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THE COMPLEX ORIENTED COHOMOLOGY OF EXTENDED POWERS

by John Robert HUNTON

1. Introduction.

Suppose p is an odd prime, $G^*(-)$ an unreduced, multiplicative, complex oriented cohomology theory and X a space. Then an important problem is to describe the behaviour of $G^*(-)$ on the extended power space

$$D_p(X) = EC_p \underset{C_p}{\times} X^p$$

in terms of the behaviour of $G^*(-)$ on X. Here C_p is the cyclic group on p elements, EC_p a contractible space with free C_p action, and X^p the cartesian product of p copies of X; the C_p action on X^p is by permutation of factors. These spaces are significant as, among other uses, they form the building blocks of the infinite loop space construction QX, [23], and are fundamental to certain constructions of Dyer-Lashof and Steenrod power operations; see, for example, [3].

In the case $G = H\mathbb{F}_p$, the mod p Eilenberg-MacLane theory, the computation of $G^*(D_p(X))$ is completely described in Nakaoka's celebrated paper [19]. The Serre spectral sequence of the fibration

$$X^p \xrightarrow{i} D_p(X) \xrightarrow{j} BC_p \tag{1.1}$$

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collapses and there is a natural isomorphism of rings between

$$H^*(D_p(X);\mathbb{F}_p) \quad \text{and} \quad H^*(BC_p;H^*(X^p;\mathbb{F}_p)).$$

In this paper we investigate to what extent such simple descriptions of $G^*(D_p(X))$ exist for other complex oriented theories G.

An arbitrary cohomology theory will generally fail to have such straightforward behaviour. For example, McClure and Snaith in [17], and McClure again in [3], study the case of mod p K-theory $K\mathbb{F}_p$. The corresponding spectral sequence of (1.1) has two potentially non-zero differentials. The first is related to the mod p Bockstein acting on $K\mathbb{F}_p{}^*(X)$, while the second may be viewed as being forced by the differential in the Atiyah-Hirzebruch spectral sequence for $K\mathbb{F}_p{}^*(BC_p)$.

In this paper we consider two main classes of cohomology theories $G^*(-)$, restricting attention initially to those spaces X with $G_*(X)$ free as a G_* module: without some restriction the spectral sequence of (1.1) becomes extremely hard to say anything about in general, however, we shall see in §4 that this restriction can be relaxed to allow some particularly interesting examples of spaces X and theories G with $G_*(X)$ definitely not free. In §3 we study the case of Morava K-theory $K(n)$; of course $K(n)_*(X)$ is free over $K(n)_*$ for every space X. Here some quite complicated phenomena can arise and the spectral sequence of (1.1) can have many non-trivial differentials. We give a qualitative description (3.7), (3.10) and (3.11) of this spectral sequence which, when specialised (3.12) to the case $n=1$, reduces to the work just mentioned on mod p K-theory [17], [3]. It also reproduces (3.15) the results of [9], [10] and [1] on $K(n)^*(D_p(X))$ for spaces X satisfying $K(n)^{\text{odd}}(X)=0$.

In §4 we consider a range of other theories $G^*(-)$, principally the Landweber exact theories such as complex cobordism MU, the Brown-Peterson theory BP, complex K-theory, elliptic cohomology or the Johnson-Wilson theories $E(n)$. In contrast to the case of Morava K-theory, the spectral sequence of (1.1) always collapses for these theories, providing in a sense the best possible analogue (4.6) of Nakaoka's theorem. We use these cases to discuss more general cohomology theories (4.8), (4.9) including results which limit further the behaviour of the differentials in the Morava K-theory spectral sequence discussed in §3.

The nature of our main argument allows us to weaken our original condition that $G_*(X)$ must be free over G_* to include any space X with $G^*(X)$ satisfying our condition (4.1) of *regularity.* This greatly extends the number of spaces X to which our result applies. For example, for the theory

G taken to be p complete BP theory, it includes all spaces X with Morava K-theory concentrated in even dimensions; as noted in [22], this contains an extraordinary number of interesting space such as the classifying spaces of many finite groups, Eilenberg MacLane spaces, QS^{2n}, $BO(n)$, $MO(n)$, BO, ImJ, and so on – see [22] for details. There are other examples of spaces and theories with $G^*(X)$ regular and this in turn allows our Theorem (4.6) to provide one of the technical ingredients for Kashiwabara's description of $BP^*(QX)$ [12].

