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# A GENERALIZATION OF JAEGER-NOMURA'S BOSE MESNER ALGEBRA ASSOCIATED TO TYPE II MATRICES

by Makoto MATSUMOTO

## 1. Spin model and its Bose-Mesner algebra.

Let  $X$  be a finite set with cardinality  $n$ , and  $\mathbf{C}[X]$  be the vector space with basis  $X$  over the complex number field  $\mathbf{C}$ . We consider the set of matrices  $M_X := \text{Hom}(\mathbf{C}(X), \mathbf{C}(X))$ , which is identified with the algebra on  $n \times n$  matrices, by identifying  $X = \{1, 2, \dots, n\}$ . An  $n \times n$  matrix  $W \in M_X$  is said to be a (two-weight<sup>1</sup>) *spin model*, if it satisfies three algebraic conditions called type I, type II, and type III conditions. These names came from the corresponding Reidemeister moves in the Knot theory, see Jones [6] and Bannai-Bannai [1]. Spin models were introduced to describe a link invariant. For a matrix  $M \in M_X$ , we denote by  $M(a, b)$  its  $(a, b)$ -component for every  $a, b \in X$ . The usual product of matrices, denoted by  $M'M$  or  $M' \cdot M$ , is defined by  $M'M(a, b) = \sum_{x \in X} M'(a, x)M(x, b)$ , while the *Hadamard product*  $M' \circ M$  is defined by  $(M' \circ M)(a, b) := M'(a, b)M(a, b)$ , which is commutative. We denote by  $J$  a matrix whose components are all 1, which is the unit for Hadamard product. If  $M$  is invertible with respect to the Hadamard product, its inverse is denoted by  $M^\circ$ . Then, the type II condition is described by

$$M \cdot {}^tM^\circ = nI,$$

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<sup>1</sup> Here a spin model means two weight, unless otherwise stated.

where  $I$  denote the unit matrix of size  $n$ . The type I condition is automatically satisfied if  $M$  has both type II and type III properties. Type III condition will be explained later.

### 1.1. A Bose-Mesner algebra associated with a type II matrix.

A matrix  $B$  which satisfies the type II condition above is called a *type II matrix*. For a type II matrix, we introduced [4] after Jaeger [5] and Nomura [7] an algebra  $N(W)$  as follows. For each  $b, c \in X$ , define a vector

$$\mathbf{y}_{b,c} := \left( \frac{W(b,x)}{W(c,x)} \right)_{x \in X} \in \mathbf{C}[X].$$

Then we define an algebra (possibly called Jaeger-Nomura algebra of  $W$ )

$$N(W) := \{M \in M_X \mid \mathbf{y}_{b,c} \text{ is } M\text{'s eigen vector for all } b, c \in X\}.$$

It can be proved that  $N(W)$  is a Bose-Mesner (BM) algebra, i.e. a commutative subalgebra of  $M_X$  closed under Hadamard product and transpose with identity  $I \in M_X$  for usual product and identity  $J \in M_X$  for Hadamard product. (Thus, it corresponds to an association scheme.)

## 2. Spin models of index two.

Jaeger-Nomura [5] introduced the notion of index of a spin model. I don't recall the definition, but according to their result we may define a spin model  $W$  of index two to be a spin model with the following form:

$$(1) \quad W = \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -B & B & C & C \\ B & -B & C & C \end{pmatrix},$$

which was fully used in [5] to construct a spin model with *chiral* link invariant, i.e. one which distinguishes some link from its mirror image.

**PROPOSITION 1** ([5], Proposition 8). — *Let  $A, C$  be a symmetric matrix of the same size. Then, the matrix  $W$  in (1) is a spin model if and only if the following conditions are satisfied:*

(A)  $A, C$  are spin models and  $B$  is a type II matrix.

