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HENKIN MEASURES, RIESZ PRODUCTS AND SINGULAR SETS

by Evgueni DOUBTSOV

1. INTRODUCTION

The principal objects of the present paper are measures defined on the complex sphere $S = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$, $n \geq 2$. The role of S in the function theory is twofold. First, S is the boundary of the unit ball B, the simplest pseudoconvex domain. Second, let $\mathcal{U}(n)$ be the group of unitary operators on $\mathbb{C}^n$, then $S = \mathcal{U}(n)/\mathcal{U}(n-1)$, in other words, S is a homogeneous space. In particular, it is possible to develop the spectral function theory on S in terms of $H(p,q)$, the spaces of complex spherical harmonics.

DEFINITION. — *Fix a dimension n. Let $(p,q) \in \mathbb{Z}^2_+$, then $H(p,q)$ is the vector space of all harmonic homogeneous polynomials in $\mathbb{C}^n$ of total degree $p+q$, of degree p in $z_1, \dots, z_n$, and of degree q in $\overline{z}_1, \dots, \overline{z}_n$. We use the same symbol for the restriction of $H(p,q)$ on S.*

The spectrum of a measure $\mu \in M(S)$ is defined by the equality

$$\operatorname{spec}(\mu) = \left\{ (p,q) \in \mathbb{Z}^2_+ : \mu_{pq}(z) = \int_S K_{pq}(z,\zeta)\, d\mu(\zeta) \not\equiv 0, \quad z \in S \right\},$$

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where $K_{pq}(z,\zeta)$ is the reproducing kernel for $H(p,q) \subset L^2(S)$. We address the reader to the monograph [Ru], Chapter 12, for a systematic presentation of the harmonic analysis on S.

Given a set (a spectrum) $\Lambda \subset \mathbb{Z}_+^2$, the general problem is to investigate the properties of the space $M_\Lambda(S) = \{\mu \in M(S) : \operatorname{spec}(\mu) \subset \Lambda\}$.

Put $M_\Lambda^s(S) = \{\mu^s :$ there exists a $\mu \in M_\Lambda(S)$ such that μ^s is the singular part of $\mu\}$ (here and in what follows, "singular" means "singular with respect to the corresponding Lebesgue measure"). It is interesting, in particular, to find the sets with the following quite rare property.

DEFINITION. — *A set $\Lambda \subset \mathbb{Z}_+^2$ is said to be* singular *if*

1) *$M_\Lambda^s(S)$ and $M_{\mathbb{Z}_+^2\setminus\Lambda}^s(S)$ are not trivial;*

2) *if $\mu \in M_\Lambda^s(S)$ and $\nu \in M_{\mathbb{Z}_+^2\setminus\Lambda}^s(S)$, then $\mu\perp\nu$ (are mutually singular).*

Let $d \in \mathbb{Z}_+$. Define $\Lambda(d) = \{(p,q) \in \mathbb{Z}_+^2 : (p-d)(q-d) = 0, p \geq d, q \geq d\}$ (often we add the point $(0,0)$ to the set $\Lambda(d)$; clearly this does not affect the properties under the question). Note that the spectrum $\Lambda(0)$ is a natural object in the complex analysis, since $\operatorname{spec}(\mu) \subset \Lambda(0)$ if and only if the Poisson integral $P[\mu]$ is a pluriharmonic function. Such a measure is said to be pluriharmonic (remark that in the pluripotential theory the term "pluriharmonic measure" is used for a completely different object). By analogy, if $\operatorname{spec}(\mu) \subset \Lambda(d)$ or $\operatorname{spec}(\mu) \subset \Lambda(d) \cup \{(0,0)\}$, then we say that μ is d-pluriharmonic.

It is shown in [D1] that $\Lambda(0)$ is singular. The first aim of the present paper is to generalize this result.

THEOREM A. — *The set $\Lambda(d)$ is singular for all $d \in \mathbb{Z}_+$.*

To prove the mutual singularity property, we use some properties of the Henkin measures (see §2) and an asymptotic formula of the Boole-Hruščëv-Vinogradov type (§3, Theorem 3.1).

Actually the non-triviality part from the definition of a singular set is known for all $\Lambda(d)$. Indeed, it is shown in [D3] that the corresponding triple $(A_d(S), S, \sigma)$ is regular in the sense of [A1]. On the other hand, given a d-plh measure μ, the slice measure μ_ξ is defined (on the circle) for $\hat{\sigma}$-almost all $\xi \in \mathbb{C}P^{n-1}$. Moreover, in the weak sense, we have the integral representations

$$\mu = \int_{\mathbb{C}P^{n-1}} \mu_\xi \, d\hat{\sigma}(\xi), \quad \mu^s = \int_{\mathbb{C}P^{n-1}} \mu_\xi^s \, d\hat{\sigma}(\xi),$$

where μ^s (μ_ξ^s) is the singular part of μ (μ_ξ) (see §3 for details). So a natural problem is to understand the properties of the family $\{\mu_\xi^s\}_{\xi\in\mathbb{C}P^{n-1}}$.

In §4 we introduce the d-pluriharmonic Riesz products to give examples of probability singular d-plh measures. Probably, such product measures are of independent interest. On the other hand, the Riesz product idea yields, for example, the following existence result.

THEOREM B. — *There exists a probability singular d-plh measure μ such that the slice measures μ_ξ live on sets of Hausdorff dimension* 1 *for all $\xi \in \mathbb{C}P^{n-1}$.*

To illustrate the results about $M_\Lambda(S)$ and $M_\Lambda^s(S)$, in §5 we discuss the peak, interpolation and null sets for the unitarily invariant spaces of continuous functions $A_{k\ell}(S)$.

Notation. — Throughout this paper σ is Lebesgue measure on S, $\sigma(S) = 1$; the unit disc is $\mathbb{D}$, and m is Lebesgue measure on the unit circle $\mathbb{T}$, $m(\mathbb{T}) = 1$. $\mathbb{C}P^{n-1}$ is the projective space, $\mathrm{pr} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ is the canonical projection, and $\hat{\sigma} = \mathrm{pr}(\sigma)$.

Given a $\mu \in M(S)$, the symbol $P[\mu]$ denotes the classical Poisson integral:

$$P[\mu](z) = \int_S P(z,\zeta)\, d\mu(\zeta) = \int_S \frac{1-|z|^2}{|z-\zeta|^{2n}}\, d\mu(\zeta) \qquad (z \in B).$$

We identify a function $f \in L^1(S)$ and a measure $f\sigma \in M(S)$.

It is useful to imagine $\mathbb{Z}_+^2$ as the first quadrant of the integer lattice. In particular, we say that $\{(p,k) : p \in \mathbb{Z}_+\}$, $k \in \mathbb{Z}_+$, is a horizontal ray (respectively $\{(\ell,q) : q \in \mathbb{Z}_+\}$, $\ell \in \mathbb{Z}_+$, is a vertical one).

2. HENKIN MEASURES

Motivated by Bourgain's investigations of the Dunford-Pettis property, Cima and Timoney introduced in [CT] the following notion.

DEFINITION. — *Let A be a Banach algebra and $X \subset A$ be a linear subspace. The* Bourgain algebra $X_{\mathcal{B}}$ *is the set of $f \in A$ such that*

$$\text{if } f_j \to 0 \text{ weakly in } X, \text{ then } \|ff_j + X\| \to 0.$$

In fact, Bourgain showed in [B1] that a subspace X of $C(K)$ had the Dunford-Pettis property if $X_{\mathcal{B}} = C(K)$. It happens that a very similar abstract notion (we put the weak* convergence in place of the weak one) is useful in the study of the Henkin measures corresponding to a subspace $X \subset C(K)$.

2.1. Henkin algebras.

In the definitions below we suppose that K is a compact Hausdorff space, ρ is a positive regular Borel measure on K, the closed support of ρ is K, and $X \subset C(K)$ is a closed subspace.

DEFINITION. — *A function sequence $\{f_j\}_{j=1}^{\infty} \subset X$ is called an (X, ρ)-sequence (or a ρ-sequence) if*

$$\int_K f_j g \, d\rho \to 0 \quad \text{for all } g \in L^1(\rho).$$

Remark. — In other words, $f_j \to 0$ weakly* with respect to the duality $(L^1(\rho), L^\infty(\rho))$ (where $X \subset C(K) \subset L^\infty(\rho)$). In particular, $\|f_j\|_{C(K)} = \|f_j\|_\infty \leq \text{const}$.

DEFINITION. — *Let $X \subset C(K)$ be a closed subspace, then the* Henkin algebra $X_{\mathcal{H}}(\rho)$ *(with respect to ρ) is the set of $\varphi \in C(K)$ such that*

$$\|\varphi f_j + X\|_\infty \to 0 \quad \text{as } j \to \infty$$

for every ρ-sequence $\{f_j\}_{j=1}^{\infty} \subset X$.

The following standard observation (compare with [CT]) justifies the word *algebra* in the above definition.

PROPOSITION 2.1. — *The space $X_{\mathcal{H}}(\rho)$ is a closed subalgebra of $C(K)$.*

Proof. — Suppose that $\{f_j\}_{j=1}^{\infty}$ is a ρ-sequence.

1. Let $\varphi_1, \varphi_2 \in X_{\mathcal{H}}(\rho)$, then there exist $g_j \in X$ such that $\|\varphi_1 f_j + g_j\|_\infty \to 0$ as $j \to \infty$. Note that g_j is a ρ-sequence, therefore, there exist $h_j \in X$ such that $\|\varphi_2 g_j + h_j\|_\infty \to 0$ as $j \to \infty$. In sum we obtain

$$\|\varphi_1\varphi_2 f_j - h_j\|_\infty \le \|\varphi_2\|_\infty \|\varphi_1 f_j + g_j\|_\infty + \|\varphi_2 g_j + h_j\|_\infty \to 0.$$

In other words $\varphi_1\varphi_2 \in X_{\mathcal{H}}(\rho)$.

2. Without loss of generality $\|f_j\|_\infty \le 1$. Let $\{\varphi_k\}_{k=1}^\infty \subset X_{\mathcal{H}}(\rho)$ and $\|\varphi_k - \varphi\|_\infty \to 0$ as $k \to \infty$. Take $k \in \mathbb{N}$ such that $\|\varphi_k - \varphi\|_\infty < \varepsilon$, then $\|f_j\varphi + X\|_\infty \le \varepsilon + \|f_j\varphi_k + X\|_\infty < 2\varepsilon$ for j large enough. □

DEFINITION. — *A measure $\mu \in M(K)$ is called an (X,ρ)-measure (or a Henkin measure) if*

$$\lim_{j\to\infty} \int_K f_j \, d\mu = 0$$

for every ρ-sequence $\{f_j\}_{j=1}^\infty \subset X$.

Remark. — Clearly, the set of the Henkin measures is norm-closed.

