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TOSHIZUMI FUKUI
SATOSHI KOIKE
MASAHIRO SHIOTA

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MODIFIED NASH TRIVIALITY OF A FAMILY OF ZERO-SETS OF REAL POLYNOMIAL MAPPINGS

by T. FUKUI, S. KOIKE and M. SHIOTA

0. Introduction.

Studying classification or stability problems on real singularities, it becomes important to show triviality theorems. In this paper, we consider triviality of a family of zero-sets of real polynomial mappings. Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in J)$ be a polynomial mapping. Then we define $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ by $F(x;t) = f_t(x)$. The notion of C^r triviality, $r = 0, 1, 2, \ldots, \infty, \omega$, is natural for trivializing $(\mathbf{R}^n \times J, F^{-1}(0))$. We recall two typical results on C^r triviality.

Example (0.1) (H. Whitney [36]). — Let $J = (1, \infty)$, and let $f_t : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(t \in J)$ be the homogeneous polynomial function defined by $f_t(x, y) = xy(x - y)(x - ty)$. Then $f_t^{-1}(0)$ consists of 4 lines for any $t \in J$, and $(\mathbf{R}^2 \times J, F^{-1}(0))$ is C^0 trivial. But $(\mathbf{R}^2, f_{t_1}^{-1}(0))$ and $(\mathbf{R}^2, f_{t_2}^{-1}(0))$ are not locally C^1 equivalent for $t_1 \neq t_2$. So of course, $(\mathbf{R}^2 \times J, F^{-1}(0))$ is not locally C^1 trivial along $\{0\} \times J$.

Let $P_{[r]}(n, p)$ denote the set of real polynomial mappings: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ of degree not exceeding r, and let $\{(\mathbf{R}^n, w^{-1}(0)) : w \in P_{[r]}(n, p)\}/\underset{C^0}{\sim}$ denote the quotient set of $\{(\mathbf{R}^n, w^{-1}(0)) : w \in P_{[r]}(n, p)\}$ by C^0 equivalence.

THEOREM (0.2) (T. Fukuda [11], A.N. Varčenko [35]). — *The cardinal number of* $\{(\mathbf{R}^n, w^{-1}(0)) : w \in P_{[r]}(n, p)\}/\underset{C^0}{\sim}$ *is finite.*

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The Whitney example shows that C^1 triviality is too strong even when we consider the local triviality of algebraic curves with isolated singularities. On the other hand, the Fukuda-Varčenko theorem tells of the existence of a finite stratification of $P_{[r]}(n,p)$ such that over each stratum the family of zero-sets is C^0 trivial. But C^0 triviality preserves only topology. Then we have the following natural question:

QUESTION (0.3). — *Does a natural and strong triviality which clarifies the structure of C^0 triviality hold under some generic condition?*

In this paper, we introduce the notion of *modified Nash triviality* (in the local sense) as such a triviality, and give some fundamental results to construct a local theory for it. This kind of direction was first tackled by T.-C. Kuo ([24], [25], [26]). He introduced the notion of *blow-analytic triviality* (or *modified analytic triviality*) for a family of real analytic functions. But it is not natural in the polynomial case. Furthermore analytic equivalence is definitely weaker than Nash equivalence even in the nonsingular algebraic case (see Theorem (1.4) in §1). This is the reason why we introduce the notion of modified Nash triviality for a family of zero-sets of real polynomial mappings. Actually, the second author has shown the following result from this viewpoint.

THEOREM (0.4) ([22]). — *Let J be an open interval, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a weighted homogeneous polynomial mapping. If $f_t^{-1}(0)$ has an isolated singularity at the origin for any $t \in J$, then $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a (finite) modified Nash trivialization in the global sense.*

Remark (0.5). — Applying this result to the Whitney example above, we see that $(\mathbf{R}^2 \times J, F^{-1}(0))$ admits a β-modified Nash trivialization along $\{0\} \times J$, where $\beta : M \to \mathbf{R}^2$ denotes the blow-up at the origin.

Two main parts of the proof of Theorem (0.4) consist of

(i) constructing a Nash modification so that $F^{-1}(0)$ is modified C^∞ trivial, and

(ii) showing the Nash triviality theorem for a family of pairs of compact Nash manifolds and compact Nash submanifolds.

In general, the latter type of Nash triviality theorem plays a very important role in the proof of modified Nash triviality theorems. This kind of tool was first developed by the third author ([33], [34]), and M. Coste and the third author have made progress in this direction ([4], [5]). In this paper, it becomes necessary to show the Nash triviality theorem for a family of pairs of compact Nash manifolds with boundary and compact Nash submanifolds with boundary (Theorem I), in order to prove local modified Nash triviality theorems. Once this is done, we apply this tool to the following cases:

(i) Each f_t satisfies some conditions on the Newton polyhedron.

(ii) The zero-set of the weighted initial form of each f_t admits an isolated singularity.

In fact, it is well-known that $(\mathbf{R}^n \times J, F^{-1}(0))$ (or $(\mathbf{C}^n \times J, F^{-1}(0))$) is (locally) C^0 trivial under one of these conditions (e.g. M. Buchner-W. Kucharz [3], J. Damon [6], [7], T. Fukuda [12], H. King [20], A.G. Kouchnirenko [23], M. Oka [29]). In particular, the first author has shown that $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a modified analytic trivialization under condition (i) ([13]). In the function case, F admits a modified analytic trivialization (the first author and E. Yoshinaga [15]). By using our tool, however, we can show that $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a modified Nash trivialization under these conditions (Theorems II, IV). We further give the classification theorem for modified Nash triviality (Theorem V) corresponding to the aforementioned Fukuda-Varčenko theorem.

We shall describe the main results in §2, and give their proofs in §§3–6. In §7, we present how our method applies to polynomial families which are explicitly given. The authors would like to thank the referee for reading the previous version of the paper carefully.

1. Preliminaries.

1.1. Some properties of Nash manifolds.

We first recall some important results about Nash manifolds. A *semialgebraic set* of $\mathbf{R}^n$ is a finite union of the form

$$\{x \in \mathbf{R}^n : f_1(x) = \cdots = f_k(x) = 0, g_1(x) > 0, ..., g_m(x) > 0\}$$

where $f_1, \dots, f_k, g_1, \dots, g_m$ are polynomial functions on $\mathbf{R}^n$. Let $r = 1, 2, \dots, \infty, \omega$. A semialgebraic set of $\mathbf{R}^n$ is called a C^r (affine) *Nash manifold* if it is a C^r submanifold of $\mathbf{R}^n$. Let $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$

be C^r Nash manifolds. A C^s mapping $f : M \to N$ $(s \leq r)$ is called a C^s *Nash mapping* if the graph of f is semialgebraic in $\mathbf{R}^m \times \mathbf{R}^n$.

THEOREM (1.1) (B. Malgrange [28]).

(1) *A C^∞ Nash manifold is a C^ω Nash manifold.*

(2) *A C^∞ Nash mapping between C^ω Nash manifolds is a C^ω Nash mapping.*

After this, a Nash manifold and a Nash mapping mean a C^ω Nash manifold and a C^ω Nash mapping, respectively.

THEOREM (1.2) ([33]). — *Let $M_1 \supset N_1$, $M_2 \supset N_2$ be compact Nash manifolds and compact Nash submanifolds. If the pairs (M_1, N_1) and (M_2, N_2) are C^∞ diffeomorphic, then they are Nash diffeomorphic.*

Remark (1.3). — In Theorem (1.2), we can replace the assumption of "C^∞ diffeomorphic" by "C^1 diffeomorphic" ([34]).

THEOREM (1.4) ([34]). — *There exist two affine nonsingular algebraic varieties which are C^ω diffeomorphic but not Nash diffeomorphic.*

Therefore Nash diffeomorphism is essentially stronger than C^ω diffeomorphism in the non-compact case. The next theorem allows one to reduce arguments in the Nash category to those in the algebraic one.

THEOREM (1.5) (Artin-Mazur Theorem [1], [34]). — *Let M be a Nash manifold, and let $f : M \to \mathbf{R}^p$ be a Nash mapping. Then there exist a union M' of connected components of some nonsingular algebraic variety and a Nash diffeomorphism $\phi : M' \to M$ such that $f \circ \phi$ is a polynomial mapping.*

Remark (1.6) ([34]). — Let M be a Nash manifold. Then there exists a nonsingular algebraic variety M' which is Nash diffeomorphic to M.

Concerning the above fact, we have a

QUESTION (1.7). — *Can we replace "a union M' of connected components of some nonsingular algebraic variety" by "a nonsingular algebraic variety M'" in Theorem (1.5)?*

1.2. Modified Nash triviality.

Secondly we define the notion of modified Nash triviality.

DEFINITION (1.8). — *Let M be a Nash manifold of dimension n, and let $\pi : M \to \mathbf{R}^n$ be a proper onto Nash mapping. We call π a* Nash modification *if there is a semialgebraic set N in $\mathbf{R}^n$ of dimension less than n such that $\pi|(M - \pi^{-1}(N)) : M - \pi^{-1}(N) \to \mathbf{R}^n - N$ is a Nash isomorphism.*

Remark (1.9).

(1) We can define the notion of C^r modification in the C^r category similarly as above. Then we replace a semialgebraic set of dimension less than n by a thin set.

(2) We can generally define the notion of a Nash modification for a proper Nash mapping between Nash manifolds.

Let J be a Nash manifold, $t_0 \in J$, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in J)$ be a polynomial mapping (or a Nash mapping). We define $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ by $F(x;t) = f_t(x)$. Assume that F is a Nash mapping.

DEFINITION (1.10).

(1) *Let $\pi : M \to \mathbf{R}^n$ be a Nash modification. We say that $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a* π-modified Nash trivialization *along J, if there is a t-level preserving Nash diffeomorphism*

$$\Phi : (W_1, \pi^{-1}(0) \times J) \to (W_2, \pi^{-1}(0) \times J)$$

which induces a t-level preserving homeomorphism

$$\phi : (U, \{0\} \times J) \to (V \times J, \{0\} \times J)$$

such that $\phi((U, F^{-1}(0) \cap U)) = (V, f_{t_0}^{-1}(0) \cap V) \times J$, where W_1, W_2 are some semialgebraic neighborhoods of $\pi^{-1}(0) \times J$ in $M \times J$, and U is some neighborhood of $\{0\} \times J$ in $\mathbf{R}^n \times J$, and V is some neighborhood of 0 in $\mathbf{R}^n$.

(2) *Let $\Pi : M \to \mathbf{R}^n \times J$ be a Nash modification such that for each $t \in J$, $\pi_t = \Pi|M_t : M_t \to \mathbf{R}^n_t$ is also a Nash modification where $M_t =$*

$\Pi^{-1}(\mathbf{R}^n \times \{t\})$ and $\mathbf{R}^n_t = \mathbf{R}^n \times \{t\}$. We say that $(\mathbf{R}^n \times J, F^{-1}(0))$ admits a Π-modified Nash trivialization *along J, if there is a Nash diffeomorphism*

$$\Phi : (W_1, \Pi^{-1}(\{0\} \times J)) \to (W_2, \Pi^{-1}(\{0\} \times J))$$

which induces a t-level preserving homeomorphism

$$\phi : (U, \{0\} \times J) \to (V \times J, \{0\} \times J)$$

such that $\phi((U, F^{-1}(0) \cap U)) = (V, f_{t_0}^{-1}(0) \cap V) \times J$, where W_1, W_2 are some semialgebraic neighborhoods of $\Pi^{-1}(\{0\} \times J)$ in M, and U is some neighborhood of $\{0\} \times J$ in $\mathbf{R}^n \times J$, and V is some neighborhood of 0 in $\mathbf{R}^n$.

1.3. Resolution.

Lastly in this section, we explain the notion of resolution for embedded varieties which we use in this paper. Throughout the paper, we denote by $S(f)$ the set of singular points of f, and by $D(f)$ the set of singular values of f for a smooth mapping f.

Let U be a real-analytic manifold, and let V be a real-analytic subvariety of U. We say that V has *normal crossings*, if for each $x \in V$ there exists a local coordinate system $(z_1, \dots, z_n)$ at x such that V is a union of some coordinate spaces near x.

Let U be a real-analytic manifold, and let V be a real-analytic subspace of U. Let $\pi : M \to U$ be a proper real-analytic modification. We say that π gives a *resolution* of V in U, if the following conditions are satisfied:

(i) M is nonsingular.

(ii) The critical set of π is a union of smooth divisors $D_1, \dots, D_d$, which have normal crossings.

(iii) $\pi^{-1}(V)$ is a union of real-analytic submanifolds $V_1, \dots, V_k$ of M, which intersect transversely with each other and with $D_{j_1} \cap \dots \cap D_{j_q}$, for $1 \le j_1, \dots, j_q \le d$.

(iv) There is a thin set T in V so that $\pi|\pi^{-1}(V - T) : \pi^{-1}(V - T) \to V - T$ is an isomorphism.

Let $\mathcal{U}, I$ be real-analytic manifolds, and let $p : \mathcal{U} \to I$ be a submersion. We set $U_t = p^{-1}(t)$, for $t \in I$. For a real-analytic subspace $\mathcal{V}$ of $\mathcal{U}$, we set $V_t = \mathcal{V} \cap U_t$. For a proper real-analytic modification $\Pi : \mathcal{M} \to \mathcal{U}$, we set

$M_t = (p \circ \Pi)^{-1}(t)$. We say that Π gives a *simultaneous resolution* of $\mathcal{V}$ in $\mathcal{U}$ over I (or V_t in U_t for $t \in I$), if the following conditions are satisfied:

(i) $\mathcal{M}$ is nonsingular.

(ii) The critical set of Π is a union of smooth divisors $\mathcal{D}_1, \dots, \mathcal{D}_d$, which have normal crossings, and $p \circ \Pi | \mathcal{D}_{i_1} \cap \dots \cap \mathcal{D}_{i_q}$ is a submersion, for each $1 \leq i_1, \dots, i_q \leq d$.

(iii) $\Pi^{-1}(\mathcal{V})$ is a union of real-analytic submanifolds $\mathcal{V}_1, \dots, \mathcal{V}_k$ of $\mathcal{M}$, which intersect transversely with each other and with $\mathcal{D}_{j_1} \cap \dots \cap \mathcal{D}_{j_q}$, for $1 \leq j_1, \dots, j_q \leq d$. Moreover $p \circ \Pi | \mathcal{V}_{i_1} \cap \dots \cap \mathcal{V}_{i_s}$ are submersions for each $1 \leq i_1, \dots, i_s \leq k$.

(iv) There exists a thin set $\mathcal{T}$ in $\mathcal{V}$ so that $\mathcal{T} \cap \mathcal{V}_t$ is a thin set in $\mathcal{V}_t$, for each $t \in I$, and so that $\Pi | \Pi^{-1}(\mathcal{V} - \mathcal{T}) : \Pi^{-1}(\mathcal{V} - \mathcal{T}) \to \mathcal{V} - \mathcal{T}$ is an isomorphism.

Then, $\pi_t := \Pi | M_t : M_t \to U_t$ gives a resolution of V_t in U_t, for $t \in I$.

Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in I)$ be a family of real analytic mappings. Let $\pi : M \to \mathbf{R}^n$ be a proper real-analytic modification. We say that π *induces a simultaneous resolution* of $(\mathbf{R}^n, f_t^{-1}(0))$ near 0, if $\pi \times \mathrm{id}$ gives a simultaneous resolution of $\bigcup_{t \in I} f_t^{-1}(0)$.

2. Statements of theorems.

In this section, we describe the main results of this paper.

2.1. Nash triviality theorem.

Let $M \subset \mathbf{R}^{m'}$ be a Nash manifold possibly with boundary of dimension m, and let $N_1, \dots, N_q$ be Nash submanifolds of M possibly with boundary, which together with $N_0 = \partial M$ have normal crossings. Assume that $\partial N_i \subset N_0$, $i = 1, \dots, q$. Then we have

THEOREM I. — *Let $\varpi : M \to \mathbf{R}^k$, $k > 0$, be a proper onto Nash submersion such that for every $0 \leq i_1 < \cdots < i_s \leq q$, $\varpi | N_{i_1} \cap \cdots \cap N_{i_s} : N_{i_1} \cap \cdots \cap N_{i_s} \to \mathbf{R}^k$ is a proper onto submersion. Then there exists a Nash diffeomorphism*

$$\varphi = (\varphi', \varpi) : (M; N_1, \dots, N_q) \to (M^*; N_1^*, \dots, N_q^*) \times \mathbf{R}^k$$

such that $\varphi|M^* = \mathrm{id}$, *where* Z^* *denotes* $Z \cap \varpi^{-1}(0)$ *for a subset* Z *of* M.