(B) Let  $n$  be the size of  $W$ , and  $D$  be one of  $\pm\sqrt{n}$ . Then the following four equalities hold:

$$(2) \quad \sum_y A(a, y) \frac{B(y, b)}{B(y, c)} = DC(b, c)^{-1} \frac{B(a, b)}{B(a, c)}$$

$$(3) \quad \sum_y C(a, y) \frac{B(b, y)}{B(c, y)} = DA(b, c)^{-1} \frac{B(b, a)}{B(c, a)}$$

$$(4) \quad \sum_y A(a, y)^{-1} B(y, b) B(y, c) = -DC(b, c)^{-1} B(a, b)^{-1} B(a, c)^{-1}$$

$$(5) \quad \sum_y C(a, y)^{-1} B(b, y) B(c, y) = -DA(b, c)^{-1} B(b, a)^{-1} B(c, a)^{-1}.$$

The concrete result of this article is just the following.

**THEOREM 2.** — Let  $A, B, C, D$  be type II matrices. Then the conditions (2) and (3) are equivalent, and the conditions (4) and (5) are equivalent.

The first equivalence is a direct consequence of [4], Theorem 1, as shown below, but the second equivalence is more delicate and I prove it by introducing something like a duality on something like BM algebra.

*Proof of (2)  $\Leftrightarrow$  (3).* — Let us define  $\mathbf{y} = b, c$  as in §1.1. Then (2) for all  $a \in X$  is equivalent to

$$A\mathbf{y}_{b,c} = DC(b, c)\mathbf{y}_{b,c},$$

and thus  $A \in N(B)$ . Also this shows that  $\Phi_W(A) = D \cdot C^\circ$ , where  $\Phi_W$  is defined as follows.

**THEOREM 3** ([4], Theorem 1). — Let  $W$  be a type II matrix of size  $n$ . Let  $M \in N(W)$ . Write  $\Phi_W(M)(a, b)$  for the eigenvalue of  $M$  with respect to  $\mathbf{y}_{a,b}$ . Thus,  $\Phi_W(M) \in M_X$ . Then,  $\Phi_W$  induces a dual pair

$$\Phi_W : N(W) \rightarrow N({}^tW),$$

i.e. a linear isomorphism satisfying  $\Phi_W(M'M) = \Phi_W(M') \circ \Phi_W(M)$ ,  $\Phi_W(M' \circ M) = \frac{1}{n} \Phi_W(M') \Phi_W(M)$ , and  $\Phi_{{}^tW}(\Phi_W(M)) = n \cdot {}^tM$ .

By this, the condition (2) is equivalently transformed to  $\Phi_W(A^{-1}) = D^{-1}C$  (since inverse is mapped to Hadamard inverse), then by taking  $\Phi_{{}^tW}$  we have  $\Phi_{{}^tW}(\Phi_W(A^{-1})) = D^{-1}\Phi_{{}^tW}(C)$ . The left hand side is  $n {}^t(A^{-1})$ ,

which is  $A^\circ$ , since by type II condition on  $A$  we have  ${}^tA^\circ = n \cdot A^{-1}$ . Thus it is equivalent to  $\Phi_{tW}(C) = DA^\circ$ , i.e. (3).

For the rest of Theorem 2, we use a similar generalization according to a categorical concept.

### 3. A category $N$ .

We shall define a category  $N$  (possibly called Jaeger-Nomura category), which is a generalization of Jaeger-Nomura algebra  $N(W)$ .  $N$  has a beautiful duality  $\Phi$ , which is not a functor unfortunately.