PROPOSITION 2.2. — *Suppose that $X_{\mathcal{H}}(\rho) = C(K)$. Let μ be an (X,ρ)-measure and $\lambda \ll \mu$. Then λ is an (X,ρ)-measure.*

Proof. — Fix a ρ-sequence $\{f_j\}_{j=1}^\infty$ and a $\varphi \in C(K)$. The definition of the Henkin algebra yields a sequence $\{g_j\}_{j=1}^\infty \subset X$ such that $\|\varphi f_j + g_j\|_\infty \to 0$ as $j \to \infty$. Remark that $\{g_j\}_{j=1}^\infty$ is a ρ-sequence. Therefore

$$\int_K f_j\varphi \, d\mu = \int_K (f_j\varphi - g_j)\, d\mu + \int_K g_j \, d\mu \to 0,$$

since μ is a Henkin measure. So $\varphi\mu$ is a Henkin measure for all $\varphi \in C(K)$. On the other hand $\lambda = \psi\mu$ with $\psi \in L^1(|\mu|)$. Since the set of the Henkin measures is closed, λ is a Henkin measure. □

There is a large number of results about the Bourgain algebras generated by subspaces of different uniform algebras (see, for example, [I] for the $\mathbb{C}^n$-setting, see also references therein). Many of such statements have their analogues in terms of the Henkin algebras. However, in the present paper, we concentrate our attention on the case $K = S$ and $\rho = \sigma$.

2.2. Henkin measures on the sphere.

PROPOSITION 2.3. — *A sequence $\{f_j\}_{j=1}^{\infty} \subset X \subset C(S)$ is a σ-sequence if and only if 1) $P[f_j](z) \to 0$ as $j \to \infty$ for all $z \in B$, and 2) $\|f_j\|_\infty \leq \text{const}$ for all $j \in \mathbb{N}$.*

Proof. — Assume that 1) and 2) hold and $g \in L^1(\sigma)$. Put $P_r[g](\zeta) = P[g](r\zeta)$, $0 \leq r < 1$, $\zeta \in S$. Fubini's theorem yields

$$\begin{aligned}\int_S f_j g\, d\sigma &= \int_S f_j \left(g - P_r[g]\right)\, d\sigma + \int_S P_r[f_j] g\, d\sigma \\ &\leq \|f_j\|_\infty \|g - P_r[g]\|_1 + \|P_r[f_j]\|_\infty \|g\|_1.\end{aligned}$$

Note that $\|g - P_r[g]\|_1 \to 0$ as $r \to 1-$ and $P[f_j]$ tends to zero uniformly on compact subsets of the ball, therefore $\int_S f_j g\, d\sigma \to 0$ as $j \to \infty$.

Let now $\{f_j\}_{j=1}^{\infty}$ be a σ-sequence. If $z \in B$, then $P(z, \cdot) \in L^1(\sigma)$, thus 1) holds. On the other hand, 2) holds for any ρ-sequence. □

In what follows, we assume that $f \in X$ implies $P_r[f] \in X$.

The above proposition enables us to relate the (X, σ)-measures and the annihilator $X^\perp = \left\{\mu \in M(S) : \int_S f\, d\mu = 0 \text{ for all } f \in X\right\}$. If $X = A(S)$ (the ball algebra), then the next statement is Valskii's theorem (see [Ru], 9.2). We can use the argument of Valskii in the general case (for reader's convenience, we reproduce the proof here, since this result is given in [D1] without proof).

THEOREM 2.4. — *Let μ be an (X, σ)-measure and $\varepsilon > 0$. Then there exists a function $g \in L^1(\sigma)$ such that $||g||_1 \leq ||\mu||_{X^*} + \varepsilon$ and $\mu - g\sigma \in X^\perp$.*

Proof. — First, we establish an auxiliary result.

CLAIM. — *Let λ be an (X, σ)-measure and $\varepsilon > 0$. Then there exists an $h \in L^1(\sigma)$ such that $||h||_1 \leq ||\lambda||$ and $||\lambda - h\sigma||_{X^*} < \varepsilon$.*

Put $u_r = P_r[\lambda]$, $0 < r < 1$. It is sufficient to verify that $\lim\limits_{r \to 1-} ||\lambda - u_r\sigma||_{X^*} = 0$. (We can define $h = u_r$ with r sufficiently large.) Assume that the latter limit is not zero. Then there exist $\delta > 0$, $r_j \to 1$ and $f_j \in X$, $||f_j||_\infty \leq 1$, such that

$$\left| \int_S f_j\, d\lambda - \int_S f_j u_{r_j}\, d\sigma \right| \geq \delta \quad \text{for all } j \in \mathbb{N}.$$

By Fubini's theorem $\int_S fu_r\,d\sigma = \int_S f_r\,d\lambda$, thus

$$\left|\int_S [f_j(\zeta) - f_j(r_j\zeta)]\,d\lambda(\zeta)\right| \geq \delta \quad \text{for all } j \in \mathbb{N}.$$

Define $g_j(z) = f_j(z) - f_j(r_j z)$, $z \in \overline{B}$. Remark that $g_j(z) \to 0$ for all $z \in B$; hence, by Proposition 2.3, $\{g_j\}_{j=1}^{\infty}$ is a σ-sequence. Therefore, by the definition of a Henkin measure, $\int_S g_j\,d\lambda \to 0$ as $j \to \infty$. This contradiction proves the claim.

Now choose $\varepsilon_j > 0$ such that $\varepsilon - \sum\limits_{j=2}^{\infty} \varepsilon_j > \varepsilon_1 - ||\mu||_{X^*} > 0$. Put $\mu_1 = \mu$ and suppose, as induction hypothesis, that $\ell \geq 1$, μ_ℓ is an (X,σ)-measure and $||\mu_\ell||_{X^*} < \varepsilon_\ell$. By the Hahn-Banach theorem $||\mu_\ell - \rho_\ell|| < \varepsilon_\ell$ for some $\rho_\ell \in X^\perp$. By the claim (with $\lambda = \mu_\ell - \rho_\ell$), there exists a $g_\ell \in L^1(\sigma)$ such that $||g_\ell||_1 < \varepsilon_\ell$ and $||\mu_\ell - \rho_\ell - g_\ell\sigma||_{X^*} < \varepsilon_{\ell+1}$. Define $\mu_{\ell+1} = \mu_\ell - \rho_\ell - g_\ell\sigma$. Note that $\mu_{\ell+1}$ is a Henkin measure, so the induction construction proceeds.

Define $g = \sum\limits_{j=1}^{\infty} g_j$, then $g \in L^1(\sigma)$ and $||g||_1 \leq \sum\limits_{j=1}^{\infty} \varepsilon_j$.

We have $\mu = \mu_{\ell+1} + \sum\limits_{j=1}^{\ell} \rho_j + \sum\limits_{j=1}^{\ell} g_j\sigma$ for every $\ell \in \mathbb{N}$, thus

$$\mu - g\sigma = \mu_{\ell+1} + \sum_{j=1}^{\ell} \rho_j - \sum_{j=\ell+1}^{\infty} g_j\sigma.$$

Since $\rho_j \in X^\perp$, we obtain

$$||\mu - g\sigma||_{X^*} \leq ||\mu_{\ell+1}||_{X^*} + \sum_{j=\ell+1}^{\infty} ||g_j||_1 \leq \varepsilon_{\ell+1} + \sum_{j=\ell+1}^{\infty} \varepsilon_j \to 0 \quad \text{as } \ell \to \infty.$$

In other words $\mu - g\sigma \in X^\perp$. □

Now we consider the $\mathcal{U}$-invariant subspaces of $C(S)$. More precisely, put $X = C_\Lambda(S) = \{f \in C(S) : \operatorname{spec}(f) \subset \Lambda\}$, $\Lambda \subset \mathbb{Z}_+^2$.

Let $K_\Lambda : L^2(S) \to L^2_\Lambda(S)$ be the orthogonal (Cauchy-Szegö) projection (as above, $L^2_\Lambda(S) = \{f \in L^2(S) : \operatorname{spec}(f) \subset \Lambda\}$).

DEFINITION. — *Given a $\varphi \in L^\infty(S)$, the Λ-Hankel operator (more precisely, the Λ-spectral Hankel type operator) $V_{\Lambda,\varphi} : L^2(S) \to L^2(S)$ is defined by the formula $V_{\Lambda,\varphi}[f] = \varphi K_\Lambda[f] - K_\Lambda[\varphi f]$.*

Note that $V_{\Lambda,\varphi} + V_{\mathbb{Z}_+^2\setminus\Lambda,\varphi} \equiv 0$.

DEFINITION (see [D3]). — *A spectrum Λ is said to have the* Compact Hankel *property (we write $\Lambda \in (CH)$) if $V_{\Lambda,\varphi} : C(S) \to C(S)$ is a compact operator for every polynomial φ on S.*

PROPOSITION 2.5. — *Suppose that $\Lambda \in (CH)$, $\rho \in M(S)$ is a positive measure, the closed support of ρ is S, and K_Λ is bounded in $L^2(\rho)$-norm. Then $C_\Lambda(S)_{\mathcal{H}}(\rho) = C(S)$.*

Proof. — Let φ be a polynomial and $\{f_j\}_{j=1}^\infty$ be a $(C_\Lambda(S), \rho)$-sequence. Note that $K_\Lambda[\varphi f_j] \in C_\Lambda(S)$, so the property $\|\varphi f_j - K_\Lambda[\varphi f_j]\|_{C(S)} \to 0$ yields $\varphi \in C_\Lambda(S)_{\mathcal{H}}(\rho)$.

Assume that $\|H_{\Lambda,\varphi}[f_j]\|_{C(S)} \not\to 0$. Since $\|f_j\|_{C(S)} \leq 1$ and $\Lambda \in (CH)$, there exists a subsequence $\{j_k\}_{k=1}^\infty$ such that $H_{\Lambda,\varphi}[f_{j_k}] \to g$ in $C(S)$ for some $g \not\equiv 0$. On the other hand $f_j \to 0$ weakly in $L^2(\rho)$ and K_Λ is $L^2(\rho)$-bounded, thus $H_{\Lambda,\varphi}[f_j] \to 0$ weakly in $L^2(\rho)$, a contradiction.

Recall that the Henkin algebras are closed, so the proof is complete.

□

For $E, F \subset \mathbb{Z}_+$, define

$$\Lambda = \Lambda(E, F) = \{(p, q) \in \mathbb{Z}_+^2 : q \in E \text{ or } p \in F\}.$$

(A union of horizontal and vertical rays.)

COROLLARY 2.6. — *Let $E, F \subset \mathbb{Z}_+$ be finite sets, $\Lambda = \Lambda(E, F)$ or $\Lambda = \mathbb{Z}_+^2 \setminus \Lambda(E, F)$. Suppose that μ is a $(C_\Lambda(S), \sigma)$-measure and $\nu \ll \mu$. Then ν is a $(C_\Lambda(S), \sigma)$-measure.*

Proof. — The property $\Lambda(E, F) \in (CH)$ is obtained in [D3]. So we apply Proposition 2.5 and Proposition 2.2. □

COROLLARY 2.7. — *Let Λ be as in Corollary 2.6. Suppose that $\mu \in L^1(S) + C_\Lambda(S)^\perp$ and $\nu \ll \mu$. Then $\nu \in L^1(S) + C_\Lambda(S)^\perp$.*

Proof. — We apply Corollary 2.6 and Theorem 2.4. □

To finish this section, we give other simple examples of the Henkin algebras generated by $\mathcal{U}$-invariant subspaces.

Example 2.8. — For $\ell \in \mathbb{Z}_+$, put $D(\ell) = \{(p, q) \in \mathbb{Z}_+^2 : p - q = \ell\}$.

Let d be a prime number and

$$\Lambda = \bigcup_{\ell=-\infty}^{\infty} D(\ell d), \quad \text{or } \Lambda = \bigcup_{\ell=0}^{\infty} D(\ell), \quad \text{or } \Lambda = \bigcup_{\ell=-\infty}^{0} D(\ell).$$

Then $C_\Lambda(S)_{\mathcal{H}}(\sigma) = C_\Lambda(S)$.