Furthermore, if previously given are Nash diffeomorphisms $\varphi_{i_j} : N_{i_j} \to N^*_{i_j} \times \mathbf{R}^k$, $0 \leq i_1 < \cdots < i_a \leq q$, *such that* $\varpi \circ \varphi_{i_j}^{-1}$ *is the natural projection, and* $\varphi_{i_s} = \varphi_{i_t}$ *on* $N_{i_s} \cap N_{i_t}$, *then we can choose a Nash diffeomorphism* φ *which satisfies* $\varphi|N_{i_j} = \varphi_{i_j}$, $j = 1, \ldots, a$.

Remark (2.1). — In Theorem I, we can replace $\mathbf{R}^k$ by one of the followings:

(i) an open cuboid $\prod_{i=1}^{k} (a_i, b_i)$,

(ii) a closed cuboid $\prod_{i=1}^{k} [a_i, b_i]$,

(iii) a Nash manifold which is Nash diffeomorphic to an open simplex.

Remark that the integration of a Nash vector field is not necessarily of Nash class. For instance, that of $x\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial x}$ is not of Nash class. Moreover, the diffeomorphism from $\{x = 0\}$ to $\{x = 1\}$ given by the flows of the vector field $(x - y + 1)\dfrac{\partial}{\partial x} + (x - y - 1)\dfrac{\partial}{\partial y}$ is not of Nash class. Therefore, we cannot use the integration method to show Nash triviality theorems. Because of this, Theorem I is a very effective tool to show Nash triviality theorems, and consequently modified Nash triviality theorems.

2.2. Theorem on modified Nash triviality.

Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in I)$ be a polynomial family of polynomial maps. We assume that there is a modification $\pi : M \to \mathbf{R}^n$ which induces a simultaneous resolution of $(\mathbf{R}^n, f_t^{-1}(0))$ for $t \in I$. Let $F : \mathbf{R}^n \times I \to \mathbf{R}^p$ be the map defined by $F(x, t) = f_t(x)$, and set $\Pi := \pi \times \mathrm{id} : (M \times I, \pi^{-1}(0) \times I) \to (\mathbf{R}^n \times I, \{0\} \times I)$. Put $\mathcal{V} = (F^{-1}(0), \{0\} \times I)$. Then $\Pi^{-1}(\mathcal{V}) \cup S(\Pi)$ is a union of Nash submanifold-germs, which have normal crossings. These Nash submanifolds define a stratification of $\Pi^{-1}(\mathcal{V}) \cup S(\Pi)$ denoted by $\mathcal{S}$.

We say that f_t $(t \in I)$ satisfies *condition (C)*, if there exist a Nash diffeomorphism germ

$$\phi : (S(\Pi), \Pi^{-1}(\{0\} \times I)) \to (S(\pi), \pi^{-1}(0)) \times I$$

and a homeomorphism germ

$$\phi' : (D(\Pi), \{0\} \times I) \to (D(\pi), 0) \times I$$

such that ϕ preserves the stratification $\mathcal{S}|S(\Pi)$ and $\phi' \circ \Pi = \Pi \circ \phi$.

The notion of Nash diffeomorphism will be defined in the first paragraph of §3.

THEOREM II. — *Assuming that the modification $\pi : M \to \mathbf{R}^n$ induces a simultaneous resolution of $(\mathbf{R}^n, f_t^{-1}(0))$ for $t \in I$, and satisfies condition (C), then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization along I.*

Remark that Condition (C) is automatically satisfied if $D(\pi)=\{0\}$, because of Theorem I. But this condition (C) does not always follow in the general case. Let Z_X denote the connected component of the Zariski closure of X so that $Z_X \supset X$, for a stratum X of $\mathcal{S}$. Remark that Z_X is nonsingular for each stratum $X \in \mathcal{S}$.

PROPOSITION (2.2). — *The following three conditions imply Condition (C):*

(i) *For each $P \in D(\Pi)$, there exists a Nash coordinate system centered at P so that for any stratum $X \in \mathcal{S}|S(\Pi)$ with $\Pi(Z_X) \ni P$, $\Pi(Z_X)$ is a coordinate space near P with respect to the coordinate system.*

(ii) *The restriction of Π to Z_X is a submersion of Z_X to $\Pi(Z_X)$, for each stratum $X \in \mathcal{S}|S(\Pi)$.*

(iii) *The restriction of the natural projection $q : \mathbf{R}^n \times I \to I$ to $\Pi(Z_X)$ is a submersion of $\Pi(Z_X)$ to I, for each stratum $X \in \mathcal{S}|S(\Pi)$.*

Here, we consider any sets as germs at $\Pi^{-1}(\{0\} \times I)$ or $\{0\} \times I$ in (i)–(iii).

2.3. Modified Nash triviality via toric modifications.

We next consider which families admit a modified Nash trivialization by a projective toric modification in the case when the family of polynomials is explicitly given. To do this, we review several definitions and facts of the theory of toric varieties. See V.I. Danilov [8], [9], V.I. Danilov - A.G. Khovanskii[10], and M. Oka [31] for details.

Let Δ be a convex polyhedron in $\mathbf{R}^n$, which means the intersection of finitely many affine half-spaces defined over $\mathbf{Q}$. For each face F of Δ, we set

$$\sigma_F = \mathrm{Cone}(\Delta; F) = \bigcup_{r \geq 0} r \cdot (\Delta - m)$$

where m is a point lying inside the face F. Let R_F be the $\mathbf{R}$-algebra generated by the semi-group $\sigma_F \cap \mathbf{Z}^n$. Let U_F denote the set of real points of $\mathrm{Spec}(R_F)$, that is, the set of $\mathbf{R}$-algebra morphisms of R_F to $\mathbf{R}$. If F_1 is a face of F_2, then U_{F_2} is identified with an open subset of U_{F_1}, using the canonical inclusion $R_{F_1} \subset R_{F_2}$, since R_{F_1} and R_{F_2} have a same quotient field. These identifications allow us to glue U_F 's together. Gluing them we obtain a real algebraic variety denoted by P_Δ. Let F be a k-dimensional face of Δ and m a point inside the face F. Setting $\overline{F} = F - m$, we can understand $\overline{F}$ a convex polyhedron in some k-dimensional vector subspace of $\mathbf{R}^n$. By the construction on the polyhedron $\overline{F}$ similar to the above, we have a k-dimensional toric variety denoted by P_F. This P_F is canonically embedded in P_Δ and we have $P_{F_1} \cap P_{F_2} = P_{F_1 \cap F_2}$ for two faces F_1, F_2 of Δ.

A polyhedron Δ is *regular at a vertex* P if $\sigma_P \cap \mathbf{Z}^n$ is generated by a basis of $\mathbf{Z}^n$. A polyhedron is *regular* if it is regular at all vertices. If Δ is regular, then P_Δ is a nonsingular real algebraic variety.

We say a polyhedron Δ_1 *majorizes* another polyhedron Δ_2, if there exists a map β from the set of faces of Δ_1 to that of Δ_2 which satisfies the following two conditions:

(i) $\beta(F_1)$ is a face of $\beta(F_2)$ if F_1 is a face of F_2 for each faces F_1, F_2 of Δ_1,

(ii) $\mathrm{Cone}(\Delta_2; \beta(F)) \subset \mathrm{Cone}(\Delta_1; F)$ for each face F of Δ_1.

The inclusion in (ii) induces a map of P_{Δ_1} to P_{Δ_2} which is also denoted by β. If Δ is a convex polyhedron majorizing the positive orthant $\mathbf{R}^n_{\geq}$, then there exists a map $P_\Delta \to P_{\mathbf{R}^n_{\geq}} = \mathbf{R}^n$ which is an algebraic modification. We often denote it by π_Δ instead of β. Here $\mathbf{R}_{\geq}$ denote the set of non-negative real numbers. This $P_\Delta \to P_{\mathbf{R}^n_{\geq}} = \mathbf{R}^n$ is proper, if $\beta^{-1}(0)$ is the set of compact faces of Δ. If Δ is regular, then $S(\beta)$ is a union of submanifolds of P_Δ which have normal crossings. We fix such Δ, that is, Δ is a regular polyhedron majorizing $\mathbf{R}^n_{\geq}$, which induces a proper algebraic modification $\beta = \pi_\Delta : P_\Delta \to \mathbf{R}^n$, called *a projective toric modification*.

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a polynomial map. We say that f is Δ-*regular*, if the following conditions are satisfied:

(i) The strict transform of $(f^{-1}(0),0)$ by the projective toric modification $P_\Delta \to \mathbf{R}^n$ is nonsingular.

(ii) The projective toric modification $P_\Delta \to \mathbf{R}^n$ gives a resolution of $(\mathbf{R}^n, f^{-1}(0), 0)$

This Δ-regularity is a weaker condition than non-degeneracy, which was treated by several authors (e.g. A.G. Khovanskii [18], [19], M. Oka [31]). In fact, non-degeneracy is equivalent to transversality of the strict transform of $f^{-1}(0)$ by β to the toric stratification of P_Δ for a toric modification $\beta : P_\Delta \to \mathbf{R}^n$ which majorizes the Newton polyhedron of f. This treatment of non-degeneracy was found in §2 in [18]. It is possible to derive topological triviality of a family of $(\mathbf{R}^n, f_t^{-1}(0), 0)$ for simultaneously non-degenerate systems $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ with a constant Newton polyhedron $\Gamma_+(f_t) = \Gamma_+(f_0)$. But the constancy of Newton polyhedra is strong as a sufficient condition for topological triviality. For example, the Briançon-Speder family (Example 7.5) is topologically trivial, but their Newton polyhedra are not constant. We will see that a weighted homogeneous polynomial in three variables which defines an isolated singularity at the origin is Δ-regular, and a family of such polynomials with same weights admits a simultaneous resolution using some toric modification (Proposition (7.3)). Using our method, we can analyse many examples, not only weighted homogeneous ones but also polynomials with generic coefficients in the given Newton polyhedra. We discuss more about Δ-regularity in §7.

Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a polynomial mapping for $t \in I = \prod_{i=1}^{k} [a_i, b_i]$, and define $F : (\mathbf{R}^n \times I, \{0\} \times I) \to (\mathbf{R}^p, 0)$ by $F(x;t) = f_t(x)$. Assume that F is a polynomial mapping. Let Δ be a regular polyhedron majorizing $\mathbf{R}^n_{\geq}$, and let π denote the proper modification $\pi_\Delta : P_\Delta \to \mathbf{R}^n$. Then, by Theorem II, we have

THEOREM II′. — *If f_t is Δ-regular for $t \in I$ and satisfies Condition (C), then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization along I.*

As a corollary of Theorem II′, we have

COROLLARY III.

(1) *Let f_t $(t \in I)$ be a polynomial family of non-degenerate systems of polynomial mappings. If f_t is convenient, i.e. each $\Gamma_+(f_t)$ meets all coordinate axes, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization.*

(2) *Let f_t $(t \in I)$ be a polynomial family of non-degenerate systems*

of polynomial mappings. If $(f_t)^a$ is independent of t for each vector $a(\neq e^i)$ supporting a non-compact face of $\Gamma_+(f_t)$, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization. (See §3 for notations not defined yet.)

(3) *Let f_t $(t \in I)$ be a polynomial family of polynomial mappings, let Δ be a regular polyhedron which is equal to $\mathbf{R}^n_{\geq}$ outside some compact set, and let π denote the modification $P_\Delta \to \mathbf{R}^n$. If each f_t is Δ-regular for $t \in I$, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization along I.*

(4) *Let $f_t(x)$ $(t \in I)$ be a polynomial family of polynomial mappings, π the blow-up of $\mathbf{R}^n$ centered at 0. We write $f_t(x) = F(x;t) = H_d(x;t) + H_{d+1}(x;t) + \cdots$, where $H_j(x;t)$ is a homogeneous polynomial mappings of degree j in x. If the zero locus of $H_d(-;t)$ $(t \in I)$ has an isolated singularity, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization along I.*

2.4. Modified Nash triviality theorem in the weighted case.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of positive integers, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a polynomial mapping for $t \in I = \prod_{i=1}^{k} [a_i, b_i]$. Assume that F is a polynomial mapping. For each $t \in I$, we write $f_t(x) = Q_t(x) + G_t(x)$, where $Q_{t,i}(x)$ is the weighted initial form of $f_{t,i}$ with respect to α $(1 \leq i \leq p)$. Then we define a mapping $\pi : S^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ by

$$\pi(x_1, \dots, x_n; u) = (u^{\alpha_1} x_1, \dots, u^{\alpha_n} x_n).$$

We set $E = S^{n-1} \times \mathbf{R}$ and $E_0 = \pi^{-1}(0) = S^{n-1} \times \{0\}$. By definition, E is a Nash manifold and E_0 is a Nash submanifold of E. The restriction mapping $\pi|(E - E_0) : E - E_0 \to \mathbf{R}^n - \{0\}$ is a 2 to 1 mapping. We call this proper Nash mapping π a *weighted double oriented blowing-up* of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$. This is a weighted version of double oriented blowing-up. (See Example (a) in page 221 in H. Hironaka [17], for its definition.) For this π, we define the notion of π-modified Nash triviality in a way similar to Definition (1.10.1). Concerning this weighted double oriented blowing-up, we have

THEOREM IV. — *If $Q_t^{-1}(0) \cap S(Q_t) = \{0\}$ for any $t \in I$, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified Nash trivialization along I.*

Remark (2.4). — In [3], M. Buchner and W. Kucharz showed that $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified C^1 trivialization along I under the same assumption as above. They treated a more general case than the polynomial case.

2.5. Classification theorems on modified Nash triviality.

We first prepare notation. Let $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ be a mapping, and let $Q \subset J$. Then we denote by F_Q the restriction mapping of F to $\mathbf{R}^n \times Q$.

We express $f = (f_1, \dots, f_p) \in P_{[r]}(n,p)$ as follows:

$$f_i(x) = \sum a_\alpha^{(i)} x^\alpha, \ (1 \le i \le p).$$

Then the coefficient space $\{(\cdots, a_\alpha^{(i)}, \cdots)\}$ is naturally identified with an Euclidean space $\mathbf{R}^N$. For $a = (\cdots, a_\alpha^{(i)}, \cdots) \in \mathbf{R}^N$, we write

$$f_a(x) = \left(\sum a_\alpha^{(1)} x^\alpha, \dots, \sum a_\alpha^{(p)} x^\alpha\right) \in P_{[r]}(n,p).$$

After this, we shall not distinguish $P_{[r]}(n,p)$ from $\mathbf{R}^N$. Let

$$F : (\mathbf{R}^n \times P_{[r]}(n,p), \{0\} \times P_{[r]}(n,p)) \to (\mathbf{R}^p, 0)$$

be the polynomial mapping defined by $F(x;a) = f_a(x)$. We put

$$\Sigma^* = \left\{ f \in P_{[r]}(n,p) : f^{-1}(0) \cap S(f) \underset{\neq}{\supset} \{0\} \text{ as germs at } 0 \in \mathbf{R}^n \right\}.$$

Then we have a classification of elements of $P_{[r]}(n,p)$ by modified Nash.

THEOREM V. — *There exists a partition of the space of polynomial mappings $P_{[r]}(n,p) = \Sigma^* \cup Q_1 \cup \dots \cup Q_d$ such that for $1 \le i \le d$,*

(i) *Q_i is a connected Nash manifold,*

and

(ii) *$(\mathbf{R}^n \times Q_i, F_{Q_i}^{-1}(0))$ admits a Π_i-modified Nash trivialization along Q_i, for some Π_i. Here, Π_i is a Nash modification which gives a simultaneous resolution of $F_{Q_i}^{-1}(0)$ in $\mathbf{R}^n \times Q_i$ over Q_i around $\{0\} \times Q_i$.*

By using the same argument as Theorem V, we have

THEOREM VI. — *Let J be a semialgebraic set in some Euclidean space, and let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in J)$ be a polynomial mapping. We* *define* $F : (\mathbf{R}^n \times J, \{0\} \times J) \to (\mathbf{R}^p, 0)$ *by* $F(x;t) = f_t(x)$. *Assume that* F *is a polynomial mapping. If each* $f_t^{-1}(0)$ *admits* $0 \in \mathbf{R}^n$ *as an isolated singularity, then there exists a finite filtration of* J *by semialgebraic sets* $J^{(i)}$

$$J = J^{(0)} \supset J^{(1)} \supset \cdots \supset J^{(m)} \supset J^{(m+1)} = \emptyset,$$

with the following properties:

(i) $\dim J^{(i)} > \dim J^{(i+1)}$ *and* $J^{(i)} - J^{(i+1)}$ *are Nash manifolds.*

(ii) *For each connected component* Q *of* $J^{(i)} - J^{(i+1)}$, $(\mathbf{R}^n \times Q, F_Q^{-1}(0))$ *admits a* Π*-modified Nash trivialization along* Q, *for some* Π. *Here,* Π *is a Nash modification which gives a simultaneous resolution of* $F_Q^{-1}(0)$ *in* $\mathbf{R}^n \times Q$ *over* Q *around* $\{0\} \times Q$.

