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A non-abelian tensor product of Leibniz algebra


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A NON-ABELIAN TENSOR PRODUCT
OF LEIBNIZ ALGEBRAS

by Allahtan V. GNEDBAYE

Introduction.

Let $g$ be a Lie algebra and let $M$ be a representation of $g$, seen as a right $g$-module. Given a $g$-equivariant map $\mu : M \rightarrow g$, one can endow the $K$-module $M$ with a bracket $\{[m, m'] := m^{\mu(m')}\}$ which is not skew-symmetric but satisfies the Leibniz rule of derivations:

$$[m, [m', m'']] = [[m, m'], m''] - [[m, m''], m'].$$

Such objects were baptized Leibniz algebras by Jean-Louis Loday and are studied as a non-commutative variation of Lie algebras (see [8]). One of the main examples of Lie algebras comes from the notion of derivations. For the Leibniz algebras, there is an analogue notion of biderivations (see [7]).

The aim of this article is to “integrate” the Leibniz algebra of biderivations by means of a non-abelian tensor product of Leibniz algebras as it is done for Lie algebras.

In the classical case, D. Guin (see [5]) has shown that, given crossed Lie $g$-algebras $\mathcal{M}$ and $\mathcal{N}$, the set of derivations $\text{Der}_g(\mathcal{M}, \mathcal{N})$ has a structure of pre-crossed Lie $g$-algebra. Moreover the functor $\text{Der}_g(\mathcal{N}, -)$ is right adjoint to the functor $-\otimes_g \mathcal{N}$ where $-\otimes_g -$ is the non-abelian tensor product of Lie algebras defined by G. J. Ellis (see [3]). D. Guin uses these objects

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to construct a non-abelian (co)homology theory for Lie algebras, which enables him to compare the $K$-modules $HC_1(A)$ and $K_2^{Madd}(A)$ where $A$ is an arbitrary associative algebra. We give a non-commutative version of his results, in the sense that Leibniz algebras play the role of Lie algebras, the additive Milnor $K$-theory $K_*^{Madd}(A)$ (resp. the cyclic homology $HC_*(A)$) being replaced by the Milnor-type Hochschild homology $HH_*^M(A)$ (resp. the classical Hochschild homology $HH_*(A)$).

To this end, we introduce the notion of (pre)crossed Leibniz $g$-algebra as a simultaneous generalization of notions of representation and two-sided ideal of the Leibniz algebra $g$. Given crossed Leibniz $g$-algebras $M$ and $N$, we equip the set $Bider_g(M,N)$ of biderivations with a structure of pre-crossed Leibniz $g$-algebra. On the other hand, we construct a non-abelian tensor product $M \ast N$ of Leibniz algebras with mutual actions on one another. When $M$ and $N$ are crossed Leibniz $g$-algebras, this tensor product has also a structure of crossed Leibniz $g$-algebra. It turns out that the functor $-*g N$ is left adjoint to the functor $Bider_g(N, -)$. Another characterization of this tensor product is the following. If the Leibniz algebra $g$ is perfect (and free as a $K$-module), then the Leibniz algebra $g \ast g$ is the universal central extension of $g$ (see [4]). We give also low-degrees (co)homological interpretations of these objects, which yield an exact sequence of $K$-modules

$$A/[A,A] \otimes HH_1(A) \oplus HH_1(A) \otimes A/[A,A] \to \mathcal{H}L_1(A,L(A))$$

$$\to \mathcal{H}L_1(A,[A,A]) \to HH_1(A) \to HH_1^M(A) \to [A,A]/[A,[A,A]] \to 0$$

where $L(A)$ is the $K$-module $A \otimes A/\text{im}(b_3)$ equipped with a suitable Leibniz bracket (see section 1.2).

Throughout this paper the symbol $K$ denotes a commutative ring with a unit element and $\otimes$ stands $\otimes_K$.

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1. Prerequisites on Leibniz algebras.

1.1. Leibniz algebras.

A Leibniz algebra is a $K$-module $g$ equipped with a bilinear map $[-,-] : g \times g \to g$, called bracket and satisfying only the Leibniz identity

$$[[x, y], z] - [[x, z], y]$$

for any $x, y, z \in g$. In the presence of the condition $[x, x] = 0$, the Leibniz identity is equivalent to the so-called Jacobi identity. Therefore Lie algebras are examples of Leibniz algebras.

A morphism of Leibniz algebras is a linear map $f : g_1 \to g_2$ such that

$$f([x, y]) = [f(x), f(y)]$$

for any $x, y \in g_1$. It is clear that Leibniz algebras and their morphisms form a category that we denote by $(\text{Leib})$.

A two-sided ideal of a Leibniz algebra $g$ is a submodule $h$ such that $[x, y] \in h$ and $[y, x] \in h$ for any $x \in h$ and any $y \in g$. For any two-sided ideal $h$ in $g$, the quotient module $g/h$ inherits a structure of Leibniz algebra induced by the bracket of $g$. In particular, let $([x, x])$ be the two-sided ideal in $g$ generated by all brackets $[x, x]$. The Leibniz algebra $g/([x, x])$ is in fact a Lie algebra, said canonically associated to $g$ and is denoted by $g_{\text{Lie}}$.

Let $g$ be a Leibniz algebra. Denote by $g' := [g, g]$ the submodule generated by all brackets $[x, y]$. The Leibniz algebra $g$ is said to be perfect if $g' = g$. It is clear that any submodule of $g$ containing $g'$ is a two-sided ideal in $g$.

1.2. Examples.

Let $M$ be a representation of a Lie algebra $g$ (the action of $g$ on $M$ being denoted by $m^g$ for $m \in M$ and $g \in g$). For any $g$-equivariant map $\mu : M \to g$, the bracket given by $[m, m'] := m^\mu(m')$ induces a structure of Leibniz (non-Lie) algebra on $M$. Observe that any Leibniz algebra $g$ can be obtained in such a way by taking the canonical projection $g \to g_{\text{Lie}}$ (which is obviously $g_{\text{Lie}}$-equivariant).

Let $A$ be an associative algebra and let $b_3 : A^\otimes 3 \to A^\otimes 2$ be the Hochschild boundary that is, the linear map defined by

$$b_3(a \otimes b \otimes c) := ab \otimes c - a \otimes bc + ca \otimes b, \ a, b, c \in A.$$
Then the bracket given by

\[ [a \otimes b, c \otimes d] := (ab - ba) \otimes (cd - dc), \quad a, b, c, d \in A, \]

defines a structure of Leibniz algebra on the \( \mathbb{K}\)-module \( L(A) := A^{\otimes 2} / \text{im}(b_3) \).
Moreover, we have an exact sequence of \( \mathbb{K}\)-modules

\[ 0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{b_2} A \rightarrow \text{HH}_0(A) \]

where \( \text{HH}_*(A) \) denotes the Hochschild homology groups and \( b_2(x, y) = [x, y] := xy - yx \) for any \( x, y \in A \).

### 1.3. Free Leibniz algebra.

Let \( V \) be a \( \mathbb{K}\)-module and let \( \overline{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n} \) be the reduced tensor module. The bracket defined inductively by

\[ [x, v] = x \otimes v, \text{ if } x \in \overline{T}(V) \text{ and } v \in V \]

\[ [x, y \otimes v] = [x, y] \otimes v - [x \otimes y, v], \text{ if } x, y \in \overline{T}(V) \text{ and } v \in V, \]

satisfies the Leibniz identity. The Leibniz algebra so defined is the free Leibniz algebra over \( V \) and is denoted by \( \mathcal{F}(V) \) (see [8]). Observe that one has

\[ v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots [[v_1, v_2], v_3] \cdots v_n], \quad \forall v_1, \cdots, v_n \in V. \]

Moreover, the free Lie algebra over \( V \) is nothing but the Lie algebra \( \mathcal{F}(V)_{\text{Lie}} \).

### 2. Crossed Leibniz algebras.

#### 2.1. Leibniz action.

Let \( g \) and \( M \) be Leibniz algebras. A Leibniz action of \( g \) on \( M \) is a couple of bilinear maps

\( g \times M \rightarrow M, \quad (g, m) \mapsto g \cdot m \) and \( M \times g \rightarrow M, \quad (m, g) \mapsto m^g \)

satisfying the axioms

i) \( m^{[g, g']} = (m^g)^{g'} - (m^{g'})^g \),

ii) \( [g, g']_m = (g^m)^{g'} - g(m^{g'}) \),
iii) \( g(g'm) = -g(mg') \),

iv) \( g[m,m'] = [g, m'] - [g', m] \),

v) \( [m,m']^g = [m^g, m'] + [m, m'^g] \),

vi) \( [m, g'm'] = -[m, g'm^g] \)

for any \( m, m' \in \mathfrak{M} \) and \( g, g' \in \mathfrak{g} \). We say that \( \mathfrak{M} \) is a \textit{Leibniz \( \mathfrak{g} \)-algebra}. Observe that the axiom i) applied to the triples \((m; g, g')\) and \((m; g', g)\) yields the relation

\[ m^{[g,g']} = -m^{[g',g]} \]

\[ 2.2. \text{Examples.} \]

Any two-sided ideal of a Leibniz algebra \( \mathfrak{g} \) is a Leibniz \( \mathfrak{g} \)-algebra, the action being given by the initial bracket.

A \( \mathbb{K} \)-module \( M \) equipped with two operations of a Leibniz algebra \( \mathfrak{g} \) satisfying the axioms i), ii) and iii) is called a \textit{representation of \( \mathfrak{g} \)} (see [8]). Therefore representations of a Leibniz algebra \( \mathfrak{g} \) are abelian Leibniz \( \mathfrak{g} \)-algebras.

\[ 2.3. \text{Crossed Leibniz algebras.} \]

Let \( \mathfrak{g} \) be a Leibniz algebra. A \textit{pre-crossed Leibniz \( \mathfrak{g} \)-algebra} is a Leibniz \( \mathfrak{g} \)-algebra \( \mathfrak{M} \) equipped with a morphism of Leibniz algebras \( \mu : \mathfrak{M} \rightarrow \mathfrak{g} \) such that

\[ \mu(\mathfrak{g}m) = [g, \mu(m)] \quad \text{and} \quad \mu(m^g) = [\mu(m), g] \]

for any \( g \in \mathfrak{g} \) and \( m \in \mathfrak{M} \). Moreover if the relations

\[ \mu(m)m' = [m, m'] \quad \text{and} \quad m^{\mu(m')} = [m, m'], \quad \forall \ m, m' \in \mathfrak{M}, \]

hold, then \((\mathfrak{M}, \mu)\) is called a \textit{crossed Leibniz \( \mathfrak{g} \)-algebra}. 
2.4. Examples.

