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LOCALLY CONFORMALLY KÄHLER METRICS ON HOPF SURFACES

by P. GAUDUCHON & L. ORNEA

To the memory of Franco Tricerri

1. Introduction.

Let M be an even-dimensional, oriented, smooth manifold. A Hermitian structure on M is a pair (J, g) consisting of an *integrable* almost-complex structure J , and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for any vector fields X, Y . The Hermitian structure is *Kähler* if J is parallel with respect to the Levi-Civita connection D^g of g ; equivalently, as J is integrable, (J, g) is *Kähler* if the *Kähler form* ω , defined by $\omega(X, Y) = g(JX, Y)$, is closed. More generally, (J, g) is called *locally conformally Kähler*, *l.c.K* for short, if for each point x of M there exist an open neighbourhood \mathcal{U} of x and a positive function f on \mathcal{U} so that the pair $(J, f^{-2}g)$ is Kähler, see [7] for a general overview.

When J is understood, we say that g is Kähler, l.c.K. etc. whenever the corresponding Hermitian structure (J, g) is Kähler, l.c.K. etc.

When M is four-dimensional, the only case considered in this paper, the defect for a Hermitian structure (J, g) to be Kähler is measured by the *Lee form*, the real 1-form θ determined by

$$(1) \quad d\omega = -2\theta \wedge \omega,$$

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see [18], [23], [8]. (*Warning:* The definition of the Lee form in the literature may differ from the above definition by a factor $\pm \frac{1}{2}$.)

Then, (J, g) is l.c.K if and only if the Lee form θ is closed, i.e. the defect for (J, g) to be l.c.K. is measured by the 2-form $d\theta$.

A special occurrence of l.c.K. metrics is when the Lee form θ is *parallel* with respect to the Levi-Civita connection of g . This case reduces to the Kähler case (θ identically zero) if M is compact, with even first Betti number b_1 [25]; if b_1 is odd, θ is (everywhere) non-zero and (M, J, g) is usually called a *generalized Hopf manifold*. The reason for this name is a God-given series of examples on a sub-class of Hopf surfaces (see below); however, in the present context, when the complex surfaces of interest are Hopf surfaces and not all are *generalized Hopf manifolds* (see Remark 1), we may prefer to call them *Vaisman surfaces*, as e.g. in [7].

Notice that not all complex surfaces admitting l.c.K. metrics admit l.c.K. metric with parallel Lee form, see [22], [3]. However, if the surface is compact, the Lee form of any l.c.K. structure can always be made *harmonic* by a conformal change of the metric, see [8].

The l.c.K. condition is conformally invariant i.e. concerns the *conformal Hermitian structure* $(J, [g])$, where $[g]$ denotes the conformal class of g . In particular, $d\theta$ is conformally invariant, but θ and ω are not: the Lee form $\tilde{\theta}$ and the Kähler form $\tilde{\omega}$ of the Hermitian structure $(\tilde{g} = f^{-2}g, J)$ are given by $\tilde{\theta} = \theta + \frac{df}{f}$ and $\tilde{\omega} = f^{-2}\omega$. These rules of transformation can be interpreted as follows. On any n -dimensional smooth manifold M , let L be the *bundle of real scalars of weight 1*, the oriented real line determined (via the $GL(n, \mathbb{R})$ -principal bundle of (all) frames on M) by the representation $A \in GL(n, \mathbb{R}) \mapsto |\det A|^{\frac{1}{n}}$. Then, each Riemannian metric g in the conformal class $[g]$ determines a positive section, hence a trivialization, ℓ , of L and the Kähler form ω appears as the expression of a conformally invariant L^2 -valued 2-form, say ω_{conf} , with respect to ℓ . In the same way, the Lee form θ can be viewed as the connection 1-form with respect to ℓ of a conformally invariant linear connection ∇^L on L , the so-called *Weyl derivative* determined by the conformal Hermitian structure $(J, [g])$. Finally, the identity (1) means that the *conformal Kähler form* ω_{conf} is *closed* as a L^2 -valued 2-form, when L^2 comes equipped with the linear connection ∇^{L^2} induced by ∇^L . Then, the 2-form $d\theta$ is equal (up to the sign) to the curvature 2-form of ∇^L , so that the l.c.K. condition just means that the connection ∇^L is *flat*.

In the case when M is compact and b_1 is even it is well-known that any l.c.K. Hermitian structure is actually *globally* conformal to a Kähler Hermitian structure [25]. Conversely, it is also well-known (but much more difficult to prove, see for example [20], [10], [2]) that any compact complex surface (M, J) with even first Betti number admits a Kähler metric; moreover, many explicit examples of Kähler Hermitian structures are known, many of them provided by the complex algebraic geometry.

The situation when M is compact with *odd* first Betti number is quite different: it is still unknown whether there exist compact complex surfaces with b_1 odd *not* admitting a l.c.K. metric, but, on the other hand, only few examples have been constructed by now in an explicit way: apart from the case of Hopf surfaces considered in the present paper, these are essentially the l.c.K. metrics constructed by F. Tricerri on some classes of *Inoue surfaces* [22] and examples appearing in [5], [6], [26].

A *Hopf surface* is a compact complex surface whose universal covering is $W = \mathbb{C}^2 - \{(0, 0)\}$. More precisely, *primary* Hopf surfaces have their fundamental group isomorphic to \mathbb{Z} , generated by the transformation γ defined by

$$(2) \quad x = (u, v) \mapsto (\alpha u + \lambda v^m, \beta v),$$

for any $x = (u, v) \in \mathbb{C}^2 - \{(0, 0)\}$; here, m is some integer and α, β, λ are complex numbers such that

$$(3) \quad |\alpha| \geq |\beta| > 1,$$

and

$$(4) \quad (\alpha - \beta^m)\lambda = 0.$$

All primary Hopf surfaces are diffeomorphic to the product $S^3 \times S^1$ and any Hopf surface is finitely covered by a primary one, [13], [14].

Following [11], the primary Hopf surfaces fall into two disjoint classes, according to their *Kähler rank*:

– The *Hopf surfaces of class 1*, whose fundamental group is generated by a transformation γ as above for which $\lambda = 0$ and α, β are any two complex numbers satisfying (3); the corresponding Hopf surface will be denoted by $M_{\alpha, \beta}$.

– The *Hopf surfaces of class 0*, corresponding to $\lambda \neq 0$ and $\alpha = \beta^m$ for some positive integer m ; the corresponding Hopf surface will be denoted by $\tilde{M}_{\beta, m, \lambda}$; as observed in [11], for any β, m and any two λ, μ in \mathbb{C}^* , $\tilde{M}_{\beta, m, \lambda}$ and $\tilde{M}_{\beta, m, \mu}$ are isomorphic as complex surfaces [11, Proposition 60].

For all Hopf surfaces, the complex structure induced by the natural complex structure of $\mathbb{C}^2 - \{(0, 0)\}$ is denoted by J ; a l.c.K. metric is always understood with respect to J .