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2. Spectral sequences.

In this section we recall a number of details needed to set up our calculations. Our main tool is the spectral sequence of (1.1) in G-theory, which we shall also refer to as a Serre spectral sequence. This has the form

$$(2.1) \qquad E_2^{*,*} = E_2^{*,*}(G^*(D_p(X))) = H^*(BC_p; G^*(X^p)) \Longrightarrow G^*(D_p(X)).$$

We describe the E_2 term of this sequence and set up some basic notation. We assume throughout that $G^*(-)$ is multiplicative and for the moment that the space X has $G_*(X)$ free as a G_* module. This ensures that $G^*(X^p)$ is described by a Künneth isomorphism (suitably completed in the case of X an infinite complex) with the action of C_p on $G^*(X^p)$ given by permutation of the tensor factors.

The C_p-invariant elements of $G^*(X^p)$ fall into two types. If $\mathcal{X}$ is a topological basis for $G^*(X)$ then topological bases for these two types are given by

$$\text{I} : \{a^{\otimes p} = a \otimes \overset{(p)}{\cdots} \otimes a \mid a \in \mathcal{X}\},$$

$$\text{II} : \{\textstyle\sum_{\sigma \in C_p} \sigma(a_1 \otimes \cdots \otimes a_p) \mid a_i \in \mathcal{X}, \text{ the } a_i \text{ not all equal}\}.$$

The column $E_2^{0,*}$ in (2.1) is isomorphic to the C_p-invariant subring of $G^*(X^p)$ and we denote by A^* and B^* the sets of elements spanned by types I and II respectively. The whole E_2 term is generated as a ring by $E_2^{0,*}$ and $H^*(BC_p; G^*)$ where the action, as a G^* module, of $\widetilde{H}^*(BC_p; G^*)$

is free on A^* and trivial on B^*. Note that $a^{\otimes p} + b^{\otimes p} \equiv (a+b)^{\otimes p}$ modulo elements of B^*. For more details, see [19] or [17].

As the fibration (1.1) splits, *i.e.*, we have a factorisation of the identity on BC_p

$$BC_p \xrightarrow{s} D_p(X) \xrightarrow{j} BC_p, \tag{2.2}$$

there is a splitting at the spectral sequence level and so the subring $H^*(BC_p; G^*) \subset E_2^{*,*}$ behaves precisely as the Atiyah-Hirzebruch spectral sequence for $G^*(BC_p)$. Thus the first 'interesting' differentials we must consider are those non-zero on the elements lying in $E_2^{0,*}$.

DEFINITION 2.3. — *For X and $G^*(-)$ as considered, we say that the spectral sequence $H^*(BC_p; G^*(X^p)) \implies G^*(D_p(X))$ is* simple *if $E_2^{0,*} = E_\infty^{0,*}$, i.e., if the only non-trivial differentials are those forced by the Atiyah-Hirzebruch spectral sequence for $G^*(BC_p)$ and the splitting (2.2). Simplicity is the best possible analogue of Nakaoka's theorem that we can hope for.*

3. Morava K-theory.

In this section we concentrate on the case of $G = K(n)$ at the odd prime p. Recall that $K(n)$ has coefficients $\mathbb{F}_p[v_n, v_n^{-1}]$ where v_n has cohomological dimension $-2(p^n-1)$; in particular, $K(n)_*(X)$ is free over $K(n)_*$ for every space X. To simplify notation we shall consider $K(n)$ as a $\mathbb{Z}/(2p^n-2)$-graded theory, identifying the periodicity element v_n with 1.

Following [9], we write the mod p cohomology of BC_p as the ring

$$H^*(BC_p; \mathbb{F}_p) = \Lambda(z) \underset{\mathbb{F}_p}{\otimes} \mathbb{F}_p[x],$$

where $z \in H^1$ and $x \in H^2$. We shall reserve the letters x and z for these and closely related elements.