#### 3.1. An abstract nonsense.

Although it is a bit confusing, by  $\mathbf{y}$  we denote a function  $X \times X \rightarrow \mathbf{C}[X] - \{0\}$ , or equivalently  $n \times n$  nonzero vectors in  $\mathbf{C}[X]$  indexed by  $X \times X$ , i.e.

$$\mathbf{y} = (\mathbf{y}_{a,b} \in \mathbf{C}[X], \neq 0 \mid a, b \in X).$$

(I.e.  $\mathbf{y}$  is not a single vector but an ordered set of vectors, from now on.) Let  $\mathbf{y}' := (\mathbf{y}'_{a,b} \mid a, b \in X)$  be another set of vectors. We define the set of morphisms from  $\mathbf{y}$  to  $\mathbf{y}'$  by

$$\text{Hom}_N(\mathbf{y}, \mathbf{y}') := \{M \in M_X \mid M\mathbf{y}_{a,b} = c_{a,b}\mathbf{y}'_{a,b} \text{ for some } c_{a,b} \in \mathbf{C} \text{ for all } a, b \in X\},$$

which is a subvector space of  $M_X = \text{Hom}(\mathbf{C}[X], \mathbf{C}[X])$ . It is clear that a morphism from  $\mathbf{y}$  to  $\mathbf{y}'$  can be composed with one from  $\mathbf{y}'$  to  $\mathbf{y}''$  to obtain one from  $\mathbf{y}$  to  $\mathbf{y}''$  (i.e. we defined a category  $C$  with object  $\mathbf{y}$ 's and morphisms  $\text{Hom}_N(\mathbf{y}, \mathbf{y}')$ ). We define a linear homomorphism

$$\Phi_{\mathbf{y}, \mathbf{y}'} : \text{Hom}_N(\mathbf{y}, \mathbf{y}') \rightarrow M_X$$

by  $\Phi(M) := (c(a, b))_{a,b}$ . It follows that

$$\Phi(M'M) = \Phi(M') \circ \Phi(M)$$

for  $M' \in \text{Hom}_N(\mathbf{y}', \mathbf{y}'')$ . Thus,  $\Phi$  is a functor from  $C$  to the category consisting of one object  $*$ , with homomorphisms  $M_X$ , with composition by Hadamard product and identity  $J$ . We denote this category by  $M_X^\circ$ .

If there is no fear of confusion, we omit the suffix of  $\Phi$ .

### 3.2. An object associated with a pair of type II matrices.

Let  $K, L$  be two type-II matrices on the same set  $X$ . Define a column vector by

$$\mathbf{y}_{a,b}^{K,L} := (K(a, x)L(b, x))_{x \in X}.$$

This gives a set of ordered  $n \times n$  vectors indexed by  $X \times X$ , i.e. an object of  $\mathcal{C}$ . We denoted it by  $\mathbf{y}^{K,L}$ .

*Remark.* — In this terminology,  $\mathbf{y}_{b,c}$  defined in §1.1 is  $\mathbf{y}_{b,c}^{W,W^\circ}$ . Moreover,

$$N(W) = \text{Hom}_N(\mathbf{y}^{W,W^\circ}, \mathbf{y}^{W,W^\circ}) = \text{End}_N(\mathbf{y}^{W,W^\circ}),$$

which becomes an  $\mathbf{C}$ -algebra by the general fact on additive category. Non trivial part is that it is closed also under Hadamard product.

### 3.3. A quasi duality.

The following is the main result of this article.

**THEOREM 4.** — *Let  $K, L, K', L'$  be arbitrary type II matrices of same size. We write  $\Phi_{(K,L;K',L')}$  for  $\Phi_{\mathbf{y}^{K,L}, \mathbf{y}^{K',L'}}$ .*

(1)  $\Phi_{(K,L;K',L')}$  defines a linear map

$$\text{Hom}_N(\mathbf{y}^{K,L}, \mathbf{y}^{K',L'}) \rightarrow \text{Hom}_N(\mathbf{y}^{t_{L'}, t_{L^\circ}}, \mathbf{y}^{t_{K'^\circ}, t_K}).$$

(2)  $\Phi_{(t_{L'}, t_{L^\circ}; t_{K'^\circ}, t_K)}$  is a mapping of converse direction, and

$$\Phi_{(t_{L'}, t_{L^\circ}; t_{K'^\circ}, t_K)}(\Phi_{(K,L;K',L')}(M)) = nM$$

holds. Thus,  $\Phi$  is a linear bijection.