The class 1 contains the subclass of $M_{\alpha, \beta}$'s such that $|\alpha| = |\beta|$. For these special Hopf manifolds, a l.c.K. metric is easily constructed as follows. Let ρ be the *distance function to the origin*, i.e. the (smooth) positive real function on W defined by $\rho(x) = \sqrt{|u|^2 + |v|^2}$ for any $x = (u, v)$ in $W = \mathbb{C}^2 - \{(0, 0)\}$. Then, $\frac{1}{4}dd^c\rho^2$ is the Kähler form of the natural flat Hermitian metric (here and henceforth, the operator d^c , acting on functions, is defined by $d^cf(X) = -df(JX)$ so that $dd^cf = 2i\partial\bar{\partial}f$). The 2-form $\frac{1}{4\rho^2}dd^c\rho^2$ clearly descends as the Kähler form of a well-defined, obviously l.c.K., Hermitian metric $g_{\alpha, \beta}$ on the Hopf surface $M_{\alpha, \beta}$. It is then easy to check that the Lee form is parallel.

The main goal of this paper is to prove

THEOREM 1. — *Each primary Hopf surface admits a l.c.K. metric. Moreover, each primary Hopf surface of class 1 admits a l.c.K. metric with parallel (non-zero) Lee form.*

An explicit construction of a l.c.K. metric with parallel Lee form on each (primary) Hopf surfaces of class 1, as well as an explicit description of the corresponding Sasakian geometry, are given in Sections 2 and 3 (see, in particular, Proposition 1, Corollary 1, Proposition 3 and Remark 6).

Previous attempts to write (non globally defined) l.c.K. metrics on Hopf surfaces appear in [19].

The existence of l.c.K. Hermitian metrics on $M_{\alpha, \beta}$ for $|\alpha|$ and $|\beta|$ different, but close to each other, has been proved by C. LeBrun [17]. The argument goes as follows. First, notice that the line bundle L of any Hopf surface $M_{\alpha, \beta}$ is naturally identified to the real line bundle $(W \times \mathbb{R})/\mathbb{Z}$, where the action of \mathbb{Z} on $W \times \mathbb{R}$ is described by $1 \cdot ((u, v), a) = ((\alpha u, \beta v), |\alpha|^{\frac{1}{2}}|\beta|^{\frac{1}{2}}a)$ (observe that $|\alpha|^{\frac{1}{2}}|\beta|^{\frac{1}{2}}$ is equal to $|\det \gamma_*|^{\frac{1}{4}}$, where γ_* is the differential of γ). The line bundle so defined admits a natural flat connection which coincides with the Weyl derivative ∇^L ; in particular the pullback of (L, ∇^L) on W coincides with the trivial line bundle $W \times \mathbb{R}$ equipped with the trivial connection. Finally, the 2-form $\frac{1}{4}dd^c\rho^2$ on W descends on M as a L^2 -valued 2-form on $M_{\alpha, \beta}$, which is obviously closed with respect to ∇^L . Then, a deformation argument using the identification $L = (W \times \mathbb{R})/\mathbb{Z}$

(see Section 4) shows that $M_{\alpha,\beta}$ still admits a l.c.K. Hermitian metric for $|\alpha| \neq |\beta|$, provided that $|\alpha|$ and $|\beta|$ are close enough) to each other [17].

In Section 2, following a suggestion in [17], we give an explicit formulation for these l.c.K. metrics and show that the same formulation actually provides a l.c.K. Hermitian metric on any (primary) Hopf surface of class 1, see Proposition 1 and Corollary 1.

It then appears that all the l.c.K. Hermitian metrics obtained in this way have a parallel, non-zero, Lee form, and are related to a very simple class of Sasakian structures on the sphere S^3 , of which a precise description is given in Section 3 (Proposition 3 and Remark 6).

Finally, LeBrun's argument still applies to prove the second statement of Theorem 1, see Section 4.

Remark 1. — Theorem 1 gives no information as to the existence or the non-existence of (l.c.K.) Hermitian metric with parallel Lee form on Hopf surfaces of class 0. However, this question has been solved recently by F. Belgun [3], so that Theorem 1 can actually be completed by the following statement: *Hopf surfaces of class 0 admit no (l.c.K.) Hermitian metrics with parallel Lee form.*

2. Construction of l.c.K. metrics on $M_{\alpha,\beta}$.

Fix any two complex numbers α, β satisfying (3) and let $\phi_{\alpha,\beta}$ the (smooth) function determined on W by

$$(5) \quad |u|^2 |\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2 |\beta|^{-2\phi_{\alpha,\beta}(x)} = 1,$$

for any $x = (u, v)$ in W . Notice that $\phi_{\alpha,\beta}$ is well-defined since, for x fixed, the function $t \mapsto |u|^2 |\alpha|^t + |v|^2 |\beta|^t$ is strictly increasing from 0 to $+\infty$.

Then, $\phi_{\alpha,\beta}$ satisfies the following equivariance property:

$$(6) \quad \phi_{\alpha,\beta}(\gamma \cdot x) = \phi_{\alpha,\beta}(x) + 1$$

for any x in \tilde{M} .

Indeed, we have

$$\begin{aligned} 1 &= |\alpha u|^2 |\alpha|^{-2\phi_{\alpha,\beta}(\gamma \cdot x)} + |\beta v|^2 |\beta|^{-2\phi_{\alpha,\beta}(\gamma \cdot x)} \\ &= |u|^2 |\alpha|^{2-2\phi_{\alpha,\beta}(\gamma \cdot x)} + |v|^2 |\beta|^{2-2\phi_{\alpha,\beta}(\gamma \cdot x)} \\ &= |u|^2 |\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2 |\beta|^{-2\phi_{\alpha,\beta}(x)} \end{aligned}$$

for any x in W .

We then get a diffeomorphism from the corresponding Hopf surface (of class 1) $M_{\alpha,\beta}$ onto the product $S^3 \times S^1$ as follows (here S^3 denotes the 3-dimensional sphere, viewed as the unit sphere in \mathbb{C}^2 , and S^1 denotes the circle, identified with the quotient \mathbb{R}/\mathbb{Z}). Let $\tilde{\psi}$ be the map from W to $S^3 \times S^1$ defined by

$$(7) \quad x = (u, v) \mapsto ((u \alpha^{-\phi_{\alpha,\beta}(x)}, v \beta^{-\phi_{\alpha,\beta}(x)}), \phi_{\alpha,\beta}(x) \bmod \mathbb{Z}).$$

Then, due to (6), $\tilde{\psi}$ is γ -invariant hence determines a map, ψ , from $M_{\alpha,\beta}$ to $S^3 \times S^1$, which is clearly a diffeomorphism; the inverse ψ^{-1} is given by

$$(8) \quad (z, t \bmod \mathbb{Z}) \mapsto [u = \alpha^t z_1, v = \beta^t z_2],$$

where $z = (z_1, z_2)$, $|z_1|^2 + |z_2|^2 = 1$, is a point of $S^3 \subset \mathbb{C}^2$ and $t \bmod \mathbb{Z}$ is an element of $S^1 = \mathbb{R}/\mathbb{Z}$; here, $[u, v]$ denotes the class of $(u, v) \bmod \Gamma_{\alpha,\beta}$. Observe that the diffeomorphism ψ depends on the choice of an argument for α and for β , say $\mathfrak{Arg} \alpha$ and $\mathfrak{Arg} \beta$.