The Atiyah-Hirzebruch spectral sequence of $K(n)^*(BC_p)$ can be read off from the work of [21]. It has just one non-zero differential, namely d_{2p^n-1} acting by $d_{2p^n-1}(z) = x^{p^n}$ and $d_{2p^n-1}(x) = 0$. There is thus a non-trivial differential d_{2p^n-1} in the spectral sequence

$$\begin{aligned} E_2^{*,*} = H^*(BC_p; K(n)^*(X^p)) &= \big(A^* \underset{K(n)^*}{\otimes} H^*(BC_p; K(n)^*)\big) \oplus \big(B^*\big) \\ &\implies K(n)^*(D_p(X)) \end{aligned} \tag{3.1}$$

being the sequence (2.1) for $K(n)^*(D_p(X))$. We shall see that in general there will be plenty of earlier differentials in this sequence.

The following proposition gathers together observations of the previous section, [9] and [17], and will form the start of an inductive description of the spectral sequence. In this as in many subsequent places we shall refer to 'the first non-zero differential' in (3.1); this is the differential d_r where $r \geq 2$ is the smallest integer such that there is *some* space X for which $d_r \neq 0$, thus r does not depend on the particular space considered in the proposition.

PROPOSITION 3.2.

(a) *The E_2 term of the spectral sequence (3.1) is generated as a ring by $E_2^{0,*}$ and $H^*(BC_p; K(n)^*)$.*

(b) *The elements of type II are all permanent cycles.*

(c) *The first non-trivial differential, d_r say, is either d_{2p^n-1}, with action as described above on the elements z and x and the trivial action on $E^{0,*}_{2p^n-1}$, or else acts on the elements $a^{\otimes p} \in A^*$ by some formula*

$$d_r(a^{\otimes p}) = (Q_i(a))^{\otimes p} \cdot e_r \qquad 0 \neq \lambda \in \mathbb{F}_p$$

where Q_i is the $K(n)$ Bockstein operation related to v_i torsion, as discussed in [2]. By the observations in (a) and (b) this suffices to completely describe the action of d_r.

Proof. — Part (a) follows straight from the description of the E_2 term in the last section and part (b) follows from a simple transfer argument, as outlined in [17], proposition 1.3; indeed for the sequence (3.1) the subgroup is just the image of the transfer.

For part (c), suppose first that the earliest non-trivial differential is d_r for some $r < 2p^n - 1$. Then d_r is zero on the elements z and x in $E_r^{1,0}$ and $E_r^{2,0}$ respectively and so must take the form

$$d_r(a^{\otimes p} \cdot e_s) = b^{\otimes p} \cdot e_{s+r}$$

(recall that we are working modulo $v_n - 1$). We write b as $\delta_r^1(a)$.

LEMMA 3.3. — *The assignment $b = \delta_r^1(a)$ defines δ_r^1 as a natural cohomology operation $\delta_r^1 \colon K(n)^*(X) \longrightarrow K(n)^*(X)$ of degree t where $r - 1 + pt \equiv 0 \bmod 2(p^n - 1)$. This operation is stable and satisfies the property $(\delta_r^1)^2 = 0$. Moreover, δ_r^1 is also a derivation and so is a primitive element in $K(n)^*(K(n))$.*

Proof. — The nature of the spectral sequence shows that δ_r^1 as defined is a natural cohomology operation. As d_r is of bidegree $(r, 1-r)$ we have the equation

$$\begin{aligned} &\deg((\delta_r^1(a))^{\otimes p}) - \deg(a^{\otimes p}) \equiv 1-r && \bmod \deg(v_n) \\ \textit{i.e.,}\quad & p\cdot(\deg(\delta_r^1)+\deg(a)) - p\cdot\deg(a) \equiv 1-r && \bmod \deg(v_n) \\ \text{giving}\quad & p\cdot t \equiv 1-r && \bmod 2(p^n-1). \end{aligned}$$