Any Leibniz algebra \( \mathfrak{g} \), equipped with the identity map \( \text{id}_\mathfrak{g} \), is a crossed Leibniz \( \mathfrak{g} \)-algebra.

Any two-sided ideal \( \mathfrak{h} \) of a Leibniz algebra \( \mathfrak{g} \), equipped with the inclusion map \( \mathfrak{h} \hookrightarrow \mathfrak{g} \), is a crossed Leibniz \( \mathfrak{g} \)-algebra.

Let \( \alpha : \mathfrak{c} \to \mathfrak{g} \) be a central extension of Leibniz algebras (i.e., a surjective morphism whose kernel is contained in the centre of \( \mathfrak{c} \), see [4]). Define operations of \( \mathfrak{g} \) on \( \mathfrak{c} \) by

\[
\alpha^{-1}(g) = [\alpha^{-1}(g), c] \quad \text{and} \quad \alpha(g) = [c, \alpha^{-1}(g)]
\]

where \( \alpha^{-1}(g) \) is any pre-image of \( g \) in \( \mathfrak{c} \). Then \( (\mathfrak{c}, \alpha) \) is a crossed Leibniz \( \mathfrak{g} \)-algebra.

**Proposition 2.1.** — For any pre-crossed Leibniz \( \mathfrak{g} \)-algebra \( (\mathfrak{M}, \mu) \), the image \( \text{im}(\mu) \) (resp. the kernel \( \text{ker}(\mu) \)) is a two-sided ideal in \( \mathfrak{g} \) (resp. \( \mathfrak{M} \)). Moreover, if \( (\mathfrak{M}, \mu) \) is crossed, then \( \text{ker}(\mu) \) is contained in the centre of \( \mathfrak{M} \).

**Proof.** — Let \( m \) be an element of \( \mathfrak{M} \). For any \( g \in \mathfrak{g} \), we have

\[
[g, \mu(m)] = \mu(g\mu(m)) \in \text{im}(\mu) \quad \text{and} \quad [\mu(m), g] = \mu(g^\mathfrak{g}) \in \text{im}(\mu).
\]

Thus, \( \text{im}(\mu) \) is a two-sided ideal in \( \mathfrak{g} \). Assume that \( m \in \text{ker}(\mu) \); then for any \( m' \in \mathfrak{M} \), we have

\[
\mu([m, m']) = [\mu(m), \mu(m')] = 0 = [\mu(m'), \mu(m)] = \mu([m', m]).
\]

Therefore \( \text{ker}(\mu) \) is a two-sided ideal in \( \mathfrak{M} \). Moreover if the Leibniz action of \( \mathfrak{g} \) on \( \mathfrak{M} \) is crossed, then we have

\[
[m, m'] = \mu(m)m' = 0 = m'\mu(m) = [m', m]
\]

for any \( m \in \text{ker}(\mu) \) and \( m' \in \mathfrak{M} \). Thus \( \text{ker}(\mu) \) is contained in the centre of \( \mathfrak{M} \).

\[\square\]

2.5. Morphism of pre-crossed Leibniz algebras.

Let \( \mathfrak{g} \) be a Leibniz algebra and let \( (\mathfrak{M}, \mu) \) and \( (\mathfrak{N}, \nu) \) be pre-crossed Leibniz \( \mathfrak{g} \)-algebras. A *morphism* from \( (\mathfrak{M}, \mu) \) to \( (\mathfrak{N}, \nu) \) is a Leibniz algebra morphism \( f : \mathfrak{M} \to \mathfrak{N} \) such that

\[
f(\mathfrak{g}m) = \mathfrak{g}(f(m)), \quad f(m^\mathfrak{g}) = (f(m))^\mathfrak{g} \quad \text{and} \quad \mu = \nu f
\]
for any $m \in \mathcal{M}$ and $g \in \mathcal{G}$. A morphism of crossed Leibniz $\mathcal{G}$-algebras is the same as a morphism of pre-crossed Leibniz $\mathcal{G}$-algebras. It is clear that pre-crossed (resp. crossed) Leibniz $\mathcal{G}$-algebras and their morphisms form a category that we denote by $(\text{pc-Leib}(\mathcal{G}))$ (resp. $(\text{c-Leib}(\mathcal{G}))$).

**Proposition 2.2.** — Let $f : (\mathcal{M}, \mu) \rightarrow (\mathcal{N}, \nu)$ be a crossed Leibniz $\mathcal{G}$-algebra morphism. Then $(\mathcal{M}, f)$ is a crossed Leibniz $\mathcal{N}$-algebra via the Leibniz action of $\mathcal{N}$ on $\mathcal{M}$ given by

$$n^m := \nu(n)m \quad \text{and} \quad m^n := m^{\nu(n)}, \quad \forall m \in \mathcal{M}, n \in \mathcal{N}.$$

**Proof.** — One easily checks that $\mathcal{M}$ is a Leibniz $\mathcal{N}$-algebra. For any $m, m' \in \mathcal{M}$ and $n \in \mathcal{N}$, we have

$$f(n^m) = f^{\nu(n)m} = \nu(n)f(m) = [n, f(m)],$$

$$f(m^n) = f^{m^{\nu(n)}} = f(m)^{\nu(n)} = [f(m), n];$$

thus $(\mathcal{M}, f)$ is a pre-crossed Leibniz $\mathcal{N}$-algebra. Moreover we have

$$f^m(m') = f^{(f(m))}m' = \mu(m)m' = [m, m'],$$

$$m^f(m') = m^{\nu(f(m'))} = m^{\mu(m')} = [m, m'];$$

thus $(\mathcal{M}, f)$ is a crossed Leibniz $\mathcal{N}$-algebra. \qed

2.6. Exact sequences.

We say that a sequence

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N}$$

is exact in the category $(\text{pc-Leib}(\mathcal{G}))$ (resp. $(\text{c-Leib}(\mathcal{G}))$) if the sequence

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N}$$

is exact as sequence of Leibniz algebras.

**Proposition 2.3.** — If the sequence

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N}$$

is exact in the category $(\text{pc-Leib}(\mathcal{G}))$ (resp. $(\text{c-Leib}(\mathcal{G}))$), then the map $\lambda$ is zero. Moreover if the Leibniz $\mathcal{G}$-algebra $(\mathcal{L}, \lambda)$ is crossed, then the Leibniz algebra $\mathcal{L}$ is abelian.

**Proof.** — Indeed, since $\beta \alpha = 0$, we have $\lambda = \nu \beta \alpha = 0$. From whence $\ker(\lambda) = \mathcal{L}$, and by Proposition 2.1, it is clear that the Leibniz algebra $\mathcal{L}$ is abelian. \qed

In this section, we fix a Leibniz algebra $\mathfrak{g}$.

3.1. Derivations and anti-derivations.

Let $(\mathfrak{m}, \mu)$ and $(\mathfrak{n}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A derivation from $(\mathfrak{m}, \mu)$ to $(\mathfrak{n}, \nu)$ is a linear map $d : \mathfrak{m} \to \mathfrak{n}$ such that

$$d([m, m']) = d(m)\mu(m') + \mu(m)d(m'), \forall m, m' \in \mathfrak{m}.$$ 

An anti-derivation from $(\mathfrak{m}, \mu)$ to $(\mathfrak{n}, \nu)$ is a linear map $D : \mathfrak{m} \to \mathfrak{n}$ such that

$$D([m, m']) = D(m)\mu(m') - D(m')\mu(m), \forall m, m' \in \mathfrak{m}.$$ 

3.2. Examples.

Let $(\mathfrak{m}, \mu)$ be a crossed Leibniz $\mathfrak{g}$-algebra and let $n$ be any element of $\mathfrak{m}$. By the axiom iii) (resp. i)) of 2.1, the linear map

$$g \mapsto \mathfrak{m}, g \mapsto \mathfrak{n} \quad \text{(resp.} \ g \mapsto \mathfrak{m}, g \mapsto -n^g)$$

is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \text{id})$ to $(\mathfrak{n}, \nu)$.

3.3. Biderivations.

Let $(\mathfrak{m}, \mu)$ and $(\mathfrak{n}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. We denote by $\text{Bider}_{\mathfrak{g}}(\mathfrak{m}, \mathfrak{n})$ the free $\mathbb{K}$-module generated by the triples $(d, D, g)$, where $d$ (resp. $D$) is a derivation (resp. an anti-derivation) from $(\mathfrak{m}, \mu)$ to $(\mathfrak{n}, \nu)$ and $g$ is an element of $\mathfrak{g}$ such that

$$\nu(d(m)) = \mu(m^g), \ \nu(D(m)) = -\mu(m^g),$$

$$h^d(m) = hD(m), \ D(m^h) = -D(h^m)$$

for any $h \in \mathfrak{g}$ and $m \in \mathfrak{m}$. 
PROPOSITION 3.1. — If the Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \nu)$ is crossed, then there is a Leibniz algebra structure on the $K$-module $\text{Biderg}(\mathfrak{g}, \mathfrak{g})$ for the bracket defined by

$$[[d, D, g], (d', D', g')] := (\delta, \Delta, [g, g'])$$

where

$$\delta(m) := d'(m^g) - d(m^g')$$ and $$\Delta(m) = -D(m^g') - d'(q_m), \forall m \in \mathfrak{g}.$$