Remark 2. — For any choice of $\mathfrak{Arg} \alpha$ and $\mathfrak{Arg} \beta$, the above action of \mathbb{Z} on \tilde{M} is the restriction of an action of the (additive) group \mathbb{R} defined by

$$(9) \quad t \cdot (u, v) = (\alpha^t u, \beta^t v),$$

for any t in \mathbb{R} . Then, $\phi_{\alpha,\beta}$ can be described as follows: for any $x = (u, v)$ in \tilde{M} , $\phi_{\alpha,\beta}(x)$ is the unique element of \mathbb{R} such that $(-\phi_{\alpha,\beta}(x)) \cdot x$ belongs to the unit sphere of \mathbb{C}^2 .

We denote by $\Phi_{\alpha,\beta}$ the real positive function on W defined by $\Phi_{\alpha,\beta} = e^{(k_1+k_2)\phi_{\alpha,\beta}}$; alternatively, $\Phi_{\alpha,\beta}$ is determined by

$$(10) \quad \rho_1^2 \Phi_{\alpha,\beta}^{-\frac{2k_1}{k_1+k_2}} + \rho_2^2 \Phi_{\alpha,\beta}^{-\frac{2k_2}{k_1+k_2}} = 1,$$

where k_1, k_2 are the (positive) real numbers given by

$$(11) \quad k_1 = \ln |\alpha|, \quad k_2 = \ln |\beta|,$$

and ρ_1, ρ_2 are the functions on \tilde{M} defined by

$$(12) \quad \rho_1(x) = |u|, \quad \rho_2(x) = |v|.$$

In this notations (3) translates to

$$(13) \quad k_1 \geq k_2 > 0.$$

Then, by (6), $\Phi_{\alpha,\beta}$ satisfies the following equivariance property with respect to the action of γ :

$$(14) \quad \Phi_{\alpha,\beta}(\gamma \cdot x) = |\alpha||\beta| \cdot \Phi_{\alpha,\beta}(x).$$

In other words, $\Phi_{\alpha,\beta}$ descends on $M_{\alpha,\beta}$ as a (positive) section of L^2 .

PROPOSITION 1. — For any pair of complex numbers α, β satisfying (3), the real 2-form $\frac{1}{4}dd^c\Phi_{\alpha,\beta}$ is the Kähler form of a Hermitian metric on \tilde{M} .

Proof. — For simplicity, $\Phi_{\alpha,\beta}$ will be denoted by Φ . Then, it follows readily from (10) that the differential of Φ is given by

$$(15) \quad d\Phi = \frac{1}{\Delta} \left(\Phi^{\frac{k_2-k_1}{k_1+k_2}} (u d\bar{u} + \bar{u} du) + \Phi^{\frac{k_1-k_2}{k_1+k_2}} (v d\bar{v} + \bar{v} dv) \right),$$

where Δ is the positive function defined by

$$(16) \quad \Delta = \frac{2k_1 \rho_1^2 \Phi^{\frac{-2k_1}{k_1+k_2}} + 2k_2 \rho_2^2 \Phi^{\frac{-2k_2}{k_1+k_2}}}{k_1 + k_2}.$$

From (15), we infer:

$$(17) \quad \partial_{u,\bar{u}}^2 \Phi = \frac{2\Phi^{\frac{k_2-k_1}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2} \left(k_1(k_1+k_2)\rho_1^4 \Phi^{\frac{-4k_1}{k_1+k_2}} + 2k_2^2 \rho_2^4 \Phi^{\frac{-4k_2}{k_1+k_2}} \right. \\ \left. + k_2(k_1+3k_2)\rho_1^2 \rho_2^2 \Phi^{-2} \right);$$

$$(18) \quad \partial_{v,\bar{v}}^2 \Phi = \frac{2\Phi^{\frac{k_1-k_2}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2} \left(k_2(k_1+k_2)\rho_2^4 \Phi^{\frac{-4k_2}{k_1+k_2}} + 2k_1^2 \rho_1^4 \Phi^{\frac{-4k_1}{k_1+k_2}} \right. \\ \left. + k_1(k_2+3k_1)\rho_1^2 \rho_2^2 \Phi^{-2} \right);$$

$$(19) \quad \partial_{u,\bar{v}}^2 \Phi = \frac{2\bar{u}v\Phi^{-1}}{\Delta^3(k_1+k_2)^2} (k_1 - k_2) \left(k_1 \rho_1^2 \Phi^{\frac{-2k_1}{k_1+k_2}} - k_2 \rho_2^2 \Phi^{\frac{-2k_2}{k_1+k_2}} \right).$$

The claim is that the Hermitian matrix

$$A = \begin{pmatrix} \partial_{u,\bar{u}}^2 \Phi & \partial_{u,\bar{v}}^2 \Phi \\ \partial_{v,\bar{u}}^2 \Phi & \partial_{v,\bar{v}}^2 \Phi \end{pmatrix}$$

is positive; this in turn is equivalent to the fact that the trace and the determinant are positive.

By (17) and (18), $\partial_{u,\bar{u}}^2 \Phi$ and $\partial_{v,\bar{v}}^2 \Phi$ are both positive; it then remains to check that the determinant of A is positive. By a straightforward calculation, this determinant is equal to

$$(21) \quad \det A = \frac{8}{\Delta^6(k_1+k_2)^3} \left(k_1^3 \rho_1^8 \Phi^{\frac{-8k_1}{k_1+k_2}} + k_2^3 \rho_2^8 \Phi^{\frac{-8k_2}{k_1+k_2}} \right. \\ \left. + 3k_1 k_2 (k_1+k_2) \rho_1^4 \rho_2^4 \Phi^{-4} \right. \\ \left. + k_1^2 (k_1+3k_2) \rho_1^6 \rho_2^2 \Phi^{\frac{-6k_1-2k_2}{k_1+k_2}} \right. \\ \left. + k_2^2 (k_2+3k_1) \rho_1^2 \rho_2^6 \Phi^{\frac{-2k_1-6k_2}{k_1+k_2}} \right) \\ = \frac{1}{\Delta^3},$$

which is obviously positive for any k_1, k_2 positive.