That δ_r^1 is stable follows as in [17], proposition 1.7: we examine the related space $\widetilde{D}_p(Y) = EC_p^+ \wedge_{C_p} Y^{(p)}$ where $Y^{(p)}$ indicates the p-fold smash product of the based space Y and the superscript $+$ denotes the addition of a disjoint basepoint. It is easy to show that there is a pairing of spectral sequences

$$E_*^{*,*}(\widetilde{G}^*(\widetilde{D}_p(Y_1))) \otimes E_*^{*,*}(\widetilde{G}^*(\widetilde{D}_p(Y_2))) \longrightarrow E_*^{*,*}(\widetilde{G}^*(\widetilde{D}_p(Y_1 \wedge Y_2))).$$

Of course if $Y = X^+$ then $\widetilde{D}_p(Y) = D_p(X)^+$ and so $\widetilde{G}^*(\widetilde{D}_p(Y)) = G^*(D_p(X))$. Now let $\sigma \in \widetilde{E(n)}^1(S^1)$ be the canonical generator. The spectral sequence (2.1) for $\widetilde{E(n)}^*(\widetilde{D}_p(S^1))$ collapses for dimensional reasons – $H^*(BC_p; E(n)^*)$ is entirely in even dimensions – and so the element $\sigma^{\otimes p}$ is a permanent cycle. Hence it maps to a permanent cycle, $\tau^{\otimes p}$ say, in the spectral sequence for $\widetilde{K(n)}^*(\widetilde{D}_p(S^1))$. Multiplication by $\tau^{\otimes p}$ now induces an isomorphism of (reduced cohomology) spectral sequences

$$E_*^{*,*}(\widetilde{K(n)}^*(\widetilde{D}_p(Y))) \longrightarrow E_*^{*,*+p}(\widetilde{K(n)}^*(\widetilde{D}_p(\Sigma Y))).$$

This is sufficient to show in (2.1) the stability of δ_r^1 under suspension on elements of the form $a^{\otimes p}$ for any $a \in \widetilde{K(n)}^*(X)$.

As $d_r d_r = 0$ we must have $\delta_r^1\delta_r^1 = 0$ and as the spectral sequence (2.1) is multiplicative we deduce that δ_r^1 is a derivation. □

We can now complete the proof of (3.2)(c). Recall [26] (see also [2]) that

$$K(n)_*(K(n)) = \Sigma_n \underset{K(n)_*}{\otimes} \Lambda_n$$

where Σ_n is isomorphic to $K(n)_*(E(n))$ and Λ_n is the Grassmann algebra over $K(n)_*$ on elements ϵ_i, $i = 0, 1, \ldots n-1$, with ϵ_i in dimension $2p^i - 1$. The $K(n)_*$-dual of this Hopf algebra is precisely $K(n)^*(K(n))$. We can take a basis of $K(n)^*(K(n))$ including the elements Q_i, $i = 0, 1, \ldots n-1$, which form the dual basis to the $\{\epsilon_i\}$ of Λ_n. Following [2], we call Λ_n^*, the algebra generated by the Q_i's, the *Bockstein* subalgebra of $K(n)^*(K(n))$. The primitive elements in $K(n)^*(K(n))$ are dual to $QK(n)_*(K(n))$, the indecomposable quotient of $K(n)_*(K(n))$. This splits as $Q\Sigma_n \oplus Q\Lambda_n$ where $Q\Lambda_n$ is spanned by the images of the ϵ_i and $Q\Sigma_n = QK(n)_*(E(n)) = 0$ by, for example, [26]. As δ_r^1 must be primitive it must lie in the subspace of $K(n)^*(K(n))$ spanned by the Q_i; as δ_r^1 must be of homogeneous degree it must be Q_i (up to some non-zero scalar multiple) for some particular i.

Finally, if the first non-trivial differential is not until $r = 2p^n - 1$ we must show that d_r is trivial on elements of the form $a^{\otimes p}$. The action of d_r on the elements $A^* \subset E_{2p^n-1}^{0,*}$ will still be of the form $d_r: a^{\otimes p} \mapsto (\delta_r^1(a))^{\otimes p} \cdot e_r$ where the order, t say, of δ_r^1 must now satisfy $(2p^n - 1) - 1 + pt \equiv 0 \mod 2(p^n - 1)$, *i.e.*, t must be even: but all the Bockstein operations Q_i have odd degrees and so in this case δ_r^1 must be the zero operation. □