Proof. — Let us show that the maps $\delta$ and $\Delta$ are respectively a derivation and an anti-derivation. Indeed, for any $m, m' \in \mathfrak{g}$, we have

$$\delta([m, m']) = d'([m, m']^g) - d([m, m']^g')$$

$$= d'([m^g, m']) + d'([m, m'^g]) - d([m^g', m']) - d([m, m'^g'])$$

$$= d'(m^g)\mu(m') + \mu(m)\mu(m') + d'(m)\mu(m^g) + \mu(m) d'(m^g)$$

$$- d(m^g)\mu(m') - \mu(m^g') d(m') - d(m)\mu(m^g')$$

$$- \mu(m) d(m^g')$$

$$= (d'(m^g) - d(m^g'))\mu(m') + \mu(m)\mu(m') + d'(m)\mu(m) - \mu(m) d'(m')$$

$$+ [d'(m), d(m')] - [d'(m), d(m')] - [d(m), d'(m')]$$

$$= \delta(m)\mu(m') + \mu(m)\delta(m') + [d(m), d'(m')]$$

and

$$\Delta([m, m']) = -D([m, m']^g) - d'(q[m, m'])$$

$$= -D([m^g, m']) - D([m, m'^g]) - d'(q[m, m']) + d'(q[m'^g, m'])$$

$$= -D(m^g)\mu(m') + D(m')\mu(m^g') - D(m)\mu(m'^g) + D(m^g')\mu(m)$$

$$- d'(q[m])\mu(m') - \mu(m) d'(m') + d'(q[m'^g, m'])\mu(m) + \mu(m) d'(m')$$

$$= (-D(m^g') - d'(q[m])\mu(m') - (-D(m^g') - d'(q[m'])\mu(m))$$

$$+ D(m')\nu(d'(m')) - D(m)\nu(d'(m')) + \nu(D(m)) d'(m')$$

$$- \nu(D(m')) d'(m)$$

$$= \Delta(m)\mu(m') - \Delta(m')\mu(m) + [D(m'), d'(m)]$$

$$- [D(m), d'(m')] + [D(m), d'(m')] - [D(m'), d'(m')]$$

$$= \Delta(m)\mu(m') - \Delta(m')\mu(m).$$
On the other hand, we have
\[ \nu(\delta(m)) = \nu(d'(m^g)) - \nu(d(m^{g'})) = \mu((m^g)^{g'}) - \mu((m^{g'})^g) = \mu(m^g g'), \]
\[ \nu(\Delta(m)) = -\nu(D(m^g)) - \nu(d'(g^m)) = \mu(g(m^g)) - \mu((g^m)g') = -\mu(g^m g'), \]
\[ h\delta(m) = h'd'(m^g) - h'd(m^{g'}) = h'D'(m^g) - h'D(m^{g'}) \]
\[ = - h'D'(g^m) - hD(m^{g'}) = -h'd'(g^m) - hD(m^{g'}) \]
\[ = h\Delta(m), \]
\[ \Delta(hm) = - D((hm)^{g'}) - d'(g^{hm}) \]
\[ = - D(h^{(g',g')}) - D(h(m^g)) + d'(g^{hm}) \]
\[ = D((hm)^{g'}) + d'(g^{hm}) = -\Delta(hm). \]

Therefore the triple \((\delta, \Delta, [g, g'])\) is a biderivation from \((\mathfrak{M}, \mu)\) to \((\mathfrak{N}, \nu)\).
Moreover, let \((d, D, g), (d', D', g')\) and \((d'', D'', g'')\) be biderivations from \((\mathfrak{M}, \mu)\) to \((\mathfrak{N}, \nu)\). We set
\[ (\delta_0, \Delta_0, g_0) := [(d, D, g), (\delta, \Delta, [g, g'])], \]
\[ (\delta_1, \Delta_1, g_1) := [(\delta', \Delta', [g, g']), (d', D', g')], \]
\[ (\delta_2, \Delta_2, g_2) := [(\delta'', \Delta'', [g, g'']), (d', D', g')]. \]

It is clear that \(g_0 = g_1 - g_2\). For any \(m \in \mathfrak{M}\), we have
\[ (\delta_1 - \delta_2)(m) = \delta'(m^g') - \delta'(m^{g''}) - d'(m^{g'''}), \]
\[ = \delta''((m^g)^{g''}) - d''((m^{g''})^{g'}) - d''((m^{g''})g') + d''(m^{g''})g' \]
\[ - d''((m^{g''})g') - d'(m^{g''})g' + d''((m^{g''})g') - d''(m^{g''})g' \]
\[ = \delta(m^g) - d(m^{g^{g''}}) = \delta_0(m) \]
and
\[ (\Delta_1 - \Delta_2)(m) = - \Delta'(m^{g''}) - d''(g^{g''}m) + \Delta''(m^{g'}) + d''(g^{g''}m) \]
\[ = D((m^{g''})^{g'}) + d'(g(m^{g''})) - d''((m^g)^{g'}) + d''(g(m^{g'})) \]
\[ - D((m^{g''})^{g''}) - d''(g(m^{g'})) + d''((m^g)^{g''}) - d''(g(m^{g''})) \]
\[ = - D(m^{g^{g''}}) - d''((g^m)^{g'}) + d''((g^m)^{g''}) \]
\[ = - D(m^{g^{g''}}) - d(m^{g'}) = \Delta_0(m). \]

Therefore the \(K\)-module \(\text{Bider}_g(\mathfrak{M}, \mathfrak{N})\) is a Leibniz algebra.
Let us equip the set $\text{Bider}_g(\mathfrak{M}, \mathfrak{N})$ with a Leibniz action of $g$.

**Proposition 3.2.** Let $(\mathfrak{M}, \mu)$ (resp. $(\mathfrak{N}, \nu)$) be a pre-crossed (resp. crossed) Leibniz $g$-algebra. The set $\text{Bider}_g(\mathfrak{M}, \mathfrak{N})$ is a pre-crossed Leibniz $g$-algebra for the operations defined by

$$h(d, D, g) := (h^d, hD, [h, g]) \quad \text{and} \quad (d, D, g)^h := (d^h, D^h, [g, h])$$

where

$$(h^d)(m) = d(m^h) - d(m)^h, \quad (hD)(m) := h^d(m) - d^h(m),$$

$$(d^h)(m) := d(m)^h - d(m^h), \quad (D^h)(m) := D(m)^h - D(m^h).$$

**Proof.** Everything can be smoothly checked and we merely give an example of these verifications. By definition we have

$$h[(d, D, g), (d', D', g')] = (h^\delta, h\Delta, [h, [g, g']]),$$

$$[h(d, D, g), (d', D', g')] = (\delta_1, \Delta_1, [[h, g], g']),$$

$$[h(d', D', g'), (d, D, g)] = (\delta_2, \Delta_2, [[h', g'], g]).$$

For any $m \in \mathfrak{M}$ we have

$$(\delta_1 - \delta_2)(m) = d'(m^{h, g'}) - (h^d)(m^{g'}) - d(m^{h, g'}) + (h^d')(m^g)$$

$$= d'((m^h)^g) - d'((m^g)^h) - d((m^g')^h) + d((m^g')^h)$$

$$= (d'((m^h)^g) - d((m^g)^h)) - (d'(m^g) - d(m^g))^h$$

$$= \delta(m^h) - \delta(m)^h = (h^\delta)(m)$$

and

$$(\Delta_1 - \Delta_2)(m) = -(hD)(m^g) - d'(m^{h, g'}) + (hD')(m^g) + d((h^d)(m)$$

$$= -(hD(m^g) + d(h(m^g)) - d'(m^g)^h) + d((h^d')(m^g))$$

$$+ hD'(m^g) - d'(m^g)^h + d((h^d)(m^g')$$

$$= h(D'(m^g) - D(m^g')) - (d'(h(m^g)) - d((h^d)(m)^g))$$

$$= h^\delta(m) - \delta(hm) = (h\Delta)(m).$$

Thus we get

$$h[(d, D, g), (d', D', g')] = [h(d, D, g), (d', D', g')] - [h(d', D', g'), (d, D, g)].$$

Now we can state the fundamental result which is a consequence of Propositions 3.1 and 3.2.
THEOREM 3.3. — For any pre-crossed (resp. crossed) Leibniz \( g \)-algebra \( (\mathcal{M}, \mu) \) (resp. \( (\mathcal{N}, \nu) \)), the Leibniz \( g \)-algebra \( \text{Bider}_g(\mathcal{M}, \mathcal{N}) \) is pre-crossed for the morphism
\[ \rho : \text{Bider}_g(\mathcal{M}, \mathcal{N}) \to g, \ (\mathcal{D}, g) \mapsto g. \]

3.4. Remarks.

For any element \( g \) of \( g \), the linear map \( \text{ad}_g : h \mapsto [h, g] \) (resp. \( \text{Ad}_g : h \mapsto -[g, h] \)) is a derivation (resp. an anti-derivation) of the Leibniz algebra \( g \). In the classical sense (i.e., without "crossing", see [7]) the couple \( (\text{ad}_g, \text{Ad}_g) \) is called inner biderivation of \( g \). Therefore the pre-crossed Leibniz \( g \)-algebra \( \text{Bider}_g(\mathcal{M}, \mathcal{N}) \) can be seen as the set of biderivations from \( (\mathcal{M}, \mu) \) to \( (\mathcal{N}, \nu) \) over inner biderivations of \( g \).

On the other hand, given a pre-crossed Leibniz \( g \)-algebra \( (\mathcal{M}, \mu) \), one easily checks that the map \( \text{Bider}_g(\mathcal{M}, -) \) is a functor from the category of crossed Leibniz \( g \)-algebras to the category of pre-crossed Leibniz \( g \)-algebras.