*Proof.* — All we need is to prove

$$\Phi(M)\mathbf{y}_{(g,h)}^{t_{L'}, t_{L^\circ}} = nM(g, h)\mathbf{y}_{(g,h)}^{t_{K'^\circ}, t_K},$$

or equivalently

$$(6) \quad \sum_b \Phi(M)(a, b)L'(b, g)L(b, h)^{-1} = nM(g, h)K'^{-1}(a, g)K(a, h).$$

The definition of  $\Phi(M)$  is given by

$$\sum_z M(c, z)K(a, z)L(b, z) = \Phi(M)(a, b)K'(a, c)L'(b, c)$$

holds for all  $a, b, c \in X$ . We specialize  $c$  to  $g$ , then we have

$$\Phi(M)(a, b)L'(b, g) = K'(a, g)^{-1} \sum_z M(c, z)K(a, z)L(b, z).$$

Plug this into the left hand side of (6) to have

$$\begin{aligned} & \sum_b \Phi(M)(a, b)L'(b, g)L(b, h)^{-1} \\ &= K'(a, g)^{-1} \sum_b \sum_z M(c, z)K(a, z)L(b, z)L(b, h)^{-1} \\ &= K'(a, g)^{-1} \sum_z M(c, z)K(a, z) \sum_b L(b, z)L(b, h)^{-1} \\ &= nM(g, h)K'^{-1}(a, g)K(a, h) \quad (\text{since } L \text{ is type II}), \end{aligned}$$

as desired.  $\square$

From now on, we shall denote

$$(K, L) := \mathbf{y}^{K, L},$$

$$N(K, L; K', L') := \text{Hom}_N(\mathbf{y}^{K, L}, \mathbf{y}^{K', L'}),$$

and consider only the objects of the form  $(K, L)$  in the category  $C$ . Let  $N$  be the full subcategory of  $C$  induced by these objects. In other words, we consider pairs of type II matrices  $(K, L)$ , and a morphism from  $(K, L)$  to  $(K', L')$ , denoted by  $N(K, L; K', L') := \text{Hom}_N(\mathbf{y}^{K, L}, \mathbf{y}^{K', L'})$ .

*Remark.* —  $\Phi$  is a generalization of duality between  $N(W)$  and  $N({}^tW)$ . We have  $N(W) = N(W, W^\circ; W, W^\circ)$  by definition. If we specialize to  $K = K'$  and  $L = L'$ , we have

$$N(K, L; K, L) \xrightarrow{\Phi} N({}^tL, {}^tL^\circ; {}^tK^\circ, {}^tK).$$

This shows that  $\Phi$  is *not* a functor. By the way, the left hand side is a subalgebra of  $M_X$  with respect to the ordinary product. From 2 of Theorem 4, it follows that the right hand side is closed under the Hadamard product, and these two algebras are isomorphic as an *abstract* algebra. Imposing the right hand side to be an algebra with usual product, we may require  $({}^tL, {}^tL^\circ) = ({}^tK^\circ, {}^tK)$ , i.e.  $L = K^\circ$ . Then the right hand side is

$$N({}^tK^\circ, {}^tK; {}^tK^\circ, {}^tK),$$

which is closed also by the usual product, thus we recover the Jaeger-Nomura algebra. Note that the transpose induces a linear bijection

$$(7) \quad N(K, L; K', L') \xrightarrow{t} N(L, K; L', K').$$

By composing with this transpose we have

$$N(K, K^\circ; K, K^\circ) \xrightarrow{\Phi} N({}^tK^\circ, {}^tK; {}^tK^\circ, {}^tK) \xrightarrow{t} N({}^tK, {}^tK^\circ; {}^tK, {}^tK^\circ),$$

i.e. the duality

$$N(K) \xrightarrow{\Psi} N({}^tK)$$

defined in [4], §3.1 (note that  $N({}^tK)$  is  $N'(K)$  there).