Remark (2.5). — In the case of functions, T.-C.Kuo gave a filtration similar to Theorem VI such that for t, t' in a connected component of $J^{(i)} - J^{(i+1)}$, f_t, $f_{t'}$ are blow-analytically equivalent (see [26]). But he has not given the filtration for blow-analytic triviality.

As a corollary of Theorem VI and a Generalized Artin-Mazur Theorem (Theorem (6.6) in §6), we have

COROLLARY VII. — *The statement of Theorem VI remains true if we only assume that* f_t *and* F *are Nash mappings.*

3. Proof of Theorem I.

Before starting the proof of Theorem I, we prepare some terminology. Let M (resp. M') be a Nash manifold possibly with boundary, and let $N_0, N_1, ..., N_q$ (resp. $N'_0, N'_1, ..., N'_q$) be Nash submanifolds of M (resp. M') possibly with boundary. We say that a mapping $f : \bigcup_{i=1}^{q} N_i \to \bigcup_{i=0}^{q} N'_i$ is a *Nash diffeomorphism* if the restriction $f|N_i : N_i \to N'_i$ is a Nash diffeomorphism for $0 \le i \le q$. A *Nash function* on $N_{i_1} \cup \cdots \cup N_{i_b}$ is a function whose restriction to each N_i is of Nash class. We also define a *Nash map* from $N_{i_1} \cup \cdots \cup N_{i_b}$ to a Nash set (i.e. the zero set of a Nash function) similarly. Note that each N_i is a Nash set in M (Corollary II.5.4 in [34]).

Let M be a Nash manifold possibly with boundary, and let $N_0, N_1, \ldots,$ N_q be Nash submanifolds of M possibly with boundary which have normal crossings. Then we have the following:

LEMMA (3.1). — *We can extend a Nash function on* $\bigcup_{i=0}^{q} N_i$ *to* M.

Proof. — Let $0 \le i \le q$. Assume that the following statement holds:

STATEMENT (i). — *Let* f_i *be a Nash function on* $\bigcup_{j=0}^{i} N_j$ *which vanishes on* $\bigcup_{j=0}^{i-1} N_j$. *Then there exists a Nash function* g_i *on* M *such that* $g_i = f_i$ *on* $\bigcup_{j=0}^{i} N_j$.

Then we can derive Lemma (3.1) from this statement as follows. Let f be a Nash function on $\bigcup_{j=0}^{q} N_j$. By statement (0), we have a Nash function g_0 on M such that $g_0 = f$ on N_0. Next apply statement (1) to $(f - g_0)|N_0 \cup N_1$ which vanishes on N_0. Then there is a Nash function g_1 on M such that $g_0 + g_1 = f$ on $N_0 \cup N_1$. Repeating these arguments, we obtain a Nash extension of f to M.

It remains to show statement (i). By Proposition II.5.6 in [34], for each $0 \le j < i$, there exist Nash functions $\varphi_{j1}, ..., \varphi_{jb_j}$ on M whose common zero set is N_j and whose gradients span the normal vector space of N_j in M at each point of N_j. Let $(\varphi_{j1}, \dots, \varphi_{jb_j})$ denote the ideal of the ring of Nash functions on M generated by $\varphi_{j1}, \dots, \varphi_{jb_j}$. Set

$$F_{i-1} = \bigcap_{j=0}^{i-1} (\varphi_{j1}, \dots, \varphi_{jb_j}).$$

In general, a ring of Nash functions is Noetherian (e.g. [34]). Let $\psi_1, \dots, \psi_d$ be the generators of F_{i-1}. Then, by the hypothesis that $N_0, ..., N_q$ have normal crossings, for each point $x \in N_i$ we can describe the germ $(f_i|N_i)_x$ as $\sum_{j=1}^{d} \rho_j (\psi_j|N_i)_x$, for Nash function germs ρ_j at x in N_i. Hence, by [34] I.6.5, we have globally

$$f_i|N_i = \sum_{\text{finite}} \rho_j(\psi_j|N_i)$$

for some Nash functions ρ_j on N_i. By Corollary II.5.5 in [34], we can extend ρ_j to M. Let $\widetilde{\rho}_j$ denote the extension. Then $g_i = \sum \widetilde{\rho}_j \psi_j$ is an extension of f_i. □

Now let us start the proof of Theorem I. We first show the case where all N_i's, $0 \le i \le q$, are of codimension 1 in M to make the idea of the proof clear. In this case, we describe only the proof of the first part of Theorem I, because the second part follows immediately from our proof. Let M be a Nash manifold, and let $N_0(=\partial M), N_1, \ldots, N_q$ be Nash submanifolds of M which satisfy the hypotheses of Theorem I. Set

$$X_j = \{N_{i_1} \cap \cdots \cap N_{i_b} \text{ of dimension } j : 0 \leq i_1, \ldots, i_b \leq q\},\ j = 1, \ldots, m-1.$$

Then, in the case where $X_j \neq \emptyset$, each element N of X_j is a Nash manifold possibly with boundary, and $(N, \{N' \in X_{j-1} : N' \subset N\})$ has the same properties as $(M; N_0, ..., N_q)$. If $m = k$, then $N_0 = \cdots = N_q = \emptyset$ and the theorem is clear. Set

$$Y_j = \bigcup\{N \in X_j\},\ j = k, \ldots, m-1;\ Y_m = M.$$

Let us consider the following statement for $k \leq j \leq m$: There exists a Nash diffeomorphism $\varphi_j = (\varphi'_j, \varpi) : Y_j \to Y_j^* \times \mathbf{R}^k$ such that

$$\varphi'_j | Y_j^* = \mathrm{id},\ \varphi'_j(N) = N^*,\ i = 0, \ldots, q, \quad \text{for } N \in X_k \cup \cdots \cup X_j.$$

Then the statement for $j = k$ clearly holds, since $\varpi | Y_k$ is a finite covering over $\mathbf{R}^k$, and that for $j = m$ coincides with our theorem. Therefore, in order to prove the theorem, it suffices to construct φ_j on each $N \in X_j$. Since φ_{j-1} is defined on $N \cap Y_{j-1}$, it is necessary to extend $\varphi_{j-1} | N \cap Y_{j-1}$ to N. Hence we can reduce the theorem to the following assertion:

ASSERTION (3.2). — *Let*

$$\varphi = (\varphi', \varpi) : \bigcup_{i=0}^{q} N_i \to \bigcup_{i=0}^{q} N_i^* \times \mathbf{R}^k$$

be a Nash diffeomorphism with $\varphi' | N_i^* = \mathrm{id}$, $\varphi'(N_i) = N_i^*$, $i = 0, \ldots, q$. *Then we can extend* φ *to a Nash diffeomorphism*

$$\widetilde{\varphi} = (\widetilde{\varphi}', \varpi) : (M; N_1, .., N_q) \to (M^*; N_1^*, .., N_q^*) \times \mathbf{R}^k$$

such that $\widetilde{\varphi}' | M^* = \mathrm{id}$.

We further reduce Assertion (3.2) to the following easier assertion:

ASSERTION (3.3). — *For* φ *in Assertion (3.2), there exist an open semialgebraic neighborhood* U *of* $\bigcup_{i=0}^{q} N_i$ *in* M *and a* C^r *Nash imbedding*

$$\Phi = (\Phi', \varpi) : U \to U^* \times \mathbf{R}^k,$$

which is an extension of φ, *such that* $\Phi' | U^* = \mathrm{id}$, *where* r *is a sufficiently large integer.*

Proof that Assertion (3.2) *follows from Assertion* (3.3). — Assume that Assertion (3.3) holds. First we modify Φ to be of Nash class. There exists a Nash manifold $\widetilde{M}^*(\supset M^*)$ in $\mathbf{R}^{m'}$ without boundary of dimension $m-k$. Let $v : V \to \widetilde{M}^*$ be a Nash tubular neighborhood in $\mathbf{R}^{m'}$, and let $\widehat{\varphi}' : M \to \mathbf{R}^{m'}$ be a Nash extension of φ' (cf. Lemma(3.1)). Then $\Phi' - \widehat{\varphi}'$ is a C^r Nash map from M to $\mathbf{R}^{m'}$, which vanishes on $\bigcup_{i=0}^{q} N_i$. Moreover, by generalizing Lemma (3.1) a little, we can choose $\widehat{\varphi}'$ so that $\widehat{\varphi}' = \mathrm{id}$ on M^*. It follows that $\Phi' - \widehat{\varphi}'$ vanishes on M^*. Let F_q be the family of Nash functions on M given in the proof of Lemma (3.1). Let F_* be a finite family of Nash functions on M whose common zero set is M^* and whose gradients span the normal vector space of M^* in M at each point of M^* (e.g. $\{\varpi_1, \dots, \varpi_n\}$ where $\varpi = (\varpi_1, \dots, \varpi_n)$), and set

$$F = \{f_1 f_2 : f_1 \in F_q, f_2 \in F_*\}.$$

Let r be sufficiently large. Then we have a finite number of $\eta_j \in F$ and C^1 Nash maps $\xi_j : M \to \mathbf{R}^{m'}$ such that $\Phi' - \widehat{\varphi}' = \sum \eta_j \xi_j$. Let $\xi_j^a : M \to \mathbf{R}^{m'}$ be a strong Nash approximation of each ξ_j in the C^1 topology (Theorem II.4.1 in [34]). See [34] for the definition of the C^1 topology. Then $\widehat{\varphi}' + \sum \eta_j \xi_j^a$ is a strong Nash approximation of Φ' in the C^1 topology, which equals Φ' on $M^* \cup \bigcup_{i=0}^{q} N_i$. Hence $v \circ (\widehat{\varphi}' + \sum \eta_j \xi_j^a)$ is a Nash map and keeps the properties of Φ'. Thus we can assume that Φ is of Nash class.

Next we modify U and Φ so that U is a Nash manifold with boundary, $\varpi|U$ and $\varpi|\partial U$ are proper submersions onto $\mathbf{R}^k$, and Φ is a C^r Nash diffeomorphism onto $U^* \times \mathbf{R}^k$ (here U is no longer open in M, but closed).

Let ρ be a nonnegative Nash function on U^* with zero set $\bigcup_{i=0}^{q} N_i^*$, e.g. the restriction to U^* of the square sum of the elements of F_q. Note that M^* and $\rho^{-1}(0)$ are compact, and for some neighborhood W of $\rho^{-1}(0)$, ρ is C^1 regular on $W - \rho^{-1}(0)$ because the critical value set of ρ is finite. Hence shrinking U we can assume that $\rho(U^*) = [0, 2)$, and

$$\rho|(U^* - \rho^{-1}(0)) : U^* - \rho^{-1}(0) \to (0, 2)$$

is a proper Nash submersion. Then by the Triviality Theorem in [5], there is a Nash diffeomorphism

$$\psi = (\psi', \rho) : U^* - \rho^{-1}(0) \to \rho^{-1}(1) \times (0, 2).$$

Let $0 < f \leq 1$ be a C^0 semialgebraic function (i.e. with semialgebraic graph) on $\mathbf{R}^k$ such that $f(0) = 1$, and

$$\rho^{-1}([0, f(y)]) \subset \Phi'(U \cap \varpi^{-1}(y)), \quad \text{for each } y \in \mathbf{R}^k.$$

There exists such f. In fact, we can construct it by using the semialgebraic function $g : \mathbf{R}^k \to \mathbf{R}$ defined by

$$g(y) = \begin{cases} \inf \rho(U^* - \Phi'(U \cap \varpi^{-1}(y))) & \text{if exists,} \\ 1 & \text{otherwise.} \end{cases}$$

This function is locally larger than a positive number. Now, by Theorem II.4.1 in [34], we can assume that f is of Nash class. Replace U with the set

$$\bigcup_{y \in \mathbf{R}^k} \Big(\varpi^{-1}(y) \cap {\Phi'}^{-1}\big(\rho^{-1}([0, f(y)])\big) \Big),$$

and denote it by the same notation U. Then U is a Nash manifold with boundary, $\varpi|U$ and $\varpi|\partial U$ are proper submersions onto $\mathbf{R}^k$, $\Phi : U \to U^* \times \mathbf{R}^k$ is a Nash imbedding, and we have

$$\Phi'(U \cap \varpi^{-1}(y)) = \rho^{-1}([0, f(y)]) \text{ for } y \in \mathbf{R}^k.$$

It remains to modify Φ to be a C^r Nash diffeomorphism. Set

$$A = \{(s,t) \in (0,1] \times [0,1] : 0 \le t \le s\},$$
$$A' = \{(s,t) \in A : 0 \le t \le s/2\}.$$

Let $\alpha : A \to (0,1] \times [0,1]$ be a C^r Nash diffeomorphism of the form $\alpha(s,t) = (s, \alpha'(s,t))$ such that $\alpha = \mathrm{id}$ on A'. Replace Φ' with the map

$$U \ni x \mapsto \begin{cases} \psi^{-1}(\psi' \circ \Phi'(x), \alpha'(f \circ \varpi(x), \rho \circ \Phi'(x)) & \text{for } x \in U - \bigcup_{i=0}^{q} N_i, \\ \Phi'(x) & \text{for } x \in \bigcup_{i=0}^{q} N_i. \end{cases}$$

Then Φ becomes a C^r Nash diffeomorphism onto $U^* \times \mathbf{R}^k$.

Thirdly we extend Φ to a C^r diffeomorphism from M to $M^* \times \mathbf{R}^k$, which is possible by Theorem 3 and Proposition 7 in [5]. Lastly we approximate the extension by a Nash diffeomorphism as in the above first step, which proves Assertion (3.2). □

Now we prepare the following assertion for the proof of Assertion (3.3).

ASSERTION (3.4). — *Let $0 \le i \le q$. For φ in Assertion (3.2), there exist an open semialgebraic neighborhood U_i of N_i in M and a Nash imbedding*

$$\Phi_i = (\Phi'_i, \varpi) : U_i \to U_i^* \times \mathbf{R}^k,$$

which is an extension of $\varphi|U_i \cap \bigcup_{j=0}^{q} N_j$, such that $\Phi'_i|U_i^* = \mathrm{id}$.

Proof that Assertion (3.3) *follows from Assertion* (3.4). — Assume that Assertion (3.4) holds, that is, U_i and $\Phi_i, i = 0, \ldots, q$, in Assertion (3.4) exist. By combining Φ_0 and Φ_1 and shrinking the neighborhoods U_0 and U_1 if necessary, we can define a C^r Nash imbedding

$$\Phi_{0,1} = (\Phi'_{0,1}, \varpi) : U_0 \cup U_1 \to (U_0 \cup U_1)^* \times \mathbf{R}^k$$

with the same properties as Φ_i as follows. Let β be a C^r Nash function on M such that $0 \leq \beta \leq 1$, $\beta = 0$ on a neighborhood of $M - U_0$ in M, and $\beta = 1$ on a neighborhood of N_0. Replace U_0 with $\operatorname{Int} \beta^{-1}(1)$, and denote it by the same notation U_0. Define a C^r Nash map $\Phi'_{0,1} : U_0 \cup U_1 \to M^*$ by

$$\Phi'_{0,1}(x) = \begin{cases} \Phi'_0(x) & \text{for } x \in U_0, \\ v(\beta(x)\Phi'_0(x) + (1 - \beta(x))\Phi'_1(x)) & \text{for } x \in U_1 \text{ with } \beta(x) > 0, \\ \Phi'_1(x) & \text{for } x \in U_1 \text{ with } \beta(x) = 0, \end{cases}$$

where v is the Nash tubular neighborhood in the above proof. Then for sufficiently small U_0 and U_1, the map

$$\Phi_{0,1} = (\Phi'_{0,1}, \varpi) : U_0 \cup U_1 \to (U_0 \cup U_1)^* \times \mathbf{R}^k$$

is a C^r Nash imbedding, which is an extension of $\varphi|(U_0 \cup U_1) \cap \bigcup_{j=0}^{q} N_j$ such that $\Phi'_{0,1}|(U_0 \cup U_1)^* = \mathrm{id}$. Repeating these arguments, we obtain U and Φ as required. □

Proof of Assertion (3.4). — We show the case $i = 0$. The other cases follow similarly. We shall construct a Nash tubular neighborhood $w : W \to N_0$ in M and a nonnegative Nash function γ on W such that

(i) $\varphi'(W \cap \bigcup_{i=0}^{q} N_i) \subset W^*$,

(ii) $w \circ \varphi' = \varphi' \circ w$ on $W \cap \bigcup_{i=0}^{q} N_i$,

(iii) $\varpi \circ w = \varpi$ on W,

(iv) $w^{-1}(N_i) \subset N_i$, $i = 1, \ldots, q$,

(v) $\gamma^{-1}(0) = N_0$,

(vi) γ is C^1 regular outside N_0 and, locally at each point of N_0, the square of a C^1 regular function, and

(vii) $\gamma \circ \varphi' = \gamma$ on $W \cap \bigcup_{i=0}^{q} N_i$.