4.1. Leibniz pairings.

Let \( \mathcal{M} \) and \( \mathcal{N} \) be Leibniz algebras with mutual Leibniz actions on one another. A Leibniz pairing of \( \mathcal{M} \) and \( \mathcal{N} \) is a triple \( (\mathcal{P}, \mathcal{h}_1, \mathcal{h}_2) \) where \( \mathcal{P} \) is a Leibniz algebra and \( \mathcal{h}_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{P} \) (resp. \( \mathcal{h}_2 : \mathcal{N} \times \mathcal{M} \to \mathcal{P} \)) is a bilinear map such that
\[
\begin{align*}
\mathcal{h}_1(m, [n, n']) & = \mathcal{h}_1(m^n, n') - \mathcal{h}_1(m, n^{n'}), \\
\mathcal{h}_2(n, [m, m']) & = \mathcal{h}_2(n^m, m') - \mathcal{h}_2(n, m^{m'}), \\
\mathcal{h}_1([m, m'], n) & = \mathcal{h}_2(m^n, m') - \mathcal{h}_1(m, n^{m'}), \\
\mathcal{h}_2([n, n'], m) & = \mathcal{h}_1(m^n, n') - \mathcal{h}_2(n, m^{n'}), \\
\mathcal{h}_1(m, m^n) & = -\mathcal{h}_1(m, m^{n'}), \\
\mathcal{h}_2(n, n'^m) & = -\mathcal{h}_2(n, m^{n'}), \\
\mathcal{h}_1(m^n, m^{n'}) & = [\mathcal{h}_1(m, n), \mathcal{h}_1(m', n')] = \mathcal{h}_2(m^n, m^{n'}), \\
\mathcal{h}_1(m, m'^n) & = [\mathcal{h}_2(n, m), \mathcal{h}_2(n', m')] = \mathcal{h}_2(m^n, m'^{n'}), \\
\mathcal{h}_1(m^n, m'^{m'}) & = [\mathcal{h}_1(m, n), \mathcal{h}_2(n, m')] = \mathcal{h}_2(m^n, m'^{m'}), \\
\mathcal{h}_1(m, m'^{n'}) & = [\mathcal{h}_2(n, m), \mathcal{h}_1(m', n')] = \mathcal{h}_2(n^m, m'^{n'})
\end{align*}
\]
for any \( m, m' \in \mathcal{M} \) and \( n, n' \in \mathcal{N} \).
4.2. Example.

Let $\mathcal{M}$ and $\mathcal{N}$ be two-sided ideals of a same Leibniz algebra $g$. Take $\mathfrak{P} := \mathcal{M} \cap \mathcal{N}$ and define

$$h_1(m, n) := [m, n] \quad \text{and} \quad h_2(n, m) := [n, m].$$

Then the triple $(\mathfrak{P}, h_1, h_2)$ is a Leibniz pairing of $\mathcal{M}$ and $\mathcal{N}$.

4.3. Non-abelian tensor product.

A Leibniz pairing $(\mathfrak{P}, h_1, h_2)$ of $\mathcal{M}$ and $\mathcal{N}$ is said to be universal if for any other Leibniz pairing $(\mathfrak{P}', h_1', h_2')$ of $\mathcal{M}$ and $\mathcal{N}$ there exists a unique Leibniz algebra morphism $\theta : \mathfrak{P} \to \mathfrak{P}'$ such that

$$\theta h_1 = h_1' \quad \text{and} \quad \theta h_2 = h_2'.$$

It is clear that a universal pairing, when it exists, is unique up to a unique isomorphism. Here is a construction of the universal pairing as a non-abelian tensor product.

**DEFINITION-THEOREM 4.1.** — Let $\mathcal{M}$ and $\mathcal{N}$ be Leibniz algebras with mutual Leibniz actions on one another. Let $V$ be the free $K$-module generated by the symbols $m \ast n$ and $n \ast m$ where $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Let $\mathcal{M} \ast \mathcal{N}$ be the Leibniz algebra quotient of the free Leibniz algebra generated by $V$ by the two-sided ideal defined by the relations

i) $\lambda(m \ast n) = \lambda m \ast n = m \ast \lambda n$, $\lambda(n \ast m) = \lambda n \ast m = n \ast \lambda m$,

ii) $(m + m') \ast n = m \ast n + m' \ast n$, $(n + n') \ast m = n \ast m + n' \ast m$,

$$m \ast (n + n') = m \ast n + m \ast n', \quad n \ast (m + m') = n \ast m + n \ast m',$$

iii) $m \ast [n, n'] = m^n \ast n' - m^{n'} \ast n$, $n \ast [m, m'] = n^m \ast m' - n^{m'} \ast m$,

$$[m, m'] \ast n = n^m \ast m' - m^m \ast n', \quad [n, n'] \ast m = n^m \ast n' - n \ast m^n,$$

iv) $m \ast m' \ast n = -m \ast m' \ast n$, $n \ast m' = n \ast m^n$,

v) $m^n \ast m'^n = [m \ast n, m' \ast n'] = m^n \ast m'^n$,

$m^n \ast m'^n = [m \ast n, m' \ast m'] = m^n \ast n \ast m'$,

$m^n \ast m'^n = [n \ast m, m' \ast n'] = n^m \ast m'^n$,

$m^n \ast m'^n = [n \ast m, m' \ast m'] = n^m \ast m'^n$,

for any $\lambda \in K$, $m, m' \in \mathcal{M}$, $n, n' \in \mathcal{N}$. Define maps

$$h_1 : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \ast \mathcal{N}, \quad h_1(m, n) := m \ast n.$$
and
\[ h_2 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{M} \ast \mathfrak{N}, \quad h_2(n, m) := n \ast m. \]
Then the triple \((\mathfrak{M} \ast \mathfrak{N}, h_1, h_2)\) is the universal Leibniz pairing of \(\mathfrak{M}\) and \(\mathfrak{N}\) and called the non-abelian tensor product (or tensor product for short) of \(\mathfrak{M}\) and \(\mathfrak{N}\).

**Proof.** — It is straightforward to see that the triple \((\mathfrak{M} \ast \mathfrak{N}, h_1, h_2)\) so-defined is a Leibniz pairing of \(\mathfrak{M}\) and \(\mathfrak{N}\). For the universality, notice that if \((\mathfrak{P}, h'_1, h'_2)\) is another Leibniz pairing of \(\mathfrak{M}\) and \(\mathfrak{N}\), then the map \(\theta\) is necessarily given on generators by
\[
\theta(m \ast n) = h'_1(m, n) \quad \text{and} \quad \theta(n \ast m) = h'_2(n, m)
\]
for any \(m \in \mathfrak{M}\) and \(n \in \mathfrak{N}\). \(\Box\)

As an illustration of this construction, we give now a description of the non-abelian tensor product when the actions are trivial.

**Proposition 4.2.** — If the Leibniz algebras \(\mathfrak{M}\) and \(\mathfrak{N}\) act trivially on each other, then there is an isomorphism of abelian Leibniz algebras
\[
\mathfrak{M} \ast \mathfrak{N} \cong \mathfrak{M}_{ab} \otimes \mathfrak{N}_{ab} \oplus \mathfrak{N}_{ab} \otimes \mathfrak{M}_{ab}
\]
where \(\mathfrak{M}_{ab} := \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}]\) and \(\mathfrak{N}_{ab} := \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]\).

**Proof.** — Recall that the underlying \(K\)-module of the free Leibniz algebra generated by \(V\) is
\[ \mathfrak{T}(V) = V \oplus V \otimes V \oplus \cdots \oplus V \otimes V \oplus \cdots \]
Since the actions are trivial, the definition of the bracket on \(\mathfrak{T}(V)\) and the relations v) enable us to see that \(\mathfrak{M} \ast \mathfrak{N}\) is an abelian Leibniz algebra and that the summands \(V \otimes V \otimes \cdots \otimes V \otimes V\) (for \(n \geq 2\)) are killed. Relations i) and ii) of 4.1 say that the \(K\)-module \(\mathfrak{M} \ast \mathfrak{N}\) is the quotient of \(\mathfrak{M} \otimes \mathfrak{N} \oplus \mathfrak{N} \otimes \mathfrak{M}\) by the relations iii). These later imply that \(\mathfrak{M} \ast \mathfrak{N}\) is the abelian Leibniz algebra \(\mathfrak{M}_{ab} \otimes \mathfrak{N}_{ab} \oplus \mathfrak{N}_{ab} \otimes \mathfrak{M}_{ab}\). \(\Box\)

4.4. Compatible Leibniz actions.

Let \(\mathfrak{M}\) and \(\mathfrak{N}\) be Leibniz algebras with mutual Leibniz actions on one another. We say that these actions are compatible if we have
\[
\begin{align*}
(m^n)(m') &= [m^n, m'], \quad (m^n)n' = [n^m, n'], \\
(n^m)m' &= [m, m'], \quad (m^n)n' = [m^n, n'], \\
m(m^n) &= [m, m'], \quad n(m^n) = [n, n^m], \\
m(n^m) &= [m, n^m], \quad n(n^m) = [n, m^n].
\end{align*}
\]
for any $m, m' \in \mathcal{M}$ and $n, n' \in \mathcal{N}$.

### 4.5. Examples.

If $\mathcal{M}$ and $\mathcal{N}$ are two-sided ideals of a same Leibniz algebra, then the actions (given by the initial bracket) are compatible.

Let $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ be pre-crossed Leibniz $g$-algebras. Then one can define a Leibniz action of $\mathcal{M}$ on $\mathcal{N}$ (resp. of $\mathcal{N}$ on $\mathcal{M}$) by setting

$\mu_{m} := \mu(m)_{n}$ and $\nu_{n} := \nu(n)_{m}$

(resp. $\mu_{m} := \nu(n)_{m}$ and $\nu_{n} := \nu(n)_{m}$).

If the Leibniz $g$-algebras $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are crossed, then these Leibniz actions are compatible.