COROLLARY 1. — For any pair of complex numbers α, β satisfying (3), the 2-form $\omega_{\alpha, \beta} = \frac{1}{4\Phi_{\alpha, \beta}} dd^c \Phi_{\alpha, \beta}$ is well-defined on $M_{\alpha, \beta}$ and is the Kähler form of a locally conformally Kähler structure, $(g_{\alpha, \beta}, J)$.

Proof. — The fact that $\omega_{\alpha, \beta}$ is well-defined on $M_{\alpha, \beta}$ follows readily from (14). By the above proposition, $\omega_{\alpha, \beta}$ is the Kähler form of a, clearly l.c.K., Hermitian structure. \square

Remark 3. — The 2-form $\frac{1}{4} dd^c \Phi_{\alpha, \beta}$ descends on $M_{\alpha, \beta}$ as a L^2 -valued 2-form, equal to the conformal Kähler form of the l.c.K. Hermitian structure $(g_{\alpha, \beta}, J)$. In the special case that $|\alpha| = |\beta|$ or, equivalently, $k_1 = k_2$, we recover $\Phi_{\alpha, \beta} = \rho^2$.

In the general situation, we actually get a 1-parameter family of l.c.K. Hermitian structures obtained by choosing any positive real number ℓ and by considering, instead of $\frac{1}{4\Phi_{\alpha, \beta}} dd^c \Phi_{\alpha, \beta}$, the new Kähler form $\frac{1}{4\Phi_{\alpha, \beta}^\ell} dd^c \Phi_{\alpha, \beta}^\ell$. This amounts to replacing k_1, k_2 by $\ell k_1, \ell k_2$ in the above formulae.

This can be done in particular in the case $|\alpha| = |\beta|$; then, the Kähler form on W is equal to $\frac{1}{4} dd^c \rho^{2\ell}$ and the corresponding Riemannian metric g_ℓ can be described as follows:

$$(22) \quad g_\ell(X, X) = \ell \rho^{(2\ell-2)} (\ell |X_{\text{rad}}|^2 + |X_{\text{rad}}^\perp|^2),$$

with the following notation: For any vector X at the point x of W , X_{rad} denotes the *radial component* of X , i.e. the orthogonal projection of X on the complex line $\mathbb{C} \cdot x$ (viewed as a real 2-plane), and X_{rad}^\perp denotes the *transversal component* of X , i.e. the orthogonal projection of X on the orthogonal complex line $(\mathbb{C} \cdot x)^\perp$ (here, *orthogonal* means orthogonal with respect to the natural flat metric of \mathbb{C}^2).

The Lee form $\theta_{\alpha, \beta}$ of the Hermitian structure $(g_{\alpha, \beta}, J)$ is clearly equal to $\frac{1}{2} \Phi_{\alpha, \beta}^{-1} d\Phi_{\alpha, \beta}$.

Let $V_{\alpha, \beta}$ denote the dual vector field of $\theta_{\alpha, \beta}$ with respect to $g_{\alpha, \beta}$, the so-called *Lee vector field*.

A direct computation using (17), (18), (19) shows that the pull-back vector field of $V_{\alpha,\beta}$ on W , still denoted by $V_{\alpha,\beta}$, is expressed by

$$(23) \quad V_{\alpha,\beta}(x) = \left(\frac{2k_1}{k_1 + k_2}u, \frac{2k_2}{k_1 + k_2}v \right).$$

Again, a direct, but lengthy, computation shows that $V_{\alpha,\beta}$ is of norm 1 with respect to $g_{\alpha,\beta}$ and is parallel with respect to the Levi-Civita connection of $g_{\alpha,\beta}$. These facts will however become more easily apparent in the framework of the next section.

3. Associated Sasakian structures.

3.1. Three-dimensional Sasakian structures.

We begin this section with some general considerations concerning three-dimensional Sasakian structures (see e.g. [4]) for more information).

A *Sasakian structure* on some oriented, three-dimensional smooth manifold N is a pair (g, Z) , where g is a Riemannian metric and Z a unit Killing vector field with respect to g , such that

$$(24) \quad D^g Z = *Z ;$$

here, D^g is the Levi-Civita connection of g , $*$ is the Hodge operator determined by the metric and the chosen orientation and $*Z$ is viewed as a skew-symmetric operator, also denoted by I ; we thus have $I(Z) = 0$ and the restriction of I to $Q := Z^\perp$ coincides with the uniquely defined complex structure compatible with the metric and the induced orientation.

The distribution Q constitutes a *contact structure* and the Riemannian dual 1-form of Z with respect to g , η , is a *contact 1-form* for Q .

Notice that (3) implies

$$(25) \quad g(X, Y) = \frac{1}{2}d\eta(X, IY),$$

for any sections X, Y of Q .

In general, for any contact structure Q and any choice of a contact 1-form η , the corresponding *Reeb vector field* is the vector field V determined by the two conditions: $\eta(V) = 1, i_V d\eta = 0$. In the case of a Sasakian structure as above, the Reeb vector field of the contact structure Q with respect to the contact 1-form η is clearly equal to Z .

We denote by R^g the curvature tensor of D^g . Since Z is a Killing vector field, it satisfies the *Kostant identity*: $D_X^g(D^g Z) = R_{X,Z}^g$ [16]. We thus get

$$(26) \quad R_{Z,Y}^g = Z \wedge Y,$$

for any vector field Y . In particular, the sectional curvature of g is equal to 1 for any 2-plane containing Z .

Since N is three-dimensional, R^g is entirely determined by the Ricci tensor Ric^g and it is easy to deduce from (26) that Z is an eigenvector field for Ric^g (viewed as a symmetric operator) with respect to the constant eigenvalue 2, whereas Q is an eigen-subbundle of Ric^g with respect to the (in general non-constant) eigenvalue $\frac{\text{Scal}^g}{2} - 1$, where Scal^g denotes the scalar curvature of g ; Ric^g can thus be written as follows:

$$(27) \quad \text{Ric}^g = \left(\frac{\text{Scal}^g}{2} - 1 \right) g + \left(3 - \frac{\text{Scal}^g}{2} \right) \eta \otimes \eta.$$

The Levi-Civita connection D^g can be computed by using the well-known 6-terms formula, see e.g. [12]; it is given by the following table, where X denotes any *unit* section of Q , and Y any vector field on N :

$$(28) \quad D_Y^g Z = IY,$$

$$(29) \quad \begin{aligned} D_Y^g X = & ((g([Z, X], IX) - 1) \eta(Y) - g([X, IX], Y)) IX \\ & + g(Y, IX) Z. \end{aligned}$$

It follows that the sectional curvature restricted to Q , $K^g(Q)$, is given by

$$(30) \quad \begin{aligned} K^g(Q) = & 1 - 2g([Z, X], IX) - g([X, IX], [X, IX]) \\ & + X \cdot g([X, IX], IX) - IX \cdot g([X, IX], X), \end{aligned}$$

for any *unit* section, X , of Q . Then, $\text{Scal}^g = 2(2 + K^g(Q))$ is immediately deduced from (30).