Assume the existence of such w and γ, and shrink W so that $\gamma < \varepsilon$ and

$$\gamma|(W^* - N_0^*) : W^* - N_0^* \to (0, \varepsilon)$$

is proper for some $\varepsilon > 0$. Then for each point x of W there exist two points x_j, $j = 1, 2$, of W^* such that $\varphi' \circ w(x) = w(x_j)$ and $\gamma(x) = \gamma(x_j)$. Now we map each point x of W to one of the above points x_j of W^* so that the mapping is continuous and the identity on W^*. Then the mapping is unique and of Nash class. Denote it by Φ'_0. This Φ'_0 fulfills the requirements in this assertion.

We first construct w on M^*. Let $w^* : W^* \to N_0^*$ be a Nash tubular neighborhood in M^*. The problem is only that w^* has to satisfy condition (iv). To solve this, we define w^* on each $\bigcup\{N \cap W^* : N \in X_j\}$ by induction on j, which is done by Lemma (3.1) as in the above arguments. Next, for small W, we extend w^* to $W \cap \bigcup_{i=0}^{q} N_i$ by

$$w(x) = \varpi^{-1}(\varpi(x)) \cap {\varphi'}^{-1}(w^* \circ \varphi'(x)) \quad \text{for } x \in W \cap \bigcup_{i=0}^{q} N_i,$$

which is a Nash map, and satisfies conditions (ii), (iii) and (iv). Thirdly, by Lemma (3.1) we extend w to W so that $w : W \to N_0$ is a Nash tubular neighborhood. Then we need to modify w so that condition (iii) is satisfied. (If we choose W small enough, then (i) is satisfied.) This is easy to see. Indeed the correspondence

$$W \times W \supset W^d \ni (x, y) \mapsto \chi(x, y) = \begin{pmatrix} \text{the image of } x \text{ under the} \\ \text{orthogonal projection onto} \\ N_0 \cap \varpi^{-1}(\varpi(y)) \text{ in } M \end{pmatrix} \in N_0$$

is of Nash class (where W^d is a small semialgebraic neighborhood of the diagonal of $W \times W$), and $\chi(w(x), x)$ satisfies all the conditions.

Finally we construct γ. In the same way as above, we can construct a Nash function γ' on W such that

(v)$'$ $\gamma' = 0$ on N_0,

(vii) $\gamma' \circ \varphi' = \gamma'$ on $W \cap \bigcup_{i=0}^{q} N_i$, and

(vi)$'$ $\gamma'|W \cap M^*$ is C^1 regular outside N_0^* and, locally at each point of N'_0, the square of a C^1 regular function.

By (vii) and (vi)$'$, γ' is C^1 regular at any point of $W \cap \bigcup_{i=1}^{q} N_i - N_0$ and, locally at each point of $W \cap \Big(\bigcup_{i=0}^{q} N_i\Big) \cap N_0$, the square of a C^1 regular function. Here we apply Lemma (3.1) when we extend γ' from $W \cap \bigcup_{i=0}^{q} N_i$ to W. Hence, by the proof of Lemma (3.1), we can choose γ' so that its first partial derivatives vanish on N_0. Let γ'' denote the square sum of the elements of F_q in the proof of Lemma (3.1), and shrink W. Then $\gamma = \gamma' + \gamma''$ satisfies conditions (v), (vi) and (vii), which completes the proof. □

We next show the general case. By the same reason as in the above proof of the codimension 1 case it suffices to prove the later half of the theorem for $i_0 = 0, \ldots, i_q = q$. We proceed by induction on q. Define a C^1 Nash diffeomorphism

$$\varphi = (\varphi', \varpi) : \bigcup_i N_i \to \Big(\bigcup_i N_i\Big)^* \times \mathbf{R}^k$$

to be (φ'_i, ϖ) on each N_i. We can assume that $\varphi' = \mathrm{id}$ on $(\bigcup_i N_i)^*$. Let N_1 be such that $N_1 \not\subset \bigcup_{i\neq 1} N_i$. By induction we have a C^1 Nash diffeomorphism

$$\tilde{\varphi} = (\tilde{\varphi}', \varpi) : M \to M^* \times \mathbf{R}^k$$

which is an extension of $\varphi| \bigcup_{i\neq 1} N_i$. Let $p : U \to N_1$ be a small closed C^1 Nash tubular neighborhood in M such that ∂U and $N_0, N_2, \ldots, N_q$ have normal crossings and $\tilde{\varphi}'(p^{-1}(x)) = p^{-1}(\varphi'(x))$ for each $x \in N_1 \cap \Big(\bigcup_{i\neq 1} N_i\Big)$. Existence of such p follows from a C^1 Nash partition of unity. Set $N_1' = N_1 \cap \Big(\bigcup_{i\neq 1} N_i\Big)$ and $N_1'' = p^{-1}(N_1')$.

By Theorem 8 in [4] we have a C^1 Nash diffeomorphism

$$\psi = (\psi', \varpi) : U \to U^* \times \mathbf{R}^k,$$

where $U^* = U \cap \varpi^{-1}(0)$, such that $\varphi' \circ p = p \circ \psi'$ and $\psi' = \mathrm{id}$ on U^*. Recalling the proof of codimension 1 case, we need only to modify ψ' so that

$$\psi' = \tilde{\varphi}' \text{ on } N_1'' \quad \text{and} \quad \psi' = \varphi' \text{ on } N_1.$$

Define a C^1 Nash map $\xi : (N_1 \cup N_1'')^* \times \mathbf{R}^k \to U^*$ so that

$$\xi \circ \varphi = \psi' \text{ on } N_1, \quad \text{and} \quad \xi \circ \tilde{\varphi} = \psi' \text{ on } N_1''.$$

Note that for each $(x, t) \in N_1'^* \times \mathbf{R}^k$, $\xi|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$, and for each $(x, t) \in N_1^* \times \mathbf{R}^k$, $p \circ \xi(x, t) = x$. We will show the following assertion:

ASSERTION (3.5). — *ξ is extensible to a C^1 Nash map $U^* \times \mathbf{R}^k \to U^*$ so that for each $(x,t) \in N_1^* \times \mathbf{R}^k$, $\xi|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$.*

If this assertion is true then for such an extension ξ, $(\xi, \mathrm{id})^{-1} \circ \psi$ is a required modification of ψ and hence the theorem is proved. We reduce the problem to the case where $\xi(x,t) = x$ for all $(x,t) \in N_1''^* \times \mathbf{R}^k$.

Let V be a small open semialgebraic neighborhood of $N_1'^* \times \mathbf{R}^k$ in $N_1^* \times \mathbf{R}^k$. It is easy to extend $\xi|N_1'' \times \mathbf{R}^k$ to a C^1 Nash map

$$\xi_1 : \bigcup_{(x,t) \in V} p^{-1}(x) \times t \to U^*$$

so that for each $(x,t) \in V$, $\xi_1|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$. Since we cannot always extend ξ_1, moreover, to $U^* \times \mathbf{R}^k$, we modify ξ_1 as follows. Choose V so that for any $\varepsilon > 1$ and $(x,t) \in N_1^* \times \mathbf{R}^k$, if $(x, \varepsilon t) \in V$ then $(x,t) \in V$. Then we have a C^1 Nash map $\kappa : V \to V$ of form $\kappa(x,t) = (x, \kappa'(x,t))$ such that

$$\kappa = \mathrm{id} \quad \text{on } (V \cap N_1^* \times 0) \cup N_1'^* \times \mathbf{R}^k,$$

and

$\kappa(V - V') \subset N_1^* \times 0$ for a very small neighborhood V' of $N_1'^* \times \mathbf{R}^k$ in V.

In place of ξ_1 consider

$$\xi_2 = \xi_1(x, \kappa'(p(x), t)),\ (x,t) \in \bigcup_{(y,t) \in V} p^{-1}(y) \times t,$$

which is also an extension of $\xi|N_1'' \times \mathbf{R}^k$ such that for each $(x,t) \in V$, $\xi_2|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$. Now we can extend ξ_2 to $U^* \times \mathbf{R}^k$ by setting $\xi_2(x,t) = x$ outside $\bigcup_{(y,t) \in V} p^{-1}(y) \times t$. We denote the extension by $\tilde{\xi}_2$. Clearly $\tilde{\xi}_2|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$ for any $(x,t) \in N_1^* \times \mathbf{R}^k$.

Define a C^1 Nash map $\xi_3 : (N_1 \cup N_1'')^* \times \mathbf{R}^k \to (N_1 \cup N_1'')^*$ by

$$(\xi_3, \mathrm{id}) = (\xi_2, \mathrm{id})^{-1} \circ (\xi, \mathrm{id}) : (N_1 \cup N_1'')^* \times \mathbf{R}^k \to (N_1 \cup N_1'')^* \times \mathbf{R}^k,$$

then

$$p \circ \xi_3(x,t) = x \quad \text{for each } (x,t) \in N_1^* \times \mathbf{R}^k,$$
$$\xi_3(x,t) = x \quad \text{for each } (x,t) \in N_1''^* \times \mathbf{R}^k.$$

Moreover, if we can extend ξ_3 to a C^1 Nash map

$$\tilde{\xi}_3 : U^* \times \mathbf{R}^k \to U^*$$

keeping the property that for each $(x,t) \in N_1^* \times \mathbf{R}^k$, $\tilde{\xi}_3|p^{-1}(x) \times t$ is a C^1 Nash diffeomorphism onto $p^{-1}(x)$, then $\tilde{\xi}_2 \circ (\tilde{\xi}_3, \mathrm{id})$ is an extension of ξ required in (3.5). Hence it suffices to show the extensibility of ξ_3 to such a $\tilde{\xi}_3$, which is equivalent to show Assertion (3.5) under the assumption $\xi(x,t) = x$ for all $(x,t) \in N_1''^* \times \mathbf{R}^k$.

We can state what we have to prove as follows:

(3.6). — *Let G_{n_1,n_2} be the Grassmannian of n_2-dimensional subspaces of $\mathbf{R}^{n_1}$, and let $\pi : E \to G_{n_1,n_2}$ denote the universal vector bundle, where*

$$E = \{(\lambda, t) \in G_{n_1,n_2} \times \mathbf{R}^{n_1} : t \in \lambda\}.$$

Set $\widehat{E} = \{(\lambda,t) \in E : |t| \le 1\}$. Let $q : N_1 \to \mathrm{Int}\, \widehat{E}$ be a C^1 Nash map. Let $(\pi \circ q)_* \widehat{E}$ denote the pullback of $\widehat{E}$ by $\pi \circ q$, and regard it as a subset of $N_1 \times \widehat{E}$ by the equality

$$(\pi \circ q)_* \widehat{E} = \{(x, \pi \circ q(x), t) \in N_1 \times \widehat{E}\}.$$

Then there exists a C^1 Nash diffeomorphism $r : (\pi \circ q)_* \widehat{E} \to (\pi \circ q)_* \widehat{E}$ such that for each $(x, \pi \circ q(x), t) \in (\pi \circ q)_* \widehat{E}$,

$$r(x, \pi \circ q(x), t) = (x, \pi \circ q(x), t') \quad \text{for some } t' \in \mathbf{R}^{n_1},$$
$$r(x, \pi \circ q(x), 0) = (x, q(x)),$$

and if $q(x) = (\pi \circ q(x), 0)$ then $r(x, \pi \circ q(x), t) = (x, \pi \circ q(x), t)$.

Here we easily reduce to the case where $N_1 = \mathrm{Int}\, \widehat{E}$ and $q = \mathrm{id}$. For simplicity of notation we consider, moreover, q only on $\mathrm{Int}\, \widehat{E} \cap \pi^{-1}(\lambda_0)$ for one $\lambda_0 \in G_{n_1,n_2}$. The general case is proved in the same way by more complicated notation. Then (3.6) is reduced to the following:

(3.7). — *Set $B = \{x \in \mathbf{R}^{n_1} : |x| < 1\}$. There exists a C^1 Nash map*

$$\eta : \mathrm{cl}(B) \times B \to \mathrm{cl}(B)$$

such that for each $y \in B$, $\eta|\, \mathrm{cl}(B) \times y$ is a diffeomorphism onto $\mathrm{cl}(B)$, $\eta(0,y) = y$, and $\eta|\, \mathrm{cl}(B) \times 0 = \mathrm{id}$.

It is easy to find a C^1 Nash map $\eta_1 : \mathrm{cl}(B) \times B \to \mathrm{cl}(B)$ such that for each $y \in B$, $\eta|\, \mathrm{cl}(B) \times y$ is a diffeomorphism onto $\mathrm{cl}(B)$, and $\eta_1(y,y) = y/2$. Here let us assume the following:

(3.8). — *There exists a C^1 Nash map*

$$\eta_2 : \mathrm{cl}(B) \times \mathrm{cl}(B) \to \mathrm{cl}(B)$$

such that for each $y \in \mathrm{cl}(B)$, $\eta_2 | \mathrm{cl}(B) \times y$ is a diffeomorphism onto $\mathrm{cl}(B)$, $\eta_2(0, y) = y/2$, and $\eta_2 = \eta_1$ on $\mathrm{cl}(B) \times 0$.

Then (3.7) follows and the theorem is proved. Indeed, if η_2 in (3.8) exists, η defined by

$$(\eta, \mathrm{id}) = (\eta_1, \mathrm{id})^{-1} \circ (\eta_2, \mathrm{id}) : \mathrm{cl}(B) \times B \to \mathrm{cl}(B) \times B$$

fulfills the requirements in (3.7). Therefore it remains to show (3.8). Clearly there exists a C^1 map $\mathrm{cl}(B) \times \mathrm{cl}(B) \to \mathrm{cl}(B)$ with the properties in (3.8). Let $\hat{\eta}_2$ be its C^1 Nash approximation such that the properties except $\hat{\eta}_2 = \eta_1$ on $\mathrm{cl}(B) \times 0$ hold, which is possible because $\mathrm{cl}(B)$ is compact. On the other hand, it is easy to construct a C^1 Nash map

$$\check{\eta}_2 : \mathrm{cl}(B) \times W \to \mathrm{cl}(B)$$

for a small semialgebraic neighborhood W of 0 in $\mathrm{cl}(B)$ such that all the properties in (3.8) hold with η_2 replaced by $\check{\eta}_2$. Combine $\hat{\eta}_2$ and $\check{\eta}_2$ by a C^1 Nash partition of unity. Then we obtain the η_2. This completes the proof of Theorem I.

4. Proofs of Theorem II, Proposition (2.2) and Corollary III.

Let $f_t = (f_{t,1}, \dots, f_{t,p}) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in I)$ be a polynomial family of polynomial map-germs. We set $F(x,t) = f_t(x)$, and $F_i(x,t) = f_{t,i}(x)$. Let $q : M \times I \to I$ be the natural projection, $P \in M$, $y = (y_1, \dots, y_n)$ a local coordinate system of M at P, and $t = (t_1, \dots, t_k)$ a local coordinate system of I. Let $\pi : M \to \mathbf{R}^n$ be a proper modification so that $\pi^{-1}(0)$ is a normal crossing divisor. We set $\mathcal{V}$ the strict transform of $F^{-1}(0)$ by $\pi \times \mathrm{id}$ and $V_t = \mathcal{V} \cap q^{-1}(t)$. Assume that we can locally express $\mathcal{V}$ as zero locus of some functions $g_i(y,t)$ $(i = 1, \dots, p)$.