### 4.6. First crossed structure.

Let $\mathcal{M}$ and $\mathcal{N}$ be Leibniz algebras with mutual compatible actions on one another. Consider the operations of $\mathcal{M}$ on $\mathcal{M} \ast \mathcal{N}$ given by

$m'(m' \ast n') := [m, m'] \ast n' - m' \ast m', \quad ^{(n' \ast m')} := n' \ast m' - [m, m'] \ast n',
(m \ast n)m' := [m, m'] \ast n + m \ast n \ast m', \quad (n \ast m)m' := n' \ast m + n \ast [m, m']$

and those of $\mathcal{N}$ on $\mathcal{M} \ast \mathcal{N}$ given by

$n'(m' \ast n') := [n, n'] \ast m' - n' \ast m', \quad ^{(n' \ast m')} := [n, n'] \ast m' - n' \ast m',
(m \ast n)n' := m' \ast n + m \ast [n, n'], \quad (n \ast m)n' := [n, n'] \ast m + n \ast m' n'$

for any $m, m' \in \mathcal{M}$ and $n, n' \in \mathcal{N}$. Then we have

**Proposition 4.3.** — With the above operations, the map

$\mu : \mathcal{M} \ast \mathcal{N} \rightarrow \mathcal{M}, \quad m \ast n \mapsto m^n, \quad n \ast m \mapsto \mu m$

(resp. $\nu : \mathcal{M} \ast \mathcal{N} \rightarrow \mathcal{N}, \quad m \ast n \mapsto \nu n, \quad n \ast m \mapsto \nu m$)

induces on $\mathcal{M} \ast \mathcal{N}$ a structure of crossed Leibniz $\mathcal{M}$-algebra (resp. $\mathcal{N}$-algebra).

**Proof.** — Once again everything can be readily checked thanks to the compatibility conditions. For example we have

$\mu(m \ast n)(m' \ast n') = m^n(m' \ast n') = [m^n, m'] \ast n' - (m^n)_{m'} \ast m' = (m^n)_{m'} \ast m' - m^n \ast n' \ast m' - (m^n)_{m'} \ast m' = m^n \ast m' n = [m \ast n, m' \ast n']$
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for any \( m, m' \in \mathcal{M} \) and \( n, n' \in \mathcal{N} \).

\[ \square \]

4.7. Second crossed structure.

Let \((\mathcal{M}, \mu)\) and \((\mathcal{N}, \nu)\) be pre-crossed Leibniz \( g \)-algebras, equipped with the mutual Leibniz actions given in Examples 4.5. One easily checks that the operations given by

\[
\begin{align*}
\mathcal{g}(m \ast n) := & \quad \mathcal{g} m \ast n - \mathcal{g} n \ast m, \quad \mathcal{g}(n \ast m) := \mathcal{g} n \ast m - \mathcal{g} m \ast n, \\
(m \ast n)^g := & \quad m^g \ast n + m \ast n^g, \quad (n \ast m)^g := n^g \ast m + n \ast m^g,
\end{align*}
\]

define a Leibniz action of \( g \) on \( \mathcal{M} \ast \mathcal{N} \).

**Proposition 4.4.** — Let \((\mathcal{M}, \mu)\) and \((\mathcal{N}, \nu)\) be pre-crossed Leibniz \( g \)-algebras. Then the map \( \eta : \mathcal{M} \ast \mathcal{N} \rightarrow g \) defined on generators by

\[
\eta(m \ast n) := [\mu(m), \nu(n)] \quad \text{and} \quad \eta(n \ast m) := [\nu(n), \mu(m)],
\]

confers to \( \mathcal{M} \ast \mathcal{N} \) a structure of pre-crossed Leibniz \( g \)-algebra. Moreover, if one of the Leibniz \( g \)-algebras \( \mathcal{M} \) or \( \mathcal{N} \) is crossed, then the Leibniz \( g \)-algebra \( \mathcal{M} \ast \mathcal{N} \) is crossed.

**Proof.** — It is immediate to check that the map \( \eta \) passes to the quotient and defines a Leibniz algebra morphism. Moreover we have

\[
\begin{align*}
\eta(\mathcal{g}(m \ast n)) &= [\mu(\mathcal{g}m), \nu(n)] - [\nu(\mathcal{g}n), \mu(m)] \\
&= [[g, \mu(m)], \nu(n)] - [[g, \nu(n)], \mu(m)] \\
&= [g, [\mu(m), \nu(n)]] - [g, \eta(m \ast n)]; \\
\eta(\mathcal{g}(n \ast m)) &= -\eta(\mathcal{g}(m \ast n)) = -[g, \eta(m \ast n)] \\
&= -[g, [\mu(m), \nu(n)]] = [g, [\nu(n), \mu(m)]] = [g, \eta(n \ast m)]; \\
\eta((m \ast n)^g) &= [\mu(m^g), \nu(n)] + [\mu(m), \nu(n^g)] \\
&= [[\mu(m), g], \nu(n)] + [\mu(m), [\nu(n), g]] \\
&= [[\mu(m), \nu(n)], g] = [\eta(m \ast n), g]; \\
\eta((n \ast m)^g) &= [\nu(n^g), \mu(m)] + [\nu(n), \mu(m^g)] \\
&= [[\nu(n), g], \mu(m)] + [\nu(n), [\mu(m), g]] \\
&= [[\nu(n), \mu(m)], g] = [\eta(n \ast m), g];
\end{align*}
\]

thus \((\mathcal{M} \ast \mathcal{N}, \eta)\) is a pre-crossed Leibniz \( g \)-algebra. Assume that, for instance,
the Leibniz g-algebra $\mathcal{M}$ is crossed. Then we have

$$
\eta(m * n)(m' * n') = [\mu(m), \nu(n)](m' * n') = \mu(m * n')(m' * n')
$$

$$
= \mu(m * n) n' * m' - \mu(m * n) n' * m'
$$

$$
= [m * n, m'] * n' - \mu(m * n) n' * m'
$$

$$
= \mu(m * n) n' * m' - m * \nu(n) \mu(m') - \mu(m * n) n' * m'
$$

$$
= m * \nu(n) * \mu(m') * [m * n, m' * n']
$$

and

$$
(m * n) \eta(m' * n') = (m * n)[\mu(m'), \nu(n')](m' * n') = (m * n)\mu(m' * n')
$$

$$
= m * \nu(n') * n + m * n \mu(m' * n')
$$

$$
= [m, m'] * n + m * n \mu(m' * n')
$$

$$
= \mu(m \eta * m' \nu(n')) - m * n \mu(m' \nu(n')) + m * n \mu(m' \nu(n'))
$$

$$
= [m * n, m' * n'].
$$

By the same way, one easily gets

$$
\eta(n * m)(n' * m') = [n * m, n' * m'], (n * m) \eta(n' * m') = [n * m, n' * m'],
$$

$$
\eta(n * m)(m' * n') = [n * m, m' * n'], (n * m) \eta(n' * m') = [n * m, n' * m'],
$$

$$
\eta(n * m)(n' * m') = [n * m, n' * m'], (n * m) \eta(n' * m') = [n * m, n' * m'].
$$

So we have proved that the Leibniz g-algebra $\mathcal{M} * \mathcal{N}$ is crossed. 

\[ \square \]

4.8. Remark.

It is clear that if $(\mathcal{M}, \mu)$ (resp. $(\mathcal{N}, \nu)$) is a crossed Leibniz g-algebra, then the map $\mathcal{M} * -$ (resp. $- * \mathcal{N}$) is a functor from the category of pre-crossed Leibniz g-algebras to the category of crossed Leibniz g-algebras.

Proposition 4.5. — Let $(\mathcal{N}, \nu)$ be a crossed Leibniz g-algebra. The functor $F(-) := - * \mathcal{N}$ is a right exact functor from the category of pre-crossed Leibniz g-algebras to the category of crossed Leibniz g-algebras.

Proof. — Taking into account Proposition 2.3, let

$$
0 \to (\mathfrak{P}, 0) \xrightarrow{1} (\Omega, \lambda) \xrightarrow{g} (\mathfrak{N}, \gamma) \to 0
$$

be an exact sequence of pre-crossed Leibniz g-algebras. Consider the sequence of Leibniz algebras

$$
F(\mathfrak{P}) \xrightarrow{\overline{\delta}(f)} F(\Omega) \xrightarrow{\overline{\delta}(g)} F(\mathfrak{N}) \to 0.
$$
It is clear that the morphism $F(g)$ is surjective. Since the map $F(f)$ is a morphism of crossed Leibniz $g$-algebras, by Proposition 2.2, $(F(\mathfrak{P}), \mathfrak{f}(f))$ is a crossed Leibniz $F(\Omega)$-algebra; and by Proposition 2.1, the image $\text{im } F(f)$ is a two-sided ideal in $F(\Omega)$. By composition we have $F(g)F(f) = F(gf) = 0$, which yields a factorisation

$$
\overline{F(g)} : F(\Omega)/\text{im } F(f) \rightarrow \mathfrak{f}(\mathfrak{R}).
$$

In fact, the morphism $\overline{F(g)}$ is an isomorphism. To see it, let us consider the map

$$
\Gamma : F(\mathfrak{R}) \rightarrow \mathfrak{f}(\Omega)/\text{im } F(f)
$$

given on generators by

$$
\Gamma(r \ast n) := g^{-1}(r) \ast n \mod \text{im } F(f) \quad \text{and} \quad \Gamma(n \ast r) := n \ast g^{-1}(r) \mod \text{im } F(f)
$$

where $g^{-1}(r)$ is any pre-image of $r$ in $\Omega$. Indeed, if $q$ and $q'$ are two pre-images of $r$, then $q - q' = f(p)$ for some $p$ in $\mathfrak{P}$. Therefore we have

$$
q \ast m - q' \ast n = (q - q') \ast n = f(p) \ast n = F(f)(p \ast n) \in \text{im } F(f),
$$

$$
n \ast q - n \ast q' = n \ast (q - q') = n \ast f(p) = F(f)(n \ast p) \in \text{im } F(f);
$$

thus the map $\Gamma$ is well-defined. One easily checks that $\Gamma$ is a morphism of Leibniz algebras and inverse to $\overline{F(g)}$.

5. Adjunction theorem.

In this section we show that, for any crossed Leibniz $g$-algebra $(\mathfrak{M}, \nu)$, the functor $- \ast \mathfrak{N}$ is left adjoint to the functor $\text{Bider}_g(\mathfrak{N}, -)$. For technical reasons, we assume that the relations

$$
m \ast ^\mu(m')n = -m \ast \mu(m'), \quad n \ast ^\nu(n')m = -n \ast \nu(n')
$$

defining the tensor product $\mathfrak{M} \ast \mathfrak{N}$ are extended to the relations

$$
m \ast g n = -m \ast n g, \quad n \ast g m = -n \ast m g
$$

for any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $g \in g$. To avoid confusion, we denote this later tensor product by $\mathfrak{M} \ast_g \mathfrak{N}$. For instance, the Leibniz $g$-algebras $\mathfrak{M} \ast \mathfrak{N}$ and $\mathfrak{M} \ast_g \mathfrak{N}$ coincide if the maps $\mu$ and $\nu$ are surjective.