Proof. — 1. Put $A = C_\Lambda(S)_{\mathcal{H}}(\sigma)$. Note that A is a $\mathcal{U}$-invariant subalgebra of $C(S)$. Since $C_\Lambda(S)$ is an algebra, we have $C_\Lambda(S) \subset A$.

2. Assume that $A \not\subset C_\Lambda(S)$, then $A = C(S)$ since $C_\Lambda(S)$ is a *maximal* $\mathcal{U}$-invariant subalgebra of $C(S)$ (see [Ru], 12.4.7 and 12.5.6). Fix a $\zeta \in S$. Let m_ζ be Lebesgue measure on the circle $\mathbb{T}_\zeta = \{\lambda\zeta : \lambda \in \mathbb{T}\}$. We suppose that $d > 1$, so $\lambda^k m_\zeta \in C_\Lambda(S)^\perp$, $\lambda \in \mathbb{T}$, for some $k \in \mathbb{Z}$. In particular, $\lambda^k m_\zeta$ is a Henkin measure. Since $A = C(S)$, Proposition 2.2 says that m_ζ is a Henkin measure. On the other hand, take polynomials $\{f_j\}_{j=1}^\infty$ such that $f_j \in H(j,j)$, $f(\zeta) = 1$, $|f| \le 1$ on S. Then $\{f_j\}_{j=1}^\infty$ is a σ-sequence and $\int_S f_j \, dm_\zeta = 1$ for all $j \in \mathbb{N}$. A contradiction. □

3. d-plh measures and d-Cauchy transforms.

Fix a $d \in \mathbb{Z}_+$. Recall that a measure $\mu \in M(S)$ is said to be d-plh if $\operatorname{spec}(\mu) \subset \Lambda(d) = \{(p,q) \in \mathbb{Z}_+^2 : (p-d)(q-d) = 0,\ p \ge d,\ q \ge d\}$.

First, we explain and establish integral representations for d-plh measures in terms of the slice-measures (in the pluriharmonic case such results are given in Chapter 5 of [A2]).

Convention. — Given an $f \in L^1(S)$, the standard "slice-integration formula" has the following form:

$$\int_S f \, d\sigma = \int_S \left(\int_{\mathbb{T}} f(\lambda\zeta) \, dm(\lambda) \right) d\sigma(\zeta).$$

Recall that $\mathrm{pr} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ is the canonical projection and $\hat{\sigma} = \mathrm{pr}(\sigma)$. We rewrite the above equality as a Fubini type theorem. Namely

$$\int_S f \, d\sigma = \int_{\mathbb{C}P^{n-1}} \left(\int_{\mathbb{T}} f_\xi(\lambda) \, dm(\lambda) \right) d\hat{\sigma}(\xi),$$

where $f_\xi(\lambda) = f\left(\lambda \dfrac{|\zeta_1|}{\zeta_1} \zeta\right)$, $\zeta_1 \neq 0$, if $\mathrm{pr}(\zeta) = \xi$.

In a sense, we identify S and $\mathbb{C}P^{n-1} \times \mathbb{T}$. Note that $\sigma\{\zeta \in S : \zeta_1 = 0\} = 0$, so such identification is correct from the measure theory point of view (of course this is not true topologically).

Suppose that $\mu \in M(S)$, $\mathrm{spec}(\mu) \subset \Lambda$, $\Lambda \subset \mathbb{Z}^2_+$, and $u = P[\mu]$, then

$$u(z) = \sum_{(p,q)\in\Lambda} \mu_{pq}(z), \quad z \in B,$$

where $\mu_{pq} \in H(p,q)$. Now we assume that μ is a d-plh measure. Fix $\zeta \in S$ and consider the slice function $u_\zeta(\lambda) = u(\lambda\zeta)$, $\lambda \in \mathbb{D}$, then we have

$$u_\zeta(\lambda) = |\lambda|^{2d}\mu_{dd}(\zeta) + |\lambda|^{2d} \sum_{p=d+1}^{\infty} \left(\mu_{dp}(\zeta)\overline{\lambda}^{p-d} + \mu_{pd}(\zeta)\lambda^{p-d}\right) = |\lambda|^{2d} v_\zeta(\lambda),$$

where the harmonic function v_ζ is defined (in $\mathbb{D}$) by the latter equality.

Let $u_r(\zeta) = u(r\zeta)$, then

$$\sup_{0\le r<1} \|u_r\|_{L^1(S)} = \lim_{r\to 1-} \|u_r\|_{L^1(S)} < \infty,$$

thus

$$\int_{\mathbb{C}P^{n-1}} \lim_{r\to 1-} \|(v_\xi)_r\|_{L^1(\mathbb{T})}\, d\hat\sigma(\xi) < \infty.$$

In particular, $\lim_{r\to 1-} \|(v_\xi)_r\|_{L^1(\mathbb{T})} < \infty$ for $\hat\sigma$-a.e. $\xi \in \mathbb{C}P^{n-1}$. Therefore, for $\hat\sigma$-a.e. $\xi \in \mathbb{C}P^{n-1}$, there exists $\mu_\xi \in M(\mathbb{T})$ such that $v_\xi = P[\mu_\xi]$ (the Poisson integral in dimension one).

Since $\|P_r[\rho]\|_1 \nearrow \|\rho\|$ as $r \to 1-$ for every measure ρ, we have

$$\|\mu\| = \int_{\mathbb{C}P^{n-1}} \|\mu_\xi\|\, d\hat\sigma(\xi).$$

Let μ^a be the absolutely continuous part of μ and u^* be the boundary values of u, then $\mu^a = u^*\sigma$. Analogously $\mu^a_\xi = u^*_\xi m$ when μ_ξ is correctly defined. So, by Fubini's theorem $\|\mu^a\| = \int \|\mu^a_\xi\|$. Therefore

$$\|\mu^s\| = \int_{\mathbb{C}P^{n-1}} \|\mu^s_\xi\|\, d\hat\sigma(\xi), \tag{3.1}$$

where μ^s (μ^s_ξ) denotes the singular part of μ (μ_ξ).

Remark. — Given an $f \in C(S)$, classical properties of the Poisson integral yield

$$\begin{aligned}
\int_S f\, d\mu &= \lim_{r\to 1-} \int_S u_r f\, d\sigma \\
&= \lim_{r\to 1-} r^{2d} \int_{\mathbb{C}P^{n-1}} \left(\int_{\mathbb{T}} (v_\xi)_r f_\xi\, dm\right) d\hat\sigma(\xi) \\
&= \int_{\mathbb{C}P^{n-1}} \left(\int_{\mathbb{T}} f_\xi\, d\mu_\xi\right) d\hat\sigma(\xi).
\end{aligned}$$

Fubini's theorem provides the same integral formula for the absolutely continuous parts, so we have also $\mu^s = \int \mu^s_\xi \, d\hat{\sigma}(\xi)$ in the above weak sense.

Now, we investigate so-called d-Cauchy projections. Define $H^2_d(S) = \{f \in L^2(S) : \mathrm{spec}(f) \subset \{(p,d) : p \in \mathbb{Z}_+\}\}$, $H^2_d(B) = \{P[f] : f \in H^2_d(S)\}$. In particular, $H^2_0(B)$ is the Hardy class $H^2(B)$. Let $C_d(z,\zeta)$ be the reproducing kernel for $H^2_d(B)$ (in the point $z \in B$). Then the d-Cauchy projection

$$C_d[\mu](z) = \int_S C_d(z,\zeta)\, d\mu(\zeta)$$

is defined for all $\mu \in M(S)$. Again, if $d = 0$, then we have the classical Cauchy-Szegö projection.

Our main object is $C_d[\mu]$ with d-plh μ. Take a $\zeta \in S$ such that μ_ζ is correctly defined, then

$$C_d[\mu](\lambda\zeta) = |\lambda|^{2d} C[\mu_\zeta](\lambda), \quad \lambda \in \mathbb{D}. \tag{3.2}$$

Therefore, the boundary values $C_d[\mu]^*$ exist σ-a.e., moreover, (3.1) leads to an asymptotic formula of the Boole-Hruščëv-Vinogradov type.

THEOREM 3.1. — *Suppose that μ is a d-pluriharmonic measure, $d \in \mathbb{Z}_+$, and μ^s is the singular part of μ. Then*

$$\lim_{y\to+\infty} y \cdot \sigma\{\zeta \in S : |C_d[\mu](\zeta)| > y\} = \|\mu^s\|/\pi.$$

Proof. — By (3.2), we have

$$\sigma\{\zeta \in S : |C_d[\mu](\zeta)| > y\} = \int_{\mathbb{C}P^{n-1}} m\{\lambda \in \mathbb{T} : |C[\mu_\xi](\lambda)| > y\}\, d\hat{\sigma}(\xi).$$

If we consider the Cauchy projection $C[\rho]$ in dimension one, then the formula under the question holds for all $\rho \in M(\mathbb{T})$ (see [HV]). Therefore, by (3.1), the limit under consideration is equal to $\|\mu^s\|/\pi$. □

Recall that $C_0(z,\zeta) = (1 - \langle z,\zeta\rangle)^{-n}$. If $d \in \mathbb{Z}_+$ is arbitrary, then

$$C_d(z,\zeta) = \sum_{k=0}^{d} P_k\left(z, \bar{z}, \langle z,\zeta\rangle, \langle \zeta, z\rangle\right) \frac{\left(\langle \zeta, z\rangle - |z|^2\right)^k}{\left(1 - \langle z,\zeta\rangle\right)^{n+k}},$$

where P_k are *polynomials* (see Theorem 3.4 in [D3]). So the singular integrals theory shows that the boundary values $C_d[\mu]^*$ exist σ-a.e. for *every* $\mu \in M(S)$. Moreover, $C_d : M(S) \to L^{1,\infty}(S)$ is a bounded operator. In particular, $y \cdot \sigma\{\zeta \in S : |C_d[f](\zeta)| > y\} \to 0$ as $y \to +\infty$ for every $f \in L^1(S)$. We write, in brief

$$C_d[L^1(S)] \subset L^{1,\infty}_0(S). \tag{3.3}$$

The latter observation and the Henkin measures technique give Theorem A. In fact, the following statement is even stronger than the second property from the definition of a singular set.

THEOREM 3.2. — *Let μ^s be the singular part of a d-plh measure and ν be a measure on S such that* $\operatorname{spec}(\nu) \subset \mathbb{Z}_+^2 \setminus \{(p,d) : p \in \mathbb{Z}_+\}$. *Then* $\mu^s \perp \nu$.

Proof. — Denote by ν^s the singular part of ν. Let $\nu^s = \nu_a^s + \nu_s^s$ be the Lebesgue decomposition with respect to μ^s.

Put $X = C_{\{(d,q):q\in\mathbb{Z}_+\}}(S)$ and $Y = C_{\mathbb{Z}_+^2\setminus\Lambda(d)}(S)$ (as above, $\Lambda(d) = \{(p,q) \in \mathbb{Z}_+^2 : (p-d)(q-d) = 0, p \geq d, q \geq d\}$). Since $\nu_a^s \ll \nu^s$ and $\nu_a^s \ll \mu^s$, Corollary 2.7 yields $\nu_a^s \in L^1(S) + X^\perp$ and $\nu_a^s \in L^1(S) + Y^\perp$.