LEMMA (4.1). — *If V_t is nonsingular for each $t \in I$ and transverse to each irreducible component of $\pi^{-1}(0)$, then π induces a simultaneous resolution of $(\mathbf{R}^n, f_t^{-1}(0), 0)$ for $t \in I$.*

Proof. — Since V_t is a nonsingular for each $t \in I$, the matrix $\left(\dfrac{\partial g_i}{\partial y_j}\right)_{1 \le i \le p; 1 \le j \le n}$ has the maximal rank p, and thus $q|\mathcal{V}$ is a submersion. Since π gives a resolution of $(\mathbf{R}^n, f_t^{-1}(0), 0)$, the critical locus of π is a union of divisors $D_1, \dots, D_d$ which have normal crossings. Then, we may choose a local coordinate system at P so that D_i is a coordinate hyperplane if $D_i \ni P$. Let D be the intersection of some of D_i's. Then, without loss of generality, we may assume that D is the zero locus of y_j's for $j = \ell+1, \dots, n$. Since V_t is transverse to D, the matrix $\left(\frac{\partial g_i}{\partial y_j}\right)_{1 \le i \le p; 1 \le j \le \ell}$ has the maximal rank p. For simplicity, we assume that rank $\left(\frac{\partial g_i}{\partial y_j}\right)_{1 \le i,j \le p} = p$, and $D = \mathbf{R}^\ell = \mathbf{R}^p \times \mathbf{R}^{\ell-p}$. By the implicit function theorem, $\mathcal{V} \cap (D \times I)$ is expressed around (P, t) as the graph of some smooth mapping $h : \mathbf{R}^{\ell-p} \times I \to \mathbf{R}^p$. Thus $q|\mathcal{V} \cap D \times I$ is a submersion at (P, t). This completes the proof. □

The following lemma assures the existence of Nash neighborhoods for resolved varieties.

LEMMA (4.2). — *Let $f_t = (f_{t,1}, \dots, f_{t,p}) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ ($t \in I$) be a polynomial family of polynomial map-germs, and let $\pi : M \to \mathbf{R}^n$ be a proper Nash modification which supplies a simultaneous resolution of the family of germs $V_t := f_t^{-1}(0)$ at 0. Set $\Pi = \pi \times \mathrm{id}$, and $\mathcal{M} = M \times I$. Then there is an open Nash neighborhood of $\Pi^{-1}(\{0\} \times I)$ in $\mathcal{M}$, whose closure is a Nash manifold with boundary. Moreover the boundary intersects the strict transform of $\mathcal{V}$ transversely.*

Proof. — Let $\rho : \mathbf{R}^n \to \mathbf{R}$ be a nonnegative Nash function with $\rho^{-1}(0) = 0$. We use the notation in the definition of simultaneous resolution. Set $V_{j,t} = \mathcal{V}_j \cap M_t$. Define the number ε_t be the supremum of the set of numbers ε which satisfies the following condition: δ is not critical value of $\rho \circ \pi | \bigcap_{j \in J} V_{j,t}$ and $\rho \circ \pi | E_F \cap \bigcap_{j \in J} V_{j,t}$ for each $J \subset \{1, \dots, k\}$, $E_F \in \mathcal{E}$, and for $0 < \delta < \varepsilon$. Locally, ε_t is larger than a positive constant. Thus, there is a positive constant ε such that $\varepsilon < \varepsilon_t$ for any $t \in I$. Then $\Pi^{-1}(\{\rho(x) < \varepsilon\})$ is the desired neighborhood. □

Proof of Theorem II. — By supposition, $\pi : M \to \mathbf{R}^n$ induces a simultaneous resolution of $(f_t^{-1}(0), 0)$, for $t \in I$. By Theorem I and Lemma (4.2) we have a Nash diffeomorphism germ

$$\Phi : (M \times I, \Pi^{-1}(\{0\} \times I)) \to (M, \pi^{-1}(0)) \times I$$

satisfying $p \circ \Phi = p$, $\Phi | S(\pi) = \phi$, and trivializing the strict transforms of $(f_t^{-1}(0), 0)$. By Condition (C) and properness of π, Φ induces a homeomorphism germ

$$\Phi' : (\mathbf{R}^n \times I, \{0\} \times I) \to (\mathbf{R}^n, 0) \times I.$$

This completes the proof. □

By the definition of Δ-regularity, we obtain the following, which shows Theorem II$'$.

PROPOSITION (4.3). — *Let I be a closed cuboid $\prod_{i=1}^{k}[a_i, b_i]$ in $\mathbf{R}^k$, and let $f_t = (f_{t,1}, \dots, f_{t,p}) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(t \in I)$ be a polynomial family of polynomial mappings. Let Δ be a regular polyhedron majorizing $\mathbf{R}^n_{\geq}$. If f_t is Δ-regular for each $t \in I$, then $\pi : P_\Delta \to \mathbf{R}^n$ induces a simultaneous resolution of the family of germs of $f_t^{-1}(0)$ at 0.*

Proof of Corollary III.

(1), (3): Since $D(\pi) = 0$, Condition (C) is trivial.

(2): Let D be a component of $\pi^{-1}(0)$ with $\pi(D) \supsetneqq \{0\}$. By supposition, $F|D \times I$ does not depend on t, and Condition (C) is clear.

Therefore (1)–(3) are immediate consequences of Theorem II.

(4): Set $\Delta = \{\nu \in \mathbf{R}^n_{\geq} : \sum_i \nu_i \geq 1\}$. Since $P_\Delta \to \mathbf{R}^n$ is the blow-up at 0, this is an immediate consequence of (3). □

Proof of Proposition (2.2). — Let $t_0 \in I$ and let $\rho : \mathbf{R}^n \to \mathbf{R}$ be a non-negative Nash function with $\rho^{-1}(0) = \{0\}$. By abuse of notation, we denote by A $A \cap \rho^{-1}([0, \varepsilon) \times I)$ (resp. $A \cap ((q \circ \Pi)^{-1}([0, \varepsilon) \times I))$ for a subset A in $\mathbf{R}^n \times I$ (resp. $P_\Delta \times I$). For any $X \in \mathcal{S}$, there is a positive number $\varepsilon_0(X)$ so that $(\rho^{-1}([0, \varepsilon)) \times I) \cap q^{-1}(t_0) \cap \Pi(Z_X)$ is Nash diffeomorphic to an open simplex, because of (i), (iii), and Theorem I. Let ε_0 be the minimum of $\varepsilon_0(X)$ for $X \in \mathcal{S}$. Let A_i denote the union of strata in $\mathcal{S}$ whose dimension is less than or equal to i. Set

$$Z_X^* = (q \circ \Pi)^{-1}(t_0) \cap Z_X, \quad \Pi(Z_X)^* = q^{-1}(t_0) \cap \Pi(Z_X),$$
$$A_i^* = (q \circ \Pi)^{-1}(t_0) \cap A_i,$$

and so on. We construct Nash diffeomorphism germs

$$\phi_i : (A_i, \Pi^{-1}(\{0\} \times I)) \to (A_i^*, \pi^{-1}(0)) \times I \text{ and}$$
$$\phi_i' : (\Pi(A_i), \{0\} \times I) \to (\Pi(A_i^*), 0) \times I$$

with $\phi_i' \circ \Pi = \Pi \circ \phi_i$, by induction on i. The first step of the induction is trivial. Assume that such ϕ_i and ϕ_i' exist. Let X be a stratum of $\mathcal{S}|S(\Pi)$ with $\dim X = i + 1$. By Theorem I, there exists a Nash diffeomorphism

$\phi'_X : \Pi(Z_X) \to \Pi(Z_X)^* \times I$, which extends $\phi'_i|\Pi(Z_X) \cap A_i$, and satisfies $q \circ \phi'_X = q$. Applying Theorem I again, there exists a Nash diffeomorphism $\phi_X : Z_X \to Z^*_X \times I$ which extends $\phi_i|Z_X \cap A_i$ and satisfies $\Pi \circ \phi_X = \phi'_X \circ \Pi$. Repeating this, we obtain the desired ϕ_{i+1} and ϕ'_{i+1}. □

5. Proof of Theorem IV.

Suppose that $Q_t^{-1}(0) \cap S(Q_t) = \{0\}$ for any $t \in I$. Then it follows from the proof of the Buchner-Kucharz theorem (Remark (2.4)) that $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a π-modified C^1 trivialization along I. Namely, there is a t-level preserving C^1 diffeomorphism $\Phi : (W_1, E_0 \times I) \to (W_2, E_0 \times I)$ which induces a t-level preserving homeomorphism $\phi : (U, \{0\} \times I) \to (V \times I, \{0\} \times I)$ such that

$$\phi : ((U, F^{-1}(0) \cap U)) = (V, f_{t_0}^{-1}(0) \cap V) \times I,\ t_0 \in I,$$

where W_1, W_2 are some neighborhoods of $E_0 \times I$ in $E \times I$, U is some neighborhood of $\{0\} \times I$ in $\mathbf{R}^n \times I$, and V is some neighborhood of 0 in $\mathbf{R}^n$. For $t \in I$, we define a mapping $\Phi_t : (W_{1,t}, E_0) \to (W_{2,t}, E_0)$ by $\Phi_t((x;u)) = \Phi((x;u),t)$, where $W_{j,t} = \Pi^{-1}(\mathbf{R}^n \times \{t\}) \cap W_j$, $j = 1,2$. Here we remark that for any $t \in I$,

(5.1)
$$\pi \circ \Phi_t((x_1, \dots, x_n; u)) = \pi \circ \Phi_t(((-1)^{\alpha_1} x_1, \dots, (-1)^{\alpha_n} x_n; -u)),\ u \neq 0.$$

Now we can make the Buchner-Kucharz result slightly clear as follows:

LEMMA (5.2). — *There exist $\varepsilon_0 > 0$ and a t-level preserving C^1 diffeomorphism*

$$\Psi : (S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I, E_0 \times I) \to (S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I, E_0 \times I)$$

such that Ψ induces a π-modified C^1 triviality of $(\mathbf{R}^n \times I, F^{-1}(0))$ along I.

Proof. — For each $t \in I$, let T_t (resp. K_t) denote the strict transform of $f_t^{-1}(0)$ (resp. $Q_t^{-1}(0)$) in $S^{n-1} \times \mathbf{R}$. Then T_t and K_t are Nash submanifolds of $S^{n-1} \times \mathbf{R}$. For $\varepsilon \neq 0$, put

$$\Gamma_\varepsilon = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n : \left(\frac{x_1}{\varepsilon^{\alpha_1}}\right)^2 + \cdots + \left(\frac{x_n}{\varepsilon^{\alpha_n}}\right)^2 = 1 \right\}.$$

Then there is $\varepsilon_0 > 0$ such that $f_t^{-1}(0)$ is transverse to Γ_ε for $t \in I$ and $0 < |\varepsilon| \leq \varepsilon_0$. Therefore T_t is transverse to $S^{n-1} \times \{\varepsilon\}$ for $t \in I$ and

$0 < |\varepsilon| \leq \varepsilon_0$. On the other hand, each K_t is perpendicular to the exceptional variety $S^{n-1} \times \{0\}$. It follows from Remark (2.4) that each T_t is transverse to $S^{n-1} \times \{0\}$. Therefore T_t is transverse to $S^{n-1} \times \{\varepsilon\}$ for $t \in I$ and $|\varepsilon| \leq \varepsilon_0$. Let T denote the strict transform of $F^{-1}(0)$ in $S^{n-1} \times \mathbf{R} \times I$. Then T is a Nash submanifold of $S^{n-1} \times \mathbf{R} \times I$, and

(5.3) $\qquad T$ is transverse to $S^{n-1} \times \{\varepsilon\} \times I$ for $|\varepsilon| \leq \varepsilon_0$.

Let v be a C^1 vector field whose flow gives a t-level preserving C^1 diffeomorphism inducing a π-modified C^1 triviality of $(\mathbf{R}^n \times I, F^{-1}(0))$ along I. We can assume that $S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I \subset U$. Then, by transverse condition (5.3) and a partition of unity, we can modify v so that v is tangent to $S^{n-1} \times \{\varepsilon\} \times I$ ($|\varepsilon| \leq \varepsilon_0$) and T. By construction we can require that the flow of this vector field gives a t-level preserving C^1 diffeomorphism

$$\Psi : (S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I, E_0 \times I) \to (S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I, E_0 \times I)$$

such that $\Psi(T \cap S^{n-1} \times [-\varepsilon_0, \varepsilon_0] \times I) = (T_{t_0} \cap S^{n-1} \times [-\varepsilon_0, \varepsilon_0]) \times I$ and each Ψ_t satisfies condition (5.1). This Ψ induces a π-modified C^1 triviality of $(\mathbf{R}^n \times I, F^{-1}(0))$ along I. □

Theorem IV follows from this lemma and the proof of Theorem I. □

Remark (5.4). — In Lemma (5.2), we have explicitly constructed a uniform Nash neighborhood and a t-level preserving C^1 diffeomorphism on it inducing a π-modified C^1 triviality, in order to make the structure of π-modified Nash triviality comprehensible. But the proof can be shortened for the reader who is interested only in the existence of a uniform Nash neighborhood satisfying the hypotheses of Theorem I.

6. Proofs of Theorem V and Corollary VII.

We first recall two important properties on semialgebraic sets.

THEOREM (6.1) (Tarski-Seidenberg Theorem [32]). — *Let A be a semialgebraic set in $\mathbf{R}^k$, and let $f : \mathbf{R}^k \to \mathbf{R}^m$ be a polynomial mapping. Then $f(A)$ is semialgebraic in $\mathbf{R}^m$.*

THEOREM (6.2) (Semialgebraic Triangulation Theorem [27]). — *Given a finite system of bounded semialgebraic sets $\{X_\alpha\}$ in $\mathbf{R}^n$, there exist a simplicial decomposition $\mathbf{R}^n = \bigcup_a \sigma_a$ and a semialgebraic automorphism τ of $\mathbf{R}^n$ such that*

(i) *each X_α is a finite union of some of the $\tau(\sigma_a)$,*

(ii) *$\tau(\sigma_a)$ is a Nash manifold in $\mathbf{R}^n$ and τ induces a Nash diffeomorphism $\sigma_a \to \tau(\sigma_a)$, for every a.*

Remark (6.3). — In Theorem (6.2), the boundedness is not essential. In fact, there is a Nash imbedding of $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$ via $\mathbf{R}^n \subset S^n$. Then every semialgeraic set in $\mathbf{R}^n$ can be considered as a bounded semialgebraic set in $\mathbf{R}^{n+1}$.