**Theorem 5.1.** — Let $(\mathfrak{M}, \mu)$ be a pre-crossed Leibniz $g$-algebra and let $(\mathfrak{N}, \nu)$ and $(\mathfrak{P}, \lambda)$ be crossed Leibniz $g$-algebras. There is an isomorphism of $K$-modules

$$
\text{Hom}_{(pc-Leib(g))}(\mathfrak{M}, \text{Bider}_g(\mathfrak{N}, \mathfrak{P})) \cong \text{Hom}_{(c-Leib(g))}(\mathfrak{M} \ast_g \mathfrak{N}, \mathfrak{P}).
$$
Proof. — Let $\phi \in \text{Hom}(\text{pc-Leib}(g))(\mathcal{M}, \text{Bider}_g(\mathcal{N}, \mathcal{P}))$ and put $(d_m, D_m, g_m) := \phi(m)$ for $m \in \mathcal{M}$. Notice that we have $g_m = \mu(m)$ thanks to the relation $\rho \phi = \mu$, where $\rho : \text{Bider}_g(\mathcal{N}, \mathcal{P}) \to g$ is the crossing morphism. We associate to $\phi$ the map $\Phi : \mathcal{M} \times_g \mathcal{N} \to \mathcal{P}$ defined on generators by

$$\Phi(m \ast n) := -D_m(n) \quad \text{and} \quad \Phi(n \ast m) := d_m(n), \ \forall \ m \in \mathcal{M}, n \in \mathcal{N}.$$ 

**Lemma 5.2.** — The map $\Phi$ is a morphism of crossed Leibniz $g$-algebras.

Conversely, given an element $\sigma \in \text{Hom}(\text{c-Leib}(g))(\mathcal{M} \times_g \mathcal{N}, \mathcal{P})$, we associate the map $\Sigma : \mathcal{M} \to \text{Bider}_g(\mathcal{N}, \mathcal{P})$ defined by

$$\Sigma(m) := (\delta_m, \Delta_m, \mu(m)), \ \forall \ m \in \mathcal{M},$$

where

$$\delta_m(n) := \sigma(n \ast m) \quad \text{and} \quad \Delta_m(n) := -\sigma(m \ast n), \ \forall \ n \in \mathcal{N}.$$ 

**Lemma 5.3.** — The map $\Sigma$ is a morphism of pre-crossed Leibniz $g$-algebras.

It is clear that the maps $\phi \mapsto \Phi$ and $\sigma \mapsto \Sigma$ are inverse to each other, which proves the adjunction theorem. $\square$

**Proof of Lemma 5.2.** — There is a lot of things to check in order to show that the map $\Phi$ is well-defined. Let us give some examples of these verifications. For any $m, m' \in \mathcal{M}$, $n, n' \in \mathcal{N}$ and $h \in g$, we have

$$\Phi(n \ast m' - n \ast m') = -D_{\nu(m)n}(n') - d_{m\nu(n')}(n)$$

$$= -\nu(n)D_m(n') - ((d_m)\nu(n'))(n)$$

$$= -\nu(n)D_m(n') + d_m(\nu(n)n') - d_m(n)\nu(n') + d_m(n\nu(n'))$$

$$= -\nu(n)d_m(n') + d_m([n, n']) - d_m(n)\nu(n') + d_m([n, n'])$$

$$= d_m([n, n']) = \Phi([n, n'] \ast m).$$

We also compute

$$\Phi(m \ast h_n) = -D_m(h_n) = D_m(n^h) = -\Phi(m \ast n^h),$$

$$\Phi(n \ast h_m) = d_m(n) = (h_m)(n) = -(d_m)^h(n) = -d_m^h(n) = -\Phi(n \ast m^h).$$
and
\[
\Phi(m^n \ast m'n') = - D_{m^n}(\mu(m')n') = -((D_m)^{\nu(n)}(\mu(m')n')) \\
= - D_{m^n}(\mu(m')n')^{\nu(n)} + D_m((\mu(m')n')^{\nu(n)}) \\
= - D_{m^n}(\mu(m')n')^{\nu(n)} + D_m([\mu(m')n', n]) \\
= - D_{m^n}(n)^{\nu(n)(m')n'} = D_{m^n}(n)^{\lambda(D_m^{\nu(n)})} \\
= [D_{m^n}(n), D_{m^n'}(n')] = [\Phi(m \ast n), \Phi(m' \ast n')] \\
= \Phi([m \ast n, m' \ast n']).
\]
Now let \( m \in \mathfrak{M}, n \in \mathfrak{N} \) and \( g \in \mathfrak{g} \). One has successively
\[
\Phi(g(m \ast n)) = \Phi(g(m \ast n) - \Phi(g(n \ast m)) = -D_{m^n}(n) - d_{m^n}(g(n)) \\
= (D_{m^n})(n) - d_{m^n}(g(n)) = -gD_{m^n}(n) = g\Phi(m \ast n), \\
\Phi(g(n \ast m)) = -\Phi(g(m \ast n)) = -g\Phi(m \ast n) = gD_{m^n}(n) = d_{m^n}(g(n)) = g\Phi(n \ast m), \\
\Phi((m \ast n)g) = \Phi(m^g \ast n) + \Phi(m \ast n^g) = -D_{m^n}(n) - D_{m^n}(n^g) \\
= -((D_{m^n})(n) - D_{m^n}(n^g) = -D_{m^n}(n) = \Phi(m \ast n)^g, \\
\Phi((n \ast m)^g) = \Phi(n^g \ast m) + \Phi(n \ast m^g) = d_{m^n}(n^g) + d_{m^n}(n) \\
= d_{m^n}(n^g) + ((d_{m^n})^g(n) = d_{m^n}(n^g) = \Phi(n \ast m)^g; \\
\lambda\Phi(m \ast n) = -\lambda(D_{m^n}(n)) = \nu(\mu(m)n) = [\mu(m), \nu(n)] = \eta(m \ast n), \\
\lambda\Phi(n \ast m) = \lambda(d_{m^n}(n)) = \nu(\mu(m)n) = [\nu(n), \mu(m)] = \eta(n \ast m).
\]
Therefore the map \( \Phi \) is a morphism of crossed Leibniz \( \mathfrak{g} \)-algebras. \( \Box \)

**Proof of Lemma 5.3.** — Let us first show that \( \Sigma(m) \) is a well-defined biderivation. For any \( n, n' \in \mathfrak{N} \), we have
\[
\delta_{m^n}(n)^{\nu(n')} + \nu(n)\delta_{m^n}(n') \\
= \sigma(n \ast m)^{\nu(n')} + \nu(n)\sigma(n' \ast m) = \sigma((n \ast m)^{\nu(n')}) + \sigma(\nu(n)(n' \ast m)) \\
= \sigma(n^{\nu(n') \ast m}) + \sigma(n \ast m^{\nu(n')}) + \sigma(\nu(n)^{\nu(n')} \ast m \ast n') \\
= 2\sigma([n, n'] \ast m) - \sigma(\nu(n)m \ast n' - n \ast m^{\nu(n')}) \\
= 2\sigma([n, n'] \ast m) - \sigma([n, n'] \ast m) = \sigma([n, n'] \ast m) = \delta_{m^n}([n, n']), \\
\]
thus \( \delta_m \) is a derivation. Moreover, we have
\[
\Delta_{m^n}(n)^{\nu(n')} - \Delta_{m^n}(n')^{\nu(n)} \\
= - \sigma(m \ast n)^{\nu(n')} + \sigma(m \ast n')^{\nu(n)} = \sigma((m \ast n)^{\nu(n')}) - \sigma((m \ast n)^{\nu(n')}) \\
= \sigma(m^{\nu(n)} \ast n') + \sigma(m \ast n^{\nu(n')}) - \sigma(m^{\nu(n')} \ast n) - \sigma(m \ast n^{\nu(n')}) \\
= \sigma(m^{\nu(n)} \ast n' - m^{\nu(n') \ast n}) - \sigma(m \ast n^{\nu(n')} \ast n) - \sigma(m \ast n^{\nu(n')}) \\
= \sigma(m \ast [n, n']) - \sigma(m \ast [n, n']) = \sigma(m \ast [n, n']) = \sigma(m \ast [n, n']) \\
= - \sigma(m \ast [n, n']) = \Delta_{m^n}([n, n']),
\]
thus $\Delta_m$ is an anti-derivation. We have also

$$\lambda(\delta_m(n)) = \lambda(\sigma(n * m)) = \eta(n * m) = [\nu(n), \mu(m)] = \nu(n^\mu(m)),$$

$$\lambda(\Delta_m(n)) = -\lambda(\sigma(m * n)) = -\eta(m * n) = -[\mu(m), \nu(n)] = -\nu(\mu^*(m) n),$$

$$h^\delta_m(n) = h^\sigma(n * m) = \sigma(h(n * m)) = -\sigma(h(m * n)) = -h^\Delta_m(n),$$

$$\Delta_m(h^\nu m) = -\sigma(m * h^\nu) = \sigma(m * n^h) = -\Delta_m(n^h).$$

Therefore $\Sigma(m) = (\delta_m, \Delta_m, \mu(m))$ is a biderivation from $(\mathfrak{N}, \nu)$ to $(\mathfrak{P}, \lambda)$.