The first inclusion and (3.3) provide $C_d[\nu_a^s] \in L_0^{1,\infty}(S)$. On the other hand, the second inclusion and (3.3) give

$$\lim_{y\to+\infty} y \cdot \sigma\{\zeta \in S : |C_d[\nu_a^s](\zeta)| > y\} = \|\nu_a^s\|/\pi.$$

Thus $\|\nu_a^s\| = 0$. □

Remark. — Note that the singular sets have to be asymmetric: for $\Lambda \subset \mathbb{Z}_+^2$, put $\overline{\Lambda} = \{(q,p) : (p,q) \in \Lambda\}$. Assume that $\Lambda \cap \overline{\Lambda}$ is finite, then Λ is not singular. Indeed, if $\mu \in M_\Lambda^s(S)$, then $\overline{\mu} \in M_{\mathbb{Z}_+^2\setminus\Lambda}^s(S)$.

Moreover, suppose that $E, F \subset \mathbb{Z}_+$ are finite, $E \cap F = \emptyset$ and $\Lambda = \Lambda(E,F) = \{(p,q) \in \mathbb{Z}_+^2 : q \in E \text{ or } p \in F\}$. Then, by Corollary 2.7, we have $M_\Lambda^s(S) \subset M_{\mathbb{Z}_+^2\setminus\Lambda}^s$.

4. Riesz products.

In this section we give a construction based on the Riesz product idea. This construction yields examples of positive singular d-pluriharmonic measures discussed above. Moreover, we can force the corresponding slice measures to have large supports. As a corollary, we obtain peak sets (for the ball algebra $A(S)$) of maximal Hausdorff dimension (see Subsection 5.3).

Recall that the classical Riesz product on the unit circle $\mathbb{T}$ is

$$\mu := \prod_{k=1}^{\infty} \left(\frac{\overline{a}_k \overline{z}^{j_k}}{2} + 1 + \frac{a_k z^{j_k}}{2} \right), \qquad z \in \mathbb{T},\ |a_k| \leq 1,\ j_{k+1}/j_k \geq 3.$$

Zygmund's theorem (see [Z]) establishes the following dichotomy:

(i) if $\sum\limits_{k=1}^{\infty} |a_k|^2 = \infty$, then μ is singular (with respect to m);

(ii) if $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$, then $\mu \ll m$ and $d\mu/dm \in L^2(\mathbb{T})$.

We are looking for Riesz type products on the complex sphere which are *not absolutely continuous*. So it is reasonable to substitute the characters z^{j_k} by polynomials of the Ryll-Wojtaszczyk type (see [RW], see also [A2] and [D3] for the non-holomorphic case):

DEFINITION. — *We say that $\{R_j\}_{j=1}^{\infty}$ is an RW-sequence (on a level $d \in \mathbb{Z}_+$ and with a constant $\delta \in (0,1)$) if $R_j \in H(j,d)$, $\|R_j\|_{L^\infty(S)} = 1$, and $\|R_j\|_{L^2(S)} \geq \delta$ for all $j \in \mathbb{N}$.*

There are two obstacles for a pluriharmonic Riesz product construction on the sphere:

1. If P is a *polynomial* and $\|P\|_{L^\infty(S)} = \|P\|_{L^2(S)}$, then $P = \text{const}$ (see [D3]). In other words, there are no RW-sequences with the constant $\delta = 1$.

2. The multiplication rule for the spherical harmonics: If $f \in H(p,q)$ and $g \in H(r,s)$, then the product fg is in $\sum\limits_{\ell=0}^{L} H(p+r-\ell, q+s-\ell)$, where $L = \min(p,s) + \min(q,r)$.

If we do not bother about the second obstacle, then we have the following "standard" analogue of the classical construction.

DEFINITION. — *Let $R = \{R_j\}_{j=1}^{\infty}$ be a Ryll-Wojtaszczyk sequence, $J = \{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$, $j_1 > d$, $j_{k+1}/j_k \geq 3$, and $a = \{a_k\}_{k=1}^{\infty}$, $|a_k| \leq 1$. The standard Riesz product $\Pi(R,J,a)$ is defined by the formal equality*

$$\Pi(R,J,a) = \prod_{k=1}^{\infty} \left(\frac{\overline{a}_k \overline{R}_{j_k}}{2} + 1 + \frac{a_k R_{j_k}}{2} \right).$$

We write $\Pi_\ell(R,J,a)$ or Π_ℓ for $\prod\limits_{k=1}^{\ell}$, and sometimes R_k for R_{j_k}.

Fix a polynomial P on S. Since $j_{k+1}/j_k \geq 3$, we have

$$\operatorname{spec}(\Pi_{\ell+q} - \Pi_\ell) \cap \operatorname{spec} P = \emptyset \quad \text{for all } q \in \mathbb{N} \text{ if } \ell \text{ is sufficiently large.}$$

Remark also that $\Pi_\ell \geq 0$ and $\|\Pi_\ell\|_{L^1(S)} = 1$. Therefore, the products $\Pi_\ell(R, J, a)\sigma$ converge weakly* to a probability measure (we use the above symbol $\Pi(R, J, a)$ for this limit).

Fix a $\zeta \in S$. Clearly the slice product $\Pi(R(\lambda\zeta), J, a)$, $\lambda \in \mathbb{T}$, is the classical Riesz product on $\mathbb{T}$ based on the pair $(\{a_k R_{j_k}(\zeta)\}, \{j_k - d\})$. Often this observation reduces a problem about standard Riesz products on the sphere to that about classical Riesz products. However, the spectrum of a standard product is quite far from being d-pluriharmonic. So we move to the main objects of the present section.

4.1. d-pluriharmonic Riesz products.

Put $PLH^2_d(S) = \{f \in L^2(S) : \operatorname{spec}(f) \subset \Lambda(d) \cup \{(0,0)\}\}$. Clearly, $f \in PLH^2_0(S)$ if and only if $P[f]$ is a plh function. Let $K = K_d : L^2(S) \to PLH^2_d(S)$ be the orthogonal projection (often we omit the index d, if there is no confusion). Given a polynomial φ on S (a symbol), recall that the corresponding $\Lambda(d)$-Hankel operator is $H_\varphi[f] = \varphi K[f] - K[\varphi f]$, $f \in L^2(S)$.

Remark that $H_\varphi : C(S) \to C(S)$ is a compact operator (see [D3]; we used this property in the proof of Corollary 2.6). Therefore

$$\|H_\varphi f_j\|_{C(S)} \to 0 \quad \text{if} \quad f_j \in C(S), \|f_j\|_{C(S)} \leq 1, \quad \text{and } f_j \to 0 \text{ weakly in } L^2(S) \tag{4.1}$$

(compare with Proposition 2.5).

The last observation leads to the notion of a d-pluriharmonic Riesz product. Our definition is a variant of the L^p-argument, $0 < p < 1$, given in [A1]; a similar construction (based on a bounded orthonormal basis in the Hardy class $H^2(B)$) is also outlined in [B2].

DEFINITION. — *Let $\{R_j\}_{j=1}^\infty$ be an RW-sequence on the level d and $a = \{a_k\}_{k=1}^\infty \subset \mathbb{D}$ (note that $|a_k| = 1$ is not allowed now and the index set J is not given a priori).*

Step 1. — Fix $j_1 > d$ and put $\varphi_1 = 1 + \operatorname{Re}(a_1 R_{j_1}) > 0$.

Step $k+1$. — Suppose that a d-plh polynomial φ_k, $\varphi_k > 0$, is constructed. For $\ell \geq 3j_k$, put

$$\begin{aligned}\varphi_{k+1}(\ell) :&= K\left(\varphi_k[1 + \operatorname{Re}(a_{k+1}R_\ell)]\right)\\ &= \varphi_k[1 + \operatorname{Re}(a_{k+1}R_\ell)] - H_{\varphi_k}[\operatorname{Re}(a_{k+1}R_\ell)].\end{aligned}$$

Since $\varphi_k > 0$, $\|R_\ell\|_{C(S)} \leq 1$, and $R_\ell \to 0$ weakly in $L^2(S)$ as $\ell \to \infty$, we have $\varphi_{k+1}(\ell) > 0$ for all $\ell \in \mathbb{N}$ large enough. Fix such an ℓ and define $j_{k+1} = \ell$, $\varphi_{k+1} = \varphi_{k+1}(\ell)$.

Now the induction construction proceeds.

As in the standard case, given a polynomial P, $\operatorname{spec}(\varphi_{k+\ell} - \varphi_k) \bigcap \operatorname{spec} P = \emptyset$ for all $\ell \in \mathbb{N}$ if k is sufficiently large. We have also $\|\varphi_k\|_{L^1(S)} = 1$, $\varphi_k > 0$, so $\varphi_k \sigma \xrightarrow{w^*} \pi$ for some probability measure π. The measure $\pi = \pi(R, J, a)$ (here $J = \{j_k\}_{k=1}^\infty$) is said to be a d-plh (or just pluriharmonic) product based on the Riesz pair (R, a).

Given a polynomial sequence $R = \{R_j\}_{j=1}^\infty$ and a sequence of unitary operators (on $\mathbb{C}^n$) $U = \{U_j\}_{j=1}^\infty$, we put $R \circ U = \{R_j \circ U_j\}_{j=1}^\infty$.

THEOREM 4.1. — *Let (R, a) be a Riesz pair. Then*

(i) *if $\sum\limits_{k=1}^{\infty} |a_k|^2 < \infty$, then all pluriharmonic Riesz products based on (R, a) are absolutely continuous with respect to σ;*

(ii) *if $\sum\limits_{k=1}^{\infty} |a_k|^2 = \infty$, then there exist an index set $J \subset \mathbb{N}$, a sign sequence $\beta = \{\beta_k\}_{k=1}^\infty$, $\beta_k \in \{\pm 1\}$, and a sequence $U = \{U_j\}_{j=1}^\infty$ of unitary operators such that $\pi(R \circ U, J, \beta a) \perp \sigma$.*

Proof of part (i). — Given a triple (R, J, a), $a \in \ell^2$, we claim that $\pi(R, J, a) \in L^2(S)$. Indeed, let $\Pi(R, J, a)$ be the corresponding standard Riesz product. Note that $\varphi_{k+1} = \varphi_k + K\left[\varphi_k \operatorname{Re}(a_{k+1} R_{j_{k+1}})\right]$, so we have the estimate

$$\begin{aligned} \|\varphi_{k+1}\|_{L^2(S)}^2 &= \|\varphi_k\|_2^2 + \|K\left[\varphi_k \operatorname{Re}(a_{k+1} R_{j_{k+1}})\right]\|_2^2 \\ &\leq \|\varphi_k\|_2^2 + \|\varphi_k \operatorname{Re}(a_{k+1} R_{j_{k+1}})\|_2^2 \leq \|\Pi_{k+1}\|_2^2. \end{aligned}$$

On the other hand, let Π_{pq} denote the $H(p,q)$-projection of Π. Then

$$\begin{aligned} \|\Pi_{k+1}\|_2^2 \leq \|\Pi\|_2^2 = \sum_{(p,q)\in\mathbb{Z}_+^2} \|\Pi_{pq}\|_2^2 &\leq 1 + \sum_{t=1}^{\infty} \frac{1}{t!} \sum_{k_\ell \neq k_s \in \mathbb{N}} \left\| \prod_{\ell=1}^{t} a_{k_\ell} R_{k_\ell} \right\|_2^2 \\ &\leq 1 + \exp\left(\sum_{k=1}^{\infty} |a_k|^2 \right) \end{aligned}$$

which proves (i). □

To establish part (ii), we will need a known result about Fourier series of the measures with a lacunary spectrum. Given a measure $\mu \in M(\mathbb{T})$, put

$s_k[\mu](\lambda) = \sum\limits_{j=-k}^{k} \hat{\mu}(j)\lambda^j$, $\lambda \in \mathbb{T}$ (the k-th Fourier partial sum). Define also

$$\mathcal{D}\mu(\lambda) = \lim_{|\theta|\to 0} \frac{\mu}{m}(\lambda, \lambda e^{i\theta}), \quad \lambda \in \mathbb{T},$$

if the latter limit exists. Recall that $\mathcal{D}\mu(\lambda) = f(\lambda)$ for m-a.e. $\lambda \in \mathbb{T}$, where $f\sigma$ is the absolutely continuous part of μ.