## 4. Application of the category $N$ .

### 4.1. Proof of (4) $\Leftrightarrow$ (5).

(4) is equivalent to

$$A^\circ \in N({}^tB, {}^tB; {}^tB^\circ, {}^tB^\circ) \text{ and } \Phi(A^\circ) = -DC.$$

We calculate

$$\Phi(A^\circ \cdot A) = \Phi(A^\circ) \circ \Phi(A).$$

The left hand side is  $\Phi(nI)$  by type II property of  $A$ , then by definition of  $\Phi$ , it is  $nJ$ . The right hand side is, by condition (4),  $-DC \circ \Phi(A)$ . By moving  $DC$  to the left hand side, we have

$$-DC^\circ = \Phi(A).$$

Applying  $\Phi$ , we have

$$\Phi(C^\circ) = -D^{-1}\Phi(\Phi(A)) = DA,$$

which is nothing but the condition (5).

### 4.2. Four-weight spin model.

Four-weight spin model was introduced in [1] as a generalizaion of two-weight model, which was fully used in determination of the link invariant in [5]. We can paraphrase their definition in terms on  $N$ . For the original definition, see [1].

**PROPOSITION 5.** — *A pair of  $n \times n$  matrix  $(W_1, W_2)$  is a four weight spin model if and only if the following conditions hold:*

1.  $W_1, W_2$  are type II matrices.



2.  $W_1 \in N({}^tW_2^\circ, {}^tW_1; {}^tW_2^\circ, {}^tW_1)$  and  $\Phi_{{}^tW_2^\circ, {}^tW_1; {}^tW_2^\circ, {}^tW_1}(W_1) = D \cdot {}^tW_2^\circ$   
(which is automatically in  $N(W_1, W_1^\circ; W_2, W_2^\circ)$ ).
3.  $W_1 \in N(W_2, W_1^\circ; W_2, W_1^\circ)$  and  $\Phi_{W_2, W_1^\circ; W_2, W_1^\circ}(W_1) = DW_2^\circ$   
(which is automatically in  $N({}^tW_1^\circ, {}^tW_1; {}^tW_2^\circ, {}^tW_2)$ ).

*Proof.* — In the terminology in [1],  $Y^{i,j} = \mathbf{y}^{W_i, {}^tW_j}$ . Their condition III<sub>1</sub> is equivalent to 2, and their condition III<sub>9</sub> is equivalent to 3.  $\square$

*Remark.* — It is shown that III<sub>1</sub> to III<sub>8</sub> are equivalent, and III<sub>9</sub> to III<sub>16</sub> are equivalent in [1], where these 16 conditions naturally arise from Reidemeister move of type III with various orientations. These equivalences are easily proved by  $N$  and  $\Phi$ . For example, note that III<sub>1</sub> is equivalent to  $W_4 = {}^tW_2^\circ = \frac{1}{D}\Phi(W_1)$ , and by taking  $\Phi$  again, we have the condition III<sub>2</sub>. Then,  ${}^tW_3 = W_1^\circ$  and  $\Phi(A^\circ) = n \cdot (\Phi(A)^\circ)$  implies III<sub>3</sub>, etc.

*Remark.* — Huang-Guo [2] proved that

$${}^tW_1^\circ \cdot W_1^\circ \in N({}^tW_2).$$

This can be interpreted in  $N$  by

$$W_1^\circ \in N({}^tW_2, {}^tW_2^\circ; {}^tW_1, {}^tW_1^\circ) \quad (\text{III}_8)$$

and

$${}^tW_1^\circ \in N({}^tW_1, {}^tW_1^\circ; {}^tW_2, {}^tW_2^\circ) \quad (\text{III}_2)$$

and thus

$${}^tW_1^\circ \cdot W_1^\circ \in N({}^tW_2, {}^tW_2^\circ; {}^tW_2, {}^tW_2^\circ) = N({}^tW_2).$$

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