For any $h \in \mathfrak{g}$, $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$\begin{align*}
(h^\delta_m)(n) = & \delta_m(n^h) - \delta_m(n)^h = \sigma(n^h * m) - \sigma(n * m^h) \\
= & -\sigma(n * m^h) = \sigma(n * h^\nu m) = \delta_m(n),
\end{align*}$$

$$\begin{align*}
(h^\Delta_m)(n) = & h^\Delta_m(n) - \delta_m(h^\nu n) = h^\sigma(m * n) - \sigma(h^\nu n * m) \\
= & \sigma(h^\nu m * n) = \Delta_m(h^\rho m(n));
\end{align*}$$

and obviously $[[h, \mu(m)] = \mu(h^\nu m)$, thus we have $\Sigma(h^\nu m) = h^\Sigma(m)$. On the other side, we have

$$\begin{align*}
((\delta_m)^h)(n) = & \delta_m(n^h) - \delta_m(n)^h = \sigma(n * m^h) - \sigma(n^h * m) \\
= & \sigma(n * m^h) = \delta_m^h(n),
\end{align*}$$

and

$$\begin{align*}
((\Delta_m)^h)(n) = & \Delta_m(n^h) - \delta_m(n^h) = -\sigma(m * n^h) + \sigma(m * n^h) \\
= & -\sigma(m^h * n) = \Delta_m^h(n).
\end{align*}$$

Since $[\mu(m), h] = \mu(m^h)$, we get $\Sigma(m^h) = \Sigma(m)^h$. By definition of the map $\Sigma$, we have $\rho \Sigma(m) = \mu(m)$. Therefore the map $\Sigma$ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras.

6. Cohomological characterizations.

6.1. Non-abelian Leibniz cohomology.

Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \text{id}_\mathfrak{g})$, and let $(\mathfrak{M}, \mu)$ be a crossed Leibniz $\mathfrak{g}$-algebra. Given an element $m \in \mathfrak{M}$, we denote by $d_m$ (resp. $D_m$) the derivation (resp. anti-derivation) $g \mapsto gm$ (resp. $g \mapsto -m^g$) from $(\mathfrak{g}, \text{id}_\mathfrak{g})$ to $(\mathfrak{M}, \mu)$, and by $\overline{\mu(m)} := \mu(m) \mod Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. One easily checks that the triple $(d_m, D_m, \overline{\mu(m)})$ is a well-defined element of $\text{Bider}_\mathfrak{g}(\mathfrak{g}, \mathfrak{M})$. 

\[\square\]
DEFINITION-PROPOSITION 6.1. — Let $\mathcal{J}$ be the $K$-module freely generated by the biderivations $(d_m, D_m, \mu(m))$, $m \in \mathcal{M}$. Then $\mathcal{J}$ is a two-sided ideal of $\text{Bider}_g(g, \mathcal{M})$. The Leibniz algebra $\text{Bider}_g(g, \mathcal{M})/\mathcal{J}$ is denoted by $\mathcal{L}^1(g, \mathcal{M})$.

Proof. — For any $m \in \mathcal{M}$ and $(d, D, g) \in \text{Bider}_g(g, \mathcal{M})$, we have

$[(d, D, g), (d_m, D_m, \mu(m))] = (\delta_m, \Delta_m, [g, \mu(m)])$

with

$\delta_m(x) = d_m([x, g]) - d([x, \mu(m)]) = [x, g]_m - d([x, \mu(m)])$

$= \mu(d(x))_m - d(x)\mu(m) - \varepsilon d(\mu(m))$

$= [d(x), m] - [d(x), m] - \varepsilon D(\mu(m))$

$= d_m(x)$

where $m_1 := -D(\mu(m))$, $\Delta_m(x) = -D([x, \mu(m)]) - d_m([g, x]) = -D([x, \mu(m)]) - [g, x]_m$

$= -D(x)\mu(m) - D(\mu(m))x + \mu(D(x))_m$

$= -[D(x), m] + D(\mu(m))x + [D(x), m]$

$= D_{m_1}(x)$,

$\mu(m_1) = -\mu(D(\mu(m))) = [g, \mu(m)] = [g, \mu(m)]$; thus we have $[(d, D, g), (d_m, D_m, \mu(m))] \in \mathcal{J}$. On the other side, we have

$[(d_m, D_m, \mu(m)), (d, D, g)] = (\delta_m, \Delta'_m, [\mu(m), g])$

with

$\delta'_m(x) = d([x, \mu(m)]) - d_m([x, g]) = d([x, \mu(m)]) - [x, g]_m$

$= d(x)\mu(m) + \varepsilon d(\mu(m)) - \mu(d(x))_m$

$= [d(x), m] + \varepsilon d(\mu(m)) - [d(x), m]$

$= d_{m_2}(x)$

where $m_2 := d(\mu(m))$, $\Delta'_m(x) = -D_m([x, g]) - d([\mu(m), x]) = m^{[x, g]} - d([\mu(m), x])$

$= m^{\mu(d(x))} - d(\mu(m))x - \mu(m)d(x)$

$= [m, d(x)] - d(\mu(m))x - [m, d(x)]$

$= D_{m_2}(x)$,

$\mu(m_2) = \mu(d(\mu(m))) = [\mu(m), g] = [\mu(m), g]$;
thus we have \([\langle d_m, D_m, \mu(m) \rangle, (d, D, g)] \in \mathcal{I}\). Therefore the set \(\mathcal{I}\) is a two-sided ideal of \(\text{Bider}_g(g, M)\).

Similarly, given a crossed Leibniz \(g\)-algebra \((M, \mu)\), one defines
\[
\mathcal{H}L^0(g, M) := \{m \in M : \mu m = m = 0, \forall g \in g\}
\]
that is, the set of invariant elements of \(M\). From the relations
\[
[m, m'] = m^{\mu(m')} = 0 = \mu(m')m = [m', m], \quad m \in \mathcal{H}L^0(g, M), \quad m' \in M,
\]
it is clear that \(\mathcal{H}L^0(g, M)\) is contained in the centre of the Leibniz algebra \(M\).

**Proposition 6.2.** — For any exact sequence of crossed Leibniz \(g\)-algebras
\[
0 \to (A, \alpha) \xrightarrow{\alpha} (B, \lambda) \xrightarrow{\beta} (C, \mu) \to 0,
\]
there exists an exact sequence of \(K\)-modules
\[
0 \to \mathcal{H}L^0(g, A) \to \mathcal{H}L^0(g, B) \to \mathcal{H}L^0(g, C) \to \mathcal{H}L^1(g, A)
\]
\[
\to \mathcal{H}L^1(g, B) \xrightarrow{\beta^1} \mathcal{H}L^1(g, C)
\]
where \(\beta^1\) is a Leibniz algebra morphism.

**Proof.** — Everything goes smoothly except the definition of the connecting homomorphism \(\partial\). Given an element \(c \in \mathcal{H}L^0(g, C)\), let \(b \in B\) be any pre-image of \(c\) in \(B\). For any \(x \in g\), we have
\[
\beta(xb) = xc = 0 = c^x = \beta(b^x).
\]
Thus the element \(xb\) (resp. \(b^x\)) is in \(\ker(\beta) = \text{im}(\alpha)\). Since the morphism \(\alpha\) is injective, the map \(d^c : x \mapsto \alpha^{-1}(xb)\) (resp. \(D^c : x \mapsto \alpha^{-1}(b^x)\)) is a derivation (resp. an anti-derivation) from \((g, \text{id}_g)\) to \((A, \alpha)\). One easily checks that the triple \((d^c, D^c, 0)\) is a well-defined element of \(\text{Bider}_g(g, A)\) whose class in \(\mathcal{H}L^1(g, A)\) does not depend on the choice of the pre-image \(b\). We put
\[
\partial(c) := \text{class}(d^c, D^c, 0).
\]

**6.2. Non-abelian Leibniz homology.**

Let \(g\) be a Leibniz algebra viewed as the crossed Leibniz \(g\)-algebra \((g, \text{id}_g)\), and let \((M, \nu)\) be a crossed Leibniz \(g\)-algebra.
DEFINITION-PROPOSITION 6.3. — The map $\Psi_\mathfrak{N} : \mathfrak{N} \ast \mathfrak{g} \to \mathfrak{N}$ given on generators by

$$
\Psi_\mathfrak{N}(n \ast g) := n^g \quad \text{and} \quad \Psi_\mathfrak{N}(g \ast n) := g_n, \quad g \in \mathfrak{g}, \quad n \in \mathfrak{N},
$$

is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. We define the low-degrees non-abelian homology of $\mathfrak{g}$ with coefficients in $\mathfrak{N}$ to be

$$
\mathfrak{H}_0(\mathfrak{g}, \mathfrak{N}) := \text{coker} \Psi_\mathfrak{N} \quad \text{and} \quad \mathfrak{H}_1(\mathfrak{g}, \mathfrak{N}) := \ker \Psi_\mathfrak{N}.
$$

Proof. — To see that the map $\Psi_\mathfrak{N}$ is a Leibniz algebra morphism is equivalent to the fact that the Leibniz action of $\mathfrak{N}$ on $\mathfrak{g}$ is well-defined. The definition of the crossing homomorphism $\eta_\mathfrak{N} : \mathfrak{N} \ast \mathfrak{g} \to \mathfrak{g}$ implies that $\Psi_\mathfrak{N}$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. \qed

PROPOSITION 6.4. — For any exact sequence of crossed Leibniz $\mathfrak{g}$-algebras

$$
0 \to (\mathfrak{A}, 0) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \to 0,
$$

there exists an exact sequence of $\mathbb{K}$-modules

$$
\mathfrak{H}_1(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{H}_1(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{H}_1(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{H}_0(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{H}_0(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{H}_0(\mathfrak{g}, \mathfrak{C}) \to 0.
$$