LEMMA 4.2 (see [Z], Chapter 3, Theorems 8.1 and 1.27). — *Assume that* $\mu \in M(\mathbb{T})$, $j_k \nearrow +\infty$, *and* $\hat{\mu}(j) = 0$ *for all* $|j| \in (j_k, 2j_k]$, $k \in \mathbb{N}$. *Then* $s_{j_k}[\mu](\lambda) \to \mathcal{D}\mu(\lambda)$ *for* m*-a.e.* $\lambda \in \mathbb{T}$ *as* $k \to \infty$.

Now, we are ready to investigate the singular pluriharmonic Riesz products.

Proof of Theorem 4.1, part (ii). — We have $\sum |a_k|^2 = \infty$ and proceed as in the definition. So on the step $k+1$ we assume that a d-plh polynomial φ_k, $\varphi_k > 0$, is constructed. Let $\ell \geq 6j_k$.

Given a polynomial $h \in H(p,q)$, $p \neq q$, we have $\|\operatorname{Re} h\|_{L^2(S)} = \|\operatorname{Im} h\|_{L^2(S)}$ because $\|\operatorname{Re} h_\zeta\|_{L^2(\mathbb{T})} = \|\operatorname{Im} h_\zeta\|_{L^2(\mathbb{T})}$ for all $\zeta \in S$. Therefore, by the RW-property, we can assume that

$$\int_S [\operatorname{Re}(a_{k+1}R_\ell)]^2 \, d\sigma \geq \gamma |a_{k+1}|^2$$

with $\gamma > 0$. Given $f, g \in L^1_{\mathbb{R}}(S)$, we have

$$\int_{\mathcal{U}} \int_S f \cdot (g \circ U) \, d\sigma \, dU = \int_S f \, d\sigma \int_S g \, d\sigma,$$

therefore, we can choose $U^\ell_{k+1} \in \mathcal{U}$ such that

$$\int_S \varphi_k^{1/2} \left[\operatorname{Re}(a_{k+1}R_\ell \circ U^\ell_{k+1})\right]^2 d\sigma \geq \int_S \varphi_k^{1/2} d\sigma \int_S \left[\operatorname{Re}(a_{k+1}R_\ell \circ U^\ell_{k+1})\right]^2 d\sigma.$$

Remark that $(1+x)^{1/2} + (1-x)^{1/2} \leq 2(1 - x^2/8)$ if $|x| \leq 1$, thus

$$\begin{aligned}
\int_S \Big(\varphi_k^{1/2} &\left[1 + \operatorname{Re}\left(a_{k+1}R_\ell \circ U^\ell_{k+1}\right)\right]^{1/2} \\
&\qquad + \varphi_k^{1/2} \left[1 - \operatorname{Re}\left(a_{k+1}R_\ell \circ U^\ell_{k+1}\right)\right]^{1/2}\Big) \, d\sigma \\
&\leq 2 \int_S \varphi_k^{1/2} \left(1 - \frac{\left[\operatorname{Re}\left(a_{k+1}R_\ell \circ U^\ell_{k+1}\right)\right]^2}{8}\right) d\sigma \\
&\leq 2 \left(1 - \frac{\gamma |a_{k+1}|^2}{8}\right) \int_S \varphi_k^{1/2} \, d\sigma.
\end{aligned}$$

So we take $\beta^\ell_{k+1} \in \{\pm 1\}$ to ensure that

$$\int_S \varphi_k^{1/2}\left[1 + \mathrm{Re}\left(\beta^\ell_{k+1} a_{k+1} R_\ell \circ U^\ell_{k+1}\right)\right]^{1/2} d\sigma \le \left(1 - \frac{\gamma |a_{k+1}|^2}{8}\right)\int_S \varphi_k^{1/2} d\sigma.$$

Define

$$\varphi_{k+1}(\ell) = K\left(\varphi_k[1 + \mathrm{Re}(\beta^\ell_{k+1} a_{k+1} R_\ell \circ U^\ell_{k+1})]\right). \tag{4.2}$$

Then $\|\varphi_{k+1}(\ell) - \varphi_k[1 + \mathrm{Re}(\beta^\ell_{k+1} a_{k+1} R_\ell \circ U^\ell_{k+1})]\|_{C(S)} \to 0$ as $\ell \to \infty$. Therefore, for all $\ell \in \mathbb{N}$ large enough, we have $\varphi_{k+1}(\ell) > 0$ and

$$\int_S \varphi_{k+1}^{1/2}(\ell)\, d\sigma \le \left(1 - \frac{\gamma |a_{k+1}|^2}{16}\right)\int_S \varphi_k^{1/2}\, d\sigma. \tag{4.3}$$

Fix such an ℓ and define $j_{k+1} = \ell$, $U_{j_{k+1}} = U^\ell_{k+1}$, $\beta_{k+1} = \beta^\ell_{k+1}$, and $\varphi_{k+1} = \varphi_{k+1}(\ell)$. Now the induction construction proceeds.

We claim that the resulting measure $\pi(R \circ U, J, \beta a)$ is singular. First, remark that π is a positive d-plh measure, so the slices π_ξ are defined for all $\xi \in \mathbb{C}P^{n-1}$. In fact, we can use also the Riesz product nature of π to obtain the same family of slice measures. Indeed, for $\zeta \in S$, put $(\varphi_\zeta)_\ell(\lambda) = \varphi_\ell(\lambda\zeta)$, $\lambda \in \mathbb{T}$, $\ell \in \mathbb{N}$. Then the sequence $\{(\varphi_\zeta)_\ell m\}_{\ell=1}^\infty$ converges weakly* in $M(\mathbb{T})$ to a probability measure.

Fix a $\xi \in \mathbb{C}P^{n-1}$. We have $j_{k+1} \ge 6j_k$ in the above construction, therefore, $\hat{\pi}_\xi(j) = 0$ for all $|j| \in (2j_k, 4j_k]$. Thus, Lemma 4.2 gives $(\varphi_\xi)_k(\lambda) = s_{2j_k}[\pi_\xi](\lambda) \to \mathcal{D}\pi_\xi(\lambda)$ for m-a.e. $\lambda \in \mathbb{T}$.

On the other hand, $\sum |a_k|^2 = \infty$, thus $\varphi_k \to 0$ in $L^{1/2}(S)$ by (4.3). Since $\varphi_k(\zeta)$ converges for σ-a.e. $\zeta \in S$, we obtain $\varphi_k \to 0$ σ-a.e. In other words, for $\hat{\sigma}$-a.e. $\xi \in \mathbb{C}P^{n-1}$, $\mathcal{D}\pi_\xi(\lambda) = 0$ for m-a.e. $\lambda \in \mathbb{T}$. Therefore π_ξ is singular for almost all $\xi \in \mathbb{C}P^{n-1}$. So π is singular. □

In the next subsections we obtain singular d-plh products with some special properties. In the corresponding constructions we always assume that the following restrictions hold.

General restrictions on the step $k+1$. — We assume that $a \notin \ell^2$, $a_k \ne 0$, and proceed as in the plh definition and in Theorem 4.1. Namely, we choose unitary operators U^ℓ_{k+1} and signs β^ℓ_{k+1} such that the estimate (4.3) holds for the d-plh polynomial $\varphi_{k+1}(\ell)$ defined by (4.2). Often we abuse the notation and omit these auxiliary sequences U and β in the corresponding equalities.

4.2. Small $H(p,q)$-projections.

As indicated in the title, we are going to control the size of π_{pd} and π_{dp}, $p \in \mathbb{Z}_+$.

Put, as in the definition, $\varphi_0 = 1$, $\varphi_1 = \overline{a}_1\overline{R}_{j_1}/2 + 1 + a_1 R_{j_1}/2$. Let $k \in \mathbb{N}$ and let $\varphi_k = \sum\limits_{p\in\mathbb{Z}} f_k(p)$ be the homogeneous expansion (here $f_k(p) \in H(p,d)$ if $p - d \in \mathbb{Z}_+$, and $f_k(p) \in H(d,-p)$ if $-p-d \in \mathbb{Z}_+$). We suppose, as induction hypothesis, that $\|f_k(p)\|_{C(S)} \le |a_k| \le 1$, $(p,d) \in \operatorname{spec}(\varphi_k) \setminus \operatorname{spec}(\varphi_{k-1})$ (clearly, this estimate holds for $k = 1$).

Step $k+1$. — Define, as in (4.2),

$$\varphi_{k+1}(\ell) = \varphi_k + K\left[\varphi_k(a_{k+1}R_\ell + \overline{a}_{k+1}\overline{R}_\ell)\right]/2$$

(we abuse the notation and omit U and β). Since the $H(p,q)$-projections of φ_k are symmetric with respect to the diagonal $\{p = q\}$, it is sufficient to investigate the projections $K\left[a_{k+1}f_k(p)R_\ell\right]/2$, $p \in \mathbb{Z}$. By (4.1), for all ℓ sufficiently large, we have

$$\|a_{k+1}K\left[f_k(p)R_\ell\right]/2\|_{C(S)} \le \|a_{k+1}f_k(p)R_\ell\|_{C(S)} \le |a_{k+1}| \tag{4.4}$$

for all $p \in \mathbb{Z}$ (of course $|p| \le 3j_k/2$). We fix ℓ so large that (4.4) and the "general restrictions on the step $k+1$" hold. By definition $j_{k+1} = \ell$ and $\varphi_{k+1} = \varphi_{k+1}(\ell)$, thus, for $0 \ge p \ge -3j_k/2$,

$$\|f_{k+1}(j_{k+1}+p+d)\|_{C(S)} = \|a_{k+1}K\left[f_k(p)R_{j_{k+1}}\right]/2\|_{C(S)}.$$

(If $p > 0$, then the above equality holds for $f_{k+1}(j_{k+1}+p-d)$.) Therefore $\|f_{k+1}(q)\|_{C(S)} \le |a_{k+1}| \le 1$, $(q,d) \in \operatorname{spec}(\varphi_{k+1}) \setminus \operatorname{spec}(\varphi_k)$, and the induction construction works. □

We take the projection on $PLH^2_d(S)$ on every induction step, therefore, the slice measures are not exactly the classical Riesz products (as it was in the standard construction). Nevertheless, (4.4) guarantees that the situation is sufficiently close to the classical one.

In particular, take a sequence a such that $\sum\limits_{k=1}^{\infty} |a_k|^2 = \infty$ and $a_k \to 0$, then the above modification yields an S-version of Menchoff's example on $\mathbb{T}$ (see [M]).