Remark 4. — The Levi-Civita connection D^g does not preserve the sub-bundle Q , but induces a linear connection ∇ on Q by orthogonal projection

$$(31) \quad \nabla_Y X = ((g([Z, X], IX) - 1) \eta(Y) - g([X, IX], Y)) IX,$$

for any unit section X of Q and for any vector field Y on N . This connection is clearly I -linear and preserves the metric g , i.e. is a Hermitian connection when Q is viewed as a Hermitian complex line bundle over N . The (real)

connection 1-form of ∇ with respect to X , viewed as a (unit) *gauge* of the Hermitian line bundle Q , is then the real 1-form ζ defined by

$$(32) \quad \zeta = (g([Z, X], IX) - 1) \eta - [X, IX]^b,$$

where $[X, IX]^b$ denotes the dual 1-form of the vector field $[X, IX]$. Then, (30) can be written as follows:

$$(33) \quad \begin{aligned} K(Q) &= -d\zeta(X, IX) - 1 \\ &= \Omega^\nabla(X, IX) - 1, \end{aligned}$$

where $\Omega^\nabla = -d\zeta$ is the (real) curvature form of ∇ .

3.2. Sasakian versus Hermitian geometry.

We here describe the well-known correspondence between three-dimensional Sasakian manifolds and l.c.K. Hermitian complex surfaces with parallel unit Lee form, [25].

First, start from a three-dimensional Sasakian manifold (N, g, Z) as above and consider the product manifold $M = N \times \mathbb{R}$; let M be equipped with the product Riemannian metric, still denoted by g , of g and the standard metric of the factor \mathbb{R} , and with the almost complex structure J defined as follows:

$$(34) \quad J|_Q = I|_Q, \quad JZ = T,$$

where $T := \partial/\partial t$ is the vector field determined by the natural parameter, t , of the factor \mathbb{R} . Let again D^g denote the Levi-Civita connection of g on M . Then, it follows from (24) that

$$(35) \quad D_U^g J = dt \wedge JU - \eta \wedge U,$$

for any vector field U on M . This implies that J is integrable and that the Lee form θ of the Hermitian structure (J, g) is the 1-form dt , see e.g. [23] or [1]. In particular, θ is D^g -parallel, of norm 1.

This construction can be compactified in the following manner. Let σ be any Sasakian transformation of (N, g, Z) , i.e. any (direct) diffeomorphism of N preserving g and Z , and choose a positive real number ℓ ; then the (Riemannian) *suspension* $M_{\sigma, \ell}$ of σ over the circle of length ℓ is obtained by identifying $N \times \{0\}$ and $N \times \{1\}$ via σ in the product $N \times [0, \ell]$ of N by the closed segment $[0, \ell]$.

The natural projection π from $M_{\sigma, \ell}$ to the circle $S_\ell^1 = \mathbb{R}/\mathbb{Z} \cdot \ell$ is thus a Riemannian submersion and the natural vector field d/dt on S_ℓ^1 admits

a natural unit lift, \tilde{T} on $M_{\sigma,\ell}$, orthogonal to the fibers of π , whose flow at time 1 coincides with σ . Applying this construction to the Sasakian three-manifold (N, g, Z) for any σ and any ℓ we eventually get a Hermitian structure with parallel (unit) Lee form on $M_{\sigma,\ell}$ by putting $JZ = \tilde{T}$.

Conversely, let (M, J, g) be a Hermitian complex surface. Let θ and V denote the Lee form and the Lee vector field. Since J is integrable, we have

$$(36) \quad D_U^g J = \theta \wedge JU + J\theta \wedge U,$$

for any vector field U on M (as usual, the RHS of (36) has to be considered as a skew-symmetric operator). If, moreover, θ , hence also V , are D^g -parallel, then the metric g splits locally as a Riemannian product $N \times \mathbb{R}$, where (N, g_N) is a three-dimensional Riemannian manifold and $V = d/dt$. By rescaling g if necessary, we can assume that V is of norm 1, as well as the vector field $Z := JV$; now, Z can be viewed as a vector field on N and (36) directly implies (24), showing that (g_N, Z) is a Sasakian structure on N .

3.3. Deformation of Sasakian structures.

Start from any Sasakian structure (g, Z) on N , fix any real positive function f on N and consider the new contact 1-form $\eta_f = \frac{1}{f}\eta$. Denote by Z_f the Reeb vector field corresponding to the same contact structure Q and the contact 1-form η_f ; then,

$$(37) \quad Z_f = fZ + Z_f^Q,$$

where Z_f^Q is a section of Q , uniquely determined by the identity

$$(38) \quad df + i_{Z_f^Q} d\eta = 0.$$

It follows that

$$(39) \quad Z_f^Q = \frac{1}{2}I(df|_Q)^\sharp,$$

where $df|_Q$ denotes the restriction of df to Q and $df|_Q^\sharp$ the section of Q dual to $df|_Q$ with respect to the restriction of g to Q .

Let g_f be the Riemannian metric on N defined as follows:

1. Z_f is of norm 1 w.r.t g_f ;

2. Q and Z_f are orthogonal with respect to g_f ;

3. $g_f(X, Y) = \frac{1}{2} d\eta_f(X, IY) = \frac{1}{f} g(X, Y)$, for any sections X, Y of Q .

We shall refer to the metric g_f as *the metric obtained by deforming the Sasakian structure (g, Z) by the function f* , see [21].

Notice that η_f is the dual 1-form of Z_f with respect to g_f .

Now we have

PROPOSITION 2. — *The pair (g_f, Z_f) is a Sasakian structure on N if and only if the following condition is satisfied:*

$$(40) \quad \text{Hess}^g f(X, Y) = \text{Hess}^g f(IX, IY),$$

for any sections X, Y of Q , where $\text{Hess}^g f = D^g df$ denotes the Hessian of f with respect to g ; equivalently, the restriction of $\text{Hess}^g f$ to Q is a multiple of the restriction of g .