Proof. — We know that the functor $- \ast \mathfrak{g}$ is right exact (Proposition 4.5). Therefore Proposition 6.4 is nothing but the "snake-lemma" applied to diagram

$$
\begin{array}{ccc}
\mathfrak{A} \ast \mathfrak{g} & \xrightarrow{\Psi_\mathfrak{A}} & \mathfrak{B} \ast \mathfrak{g} & \xrightarrow{\Psi_\mathfrak{B}} & \mathfrak{C} \ast \mathfrak{g} & \to & 0 \\
\downarrow \Psi_\mathfrak{A} & & \downarrow \Psi_\mathfrak{B} & & \downarrow \Psi_\mathfrak{C} & \\
0 & \to & \mathfrak{A} & \to & \mathfrak{B} & \to & \mathfrak{C} & \to & 0
\end{array}
$$

which is obviously commutative. \qed

6.3. Universal central extension.

Let $\mathfrak{g}$ be a Leibniz algebra and let $\Psi := \Psi_\mathfrak{g}$ be the morphism defining the homolgy $\mathfrak{H}_*(\mathfrak{g}, \mathfrak{g})$. From the relations $v)$ of Definition-Theorem 4.1, it is clear that $\Psi : \mathfrak{g} \ast \mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$ is a central extension of Leibniz algebras (see [4]).
Theorem 6.5. — If the Leibniz algebra $g$ is perfect and free as a $K$-module, then the morphism $\Psi : g \ast g \to [g, g] = g$ is the universal central extension of $g$. Moreover, we have an isomorphism of $K$-modules

$$H_2 L_1(g, g) \cong H_2 L_2(g).$$

Proof. — It is enough to prove the universality of the central extension $\Psi : g \ast g \to [g, g] = g$. Let $\alpha : C \to g$ be a central extension of $g$. Since $\ker(\alpha)$ is central in $C$, the quantity $[\alpha^{-1}(x), \alpha^{-1}(y)]$ does not depend on the choice of the pre-images $\alpha^{-1}(x)$ and $\alpha^{-1}(y)$ where $x, y \in g$. One easily checks that the map $\phi : g \ast g \to C$ given on generators by

$$\phi(x \ast y) := [\alpha^{-1}(x), \alpha^{-1}(y)]$$

is a well-defined Leibniz algebra morphism such that $\alpha \phi = \Psi$. The uniqueness of the map $\phi$ follows from Lemma 2.4 of [4] since the perfectness of $g$ implies that of $g \ast g$:

$$x \ast y = \left(\sum_i [x_i, x_i']\right) * \left(\sum_j [y_j, y_j']\right) = \sum_{i,j} [x_i \ast x_i', y_j \ast y_j'].$$

By definition we have $H_2 L_1(g, g) = \ker()$. After [4] the kernel of the universal central extension of a Leibniz algebra $g$ is canonically isomorphic to $H_2 L_2(g)$. Therefore we have

$$H_2 L_1(g, g) \cong H_2 L_2(g).$$

7. The Milnor-type Hochschild homology.

Let $A$ be an associative algebra viewed as a Leibniz (in fact Lie) algebra for the bracket given by $[a, b] := ab - ba, a, b \in A$. Recall that the $K$-module $L(A) := A^{\otimes 2} / \text{im}(b_3)$ is a Leibniz (non-Lie) algebra for the bracket defined by

$$(x \otimes y, x' \otimes y') := (xy - yx) \otimes (x'y' - y'x'), \forall x, y, x', y' \in A.$$  

Proposition 7.1. — The operations given by

- $A \times L(A) \to L(A)$, $a(x \otimes y) := [a, x] \otimes y - [a, y] \otimes x$,
- $L(A) \times A \to L(A)$, $(x \otimes y)^a := [x, a] \otimes y + x \otimes [y, a]$  

confer to $L(A)$ a structure of Leibniz $A$-algebra. Moreover the map

$$\mu_A : L(A) \to A, x \otimes y \mapsto [x, y] = xy - yx$$
equips $L(A)$ with a structure of crossed Leibniz $A$-algebra.

**Proof.** — The operations are well-defined since we have

$$
\begin{align*}
\rho(b_3(x \otimes y \otimes z)) &= b_3(ax \otimes y \otimes z - a \otimes z \otimes xy - za \otimes x \otimes y \\
&\quad + a \otimes yz \otimes x + a \otimes zx \otimes y - a \otimes y \otimes zx)
\end{align*}
$$

and

$$
(b_3(x \otimes y \otimes z))^a = b_3(-ax \otimes y \otimes z + xy \otimes a \otimes z + x \otimes y \otimes za \\
- x \otimes a \otimes yz - zx \otimes a \otimes y - zx \otimes y \otimes a).
$$

One easily checks that the couple $(L(A), \mu_A)$ is a pre-crossed Leibniz $A$-algebra. Moreover we have

$$
\begin{align*}
\mu_A(x \otimes y)(x' \otimes y') - [x \otimes y, x' \otimes y'] &= b_3([x, y] \otimes x' \otimes y' - [x, y] \otimes y' \otimes x') \\
(x \otimes y)\mu_A(x \otimes y) - [x \otimes y, x' \otimes y'] &= b_3(x \otimes [x', y'] \otimes y - x \otimes y \otimes [x', y']).
\end{align*}
$$

Thus the Leibniz $A$-algebra $(L(A), \mu_A)$ is crossed. $\square$

It is clear that the inclusion map $[A, A] \hookrightarrow A$ induces a structure of crossed Leibniz $A$-algebra on the two-sided ideal $[A, A]$, and that the map $\mu_A : L(A) \rightarrow [A, A]$ is a morphism of crossed Leibniz $A$-algebras. Moreover we have an exact sequence of $k$-modules

$$
0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{\mu_A} [A, A] \rightarrow 0.
$$

**Lemma 7.2.** — The Leibniz algebra $A$ acts trivially on $\text{HH}_1(A)$.

**Proof.** — One easily checks that

$$
\begin{align*}
\rho(x \otimes y) &= a \otimes [x, y] + b_3(a \otimes x \otimes y - a \otimes y \otimes x) \equiv a \otimes [x, y] \text{ in } L(A)
\end{align*}
$$

and

$$
\begin{align*}
(x \otimes y)^a &= [x, y] \otimes a + b_3(x \otimes a \otimes y - x \otimes y \otimes a) \equiv [x, y] \otimes a \text{ in } L(A).
\end{align*}
$$

Therefore, if $\omega = \sum \lambda_i(x_i \otimes y_i) \in \text{HH}_1(A)$, that is $\sum \lambda_i[x_i, y_i] = 0$, then we have

$$
\begin{align*}
\rho \omega &= \sum \lambda_i \rho(x_i \otimes y_i) \equiv \sum \lambda_i(a \otimes [x_i, y_i]) \equiv a \otimes \sum \lambda_i[x_i, y_i] = 0
\end{align*}
$$

and

$$
\begin{align*}
\omega^a &= \sum \lambda_i(x_i \otimes y_i)^a \equiv \sum \lambda_i([x_i, y_i] \otimes a) \equiv (\sum \lambda_i[x_i, y_i]) \otimes a = 0
\end{align*}
$$

for any $a \in A$. $\square$

As an immediate consequence, we get the following
COROLLARY 7.3. — The sequence
\[ 0 \to \text{HH}_1(A) \to L(A) \xrightarrow{\mu_A} [A, A] \to 0 \]
is an exact sequence of crossed Leibniz A-algebras.

We deduce from Proposition 6.4 an exact sequence of K-modules
\[ \mathcal{H}L_1(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) \to \mathcal{H}L_1(\mathfrak{A}, L(\mathfrak{A})) \to \mathcal{H}L_0(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \to \mathcal{H}L_0(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) \to \mathcal{H}L_0(\mathfrak{A}, L(\mathfrak{A})) \to \mathcal{H}L_0(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \to 0. \]

Since A and HH_1(A) act trivially on each other, we have
\[ \mathcal{H}L_0(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) = \text{HH}_1(\mathfrak{A}) \]
and
\[ \mathcal{H}L_2(\mathfrak{A}, \text{HH}_1(\mathfrak{A})) = \mathfrak{A} \ast \text{HH}_1(\mathfrak{A}) \cong \mathfrak{A} / [\mathfrak{A}, \mathfrak{A}] \otimes \text{HH}_1(\mathfrak{A}) \oplus \text{HH}_1(\mathfrak{A}) \otimes \mathfrak{A} / [\mathfrak{A}, \mathfrak{A}]. \]

On the other hand, it is clear that
\[ \mathcal{H}L_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \cong [\mathfrak{A}, \mathfrak{A}] / [\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]]. \]

Therefore we can state

THEOREM 7.4. — For any associative algebra A with unit, there exists an exact sequence of K-modules
\[ A / [A, A] \otimes \text{HH}_1(A) \oplus \text{HH}_1(A) \otimes A / [A, A] \to \mathcal{H}L_1(\mathfrak{A}, L(\mathfrak{A})) \to \mathcal{H}L_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \]
\[ \to \text{HH}_1(A) \to \text{HH}_1^M(A) \to [A, A] / [A, [A, A]] \to 0 \]
where HH_1^M(A) denotes the Milnor-type Hochschild homology of A.

Proof. — Recall that HH_1^M(A) is defined to be the quotient of A \otimes A by the relations
\[ a \otimes [b, c] = 0, \ [a, b] \otimes c = 0, \ b_3(a \otimes b \otimes c) = 0 \]
for any a, b, c \in A (see [6, 10.6.19]). By definition L(A) = A \otimes A / \text{im}(b_3)
and from the proof of Lemma 7.2, we get
\[ \Psi_{L(A)}(a \ast (x \otimes y)) = \alpha(x \otimes y) \equiv a \otimes [x, y] \]
and
\[ \Psi_{L(A)}((x \otimes y) \ast a) = (x \otimes y)^a \equiv [x, y] \otimes a. \]
Therefore it is clear that \( \mathcal{H}L_0(\mathfrak{A}, L(\mathfrak{A})) = \text{coker}(L(\mathfrak{A})) \) is isomorphic to HH_1^M(A).
Remark. — The $\mathbb{K}$-modules $\text{HH}_1(A)$ and $\text{HH}^M_1(A)$ coincide when the associative algebra $A$ is superperfect as a Leibniz algebra that is, $A = [A, A]$ and $HL_2(A) = 0$. Also, if the associative algebra $A$ is commutative, then we have

$$\text{HH}_1(A) \cong \text{HH}^M_1(A) \cong \Omega^1_{A|\mathbb{K}}.$$ 

Let us also mention that the Milnor-type Hochschild homology appears in the description of the obstruction to the stability

$$\text{HL}_n(gl_{n-1}(A)) \rightarrow \text{HL}_n(gl_n(A)) \rightarrow \text{HH}^M_{n-1}(A) \rightarrow 0$$

where $gl_n(A)$ is the Lie algebra of matrices with entries in the associative algebra $A$ (see [2], [6, 10.6.20]).

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