq V$, choose $k \in \Gamma$ (by axiom 2) such that $k(V) = V_0$. Then $\operatorname{supp} k^{-1}hk = k^{-1}(\operatorname{supp} h) \subset V$, and by the previous case, there exists $\rho \in H$ such that $k^{-1}hk_{|V} = \rho_{|V}$, so that $h_{|V_0} = k\rho k^{-1}_{|V_0}$. Since $k\rho k^{-1} \in H$, the proof is done.

LEMMA 4.2 (variation of 1.4.6 in [5] or Lemma 2.2.7 in [1]). — Γ *still satisfies the modified axiom 1 and axiom 2. Moreover, it is supposed 2-transitive:*

$$\forall(x_1, x_2),\ \forall(y_1, y_2),\ x_1 \neq x_2 \text{ and } y_1 \neq y_2 \Rightarrow \exists\phi \in \Gamma\ \phi(x_i) = y_i,\ i = 1, 2.$$

Let h_1, $h_2 \in \Gamma$ be such that there exists $V_0 \in \mathcal{U}$ with $\operatorname{supp} h_i \subset V_0$, $i = 1, 2$. Then $[h_1, h_2]$ belongs to H.

Proof. — Let x be in X. There exist α_1, α_2 in H such that x, ${\alpha_1}^{-1}(x)$ and ${\alpha_2}^{-1}(x)$ are pairwise distinct. Indeed, since $\alpha \neq \mathrm{id} \in H$, there exists some $x \in X$ with $\alpha(x) \neq x$. So, in a neighborhood of x there exists $y \neq x$ such that $\alpha(y) \neq y$. Now one can find $\phi \in \Gamma$ with $\phi(x) = y$ and $\phi^{-1}\alpha\phi(x) \neq \alpha(x)$ (which is equivalent to $\alpha(y) \neq \phi\alpha(x)$). As for the condition $\alpha(y) \neq y$, it is equivalent to $\phi^{-1}\alpha\phi(x) \neq x$. Then one sets $\alpha_1^{-1} = \alpha$, $\alpha_2^{-1} = \phi^{-1}\alpha\phi$. So α_1 and α_2 belong to H, x, ${\alpha_1}^{-1}(x)$ and ${\alpha_2}^{-1}(x)$ are pairwise distinct. Then choose $U \in \mathcal{U}$ a neighborhood of x such that U, ${\alpha_1}^{-1}(U)$ and ${\alpha_2}^{-1}(U)$ are pairwise disjoint. One can also find g_1, g_2 in Γ, and a neighborhood $V \in \mathcal{U}$ of x such that V, $g_1^{-1}(V)$ and $g_2^{-1}(V)$ are pairwise disjoint and included in U. Suppose first that $\operatorname{supp} h_i \subset V$, $i = 1, 2$. Then apply the previous lemma to $(\alpha_i, g_i, h_i, V, W_i = g_i^{-1}V)$, $i = 1, 2$. One gets $\rho_{i|V} = h_{i|V}$. The support of ρ_i is included in $V \cup g_i^{-1}(V) \cup {\alpha_i}^{-1}(V) \cup {\alpha_i}^{-1} g_i^{-1}(V)$. The seven sets involved are disjoint, so that

$$[h_1, h_2] = [\rho_1, \rho_2].$$

To conclude, we may assume $V = V_0$, at the price of making some conjugation.

End of the proof of Theorem 4.1. — Choose $V_0 = \partial L_0$ where L_0 is some branch of the tree, and by condition 3, find two non-commuting elements h_1 and h_2 in G^+ with supports in ∂L_0. Apply Lemma 4.2 to $\Gamma = (N_p)_G$, which is 2-transitive on $\partial \mathcal{T}_p$, since $G_{p,2}$ itself is 2-transitive. Then $[h_1, h_2] \in G^+ \cap H$, so $H \supset G^+$.

Similarly, choose two non-commuting elements h_1' and h_2' in $G_p = G_{p,1} \subset G_{p,2}$ (they are supported in a branch), so that $[h_1', h_2'] \in [G_p, G_p] \cap H \subset [G_{p,2}, G_{p,2}] \cap H$, and $H \supset [G_{p,2}, G_{p,2}]$. Finally, H contains two groups that generate $(N_p)_G$, so $H = (N_p)_G$.

5. Concluding remarks.

The question of the simplicity of the group N_n is a preamble of a series of homological problems. First the result implies $H_1(N_n, \mathbb{Z}) = 0$. As for the second homology group $H_2(N_n, \mathbb{Z})$, though its complete computation could not be achieved (because the group N_n is very huge), we know it is non trivial. Indeed, the group N_n possesses a non-trivial central extension by $\mathbb{Z}/2\mathbb{Z}$, called the “Central Geometric Extension” in [9] and [10], a sort of analogue of the Bott-Virasoro extension of $\mathrm{Diff}^+(S^1)$.

On the other hand, K. Brown proved that the groups G_n are all $\mathbb{Q}$-acyclic, i.e. $H_i(G_n, \mathbb{Q}) = 0$ for all $i > 0$ (*cf.* [3]). By using a description of N_n as the automorphism group of a free object of some appropriate category, it becomes possible to define an N_n-simplicial complex, and to use it to prove the $\mathbb{Q}$-acyclicity of N_n (*cf.* [9] and [10]).

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