COROLLARY 4.3. — *There exists a probability singular d-plh measure $\mu \in M(S)$ such that $\|\mu_{pq}\|_{L^\infty(S)} \to 0$ as $p+q \to \infty$.*

Remark. — Put $a_k = 1/2$ for all $k \in \mathbb{N}$, then it is possible to ensure $\|f_k(j_k)\|_2 \geq \gamma$ for some $\gamma > 0$. So we obtain a probability singular d-plh measure $\mu \in M(S)$ such that $\|\mu_{j_k d}\|_{L^\infty(S)} \geq \|\mu_{j_k d}\|_{L^2(S)} \geq \gamma$.

If we consider arbitrary measures, then define $\mu = \delta_\zeta$ (the Dirac measure at the point $\zeta \in S$). Recall that $K_{pq}(z, w)$ denotes the reproducing kernel in $H(p, q)$, therefore $\mu_{pq}(z) = K_{pq}(z, \zeta)$ and the family $\{\|\mu_{pq}\|_{L^2(S)} : (p, q) \in \mathbb{Z}_+^2\}$ even is not bounded.

4.3. Maximal Hausdorff dimension and symmetric measures.

Let π be a singular pluriharmonic Riesz product. Then the slices π_ξ are singular for almost all $\xi \in \mathbb{C}P^{n-1}$. We are going to show that this fact is compatible with properties of an opposite nature. More precisely, we obtain π_ξ which are supported by sets of maximal Hausdorff dimension. Moreover, we construct a singular π with uniformly symmetric slices.

The following lemma is well-known (see, for example, [P], Lemma 2.1).

LEMMA 4.4. — *Suppose that $I \subset \mathbb{T}$ is an interval and $m(I)$ is the Lebesgue measure of I. Let μ be the classical Riesz product based on a pair $(\{j_k\}_{k=1}^\infty, a)$, then*

$$|\mu(I) - m(I)| \leq 2\|a\|_\infty / j_1.$$

Since the d-plh Riesz products *are not products*, our variant of the above lemma is more sophisticated.

For $\pi = \pi(R, J, a)$, define $\pi^{(p)} = \pi/\varphi_p$, where φ_p, $\varphi_p > 0$, is the polynomial from the plh definition. Put also $g_k = 1 + \operatorname{Re}(\beta_k a_k R_{j_k} \circ U_{j_k})$ and $\Pi^{(p)} = \prod\limits_{k=p+1}^{\infty} g_k$ (a tail of the standard Riesz product).

LEMMA 4.5. — *Let $a \notin \ell^2$, $\|a\|_\infty < 1/4$, and $\varepsilon_p \in (0, 1)$. Then there exists a singular d-plh Riesz product $\pi = \pi(R, \{j_k\}_{k=1}^\infty, a)$ such that the following property holds for every $\xi \in \mathbb{C}P^{n-1}$: Let $I \subset \mathbb{T}$ be an interval, then*

$$|\pi_\xi^{(p)}(I) - m(I)| \leq \varepsilon_p m(I) + 1/j_{p+1}$$

for all $p \in \mathbb{Z}_+$.

Proof. — For $(q,p) \in \mathbb{Z}_+^2$, fix $\varepsilon(q,p) > 0$ such that $\sum\limits_{q=p+1}^{\infty} \varepsilon(q,p) < \varepsilon_p$ for all $p \in \mathbb{Z}_+$.

We proceed as in the general scheme. On the step $k+1$ we use (4.1) and impose the extra restrictions (recall that $\varphi_p > 0$):

$$\left\| \frac{\varphi_{k+1} - \varphi_k g_{k+1}}{\varphi_p} \right\|_\infty < \varepsilon(k+1,p) \qquad \text{for all } p \le k. \tag{4.5}$$

We claim that the resulting measure $\pi = \pi(R \circ U, J, \beta a)$ satisfies the conditions of the lemma.

Indeed

$$\left|\pi_\xi^{(p)}(I) - m(I)\right| \le \left|\pi_\xi^{(p)}(I) - \Pi_\xi^{(p)}(I)\right| + \left|\Pi_\xi^{(p)}(I) - m(I)\right| \overset{\text{def}}{=} X_1 + X_2.$$

Since $\Pi_\xi^{(p)}$ is a classical Riesz product, Lemma 4.4 yields the estimate $2X_2 \le 1/j_{p+1}$. Now we claim that, for all p and k,

$$X(p,k) \overset{\text{def}}{=} \int_I \left| \frac{\varphi_{p+k}}{\varphi_p} - \prod_{\ell=1}^{k} g_{p+\ell} \right| dm \le \varepsilon_p(m(I) + (2j_{p+1})^{-1}).$$

(Note that the above estimate gives $X_1 \le \varepsilon_p(m(I) + (2j_{p+1})^{-1})$ and finishes the proof.)

We have

$$X(p,k) \le \sum_{\ell=0}^{k-1} \int_I \left| \frac{\varphi_{p+k-\ell} - \varphi_{p+k-\ell-1} g_{p+k-\ell}}{\varphi_p} \right| \prod_{j=0}^{\ell-1} g_{p+k-j}\, dm$$

$$\overset{\text{def}}{=} \sum_{\ell=0}^{k-1} \int_I Y_1 Y_2 \, dm.$$

By (4.5), we have $\|Y_1\|_\infty \le \varepsilon(p+k-\ell, p)$. Since Y_2 is a finite Riesz product, Lemma 4.4 yields $\int_I Y_2\, dm \le m(I) + (2j_{p+1})^{-1}$. By the definition of $\varepsilon(q,p)$, the proof is complete. $\square$

Now, we give a direct proof of Theorem B.

Proof of Theorem B. — On the step $k+1$ we repeat the construction of Lemma 4.5 with $\varepsilon_p = 1/4$, $p \in \mathbb{Z}_+$. Moreover, we ensure that

$$2^{k+1} > \varphi_{k+1} > 2^{-k-1} \tag{4.6}$$

(we can apply (4.1) since $|a_{k+1}| < 1/4$);

$$k(\log j_k)^{-1} \to 0 \text{ as } k \to \infty. \tag{4.7}$$

We claim that the Riesz product $\pi(R \circ U, J, \beta a)$ solves the problem.

Fix a $\xi \in \mathbb{C}P^{n-1}$ and put $\varphi_k = (\varphi_k)_\xi$, $\pi = \pi_\xi$. Suppose that $I \subset \mathbb{T}$ is an interval, $2/j_{k+1} \leq m(I) \leq 2/j_k$.

By definition $\pi(I) = \int_I \varphi_k \, d\pi^{(k)}$, so, by (4.6)

$$\left|\log \pi(I) - \log \pi^{(k)}(I)\right| \leq \max_{\zeta \in \mathbb{T}} |\log \varphi_k(\zeta)| \leq \text{const } k.$$

On the other hand, $m(I) j_{k+1} \geq 2$, therefore, Lemma 4.5, with $\varepsilon_k = 1/4$, provides $\left|\pi^{(k)}(I) - m(I)\right| \leq 3m(I)/4$. Hence

$$\left|\log \pi^{(k)}(I) - \log m(I)\right| \leq \text{const}\,.$$

In sum, we have $|\log \pi(I) - \log m(I)| \leq \text{const } k$. Recall that $1/m(I) \geq j_k/2$, thus, by (4.7)

$$\left|\frac{\log \pi(I)}{\log m(I)} - 1\right| \leq \frac{\text{const } k}{|\log m(I)|} \leq \frac{\text{const } k}{\log(j_k/2)} \to 0 \text{ as } k \to \infty.$$

So (see, for example, [Bi]) $\pi = \pi_\xi(R \circ U, J, \beta a)$ is supported by a set of Hausdorff dimension 1. $\square$

Finally, we obtain even more interesting example and construct a singular symmetric plh product (an analogue of the symmetric classical Riesz product obtained in [AAN]). Such a measure promises multidimensional generalizations of the results given in [AAN].

DEFINITION. — *A positive finite measure $\mu \in M(\mathbb{T})$ is said to be* symmetric *if*

$$\frac{\mu(I_+)}{\mu(I_-)} \to 1 \qquad \text{as } m(I_+) = m(I_-) \to 0,$$

where I_+ and I_- are adjacent intervals.

It is well-known and easy to see that every symmetric measure $\mu \in M(\mathbb{T})$ lives on a set of Hausdorff dimension 1, so we have Theorem B again.

THEOREM 4.6. — *There exists a probability singular d-plh measure μ such that μ_ξ, $\xi \in \mathbb{C}P^{n-1}$, are uniformly symmetric.*

Proof. — Fix a sequence $a \notin \ell^2$ such that $\|a\|_\infty < 1/4$ and $a_k \to 0$ (the latter property is crucial).

On the step $k+1$ of the induction construction we impose the restrictions from Subsection 4.2 and Lemma 4.5 with $\varepsilon_p \to 0$ (i.e. the general restrictions, (4.4), and (4.5) hold). Moreover, we ensure that

$$1 + 2|a_{k+1}| \geq \varphi_{k+1}/\varphi_k \geq 1 - 2|a_{k+1}| \tag{4.8}$$

(we apply (4.1));

$$7^k j_k / j_{k+1} \to 0 \text{ as } k \to \infty. \tag{4.9}$$

Our example is the Riesz product $\pi(R \circ U, J, \beta a)$.

Fix a $\xi \in \mathbb{C}P^{n-1}$ and put $\varphi_k = (\varphi_k)_\xi$, $\pi = \pi_\xi$. By (4.4), we have $|\hat{\pi}(\ell)| \leq 1$ for all $\ell \in \mathbb{Z}$, therefore

$$\begin{aligned} &\sum_{\ell \in \mathbb{Z}} |\hat{\varphi}_{k-1}(\ell)| \leq \text{const } 3^k \quad \text{and} \\ &\max_{z \in \mathbb{T}} |\varphi'_{k-1}(z)| \leq \text{const } 3^k j_{k-1}. \end{aligned} \tag{4.10}$$

Now we suppose that $I_+ \cup I_- = I$, $1/j_{k+1} \leq m(I) \leq 1/j_k$.

Since $m(I) \leq 1/j_k$, the estimates (4.8–4.10) provide

$$\begin{aligned} \max_{z,w \in I} |\log \varphi_{k-1}(z) - \log \varphi_{k-1}(w)| &\leq m(I) \frac{\max_I |\varphi'_{k-1}|}{\min_I |\varphi_{k-1}|} \\ &\leq \text{const } \frac{1}{j_k} \frac{3^k j_{k-1}}{2^{-k}} \to 0. \end{aligned}$$

Since $a_k \to 0$, this fact and (4.8) yield

$$\max_I \log \varphi_{k+1} - \min_I \log \varphi_{k+1} \to 0 \text{ as } k \to \infty. \tag{4.11}$$

On the other hand, $m(I) j_{k+1} \geq 1$, so Lemma 4.5 and (4.9) give

$$\left| \pi^{(k+1)}(I) - m(I) \right| \leq m(I)\varepsilon_{k+1} + 1/j_{k+2} = o(m(I)).$$

Recall that $\pi(I) = \int_I \varphi_{k+1}\, d\pi^{(k+1)}$, therefore, we obtain

$$\pi(I_\pm) = \varphi_{k+1}(\zeta_\pm)(m(I) + o(m(I))), \quad \text{where } \zeta_\pm \in I_\pm.$$

Thus, by (4.11), we have

$$\left| \frac{\pi(I_+)}{\pi(I_-)} - 1 \right| = \left| \frac{m(I)\varphi_{k+1}(\zeta_+)}{m(I)\varphi_{k+1}(\zeta_-)} - 1 \right| + o(1) \to 0$$

as $m(I) \to 0$. □

5. PEAK, INTERPOLATION AND NULL SETS

We apply the results about $M_\Lambda(S)$ to the study of the sets mentioned in the title of the section. As usual, we identify $X \subset L^1(S)$ and $X(B) = \{P[f] : f \in X\}$.