Proof. — Let D^{g_f} be the Levi-Civita connection of g_f . We first show that $D_{Z_f}^{g_f} Z_f = 0$. Indeed, $g_f(D_{Z_f}^{g_f} Z_f, Z_f) = 0$, since Z_f is of norm 1 with respect to g_f , and, for any section X of Q , we have $g_f(D_{Z_f}^{g_f} Z_f, X) = g_f([X, Z_f], Z_f) = \eta_f([X, Z_f]) = -d\eta_f(X, Z_f) = 0$. Then, for any sections X, Y of Q , we have

$$\begin{aligned} g_f(D_X^{g_f} Z_f, Y) &= -\frac{1}{2} \eta_f([X, Y]) \\ &\quad + \frac{1}{2} (Z_f \cdot g_f(X, Y) - g_f([Z_f, X], Y) - g_f([Z_f, Y], X)). \end{aligned}$$

This shows that the pair (g_f, Z_f) is Sasakian if and only if Z_f is Killing with respect to the induced metric g_f ; moreover, in the present case that Z is already a Killing vector field with respect to g , Z_f is a Killing vector field with respect to g_f if and only if

$$(41) \quad g(D_X^g Z_f^Q, Y) + g(X, D_Y^g Z_f^Q) = df(Z) g(X, Y),$$

holds for any sections X, Y of Q . We then have

$$\begin{aligned} g(D_X^g Z_f^Q, Y) &= \frac{1}{2} X \cdot g(I(df|_Q)^\sharp, Y) - \frac{1}{2} g(I(df|_Q)^\sharp, D_X^g Y) \\ &= -\frac{1}{2} X \cdot df(IY) + \frac{1}{2} df(ID_X^g Y) \\ &= -\frac{1}{2} \text{Hess}^g f(X, IY) - \frac{1}{2} df((D_X^g I)Y) \\ &= -\frac{1}{2} \text{Hess}^g f(X, IY) + \frac{1}{2} df(Z) g(X, Y), \end{aligned}$$

which shows that (41) is true if and only (40) is satisfied. The last statement comes from Q being of rank 2 (notice that, except for this last statement, the argument holds in any dimension). \square

By using (30), we get the following formulation for the scalar curvature of g_f :

$$(42) \quad \widetilde{\text{Scal}}^{g_f} = 2 \left(3 + 4(f - 1) + 4\text{Hess}^g f(X, X) - 3 \frac{((df(X))^2 + (df(JX))^2)}{f} \right),$$

for any unit section X of Q .

3.4. Sasakian structures attached to $M_{\alpha, \beta}$.

Recall that the Hopf surface $M_{\alpha, \beta}$ as a manifold has been identified to the product $S^3 \times S^1$ by $\psi : M_{\alpha, \beta} \mapsto S^3 \times S^1$, defined by (7) and its inverse $\psi^{-1} : S^3 \times S^1 \mapsto M_{\alpha, \beta}$ described by (8).

We adopt the following notations. The sphere S^3 is realized as the set of elements of \mathbb{C}^2 of norm 1: a generic element of S^3 is denoted by $z = (z_1, z_2)$, where z_1, z_2 are complex numbers such that $|z_1|^2 + |z_2|^2 = 1$. Accordingly, a generic vector X of S^3 at z is identified to a pair of complex numbers $(X_1 = \dot{z}_1, X_2 = \dot{z}_2)$ satisfying $\Re(X_1 \bar{z}_1 + X_2 \bar{z}_2) = 0$ (\Re and \Im denote respectively the real and imaginary part of a complex number). We denote by Z the vector field on S^3 generated by the natural action of S^1 , so that

$$Z = (iz_1, iz_2).$$

We denote by $Q := Z^\perp$ the rank 2 vector sub-bundle of TS^3 orthogonal to Z with respect to the standard metric, g , of S^3 (of constant sectional curvature +1). The natural complex structure of $Q = Z^\perp$ is denoted by i . We denote by E, iE the generators of Q defined by

$$E = (\bar{z}_2, -\bar{z}_1), \quad iE = (iz_2, -iz_1).$$

For any complex number μ , μE stands for the real vector field $\Re \mu E + \Im \mu iE$. The three (real) vector fields Z, E, iE are (unit) Killing vector fields with respect to g and generate the (real) Lie algebra of left-invariant vector fields of S^3 , when S^3 is identified to the Lie group $Sp(1)$ of unit quaternions, via the usual identification $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. Their brackets are given by

$$(43) \quad [Z, E] = -2iE, \quad [E, iE] = -2Z, \quad [iE, Z] = -2E.$$

Also recall that, if D^g denotes the Levi-Civita connection of g , we have

$$(44) \quad D_Z^g Z = 0, \quad D_X^g Z = iX,$$

for any X in Q , i.e. the pair (g, Z) is a *Sasakian structure*, called the *canonical* Sasakian structure of S^3 . The corresponding contact 1-form is denoted by η : $\eta(X) = g(Z, X)$, for any vector field X on S^3 .

The vector fields Z, E, iE will be also considered as vector fields on $S^3 \times S^1$ (with the same notation). As for the factor $S^1 = \mathbb{R}/\mathbb{Z}$, we denote by t the natural parameter of \mathbb{R} and by T the vector field $\partial/\partial t$, also considered as a vector field on $S^3 \times S^1$.

For later convenience, we consider the complex function, F , on $S^3 \times S^1$ defined by

$$(45) \quad \begin{aligned} F(z) &= \ln \alpha |z_1|^2 + \ln \beta |z_2|^2 \\ &= k_1 |z_1|^2 + k_2 |z_2|^2 + i (\Re \alpha |z_1|^2 + \Re \beta |z_2|^2). \end{aligned}$$

Viewed as functions on $M_{\alpha, \beta}$, $|z_1|^2$, $|z_2|^2$ and $\Re F$ are respectively equal to $\rho_1^2 \Phi^{\frac{-2k_1}{k_1+k_2}}$, $\rho_2^2 \Phi^{\frac{-2k_2}{k_1+k_2}}$ and $\frac{(k_1+k_2)}{2} \Delta$.

The image of any vector field $U = (U_1, U_2)$ of $M_{\alpha, \beta}$ by the differential ψ_* of ψ is of the form (X, aT) , where $X = (X_1, X_2)$ is tangent to S^3 and a is a real function on $S^3 \times S^1$. It is easily checked that

$$(46) \quad a = \frac{\Re(U_1(x) \bar{u} |\alpha|^{-2t} + U_2(x) \bar{v} |\beta|^{-2t})}{\Re F},$$

$$(47) \quad X_1 = \alpha^{-t} U_1 - a \ln \alpha z_1, \quad X_2 = \beta^{-t} U_2 - a \ln \beta z_2.$$

Finally, the vector field X can be written uniquely as $bZ + \mu E$, where b is a real function and μ a complex function on $S^3 \times S^1$, given by

$$(48) \quad b = -i(X_1 \bar{z}_1 + X_2 \bar{z}_2), \quad \mu = X_1 z_2 - X_2 z_1.$$

We conclude that the complex structure J of $M_{\alpha, \beta}$, transported on $S^3 \times S^1$ by ψ , is described by the following table:

$$(49) \quad \begin{aligned} JT &= \frac{1}{\Re F} (-\Im F T + |F|^2 Z + i \bar{F} (\ln \alpha - \ln \beta) z_1 z_2 E), \\ JZ &= \frac{1}{\Re F} (-T + \Im F Z + (\ln \alpha - \ln \beta) z_1 z_2 E), \\ JE &= iE. \end{aligned}$$

The Kähler form $\omega_{\alpha, \beta}$, transported on $S^3 \times S^1$ by ψ , is given by

$$(50) \quad \begin{aligned} \omega_{\alpha, \beta} &= \frac{1}{4} e^{-(k_1+k_2)t} dd^c e^{(k_1+k_2)t} \\ &= \frac{(k_1+k_2)}{4} dd^c t + \frac{(k_1+k_2)^2}{4} dt \wedge d^c t, \end{aligned}$$

where d^c refers to the operator J defined by (49).