We next show the following:

LEMMA (6.4). — *The set Σ^* is semialgebraic in $P_{[r]}(n,p)$.*

Proof. — We define a polynomial function $G : P_{[r]}(n,p) \times \mathbf{R}^n \to \mathbf{R}$ by

$$G(w,x) = \begin{cases} \sum_{i=1}^{p} w_i(x)^2 + \sum_{1\le i_1<\ldots<i_p\le n} \left| \dfrac{\partial(w_1,\ldots,w_p)}{\partial(x_{i_1},\ldots,x_{i_p})} \right|^2 & \text{if } n > p, \\ \sum_{i=1}^{p} w_i(x)^2 + \sum_{1\le j_1<\ldots<j_n\le p} \left| \dfrac{\partial(w_{j_1},\ldots,w_{j_n})}{\partial(x_1,\ldots,x_n)} \right|^2 & \text{if } n \le p. \end{cases}$$

We denote by $\mathbf{R}_+$ the set of positive real numbers. Let $\Pi_1 : P_{[r]}(n,p) \times \mathbf{R}^n \times \mathbf{R}_+ \to P_{[r]}(n,p) \times \mathbf{R}_+$, and $\Pi_2 : P_{[r]}(n,p) \times \mathbf{R}_+ \to P_{[r]}(n,p)$ be the natural projections. Here we set

$$A = \{(w,x,\alpha) \in P_{[r]}(n,p) \times \mathbf{R}^n \times \mathbf{R}_+ : G(w,x) > 0\}$$
$$B = \{(w,x,\alpha) \in P_{[r]}(n,p) \times \mathbf{R}^n \times \mathbf{R}_+ : 0 < |x| < \alpha\}$$
$$C = \{(w,\alpha) \in P_{[r]}(n,p) \times \mathbf{R}_+ : (w,x,\alpha) \in B \Rightarrow (w,x,\alpha) \in A\}.$$

Then A and B are semialgebraic in $P_{[r]}(n,p) \times \mathbf{R}^n \times \mathbf{R}_+$, and $C = P_{[r]}(n,p) \times \mathbf{R}_+ - \Pi_1(B - A)$. Therefore it follows from Theorem (6.1) that C is semialgebraic in $P_{[r]}(n,p) \times \mathbf{R}_+$. Furthermore we easily see that $P_{[r]}(n,p) - \Sigma^* = \Pi_2(C)$. Therefore it follows from Theorem (6.1) that $P_{[r]}(n,p) - \Sigma^*$ is semialgebraic in $P_{[r]}(n,p)$, and so is Σ^*. □

Proof of Theorem V. — Set

$$\Sigma^{**} = \{w \in P_{[r]}(n,p) : 0 \in \mathbf{R}^n \text{ is not in the singular locus of the zero locus of } w\}.$$

Then $\Sigma^{**} = \{w \in P_{[r]}(n,p) : G(w,0) > 0\}$, where G is the polynomial function defined in the proof of Lemma (6.4). Therefore Σ^{**} is semialgebraic in $P_{[r]}(n,p)$. It follows from Theorem (6.2) that there is a finite partition $\Sigma^{**} = Q_1 \cup \cdots \cup Q_s$ such that each Q_i is a connected Nash manifold which is Nash diffeomorphic to an open simplex of some dimension. Now we shall show that $(\mathbf{R}^n \times Q_i, F_{Q_i}^{-1}(0))$ is Nash trivial along Q_i for each i. We first consider the case $n > p$. For $x = (x_1, ...x_n) \in \mathbf{R}^n$, put $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Set $S_\varepsilon = \{x \in \mathbf{R}^n : \|x\|^2 = \varepsilon\}$ for $\varepsilon > 0$. Then there is a positive C^0 semialgebraic function $g_i : Q_i \to \mathbf{R}$ such that

$$F_{Q_i}^{-1}(0) \cap \{(x,a) \in \mathbf{R}^n \times Q_i : \|x\|^2 < g_i(a)\} \text{ is a } C^\omega \text{ manifold}$$

and $f_a^{-1}(0)$ is transverse to S_ε for $0 < \varepsilon < g_i(a)$ and $a \in Q_i$. By Theorem II.4.1 in [34], we can approximate a positive C^0 semialgebraic function $g_i/2$ by a positive Nash function h_i so that $f_a^{-1}(0)$ is transverse to S_ε for $0 < \varepsilon \leq h_i(a)$ and $a \in Q_i$. Here we set

$$M_i = \{(x,a) \in \mathbf{R}^n \times Q_i : \|x\|^2 \leq h_i(a)\}.$$

Then M_i is a Nash manifold with boundary ∂M_i which is transverse to $F_{Q_i}^{-1}(0)$. Applying Theorem I, we can easily see the Nash triviality of $(\mathbf{R}^n \times Q_i, F_{Q_i}^{-1}(0))$ along Q_i for each i. In the case $n \leq p$, $F_{Q_i}^{-1}(0) = \{0\} \times Q_i$ in the interior of the above kind of Nash manifold with boundary. Therefore Nash triviality holds in this case, too.

We next consider the space $P_{[r]}(n,p) - \Sigma^* - \Sigma^{**}$ denoted by Γ, that is

$$\Gamma = \left\{ w \in P_{[r]}(n,p) : \begin{array}{l} \text{the singular locus of the zero locus} \\ \text{of } w \text{ is } \{0\} \text{ as germs at } 0 \in \mathbf{R}^n \end{array} \right\}.$$

By Lemma (6.4), the space Γ is semialgebraic in $P_{[r]}(n,p)$. Put $b = \dim \Gamma$. For a subset A of $\mathbf{R}^n \times \Gamma$, $\mathrm{cl}(A)$ denotes the closure of A in $\mathbf{R}^n \times \Gamma$. Let $\beta : \mathbf{R}^n \times P_{[r]}(n,p) \to P_{[r]}(n,p)$ be the natural projection.

In the case $n \leq p$, we consider the space

$$\beta|\mathbf{R}^n \times \Gamma\,(\mathrm{cl}\,(\{(x,a) \in \mathbf{R}^n \times \Gamma : F_\Gamma(x;a) = 0\} - \{0\} \times \Gamma) \cap \{0\} \times \Gamma),$$

denoted by Λ. For any $a \in \Gamma$, $f_a^{-1}(0) = \{0\}$ as germs at $0 \in \mathbf{R}^n$. Therefore it follows from Theorem (6.1) that Λ is a semialgebraic set in $P_{[r]}(n,p)$ of dimension less than b, and $\Gamma - \Lambda$ is a semialgebraic set in $P_{[r]}(n,p)$ of dimension b. By Theorem (6.2), there is a finite partition

$$\Gamma - \Lambda = R_1 \cup \cdots \cup R_q$$

such that each R_i is a connected Nash manifold which is Nash diffeomorphic to an open simplex of some dimension. For each i, there is a Nash manifold M_i in $\mathbf{R}^n \times R_i$ such that $F_{R_i}^{-1}(0) = \{0\} \times R_i$ in the interior of M_i, as above. Therefore $(\mathbf{R}^n \times R_i, F_{R_i}^{-1}(0))$ is Nash trivial along R_i for $1 \leq i \leq q$. Then we put $Q_{s+i} = R_i$ for $1 \le i \le q$. Next we apply these arguments for the semialgebraic set Γ of dimension b to the semialgebraic set Λ of dimension less than b. Repeating this procedure, we can get a finite partition of $P_{[r]}(n,p)$ which satisfies conditions (i),(ii) in Theorem V.

It remains to show the case $n > p$. Let $\rho : \mathbf{R}^n \times P_{[r]}(n,p) \to \mathbf{R}$ be the polynomial function defined by $\rho(x;a) = x_1^2 + \cdots + x_n^2$. We define a polynomial function $\Psi : \mathbf{R}^n \times P_{[r]}(n,p) \to \mathbf{R}$ by

$$\Psi(x;a) = \sum_{i=1}^{p} F_i(x;a)^2 + \sum_{1 \le i_1 < \cdots < i_{p+1} \le n} \left| \frac{\partial(F_1, \ldots, F_p, \rho)}{\partial(x_{i_1}, \ldots, x_{i_{p+1}})}(x;a) \right|^2 .$$

We consider the space

$$\beta|\mathbf{R}^n \times \Gamma\Big(\mathrm{cl}\Big(\{(x,a) \in \mathbf{R}^n \times \Gamma : \Psi_\Gamma(x;a) = 0\} - \{0\} \times \Gamma\Big) \cap \{0\} \times \Gamma\Big),$$

denoted by Ω. For $a \in \Gamma$, we define $\psi_a : \mathbf{R}^n \to \mathbf{R}$ by $\psi_a(x) = \Psi(x;a)$. Since $f_a^{-1}(0)$ is an algebraic set with an isolated singularity for $a \in \Gamma$, $\psi_a^{-1}(0) = \{0\}$ as germs at $0 \in \mathbf{R}^n$. Therefore it follows from Theorem (6.1) that Ω is a semialgebraic set in $P_{[r]}(n,p)$ of dimension less than b, and $\Gamma - \Omega$ is a semialgebraic set in $P_{[r]}(n,p)$ of dimension b. By Theorem (6.2), there is a finite partition

$$\Gamma - \Omega = B_1 \cup \cdots \cup B_c$$

such that each B_i is a connected Nash manifold which is Nash diffeomorphic to an open simplex of some dimension. For simplicity, assume that

$$\begin{cases} \dim B_i = b & \text{if } 1 \le i \le d \\ \dim B_i < b & \text{if } d+1 \le i \le c. \end{cases}$$

For $1 \le i \le d$, there is a positive C^0 semialgebraic function $g_i : B_i \to \mathbf{R}$ such that $f_a^{-1}(0)$ is transverse to S_ε for $0 < \varepsilon < g_i(a)$ and $a \in B_i$. As above, we can approximate a positive C^0 semialgebraic function $g_i/2$ by a positive Nash function h_i so that $f_a^{-1}(0)$ is transverse to S_ε for $0 < \varepsilon \le h_i(a)$ and $a \in B_i$. Here we set

$$U_i = \{(x,a) \in \mathbf{R}^n \times B_i : \rho(x;a) < g_i(a)\},$$
$$M_i = \{(x,a) \in \mathbf{R}^n \times B_i : \rho(x;a) \le h_i(a)\}$$

for $1 \le i \le d$. We can assume that $M_i \subset U_i$. For $1 \le i \le d$, let $X_i \subset \mathbf{R}^n \times B_i$ be an algebraic set defined by $X_i = F_{B_i}^{-1}(0)$. Then each M_i is a Nash manifold with boundary ∂M_i which is transverse to X_i. We consider X_i to be defined in U_i. Therefore X_i is a family of algebraic sets whose singular locus is in $\{0\} \times B_i$. Then, by Hironaka's Main Theorem I in [16], there exists a blow-up $\Pi_i : M_i \to \mathbf{R}^n \times B_i$ with center Ξ_i in $\{0\} \times B_i$ such that the strict transform X_i' of X_i by Π_i is nonsingular at $X_i' \cap \Pi^{-1}(\Xi_i)$ and $\Pi_i^{-1}(\Xi_i)$ has only normal crossings. Moreover, by applying Main Theorem II in [16] to $X_i' \cap \Pi_i^{-1}(\Xi_i)$ and $\Pi^{-1}(\Xi_i)$, we can require that $\Pi^{-1}(X_i)$ has only normal crossings. Let

$$\Pi_i(\Xi_i) = E_{i1} \cup \cdots \cup E_{i\alpha(i)}$$

where E_{ij} is nonsingular, and let $X_i' = E_{i\alpha(i)+1}$. We set $W_i = \Pi_i^{-1}(M_i)$ for $1 \leq i \leq d$. Then W_i is a Nash manifold with boundary ∂W_i such that

$$\mathrm{Int}(W_i) \supset E_{i1} \cup \cdots \cup E_{i\alpha(i)}$$

and ∂W_i is transverse to $E_{i\alpha(i)+1}$. We denote by C_i the union of critical value sets of

$$\beta \circ \Pi_i | E_{ij_1} \cap \cdots \cap E_{ij_n}, \; 1 \leq j_1 < \ldots < j_n \leq \alpha(i) + 1.$$

By Sard's Theorem and Theorem (6.1), C_i is a semialgebraic set of dimension less than b. Applying Theorem (6.2) again to B_i, there is a finite partition

$$B_i = T_{i1} \cup \cdots \cup T_{i\gamma(i)}$$

such that each T_{ik} is a connected Nash manifold which is Nash diffeomorphic to an open simplex of some dimension, and the T_{ik}'s are compatible with

$$\beta \circ \Pi_i(E_{ij_1} \cap \cdots \cap E_{ij_u}), \; 1 \leq j_1 < \ldots < j_u \leq \alpha(i) + 1$$

and C_i. As above, assume that

$$\begin{cases} \dim T_{ik} = b & \text{if } 1 \leq k \leq \lambda(i) \\ \dim T_{ik} < b & \text{if } \lambda(i) + 1 \leq k \leq \gamma(i). \end{cases}$$

Remark that $T_{ik} \cap C_i = \emptyset$, $k = 1, \ldots, \lambda(i)$. For $1 \leq k \leq \lambda(i)$, set

$$\begin{aligned} M_{ik} &= (\beta \circ \Pi_i)^{-1}(T_{ik}) \cap W_i, \\ \Pi_{ik} &= \Pi_i | M_{ik} : M_{ik} \to \mathbf{R}^n \times T_{ik}, \text{ and} \\ \beta_{ik} &= \beta \circ \Pi_i | M_{ik} : M_{ik} \to T_{ik}. \end{aligned}$$

We further set

$$E_{ijk} = E_{ij} \cap (\beta \circ \Pi_i)^{-1}(T_{ik}), \; j = 1, \ldots, \alpha(i) + 1.$$

Then M_{ik} is a Nash manifold with boundary ∂M_{ik} and β_{ik} is a proper onto Nash submersion. Let $E_{i0k} = \partial M_{ik}$. Then E_{ijk}, $j = 0, 1, \ldots, \alpha(i) + 1$, are Nash submanifolds of M_{ik} possibly with boundary which have normal crossings and

$$\beta_{ik} | E_{ij_1k} \cap \cdots \cap E_{ij_uk} : E_{ij_1k} \cap \cdots \cap E_{ij_uk} \to T_{ik}, \; 0 \leq j_1 < \ldots < j_u \leq \alpha(i)+1,$$

are proper onto submersions. Here we remark that $\Pi_{ik} : M_{ik} \to \mathbf{R}^n \times T_{ik}$ is a blow-up with center $\{0\} \times T_{ik}$ (or $\emptyset$). Therefore it follows from Theorem I that $(\mathbf{R}^n \times T_{ik}, F_{T_{ik}}^{-1}(0))$ admits a Π_{ik}-modified Nash trivialization along T_{ik} (in the empty case, $(\mathbf{R}^n \times T_{ik}, F_{T_{ik}}^{-1}(0))$ is Nash trivial along T_{ik}) for $1 \leq k \leq \lambda(i)$, $i = 1, \ldots, d$. Then we set

$$Q_{s+\lambda(1)+\cdots+\lambda(i)+k} = T_{ik} \quad \text{for } 1 \leq k \leq \lambda(i+1) \text{ and } 0 \leq i \leq d-1.$$

Next we apply these arguments for the semialgebraic set Γ of dimension b to the following semialgebraic set of dimension less than b:

$$\Omega \cup \left(\bigcup_{i=d+1}^{c} B_i \right) \cup \left(\bigcup_{i=1}^{d} \bigcup_{k=\lambda(i)+1}^{\gamma(i)} T_{ik} \right).$$

Repeating this procedure, we can get a finite partition of $P_{[r]}(n,p)$ which satisfies conditions (i),(ii) in Theorem V in this case, too. □

Remark (6.5). — Subdividing the above partition of $P_{[r]}(n,p) - \Sigma^*$ if necessary, we can construct a partition in Theorem V whose elements satisfy the frontier condition.

Proof of Corollary VII. — We can show the following theorem in a similar way to the Artin-Mazur Theorem (Theorem (1.5)).

THEOREM (6.6) (Generalized Artin-Mazur Theorem). — *Let M be the product of two Nash manifolds M_1 and J. Let $f : M \to \mathbf{R}^p$ be a Nash mapping, and let $\varpi : M \to J$ be the natural projection. Then there exist a union M' of connected components of some nonsingular algebraic variety in $\mathbf{R}^k \times \mathbf{R}^m$, a union J' of connected components of some nonsingular algebraic variety in $\mathbf{R}^k$, and Nash diffeomorphisms $\phi : M' \to M$, $\psi : J \to J'$ such that $f \circ \phi$ is a polynomial mapping and $\psi \circ \varpi \circ \phi$ is the onto projection $\beta | M' : M' \to J'$, where $\beta : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}^k$ denotes the natural projection.*

Remark (6.7). — In Theorem (6.6), it is difficult to choose M' as the product of J' and some Nash manifold.

Corollary VII is an obvious consequence of Theorem VI and this generalized Artin-Mazur Theorem. □

7. Δ-regularity and examples.

7.1. Δ-regularity.