DEFINITIONS. — *Let $X \subset C(S)$ be a closed subspace and $K \subset S$ be a compact set.*

We say that K is an (N)-set (a null set for X) if $|\mu|(K) = 0$ for all $\mu \in X^\perp$ (as above, $X^\perp = \{\mu \in M(S) : \int_S f\, d\mu = 0$ for all $f \in X\}$).

K is a (PI)-set (a peak interpolation set) if given a $g \in C(K)$, $g \not\equiv 0$, there exists $f \in X(B)$ with $f|_K = g$ and $|f(z)| < \|g\|_K$ for $z \in \overline{B} \setminus K$.

K is a (P)-set (a peak set) if the previous property holds for $g \equiv 1$.

K is an (I)-set (an interpolation set) if $X|_K = C(K)$.

In the classical case of the ball algebra $A(S) = \{f \in C(S) : P[f]$ is a holomorphic function$\}$ the above properties are known to be equivalent (see [Ru], Chapter 10). Moreover, a smooth manifold $M \subset S$ has these properties with respect to $A(S)$ if and only if M is complex tangential.

Other classical $\mathcal{U}$-invariant space is $PHC(S) := \{f \in C(S) : P[f]$ is a pluriharmonic function$\}$. $PHC(S)$ is not an algebra, so the situation is more complicated. For example, given a simple smooth curve $\gamma \subset S$, the restriction $PHC(S)|_\gamma$ is a closed subspace of finite codimension in $C(\gamma)$ (in other words, γ is a set of *almost pluriharmonic interpolation*, see [AB] and references therein). On the other hand, a smooth manifold $M \subset S$ of dimension at least two can be a set of almost pluriharmonic interpolation only if M is complex tangential (see [RS]). Finally, the Henkin-Cole-Range theorem implies that (N)-sets for $PHC(S)$ and $A(S)$ coincide (see [D3]).

The principal objects of the present section are the spaces $A_{k\ell}(S) = \{f \in C(S) : \mathrm{spec}(f) \subset \{(p,q) \in \mathbb{Z}^2_+ : k \geq q \geq \ell\}\}$, $k, \ell \in \mathbb{Z}_+$, $k \geq \ell$. In particular, $A_{00}(S)$ is the ball algebra.

5.1. Equivalent properties for $A_{k\ell}(S)$.

The spectrums of $A_{k\ell}(S)$ and $A(S)$ have a similar geometry, therefore, it is reasonable to expect that $A_{k\ell}(S)$ inherit some properties of $A(S)$. In particular, $A_{k0}(S)$ are modules over the algebra $A(S)$ and permit a natural description in terms of iterated $\overline{\partial}$.

However, it is easy to see that $(P) \not\Rightarrow (N)$ for $A_{k0}(S)$, $k \geq 1$. Indeed, put $e_1 = (1, 0, \dots, 0) \in S$ and $\mathbb{T}_1 = \{\lambda e_1 : |\lambda| = 1\}$, then $\mathbb{T}_1 \in (P)$ for $A_{k0}(S)$ (consider the function $f(z) = (1 + z_1\overline{z}_1 - z_2\overline{z}_2)/2$). On the other hand, $\mathbb{T}_1 \notin (N)$ for $A_{k0}(S)$.

So we introduce a variant of the property (P).

DEFINITION. — *Let $X \subset C(S)$ and $K \subset S$. We write $K \in (SP)$ (a strong peak set) if there exists a sequence $\{f_j\}_{j=1}^{\infty} \subset X(B)$ such that $f_j\,|_K = 1$, $|f| < 1$ on B, and $1 > |f_j| \to 0$ on $S \setminus K$.*

Remark. — By Bishop's and Glicksberg's theorems, we have $(N) \Rightarrow (SP)$ for any $X \subset C(S)$ (see [D3]). Note that there exists a unitarily invariant function algebra A with $(SP) \not\Rightarrow (N)$. Indeed, put $A = C_\Delta(S)$, where $\Delta = \{(p, q) \in \mathbb{Z}_+^2 : p \geq q\}$, then $\mathbb{T}_1 \in (SP) \setminus (N)$ for A.

Now, we have the following equivalences.

PROPOSITION 5.1. — *Let $k, \ell \in \mathbb{Z}_+$, $k \geq \ell$, then $(N) \Leftrightarrow (SP)$ for $A_{k\ell}(S)$; moreover, $(N) \Leftrightarrow (PI) \Leftrightarrow (I) \Leftrightarrow (SP)$ for $A_{k0}(S)$.*

Proof. — Let $K \in (SP)$ for $A_{k\ell}(S)$. We claim that $K \in (N)$.

First, assume that $\sigma(K) > 0$. Note that $K \neq S$, so we have $(1 - f)\,|_K = 0$ and $1 - f \not\equiv 0$ for some $f \in A_{k\ell}(S)$. Since $\operatorname{spec}(1 - f) \subset \{(p, q) \in \mathbb{Z}_+^2 : k \geq q\}$, the inverse part of the F. and M. Riesz type theorem (see [Br]) yields $\sigma \ll (1 - f)\sigma$. A contradiction.

Now $\sigma(K) = 0$. By the (SP)-definition, take a sequence $\{f_j\}_{j=1}^{\infty} \subset A_{k\ell}(S)$ such that $f_j = 1$ on K and $1 \geq |f_j| \to 0$ on $S \setminus K$. Let $\mu \in A_{k\ell}(S)^\perp$ and μ_s be the singular part of μ. By Corollary 2.7, there exists $g \in L^1(\sigma)$ such that $|\mu_s| - g\sigma \in A_{k\ell}(S)^\perp$. Since $f_j \to 0$ σ-a.e., we have

$$\int_S f_j\, d|\mu_s| = \int_S f_j g\, d\sigma \to 0 \quad \text{as } j \to \infty.$$

On the other hand, $\int_S f_j\, d|\mu_s| \to |\mu_s|(K)$, thus $|\mu_s|(K) = 0$ and $|\mu|(K) = 0$. In other words $(SP) \Leftrightarrow (N)$ for $A_{k\ell}(S)$.

Since the implications $(N) \Leftrightarrow (PI) \Leftrightarrow (I)$ for $A_{k0}(S)$ are established in [D2], the proof is complete. □

5.2. Λ-Cauchy transforms of measures on smooth curves.

In this subsection we use the approach of Nagel [N] to characterize the smooth (N)-sets for $A_{k0}(S)$. This method is based on the investigation of the Cauchy type integrals.

For $\Lambda \subset \mathbb{Z}_+^2$, denote by $K_\Lambda(z,\zeta)$ the reproducing kernel for the Hilbert space $L^2_\Lambda(B)$. Clearly, $K_\Lambda(z,\cdot) \in C(S)$, $z \in B$, therefore, the Λ-*Cauchy transform*

$$K_\Lambda[\mu](z) = \int_S K_\Lambda(z,\zeta)\, d\mu(\zeta)$$

is defined for all $\mu \in M(S)$.

The following observation is standard.

LEMMA 5.2. — *Suppose that $K_\Lambda[\mu] \in L^1_\Lambda(B)$. Then $\mu - h\sigma \in C_\Lambda(S)^\perp$ for some $h \in L^1(\sigma)$.*

DEFINITION. — *Let $\phi : [0,1] \to S$ be a C^1 curve. We say that ϕ is nowhere complex tangential (NCT) if there exists $\varepsilon > 0$ such that*

$$|\langle \phi(t), \phi'(t)\rangle| \geq \varepsilon \quad \text{for all } t \in [0,1].$$

Remark. — The tangent to a curve ϕ is in the complex tangent space to S at $\phi(t)$ if and only if $\langle \phi(t), \phi'(t)\rangle = 0$.

A smooth manifold $M \subset S$ is said to be complex tangential if the real tangent space to M at ζ is in the complex tangential space to S at ζ for all $\zeta \in M$. Note that if M is not complex tangential, then there exists a NCT curve $\phi : [0,1] \to M$.

LEMMA 5.3. — *Let $E, F \subset \mathbb{Z}_+$, $\max\{E \cup F\} = k$, and $\Lambda = \Lambda(E,F) = \{(p,q) \in \mathbb{Z}_+^2 : q \in E$ or $p \in F\}$ (a finite union of rays). Let $g \in C_0^{k+1}[0,1]$ be a function with compact support on $(0,1)$ and ϕ be a nowhere complex tangential C^{k+2} curve. Suppose that a measure $\mu \in M(S)$ is defined by the equation*

$$\int_S f\, d\mu = \int_0^1 f(\phi(t))g(t)\, dt \quad \text{for } f \in C(S).$$

Then $K_\Lambda[\mu] \in L^1_\Lambda(B)$.

Proof. — Given an $x \in [0,1]$, there are neighborhoods $U_x \subset [0,1]$ of x and $V_x \subset S$ of $\phi(x)$ such that if $t \in \overline{U_x}$ and $\zeta \in V_x$, then $\phi(t) \in V_x$

and $|\langle\zeta,\phi'(t)\rangle| \geq \varepsilon/2$. Let $\{U_j\}_{j=1}^N$ be a finite subcover of $\{U_x\}$, let $V_j \subset S$ correspond to U_j, and let $\{h_j\}_{j=1}^N$ be a C^∞ partition of unity subordinate to $\{U_j\}_{j=1}^N$. Put $g_j(t) = h_j(t)g(t)$, $t \in [0,1]$.

We consider the case $\Lambda = \Lambda(\{k\},\emptyset)$ (the only ray). Define $F = K_\Lambda[\mu]$, then

$$F(r\zeta) = \sum_{j=1}^N \int_0^1 K_\Lambda(r\zeta,\phi(t))g_j(t)\,dt, \quad \zeta \in S,\ 0 \leq r < 1.$$

It is sufficient to verify that

$$\sup_{0\leq r<1} \int_S |F(r\zeta)|^p\,d\sigma(\zeta) < \infty \quad \text{for some } p > 1.$$

(Then $F = P[F^*]$ for some $F^* \in L^p_\Lambda(S)$, in other words, $F \in L^p_\Lambda(B) \subset L^1_\Lambda(B)$.)

Recall that

$$K_\Lambda(z,w) = \sum_{\ell=0}^k P^0_\ell\left(z,\overline{z},\langle z,w\rangle,\langle w,z\rangle\right)\frac{\left(\langle w,z\rangle - |z|^2\right)^\ell}{\left(1-\langle z,w\rangle\right)^{n+\ell}},$$

where P^0_ℓ are polynomials (we used this fact in §3).

If $\zeta \in S\backslash V_j$, then $K_\Lambda(r\zeta,\phi(t))g_j(t)$ is bounded uniformly with respect to $r \in [0,1)$ because $\phi(\overline{U_j}) \subset V_j$ and $\operatorname{supp}(g_j) \subset U_j$. So we have to estimate the j-th integral when $\zeta \in V_j$. Now fix a j and omit it in the notation.