From (49), we obtain

$$(51) \quad dt(T) = 1, dt(Z) = dt(E) = dt(iE) = 0,$$

$$(52) \quad d^c t(T) = \frac{\Im F}{\Re F}, d^c t(Z) = \frac{1}{\Re F}, d^c t(E) = d^c t(iE) = 0,$$

hence also, the following table for $\omega_{\alpha,\beta}$

$$(53) \quad \begin{aligned} \omega_{\alpha,\beta}(T, Z) &= \frac{(k_1 + k_2)^2}{4\Re F}, \\ \omega_{\alpha,\beta}(T, \lambda E) &= \frac{(k_1 + k_2)}{2(\Re F)^2} \Re(\bar{\lambda} z_1 z_2) (k_1 \mathfrak{A} \operatorname{rg} \beta - k_2 \mathfrak{A} \operatorname{rg} \alpha), \\ \omega_{\alpha,\beta}(Z, \lambda E) &= \frac{(k_1 + k_2)}{2(\Re F)^2} (k_1 - k_2) \Re(\bar{\lambda} z_1 z_2), \\ \omega_{\alpha,\beta}(\lambda E, \mu E) &= -\frac{(k_1 + k_2)}{2\Re F} \Im(\lambda \bar{\mu}), \end{aligned}$$

for any complex numbers λ, μ . We finally derive the following table for the metric $g_{\alpha,\beta} = \omega_{\alpha,\beta}(\cdot, J\cdot)$:

$$(54) \quad \begin{aligned} g_{\alpha,\beta}(T, T) &= \frac{(k_1 + k_2)^2 |F|^2}{4(\Re F)^2} + \frac{(k_1 + k_2)(k_1 \mathfrak{A} \operatorname{rg} \beta - k_2 \mathfrak{A} \operatorname{rg} \alpha)^2}{2(\Re F)^3} |z_1|^2 |z_2|^2, \\ g_{\alpha,\beta}(T, Z) &= \frac{(k_1 + k_2)^2 \Im H}{4(\Re F)^2} + \frac{(k_1^2 - k_2^2)(k_1 \mathfrak{A} \operatorname{rg} \beta - k_2 \mathfrak{A} \operatorname{rg} \alpha)}{2(\Re F)^2} |z_1|^2 |z_2|^2, \\ g_{\alpha,\beta}(T, \lambda E) &= \frac{(k_1 + k_2)(k_1 \mathfrak{A} \operatorname{rg} \beta - k_2 \mathfrak{A} \operatorname{rg} \alpha)}{2(\Re F)^2} \Im(\bar{\lambda} z_1 z_2), \\ g_{\alpha,\beta}(Z, Z) &= \frac{(k_1 + k_2)^2}{4(\Re F)^2} + \frac{(k_1 + k_2)(k_1 - k_2)^2}{2(\Re F)^3} |z_1|^2 |z_2|^2, \\ g_{\alpha,\beta}(E, E) &= g_{\alpha,\beta}(iE, iE) = \frac{(k_1 + k_2)}{2\Re F}, \\ g_{\alpha,\beta}(Z, \lambda E) &= \frac{(k_1^2 - k_2^2)}{2(\Re F)^2} \Im(\bar{\lambda} z_1 z_2), \\ g_{\alpha,\beta}(E, iE) &= 0. \end{aligned}$$

From (23), we infer that the Lee vector field $V_{\alpha,\beta}$, viewed as a vector field on $S^3 \times S^1$ via ψ , is written as:

$$(56) \quad V_{\alpha,\beta} = \frac{2}{(k_1 + k_2)} (T - \Im F Z - (\mathfrak{A} \operatorname{rg} \alpha - \mathfrak{A} \operatorname{rg} \beta) i z_1 z_2 E).$$

The vector field $JV_{\alpha,\beta}$ is thus equal to

$$(57) \quad JV_{\alpha,\beta} = \frac{2}{(k_1 + k_2)} \Re F Z + (k_1 - k_2) i z_1 z_2 E).$$

In particular, $JV_{\alpha,\beta}$ is independent of t and is tangent to the factor S^3 , hence can be viewed as a vector field on S^3 ; as such, it will be denoted by Z_Δ .

We then have

$$(58) \quad \begin{aligned} Z_\Delta &= \left(\frac{2k_1}{k_1 + k_2} i z_1, \frac{2k_2}{k_1 + k_2} i z_2 \right) \\ &= \frac{2}{(k_1 + k_2)} (\Re F Z + (k_1 - k_2) i z_1 z_2 E) \\ &= Z + \frac{2(k_1 - k_2)}{(k_1 + k_2)} Z_R, \end{aligned}$$

where Z_R is the vector field on S^3 defined by $Z_R = (i z_1, -i z_2)$. It can be seen that Z_R is a *right-invariant* (unit) Killing vector field for the standard metric g of S^3 . In particular, Z_Δ is itself a Killing vector field with respect to g .

We observe that the restriction of $g_{\alpha,\beta}$ on each fiber of the natural fibration $\pi : S^3 \times S^1 \rightarrow S^1$ is independent of t , hence can be considered as a Riemannian metric on the sphere S^3 ; this metric is denoted by g_Δ .

PROPOSITION 3. — *The pair (g_Δ, Z_Δ) is a Sasakian structure on S^3 , actually coincides with the Sasakian structure obtained by deforming the canonical Sasakian structure (g, Z) of S^3 by the function Δ defined by*

$$(59) \quad \Delta(z) = \frac{2}{(k_1 + k_2)} \Re F = \frac{2k_1 |z_1|^2 + 2k_2 |z_2|^2}{k_1 + k_2}.$$

Proof. — By using (55) we check that Z_Δ is of norm 1 and is orthogonal to Q with respect to the metric g_Δ . Then, with respect to the triple Z_Δ, E, iE , g_Δ is described by the following table:

$$(60) \quad \begin{aligned} g_\Delta(Z_\Delta, Z_\Delta) &= 1, \\ g_\Delta(Z_\Delta, E) &= g_\Delta(Z_\Delta, F) = g_\Delta(E, F) = 0, \\ g_\Delta(E, E) &= g_\Delta(iE, iE) = \frac{1}{\Delta}. \end{aligned}$$

Now, the vector field Z_Δ can be written

$$(61) \quad Z_\Delta = \Delta Z - \frac{2(k_1 - k_2)}{(k_1 + k_2)} \Re(z_1 z_2) E + \frac{2(k_1 - k_2)}{(k_1 + k_2)} \Im(z_1 z_2) F.$$

On the other hand, we clearly have

$$(62) \quad d\Delta(E) = \frac{4(k_1 - k_2)}{(k_1 + k_2)} \Re(z_1 z_2), \quad d\Delta(F) = \frac{4(k_1 - k_2)}{(k_1 + k_2)} \Im(z_1 z_2).$$