The first purpose of this section is to give an explicit description of Δ-regularity. Let $x = (x_1, \dots, x_n)$ be the coordinate system of $\mathbf{R}^n$. Let $f = (f_1, \dots, f_p) : \mathbf{R}^n \to \mathbf{R}^p$ be a polynomial map defined by $f_j(x) = \sum c^j_\nu x^\nu$ where $\nu = (\nu_1, \dots, \nu_n)$, $c^j_\nu \in \mathbf{R}$, $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$. We set $\Gamma_+(f) = (\Gamma_+(f_1), ..., \Gamma_+(f_p))$ where $\Gamma_+(f_j)$ is the convex hull of the union of the sets $\nu + \mathbf{R}^n_{\geq}$ for ν with $c^j_\nu \neq 0$ and call it the *Newton polyhedron* of f. For an integral vector $a = {}^t(a_1, \dots, a_n)$, $\ell^j(a)$ denotes the minimum of $\langle a, \nu \rangle$ with $c^j_\nu \neq 0$, where $\langle a, \nu \rangle = \sum_{i=1}^{n} a_i \nu_i$. Set

$$f^a(x) = (f^a_1(x), \dots, f^a_p(x)), \quad \text{where } f^a_j(x) = \sum_{\langle a,\nu \rangle = \ell^j(a)} c^j_\nu x^\nu,$$

and let $\dfrac{\partial f^a}{\partial x}(x)$ denote the jacobian matrix $\left(\dfrac{\partial f^a_j}{\partial x_i}\right)_{1 \leq i \leq n; 1 \leq j \leq p}$. We denote by e^i the i-th unit column vector ${}^t(0, \dots, 0, 1, 0, .., 0)$, $(1 \leq i \leq n)$. Then we set

$$I_0 = \left\{ i : \ell^j(e^i) > 0 \text{ for some } j \right\}, \quad \text{and} \quad S_0 = \left\{ x \in \mathbf{R}^n : \prod_{i \in I_0} x_i = 0 \right\}.$$

An integral vector $a = (a_1, \dots, a_n)$ is said to be *primitive* if $\gcd(a_1, \dots, a_n) = 1$. A face F of Δ is said to be *supported by a* if F is defined by

$$F = \left\{ \nu \in \Delta : \langle a, \nu \rangle = \min\{\langle a, \mu \rangle : \mu \in \Delta\} \right\}.$$

Let P be a vertex of Δ. It is easy to see that Δ is regular at P if and only if primitive vectors supporting $(n-1)$-dimensional faces of Δ containing P form a basis of $\mathbf{Z}^n$. Let Δ be a regular polyhedron majorizing $\mathbf{R}^n_{\geq}$, $\Delta^{(1)}$ the set of primitive vectors which support faces of Δ of codimension 1, and $\Delta^{(1)}_+$ the union of $\Delta^{(1)} - \{e^1, \dots, e^n\}$ and $\{e^i : i \in I_0\}$. Set

$$V(\Delta) = \left\{ a \in \mathbf{Z}^n : \begin{array}{l} a \text{ can be written in the form } a = a^1 + \cdots + a^k \\ \text{for some } a^j \in \Delta^{(1)}_+, \ (1 \leq j \leq k), \text{ and supports} \\ \text{a face of } \Delta \text{ of codimension } k. \end{array} \right\}.$$

Let F be a face of Δ. For a vertex P of F, we set $I(P) = \{i : \langle e^i, P\rangle > 0\}$. Then we define

$$S(F) = \bigcap_P \Big\{ x \in \mathbf{R}^n : \prod_{i \in I(P)} x_i = 0 \Big\}$$

where the intersection is taken over all vertices P of F. Let Δ be a regular polyhedron majorizing $\mathbf{R}^n_{\geq}$. Without loss of generality, we may assume that $\min\{\langle e^i, \nu\rangle : \nu \in \Delta\} = 0$, for each $i = 1, \ldots, n$.

PROPOSITION (7.1). — *If a polynomial map-germ $f = (f_1, \ldots, f_p) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ satisfies*

$$\Big\{ x : f^a(x) = 0, \ \operatorname{rank} \frac{\partial f^a}{\partial x}(x) < p \Big\} \subset S(F_a) \cup S_0, \ \textit{for every } a \in V(\Delta),$$

then f is Δ-regular. Here F_a is the face of Δ supported by a.

For the proof of Proposition (7.1), we prepare several notations. Let P be a vertex of Δ and $\mathbf{R}^n_y$ the coordinate patch of P_Δ corresponding to the cone $\mathrm{Cone}(\Delta; P)$. We denote by $y = (y_1, \ldots, y_n)$ the canonical coordinate system of $\mathbf{R}^n_y$. Let $\overline{a}^j = {}^t(\overline{a}^j_1, \ldots, \overline{a}^j_n)$, $j = 1, \ldots, n$ be a basis of $\mathrm{Cone}(\Delta; P) \cap \mathbf{Z}^n$, and let (a^j_i) be the inverse matrix of $(\overline{a}^j_i)$. Then the intersection of faces F_{a^j} of Δ supported by $a^j = {}^t(a^j_1, \ldots, a^j_n)$'s $(j = 1, \ldots, n)$ is the vertex P. And the map $\pi|\mathbf{R}^n_y$ is expressed by

$$x_i = y_1^{a^1_i} y_2^{a^2_i} \cdots y_n^{a^n_i}, \ \text{for } i = 1, \ldots, n.$$

In particular, the critical locus of π is a normal crossing divisor. This divisor generates the canonical stratification of the critical set of π, indexed by some faces of Δ. In fact, these strata are indexed by F_a with $a \in E(\Delta)$. Here F_a is the face of Δ supported by a, and

$$E(\Delta) = \left\{ a \in \mathbf{Z}^n : \begin{array}{l} a \text{ can be written in the form } a = b^1 + \cdots + b^k \\ \text{for some } b^j \in \Delta^{(1)} - \{e^1, \ldots, e^n\} \ (j = 1, \ldots, k), \\ \text{and } \operatorname{codim} F_a = k. \end{array} \right\}.$$

We consider the stratification $\mathcal{E}$ of the critical set of π generated by this stratification and S_0, also indexed by some faces of Δ. Strata in $\mathcal{E}$ are indexed by F_a with $a \in V(\Delta)$.

Let F be a face of Δ with $P \in F$ so that $F = F_a$ for some $a \in V(\Delta)$, and denote by E_F the corresponding stratum of $\mathcal{E}$. By suitably renumbering, we may assume that for all $j \in \{1, \ldots, n\}$ there exists $i \notin I_0$

with $a^j = e^i$, if and only if, $j = i \in \{1,..,t\}$. Here t is some non-negative integer less than n. Then the map π is expressed by

$$x_i = \begin{cases} y_i y_{t+1}^{a_i^{t+1}} \cdots y_n^{a_i^n}, & \text{for } i = 1, \ldots, t, \\ y_{t+1}^{a_i^{t+1}} \cdots y_n^{a_i^n}, & \text{for } i = t+1, \ldots, n. \end{cases}$$

In particular, the stratification $\mathcal{E}|\mathbf{R}_y^n$ is the stratification generated by the divisors $\{y_i = 0\}$ for $i = t+1, \ldots, n$. Let J be a subset of $\{t+1, \ldots, n\}$ such that $\mathrm{cl}(E_F) \cap \mathbf{R}_y^n = \{y_j = 0 : j \in J\}$. Here $\mathrm{cl}(E_F)$ denotes the closure of E_F in P_Δ. We consider a subset $\tilde{E}_F$ in $\mathbf{R}_y^n$ which is defined by the following condition:

$$\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n) \in \tilde{E}_F \Leftrightarrow \begin{cases} \tilde{y}_j = 1 & \text{for } j \in J, \\ \tilde{y}_j \neq 0 & \text{for } j \notin J \text{ with } t < j \leq n. \end{cases}$$

For $k = 1, \ldots, p$, we define $\tilde{f}_k(y)$, $\tilde{f}_k{}^a(y)$ by

$$f_k \circ \pi(y) = y_1^{\ell^k(a^1)} \cdots y_n^{\ell^k(a^n)} \tilde{f}_k(y),$$

and

$$f_k^a \circ \pi(y) = y_1^{\ell^k(a^1)} \cdots y_n^{\ell^k(a^n)} \tilde{f}_k{}^a(y),$$

respectively. Remember that the vector a supports the face F of Δ. By definition, the polynomial $\tilde{f}_k{}^a(y)$ does not depend on y_j for $j \in J$. We set $\tilde{f}(y) = (\tilde{f}_1(y), \ldots, \tilde{f}_p(y))$, and $\tilde{f}^a(y) = (\tilde{f}_1{}^a(y), \ldots, \tilde{f}_p{}^a(y))$.

Lemma (7.2). — *The following conditions are equivalent:*

(i) *There exists* $y \in E_F$ *so that* $\tilde{f}(y) = 0$ *and* rank $\left(\dfrac{\partial \tilde{f}}{\partial y_j}(y) : j \notin J\right) < p$.

(ii) *There exists* $y \in E_F$ *so that* $\tilde{f}^a(y) = 0$ *and* rank $\left(\dfrac{\partial \tilde{f}^a}{\partial y_j}(y) : j \notin J\right) < p$.

(iii) *There exists* $\tilde{y} \in \tilde{E}_F$ *so that* $\tilde{f}^a(\tilde{y}) = 0$ *and* rank $\left(\dfrac{\partial \tilde{f}^a}{\partial y_j}(\tilde{y}) : j \notin J\right) < p$.

(iv) *There exists* $\tilde{x} = (x_1, \ldots, x_n)$ *so that* $x_i \neq 0$ *for* $i = t+1, \ldots, n$, $f^a(\tilde{x}) = 0$ *and* rank $\left(\dfrac{\partial f^a}{\partial x}(\tilde{x})\right) < p$.

Proof.

"(i) $\Leftrightarrow$ (ii)": This is an obvious consequence of the fact $\tilde{f}|E_F = \tilde{f}^a|E_F$.

"(ii) $\Leftrightarrow$ (iii)": Since $\tilde{f}^a(y)$ does not depend on y_j, $j \in J$, conditions (ii) and (iii) are equivalent.

"(iii) $\Leftrightarrow$ (iv)": We first see that $\tilde{f_k}^a(\tilde{x}) = \prod_j \tilde{y}_j^{\ell^k(a^j)} \cdot \tilde{f_k}^a(\tilde{y})$ for $k = 1, \ldots, p$. Next we remark the following identity proved by using the fact $x_i = y_1^{a_i^1} \cdots y_n^{a_i^n}$, $i = 1, \ldots, n$:

$$\frac{\partial f^a}{\partial x}\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \ldots & a_n^n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1^a}{\partial y_1} & \cdots & \frac{\partial f_1^a}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial f_p^a}{\partial y_1} & \cdots & \frac{\partial f_p^a}{\partial y_n} \end{pmatrix}\begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix}.$$

Since $a^i = e^i$, $i = 1, \ldots, t$, each component of the matrix

$$\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \ldots & a_n^n \end{pmatrix}\begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix}^{-1}$$

is regular on $\tilde{E}_F$, and "$x_i = 0$ if and only if $y_i = 0$" for $i = 1, \ldots, t$ on $\tilde{E}_F$. Because $f_k^a(x)$ is a weighted homogeneous polynomial of weight $(a_1^j, \ldots, a_n^j)$ for $j \in J$, we obtain $\dfrac{\partial f^a}{\partial y_j}(\tilde{y}) = 0$, $j \in J$, if $\tilde{y} \in \tilde{E}_F \cap (f^a \circ \pi)^{-1}(0)$. These imply "(iii) $\Leftrightarrow$ (iv)". □

Proof of Proposition (7.1). — We continue the notation. Note that $I(P) \cup I_0 = \{t+1, \ldots, n\}$. By assumption, Condition (iv) in Lemma (7.2) does not hold. Thus, by Lemma (7.2), the negation of Condition (i) in Lemma (7.2) holds, which implies that π gives a resolution of $f^{-1}(0)$ near E_F. This completes the proof. □

7.2. Weighted homogeneous polynomials.

Here we consider weighted homogeneous polynomials in 3 variables.

PROPOSITION (7.3). — *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of type $(a_1, a_2, a_3; d)$, that is, a linear combination of monomials $x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3}$ with $a_1\nu_1 + a_2\nu_2 + a_3\nu_3 = d$. Here, we assume that a_1, a_2, a_3, d are positive integers with $\gcd(a_1, a_2, a_3) = 1$. If f has an isolated singularity at the origin, then there is a regular polyhedron Δ so that $D(\pi) = 0$ and that f is Δ-regular.*

Proof. — We set $\Gamma = \Gamma_+(f)$. For an integral vector α, we denote by $\ell(\alpha)$ the minimum of $\langle \alpha, \nu \rangle$ with $\nu \in \Gamma$, and by $\gamma(\alpha)$ the face of Γ supported by α. Let $g(x_1, x_2, x_3)$ be the polynomial so that $f = x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3} g$, $\ell_i \geq 0$, and that $g(x_1, x_2, x_3)$ is not a multiple of x_i $(i = 1, 2, 3)$. Since f defines an isolated singularity at the origin, each $\ell_i = 0$ or 1 and at most one ℓ_i is 1. Set

$$\Delta_0 = \{(\nu_1, \nu_2, \nu_3) \in \mathbf{R}^3_{\geq} : a_1\nu_1 + a_2\nu_2 + a_3\nu_3 \geq d\},$$

and π_0 the map $P_{\Delta_0} \to \mathbf{R}^3$. Let Z_i be the intersection of ${\pi_0}^{-1}(0)$ and the strict transform of $\{x_i = 0\}$ via π_0. Let Σ denote the singular locus of P_{Δ_0} and let i, j, k be integers with $\{i, j, k\} = \{1, 2, 3\}$. Then we have that "$Z_k \subset \Sigma$ if and only if $\gcd(a_i, a_j) > 1$," and that "$Z_j \cap Z_k \subset \Sigma$ if and only if $a_i > 1$." Since f defines an isolated singularity at 0, we have the followings:

(o) $\left\{\dfrac{\partial f}{\partial x} = 0\right\} = \{0\}$.

(i) If $\ell_k = 1$, then $\left\{\dfrac{\partial f^{e_k}}{\partial x} = 0\right\} = \{x_i = x_j = 0\}$.

If $\ell_k = 0$, then there are integers ν'_i, ν'_j such that $a_i\nu'_i + a_j\nu'_j = d$. Moreover, if there are integers ν_i, ν_j with $a_i\nu_i + a_j\nu_j + a_k = d$, a_k is a multiple of $\gcd(a_i, a_j)$. Since $\gcd(a_1, a_2, a_3) = 1$, we have $Z_k \not\subset \Sigma$. Thus we obtain the following:

(ii) If $Z_k \subset \Sigma$ and $\ell_k = 0$, then there are no integers ν_i, ν_j with $a_i\nu_i + a_j\nu_j + a_k = d$. Thus f contains no linear terms in x_k, and $\dfrac{\partial(f^{e_k})}{\partial x} = \left(\dfrac{\partial f}{\partial x}\right)^{e_k}$. Therefore $\left\{\dfrac{\partial(f^{e_k})}{\partial x} = 0\right\} \subset \{x_i x_j = 0\}$.

Let K be the set of numbers k so that some power of x_k appears in f. If $k \notin K$, there is a number j such that the term $x_j x_k^b$ attains a vertex of Γ for some integer b, because f defines an isolated singularity. Let K_1 be the set of numbers k such that $k \notin K$ and that there exists exactly one j satisfying the condition above. We denote this unique j by j_k. Let K_2 be the set of numbers k such that $k \notin K$ and that there exist exactly two j's satisfying the condition above.

For $k \in K_1$, we set $\Delta_0(j_k) = \Delta_0 \cap \{\nu_{j_k} = 0\}$. Let $\Delta_1(j_k)$ denote a polyhedron in $\{\nu_{j_k} = 0\}$ such that $\Delta_1(j_k) \cap \{\nu_k \geq 1\} = \Delta_0(j_k)$ and $\Delta_1(j_k)$ is regular at each point in $\{\nu_k \leq 1, \nu_{j_k} = 0\}$, and $\Delta_1(j_k)$ meets x_k-axis. We set Δ_1 the convex hull of $\Gamma \cup \bigcup_{k \in K_1} \Delta_1(j_k)$, and set π_1 the obvious map $P_{\Delta_1} \to P_{\Delta_0}$.

(iii) If $k \in K_1$ and $\ell_{j_k} = 0$, then P_{Δ_1} is regular near the strict transform of $\{x_{j_k} = 0\}$ in the exceptional set of π_1.

Proof of (iii). — Without loss of generality, we may assume that $(k,i,j_k)=(1,2,3)$. We first assume that $1\in K$. Let (a'_1,a'_2) be a primitive vector supporting an edge E of $\Delta_1(3)$. Then there is a positive rational number a'_3 so that $a'=(a'_1,a'_2,a'_3)$ supports the face generated by $(b,0,1)$ and the edge E. Since $a'_1 b+a'_3$ is an integer, a' is a primitive integral vector. Thus, the smoothness of $P_{\Delta_1(3)}$ implies the smoothness of P_{Δ_1} near $P_{\Delta_1(3)}$. The proof for the case $1\notin K$ is similar, and we omit the details. This completes the proof of (iii).