Put $z = r\zeta$ and consider the integral of the ℓ-th summand in the expansion of the kernel $K_\Lambda(z,\phi(t))$. We have

$$\begin{aligned}
(n+\ell-1)&\int_0^1 P^0_\ell\cdot\frac{\left(\langle\phi(t),z\rangle - |z|^2\right)^\ell}{\left(1-\langle z,\phi(t)\rangle\right)^{n+\ell}}\langle z,\phi'(t)\rangle\frac{g(t)\,dt}{\langle z,\phi'(t)\rangle}\\
&= \int_0^1 \frac{d}{dt}\left[P^0_\ell\left(z,\dots,\langle\phi(t),z\rangle\right)\frac{\left(\langle\phi(t),z\rangle - |z|^2\right)^\ell}{\left(1-\langle z,\phi(t)\rangle\right)^{n+\ell-1}}\right]\frac{g(t)\,dt}{\langle z,\phi'(t)\rangle}\\
&\quad - \int_0^1 \frac{d}{dt}\left[P^0_\ell\left(z,\dots,\langle\phi(t),z\rangle\right)\right]\frac{\left(\langle\phi(t),z\rangle - |z|^2\right)^\ell}{\left(1-\langle z,\phi(t)\rangle\right)^{n+\ell-1}}\frac{g(t)\,dt}{\langle z,\phi'(t)\rangle}\\
&\quad - \int_0^1 \ell\langle\phi'(t),z\rangle P^0_\ell\frac{\left(\langle\phi(t),z\rangle - |z|^2\right)^{\ell-1}}{\left(1-\langle z,\phi(t)\rangle\right)^{n+\ell-1}}\frac{g(t)\,dt}{\langle z,\phi'(t)\rangle}\\
&:= I^1_\ell + I^2_\ell + I^3_\ell.
\end{aligned}$$

Since g is compactly supported on $(0,1)$, the integration by parts yields

$$-I^1_\ell(r\zeta) = \int_0^1 P^0_\ell\frac{\left(\langle\phi(t),r\zeta\rangle - r^2\right)^\ell}{\left(1-\langle r\zeta,\phi(t)\rangle\right)^{n+\ell-1}}\frac{d}{dt}\left[\frac{g(t)}{\langle r\zeta,\phi'(t)\rangle}\right]dt.$$

Let $r \geq 1/2$. Since $\zeta \in V$, the derivative in the above integral is bounded, so Fubini's theorem gives

$$\int_S \left|I_\ell^1(r\zeta)\right|^p \, d\sigma(\zeta) \leq \text{const} \int_S \frac{\left|\langle \eta, r\zeta\rangle - r^2\right|^{\ell p}}{|1 - \langle r\zeta, \eta\rangle|^{(n+\ell-1)p}} \, d\sigma(\zeta).$$

Let $(n-1)p < n$. Since $\left|\langle \eta, r\zeta\rangle - r^2\right| \leq |1 - \langle r\zeta, \eta\rangle|$, the last integral is bounded as $r \to 1-$ (see [Ru], 1.4.10).

The integral I_ℓ^2 is even simpler. To estimate I_ℓ^3, we put

$$g_1(z,t) = \frac{g(t)}{\langle z, \phi'(t)\rangle},$$
$$P_\ell^1 = \ell\langle \phi'(t), z\rangle P_\ell^0$$

and continue the integration by parts. After the ℓ-th integration the corresponding integral is absent. □

PROPOSITION 5.4. — *Let $k \in \mathbb{Z}_+$ and let $M \subset S$ be a real submanifold of class C^{k+2}. Then every compact set $K \subset M$ is an (N)-set ((PI), (I), or (SP)-set) for $A_{k0}(S)$ if and only if M is complex tangential.*

Proof. — Suppose that M is not complex tangential, then there is a NCT curve $\phi : [0,1] \to M$ of class C^{k+2}. Take $g \in C_0^{k+1}[0,1]$ and define a measure $\mu \in M(S)$ as in Lemma 5.3. By Lemma 5.2, we obtain $\mu - h\sigma \in A_{k0}(S)^\perp$ with $h \in L^1(\sigma)$, hence $|\mu|(\phi([0,1])) = 0$ (remark that $\sigma(\phi[0,1])) = 0$). A contradiction.

Clearly, the above argument works for all sets Λ considered in Lemma 5.3.

On the other hand, if M is complex tangential, then every $K \subset M$ is an (N)-set even for the ball algebra $A(S)$. □

5.3. Pluriharmonic measures and (N)-sets for $A_{k\ell}(S)$.

Put $M_0(S) = \{\rho$ is a probability measure on S such that $\int_S f \, d\rho = f(0)$ for all $f \in A(B)\}$. It is well known that $K \in (N)$ for $A(S)$ if and only if $\rho(K) = 0$ for all $\rho \in M_0(S)$. In the proposition below (which is, probably, of independent interest), "pluriharmonic" means "0-plh".

PROPOSITION 5.5. — *Suppose that μ is the singular part of a pluriharmonic measure. Then there exist compacts $K_j \in (N)$ for $A(S)$ such that*

$$|\mu|\left(S \setminus \bigcup_{j=1}^{\infty} K_j\right) = 0.$$

Proof. — Let $\mu = \mu_a + \mu_s$ be the Glicksberg-König-Seever decomposition of μ with respect to the set of representing measures $M_0(S)$ (for this Lebesgue type decomposition see, for example, [Ru], 9.4.4). So $\mu_a \ll \mu$ and $\mu_a \ll \rho_0$ for some $\rho_0 \in M_0(S)$. By Theorem 3.2, we have $\mu \perp \rho_0$, thus $\mu_a \perp \rho_0$, hence $\mu_a = 0$. In other words, μ is concentrated on a Borel set E such that $\rho(E) = 0$ for all $\rho \in M_0(S)$.

So take compact sets $K_j \subset E$ such that $|\mu|(K_j) > ||\mu|| - 1/j$. □

Since singular plh measures do exist, we have

COROLLARY 5.6. — *There exists a compact set $K \subset S$ such that $K \in (N)$ for $A(S)$ but $K \notin (N)$ for all $A_{k\ell}(S)$, $k \geq \ell \geq 1$.*

Remark. — Recall that (N)-sets for $PHC(S)$ and $A(S)$ coincide. It is interesting to compare this fact, Proposition 5.4 and Corollary 5.6.

Finally, note that Theorem B and Proposition 5.5 give an alternative proof of Henriksen's theorem about peak sets of maximal Hausdorff dimension (the original proof is based on more geometrical ideas).

COROLLARY 5.7 (B.S. Henriksen [He]). — *There exists a compact set $E \subset S$ of Hausdorff dimension $2n - 1$ such that E is a peak set for $A(S)$.*

6. FINAL REMARKS

1. Every probability singular d-plh measure with $d > 0$ is a quite exotic example of a representing measure for the ball algebra. (Classical elements of $M_0(S)$ live on unions of smooth manifolds.)

2. It is not clear how to construct (if possible) a d-plh measure μ such that μ_ξ, $\xi \in \mathbb{C}P^{n-1}$, are supported by small sets (in the sense of the Hausdorff dimension, for example).

3. Theorem A motivates the investigation of the Riesz type products with spectrum in $\Lambda(d)$, $d \in \mathbb{Z}_+$. On the other hand, if $\Lambda \subset \mathbb{Z}_+^2$ is arbitrary, then it is not clear what is the Λ-Riesz product. First, we need the RW-polynomials with spectrum in Λ. Second, there is a lot of sets Λ without CH-property (see [D3]). Finally, often the multiplication rule for the spherical harmonics breaks the Riesz product idea (consider, for example, the diagonal $\{(p,p) \in \mathbb{Z}_+^2\}$).

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BIBLIOGRAPHY

[A1] A.B. ALEKSANDROV, Inner functions on compact spaces, Funct. Anal. Appl., 18 (1984), 87–98.

[A2] A.B. ALEKSANDROV, Function theory in the ball, in: G. M. Khenkin and A. G. Vitushkin (Eds.), Encyclopaedia Math. Sci., 8 (Several Complex Variables II), Springer, Berlin, 1994, 107–178.

[AAN] A.B. ALEKSANDROV, J.M. ANDERSON, and A. NICOLAU, Inner functions, Bloch spaces and symmetric measures, preprint, 1997.

[AB] H. ALEXANDER and J. BRUNA, Pluriharmonic interpolation and hulls of C^1 curves in the unit sphere, Rev. Mat. Iberoamericana, 11 (1995), 547–568.

[Bi] P. BILLINGSLEY, Ergodic theory and information, Wiley, New York, 1965.

[B1] J. BOURGAIN, The Dunford-Pettis property for the ball-algebras, the polydisc-algebras and the Sobolev spaces, Studia Math., 77 (1984), 245–253.

[B2] J. BOURGAIN, Applications of the spaces of homogeneous polynomials to some problems on the ball algebra, Proc. Amer. Math. Soc., 93 (1985), 277–283.

[Br] R.J.M. BRUMMELHUIS, An F. and M. Riesz theorem for bounded symmetric domains, Ann. Inst. Fourier, Grenoble, 37-2 (1987), 139–150.

[CT] J. CIMA and R. TIMONEY, The Dunford-Pettis property for certain planar uniform algebras, Mich. Math. J., 34 (1987), 99–104.

[D1] E.S. DUBTSOV (=DOUBTSOV), Singular parts of pluriharmonic measures, Zap. Nauchn. Sem. S.Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 217 (1994), 54–58 (Russian); English transl., J. Math. Sci. (New York), 85-2 (1997), 1790–1793.

[D2] E.S. DUBTSOV, Some questions of the harmonic analysis on a complex sphere, Vestnik St. Petersburg Univ. Math., 28-1 (1995), 12–16.

[D3] E. DOUBTSOV, Regular unitarily invariant spaces on the complex sphere, submitted.

[He] B.S. HENRIKSEN, A peak set of Hausdorff dimension $2n-1$ for the algebra $A(D)$ in the boundary of a domain D with C^∞-boundary in $\mathbb{C}^n$, Math. Ann., 259 (1982), 271–277.

[HV] S.V. HRUŠČËV and S.A. VINOGRADOV, Free interpolation in the space of uniformly convergent Taylor series, Lect. Notes Math., 864 (1981), 171–213.

[I] K. IZUCHI, Bourgain algebras of the disc, polydisc and ball algebras, Duke Math. J., 66 (1992), 503–519.

[M] D.E. MENCHOFF, Sur l'unicité du développement trigonométrique, C. R. Acad. Sci. Paris, Sér. A-B, 163 (1916), 433–436.

[N] A. NAGEL, Cauchy transforms of measures and a characterization of smooth peak interpolation sets for the ball algebra, Rocky Mountain J. Math., 9 (1979), 299–305.

[P] A. PEYRIÈRE, Étude de quelques propriétés des produits de Riesz, Ann. Inst. Fourier, Grenoble, 25-2 (1975), 127–169.

[RS] J.-P. ROSAY and E.L. STOUT, On pluriharmonic interpolation, Math. Scand., 63 (1988), 268–281.

[Ru] W. RUDIN, Function Theory in the Unit Ball of $\mathbb{C}^n$, Grundlehren Math. Wiss., 241, Springer, Berlin, 1980.

[RW] J. RYLL and P. WOJTASZCZYK, On homogeneous polynomials on a complex ball, Trans. Amer. Math. Soc., 276 (1983), 107–116.

[Z] A. ZYGMUND, Trigonometric Series, vol. 1, Cambridge Univ. Press, 1959.

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