These prove that Z_Δ is the Reeb vector field of the contact structure Q with respect to the contact 1-form $\frac{\eta}{\Delta}$. By (60), the metric g_Δ coincides with the Riemannian metric determined by the Reeb vector field Z_Δ . It remains to check that Δ satisfies the condition of Proposition 2, which is clear. \square

COROLLARY 2. — *For any complex numbers α, β satisfying (3), the Lee form of the l.c.K. Hermitian structure $(g_{\alpha, \beta}, J)$ is parallel with respect to $D^{g_{\alpha, \beta}}$.*

Proof. — As already observed in Section 3.2, (24) together with (36) imply that the Lee vector field $V_{\alpha, \beta}$, hence also the Lee form $\theta_{\alpha, \beta}$, is parallel with respect to $g_{\alpha, \beta}$. \square

Remark 5. — By (42) and the above proposition, we infer that the scalar curvature $\text{Scal}^{g_{\alpha, \beta}}$, which is also equal to the scalar curvature Scal^{g_Δ} of g_Δ , is given by

$$(63) \quad \text{Scal}^{g_{\alpha, \beta}} = 6 \left(1 - 4 \frac{(k_1 - k_2)}{(k_1 + k_2)} \frac{(k_1 |z_1|^2 - k_2 |z_2|^2)}{(k_1 |z_1|^2 + k_2 |z_2|^2)} \right).$$

In particular, $\text{Scal}^{g_{\alpha, \beta}}$ is not constant, except in the case that $k_1 = k_2$, i.e. $|\alpha| = |\beta|$.

Remark 6. — It follows readily from (23) that the flow $\Psi^{V_{\alpha, \beta}}$ of $V_{\alpha, \beta}$ on $S^3 \times S^1$ is given by

$$(64) \quad \Psi_s^{V_{\alpha, \beta}}((z, t)) = \left(\left(e^{-i \frac{2s}{(k_1 + k_2)} \Re \alpha} \cdot z_1, e^{-i \frac{2s}{(k_1 + k_2)} \Re \beta} \cdot z_2 \right), \right. \\ \left. t + \frac{2s}{(k_1 + k_2)} \bmod \mathbb{Z} \right).$$

This flow preserves the fibration π and induces an isometry with respect to g_Δ from each fiber to the corresponding target fiber. In particular, after one rotation over S^1 , this isometry is the isometry $\sigma_{\alpha, \beta}$ from S^3 to itself defined by

$$(65) \quad \sigma_{\alpha, \beta}((z, t)) = \left((e^{-i \Re \alpha} \cdot z_1, e^{-i \Re \beta} \cdot z_2), t \right).$$

Finally, the l.c.K. metric $g_{\alpha, \beta}$ on the Hopf surface $M_{\alpha, \beta}$ is obtained by the following procedure (see [9] for the case $k_1 = k_2$):

1. Equip the sphere S^3 with the Riemannian metric g_Δ obtained by deforming the canonical Sasakian structure (g, Z) by the function Δ defined by (59) (see Section 3.3).

2. Realize $(M_{\alpha,\beta}, g_{\alpha,\beta})$ as the suspension of the isometry $\sigma_{\alpha,\beta}$ defined by (65) over the circle of length $\frac{(k_1 + k_2)}{2}$ (see Section 3.2).

Proof of Theorem 1.

The first statement has been proved in the preceding sections, see in particular Proposition 1 and Proposition 3.

In order to prove the second statement, *i.e.* the existence of l.c.K. metrics on all Hopf surfaces of class 0, we use a specific deformation argument due to C. LeBrun [17]. Here are details. Fix any complex number β such that $|\beta| > 1$ and any positive integer m . Consider the three-dimensional complex manifold \mathcal{M} defined as the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{(0, 0)\})$ by the group $\tilde{\Gamma}_{\beta,m} \equiv \mathbb{Z}$ generated by the transformation $\tilde{\gamma}_{\beta,m} : (\lambda, (u, v)) \mapsto (\lambda, (\beta^m u + \lambda v^m, \beta v))$. Let p be the natural projection from \mathcal{M} onto \mathbb{C} which assigns λ to the class of $(\lambda, (u, v))$. Then, p is a holomorphic fibration whose fiber at $\lambda = 0$ is the Hopf surface of class 1 $M_{\beta^m,\beta}$ whereas fibers at $\lambda \neq 0$ are Hopf surfaces of class 0, all isomorphic to each other as recalled in the first section.

The bundle of scalars of weight 1 on \mathcal{M} (see Section 1) is naturally identified to the quotient of the product bundle $\mathbb{C} \times (\mathbb{C}^2 - \{(0, 0)\}) \times \mathbb{R}$ by $\tilde{\gamma}_{\beta,m} \equiv \mathbb{Z}$ acting by $1.(\lambda, (u, v), a) = (\lambda, (\beta^m u + \lambda v^m, \beta v), |\beta|^{\frac{m+1}{2}} a)$. (Notice that $|\beta|^{\frac{m+1}{2}}$ is equal to $|\det(\tilde{\gamma}_{\beta,m})|^{1/4}$.) Let \mathcal{L} denote this bundle.

As already observed, the function $\Phi_{\beta^m,\beta}$ introduced in Section 2 can be considered as a section of \mathcal{L}^2 over $p^{-1}(0) = M_{\beta^m,\beta}$. It extends to a smooth section, $\tilde{\Phi}$ of \mathcal{L}^2 on \mathcal{U} for some neighbourhood \mathcal{U} of 0 in \mathbb{C} . For any λ in \mathcal{U} , let $\tilde{\Phi}_\lambda$ be the restriction of $\tilde{\Phi}$ to the fiber $p^{-1}(\lambda)$, also viewed as a function on $W = \mathbb{C}^2 - \{(0, 0)\}$. Then, $\tilde{\Phi}_0$ is equal to $\Phi_{\beta^m,\beta}$. By Proposition 1, $\frac{1}{4} dd^c \Phi_{\beta^m,\beta}$ is a Kähler form on \tilde{M} ; by continuity, the same is true for $\frac{1}{4} dd^c \tilde{\Phi}_\lambda$, so that $\frac{1}{4\tilde{\Phi}_\lambda} dd^c \tilde{\Phi}_\lambda$ is the Kähler form of a l.c.K. metric on $p^{-1}(\lambda)$. We thus get a l.c.K. metric on $\tilde{M}_{\beta,m,\lambda}$ for any λ in \mathcal{U} , hence for any λ in \mathbb{C} . By varying β and m , we eventually get a l.c.K. metric for each primary Hopf surface of class 0.

Remark 7. — Note that the above deformation argument is specific to the Hopf surface. A general stability theorem for l.c.K. structures, as in the Kähler case, [15], is still lacking in the literature.

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