For $k\in K_2$, we set $a^k=(a+ce_j+ce_i)/a_k$, where c is the minimal positive integer such that a_j+c is a multiple of a_k. It is easy to see that this a^k is an integral vector. Set

$$\Delta_2=\Delta_1\cap\{\nu\in\Delta:\langle a^k,\nu\rangle\geq\ell(a^k),k\in K_2\}.$$

We also set $F_k=\gamma(a)\cap\gamma(a^k)$ for $k\in K_2$. Then P_{Δ_1} may be singular along P_{F_k} for $k\in K_2$. We consider a partial resolution $\pi_2:P_{\Delta_3}\to P_{\Delta_2}$ satisfying the following conditions:

(1) each 2-face of Δ_3 is supported by a vector that supports a 2-dimensional face of Δ_2 or a linear combination of a and a^k with positive coefficients for $k\in K_2$.

(2) P_{Δ_2} is regular at each point of $\pi_2^{-1}(P_{F_k})$ except codimension 3, i.e. some finite points, for $k\in K_2$.

(iv) If $k\in K_2$, then P_{Δ_3} is regular at each point in $\pi_2^{-1}(P_{F_k})$.

Proof of (iv). — Let P be a vertex of Δ_3 such that P is in the inverse image of F_k by $\Delta_3\to\Delta_2$. Let c^1 and c^2 be the primitive vectors supporting the 2-dimensional faces containing P and assume that both vectors are linear combinations of a and a^k with non-negative coefficients. Then the plane spanned by c^1 and c^2 is that spanned by a^k and e_i+e_j. Since $\det(e_i,e_i+e_j,a^k)=\pm1$, the "height" of e_i to this plane is 1. Since c^1,c^2 form a part of basis of $\mathbf{Z}^3$, we have $\det(c^1,c^2,e^i)=\pm1$. This completes the proof of (iv).

Let Δ be a regular polyhedron majorizing Δ_3 so that $P_\Delta\to P_{\Delta_3}$ is an isomorphism except over the singular locus of P_{Δ_3}. It is possible to obtain the classification table of the Newton polyhedra of weighted homogeneous f with isolated singularity (see III. §6 in [31], for example). By elementary computation in each case, (o)-(iv) shows that f is Δ-regular. □

The parameterized version of Proposition (7.3) is also true.

PROPOSITION (7.4). — *Let I be a closed cuboid. Let $f_t(x_1, x_3, x_3)$ ($t \in I$) be a family of weighted homogeneous polynomials of type $(a_1, a_2, a_3; d)$. Here, we assume that a_1, a_2, a_3, d are positive integers with $\gcd(a_1, a_2, a_3) = 1$. If f_t has an isolated singularity at the origin for each $t \in I$, then there is a regular polyhedron Δ so that $D(\pi_\Delta) = 0$ and that f_t is Δ-regular. Thus, $(\mathbf{R}^3 \times I, F^{-1}(0))$ admits a π_Δ-modified Nash trivialization.*

Proof. — If we set $\Gamma = \Gamma_+(f_t)$ for general t, the exactly same arguments as in the proof of Proposition (7.3), after obvious changing of notation, shows that f_t is Δ-regular for an appropriate Δ. We remark that $D(\pi_\Delta) = 0$. By Theorem II, we obtain the last sentence. □

7.3. Examples.

We work some examples here.

Example (7.5). — Let $f_t(x_1, x_2, x_3) = x_3^5 + tx_2^6x_3 + x_1x_2^7 + x_1^{15}$ (J. Briançon–J.-P. Speder [2]). This is a weighted homogeneous polynomial of type $(1, 2, 3; 15)$, and defines an isolated singularity at the origin, if $t \neq t_0 = -15^{1/7}(\frac{7}{2})^{4/5}/3$. Let I be a closed interval not containing t_0. Let Δ be a polyhedron whose vertices are

$(15,0,0), (1,7,0), (0,8,0), (0,6,1), (3,0,4), (1,1,4), (0,3,3), (1,0,5), (0,0,6)$,

and that coincides with the positive orthant $\mathbf{R}^3_{\geq}$ outside some compact set, i.e.

$$\Delta = \left\{ (\nu_1, \nu_2, \nu_3) \in \mathbf{R}^3_{\geq} : \begin{array}{ll} \nu_1 + 2\nu_2 + 3\nu_3 \geq 15, & \nu_1 + \nu_2 + \nu_3 \geq 6, \\ \nu_1 + 2\nu_2 + 2\nu_3 \geq 11, & \nu_1 + \nu_2 + 2\nu_3 \geq 8 \end{array} \right\}.$$

It is not difficult to see that Δ is regular. We show that $\pi = \pi_\Delta : P_\Delta \to \mathbf{R}^3$ gives a simultaneous resolution for $(\mathbf{R}^3, f_t^{-1}(0))$ ($t \in I$). In fact, setting $y = (y_1, y_2, y_3)$, for example, the coordinate system of a coordinate patch of P_Δ defined by

$$\begin{cases} x_1 = y_1y_2y_3 \\ x_2 = \quad y_2y_3^2 \\ x_3 = \quad y_2^2y_3^3, \end{cases}$$

we have

$$f_t \circ \pi(y) = y_2^8y_3^{15}(y_2^2 + t + y_1 + y_1^{15}y_2^7),$$

and the strict transform is nonsingular and transverse to each irreducible component of $\pi^{-1}(0)$ in this coordinate patch. Similar computations in the other coordinate patches show that $\pi = \pi_\Delta : P_\Delta \to \mathbf{R}^3$ gives a simultaneous resolution for $(\mathbf{R}^3, f_t^{-1}(0))$ $(t \in I)$. (This fact is also followed from Proposition (7.4), and the idea of proof of (7.4) can be explained in this way.) We here remark that $D(\pi) = 0$. Thus, $(\mathbf{R}^3 \times I, F^{-1}(0))$ admits a π-modified Nash trivialization, because of Theorem II. In this case, F admits a modified analytic trivialization as a family of functions (see T. Fukui [14] for details). However, the induced topological triviality does not preserve the tangency of analytic arcs contained in $f_t^{-1}(0)$ (see S. Koike [21]).

We should remark that, in [30], M. Oka already observed that the Briançon-Speder family admits a weak simultaneous resolution, which is a notion of a family of varieties, using appropriate toric modification.

Example (7.6). — Let $f_t(x_1, x_2, x_3) = x_3^3 + tx_2^\alpha x_3 + x_1 x_2^\beta + x_1^{3\alpha}$ ([2]), where α is an odd integer with $\alpha \geq 3$, and $2\beta + 1 = 3\alpha$. This is a weighted homogeneous polynomial of type $(1, 2, \alpha; 3\alpha)$, and defines an isolated singularity at the origin, if $t \neq t_0 = (-1/3\alpha)^{1/3\beta}(-3\beta^2/\alpha^2)^{1/3}$. Let I be a closed interval not containing t_0. Then there exists a regular polyhedron Δ so that $\pi = \pi_\Delta : P_\Delta \to \mathbf{R}^3$ gives a simultaneous resolution for $(\mathbf{R}^3, f_t^{-1}(0))$ $(t \in I)$, and so that $D(\pi) = 0$. Thus, by Theorem II, $(\mathbf{R}^3 \times I, F^{-1}(0))$ admits a π-modified Nash trivialization.

Let $I = I_\varepsilon$ denote an interval $[-\varepsilon, \varepsilon]$ for a sufficiently small positive number ε.

Example (7.7). — Let $f(x_1, x_2, x_3) = x_1^{12} + (x_2^3 + x_3^2)^2 + x_1 x_2 x_3^3$. This is a weighted homogeneous polynomial of type $(1, 2, 3; 12)$, and defines an isolated singularity at the origin. An elementary calculation shows that f is not non-degenerate. By Proposition (7.4), there is a regular polyhedron Δ so that f is Δ-regular and so that $D(\pi_\Delta) = 0$. If we define $F : (\mathbf{R}^3 \times I, \{0\} \times I) \to (\mathbf{R}, 0)$ by $F(x; t) = f(x) + tx_2^3 x_3^2$, then, by Proposition (7.4), π_Δ gives a simultaneous resolution for $(\mathbf{R}^3, f_t^{-1}(0))$ $(t \in I)$ where $f_t(x) = F(x; t)$. Therefore, because of Theorem II, $(\mathbf{R}^3 \times I, F^{-1}(0))$ admits a π_Δ-modified Nash trivialization.

Example (7.8). — Let $f(x_1, x_2, x_3) = x_1^\alpha + x_1^6(x_2^3 + x_3^2) + (x_2^3 + x_3^2)^2 + x_1 x_2 x_3^3$, where α is an integer with $\alpha > 12$. This is not a weighted homogeneous polynomial, and defines an isolated singularity near the origin. An elementary calculation shows that f is not non-degenerate. Using a discussion similar to the proof of Proposition (7.3), we can construct a regular polyhedron Δ so that f is Δ-regular and so that $D(\pi_\Delta) = 0$. If we define F by $F(x;t) = f(x) + tx_2^3x_3^2$, then, by Proposition (7.4), π_Δ gives a simultaneous resolution for $(\mathbf{R}^3, f_t^{-1}(0))$ $(t \in I)$ where $f_t(x) = F(x;t)$. Therefore, because of Theorem II, $(\mathbf{R}^3 \times I, F^{-1}(0))$ admits a π_Δ-modified Nash trivialization.

Example (7.9). — Let $f(x_1, x_2, x_3, x_4) = x_1^{12} + x_2^8 + (x_3^3 + x_4^2)^2 + x_1x_3x_4^3$. This is a weighted homogeneous polynomial of type $(2, 3, 4, 6; 24)$ and defines an isolated singularity at the origin. By elementary computation, we can show that there is a regular polyhedron Δ so that f is Δ-regular and so that $D(\pi_\Delta) = 0$. If we define $F : (\mathbf{R}^4 \times I, \{0\} \times I) \to (\mathbf{R}, 0)$ by $F(x;t) = f(x) + tx_3^3x_4^2$, then, by the same way as the proof of Proposition (7.4), we are able to show that π_Δ gives a simultaneous resolution for $(\mathbf{R}^4, f_t^{-1}(0))$ $(t \in I)$ where $f_t(x) = F(x;t)$. Therefore, because of Theorem II, $(\mathbf{R}^4 \times I, F^{-1}(0))$ admits a π_Δ-modified Nash trivialization.

This example seems to suggest that the analogy to Proposition (7.3) holds in the case $n \geq 4$. Recently L. Paunescu and the first author have given a positive answer for this in [37].

Example (7.10). — Let $f_c : \mathbf{R}^n \to \mathbf{R}^p$ be the polynomial map defined by $f_{c,j}(x) = x_1 \cdots x_n + \sum c_{j,i}x_i^{k_i}$, where $c = (c_{j,i})$, and k_i's are positive integers. Let $c(t) = (c_{j,i}(t))$ be polynomial functions defined over a closed cuboid I so that $f_{c(t)}$ is non-degenerate. If we set $F(x;t) = f_{c(t)}(x)$, then $(\mathbf{R}^n \times I, F^{-1}(0))$ admits a modified Nash trivialization. This is proved by the same way as above and we omit the details.

Appendix.

We cannot expect a similar theory for homeomorphisms coming from regular isomorphisms after some blow-up, because of the following:

PROPOSITION (A.1). — *Let f_t $(t \in I)$ be the family defined in Example* (0.1), *I an arbitrary open subinterbal of J, and $\Pi : M \times I \to \mathbf{R}^2 \times I$ a finite succession of blow-ups whose centres are mapped submersively to I by the natural induced maps. Then, no regular automorphism of $(M \times I, \Pi^{-1}(0))$ induces a t-level preserving homeomorphism of $(\mathbf{R}^2 \times I, F^{-1}(0))$.*

Proof. — Suppose that there is such a regular automorphism Φ of $(M \times I, \Pi^{-1}(0))$. Let π_t be the restriction of Π to $M \times \{t\}$ for $t \in I$. Let

$P^1(\mathbf{R})$ denote the real projective line. We first remark that $\pi_t^{-1}(0)$ is a union of $P^1(\mathbf{R})$'s. Choose $t_1, t_2 \in I$, with $t_1 \neq t_2$. Since Φ induces a regular automorphism of $\Pi^{-1}(0)$, Φ must induce regular automorphisms between corresponding components ($= P^1(\mathbf{R})$) of $\pi_{t_1}^{-1}(0)$ to $\pi_{t_2}^{-1}(0)$. So, Proposition (A.1) is an obvious consequence of the following fact. □

PROPOSITION (A.2). — *A regular automorphism of $P^1(\mathbf{R})$ is linear.*

Proof. — Let ϕ be a regular automorphism of $P^1(\mathbf{R})$. Then ϕ is expressed by

$$[x:y] \mapsto \phi([x:y]) = \left[\frac{A_1(x,y)}{B_1(x,y)} : \frac{A_2(x,y)}{B_2(x,y)}\right]$$

by some real homogeneous polynomials A_1, A_2, B_1, B_2. Here we assume that B_1, B_2 have no zeros in $P^1(\mathbf{R})$. Since $\phi([x:y]) = [A_1B_2 : A_2B_1]$, we may assume that $\phi([x:y]) = [P(x,y) : Q(x,y)]$ by some real homogeneous polynomials P and Q. Dividing the greatest common divisor of P and Q, we may assume that P and Q have no common factor. Without loss of generality, we may also assume that

$$(A1) \qquad \phi([1:0]) = [1:0], \quad \text{and} \quad \phi([0:1]) = [0:1],$$

so we can write $P(x,y) = xP_1(x,y)$ and $Q(x,y) = yQ_1(x,y)$ where P_1 and Q_1 are real homogeneous polynomials. Assuming that ϕ is not linear, we have that P_1 and Q_1 are not constant.

Since ϕ^{-1} is also a regular automorphism, we may also write

$$\phi^{-1}([X:Y]) = [\overline{P}(X,Y) : \overline{Q}(X,Y)]$$

using some real homogeneous polynomials $\overline{P}$ and $\overline{Q}$. By (A1), there are polynomials $\overline{P}_1$ and $\overline{Q}_1$ such that

$$\overline{P}(X,Y) = X\overline{P}_1(X,Y), \quad \text{and} \quad \overline{Q}(X,Y) = Y\overline{Q}_1(X,Y).$$

Since we may assume $\overline{P}$ and $\overline{Q}$ have no common factor, we may assume that

$$(A2) \qquad \overline{P}_1(1,0) \neq 0, \quad \text{and} \quad \overline{Q}_1(0,1) \neq 0.$$

By elementary computation, we have

$$\begin{aligned}
[x:y] &= \phi^{-1} \circ \phi([x:y]) \\
&= [\overline{P}(xP(x,y), yQ(x,y)) : \overline{Q}(xP(x,y), yQ(x,y))] \\
&= [xP_1(x,y)\overline{P}_1(xP_1(x,y), yQ_1(x,y)) : \\
&\qquad\qquad yQ_1(x,y)\overline{Q}_1(xP_1(x,y), yQ_1(x,y))].
\end{aligned}$$

So we obtain

$$P_1(x,y)\overline{P}_1(xP_1(x,y),yQ_1(x,y)) = Q_1(x,y)\overline{Q}_1(xP_1(x,y),yQ_1(x,y)).$$

Since P_1 and Q_1 have no common factor, P_1 must divides $\overline{Q}_1(xP_1,yQ_1)$. If we write $\overline{Q_1}(X,Y) = \sum b_i X^i Y^{d-i}$, then we have $b_0 = 0$, but this contradicts the fact $b_0 \neq 0$ that is coming from (A2). □

The following problem seems to be open.

PROBLEM. — *Is a regular automorphism of the real projective space $P^n(\mathbf{R})$ $(n \geq 2)$ linear? (Are there regular automorphisms of $P^n(\mathbf{R})$ which are not linear?)*

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T. FUKUI,
Saitama University
Department of Mathematics
255 Shimo-Okubo
Urawa 338-8570 (Japan).
tfukui@rimath.saitama-u.ac.jp

S. KOIKE,
Hyogo University of Teacher Education
Department of Mathematics
Yashiro
Hyogo 673-1415 (Japan).
koike@sci.hyogo-u.ac.jp

M. SHIOTA,
Nagoya University
Department of Mathematics
Furo-cho
Chikusa-ku
Nagoya 464-8602 (Japan).
d42806a@nucc.cc.nagoya-u.ac.jp