In fact, we can identify exactly what the first non-trivial differential is. To do this we use an argument based on the Atiyah-Hirzebruch spectral sequence for $K(n)^*(D_p(X))$. As is well known, the first non-trivial differential in the Atiyah-Hirzebruch spectral sequence for the Morava K theory of an arbitrary space is d_{2p^n-1} and acts up to an invertible multiple as Milnor's operation Q_n [18], [25], [26]. In order to avoid confusion, we shall continue to refer to the Morava K-theory Bockstein operations as Q_s, $0 \leq s < n$, and shall write Milnor's operation as Q_n^{Mil}. We can calculate the action of Q_n^{Mil} on $H^*(D_p(X); K(n)_*)$ by the inductive formulæ [18] which define Q_0^{Mil} as β, the mod p Bockstein, and put $Q_r^{\mathrm{Mil}} = \mathcal{P}^{p^{r-1}} Q_{r-1}^{\mathrm{Mil}} - Q_{r-1}^{\mathrm{Mil}} \mathcal{P}^{p^{r-1}}$.

LEMMA 3.4 [16] (see page 402). — *Suppose $a \in H^q(X)$. Then the (odd primary) Steenrod algebra action in $H^*(D_p(X); \mathbb{F}_p)$ is given by*

$$\mathcal{P}^s(a^{\otimes p} \cdot e_j) = \sum_i (s - pi, [j/2] - s + i + qm + p^t)(\mathcal{P}^i(a))^{\otimes p} \cdot e_{j+2(s-pi)(p-1)}$$

$$+ \delta(j-1)\alpha(q) \sum_i (s - pi - 1, [j/2] - s + i + qm + p^t)$$

$$(\beta\mathcal{P}^i(a)) \cdot e^{\otimes p}_{j-p+2(s-pi)(p-1)}$$

and $$\beta(a^{\otimes p} \cdot e_j) = \delta(j) a^{\otimes p} \cdot e_{j+1}$$

where $\delta(j)$ is 0 or 1 if j is even or odd, $\alpha(q) = -(-1)^{mq} m!$, $m = (p-1)/2$, and t is any sufficiently large number. Here we have used the notation e_j to mean x^k if $j = 2k$ and $x^k \cdot z$ if $j = 2k+1$. □

We need only consider the Steenrod operations on elements of this form since the remaining elements are those in the image of the transfer. As in the spectral sequence (3.1) these elements split off, performing their own independent spectral sequence and play no role in the groups we are interested in. We will at points omit mentioning these transfer images.

COROLLARY 3.5. — *Ignoring the elements in the image of the transfer, the action of Q_n^{Mil} on $H^*(D_p(X);\mathbb{F}_p)$ is generated by*

$$Q_n^{\mathrm{Mil}}(a^{\otimes p}) = \sum_{i=0}^{n} \lambda_i (Q_{n-i}^{\mathrm{Mil}} a)^{\otimes p} \cdot x^{l(i)}, \quad Q_n^{\mathrm{Mil}}(z) = x^{p^n}, \quad Q_n^{\mathrm{Mil}}(x) = 0$$

for appropriate positive integers $l(i)$ and scalars $\lambda_i \in \mathbb{F}_p$; the element λ_1 is always non-zero. □

Consider the lens space L_n, the $2p^n - 1$ skeleton of BC_p. By restriction, the mod p cohomology of L_n can be written

$$H^*(L_n;\mathbb{F}_p) = \Lambda(z) \otimes \mathbb{F}_p[x]/(x^{p^n}).$$

The following lemma records the Milnor operations on this cohomology ring. It also details the Morava K-theory of the space, the action of the Atiyah-Hirzebruch spectral sequence for it and, from [2], the associated $K(n)$ Bocksteins. We write $Q_\bullet^{\mathrm{Mil}}$ and $Q_\bullet$ for the total Milnor and total Morava Bockstein operations respectively, *i.e.*, for $\sum_{i=0}^{\infty} Q_i^{\mathrm{Mil}}$ and $\sum_{i=0}^{n-1} Q_i$.

LEMMA 3.6.

(a) *In the ring $H^*(L_n;\mathbb{F}_p)$ the total Milnor operation acts as*

$$Q_\bullet^{\mathrm{Mil}}(z) = \sum_{i=0}^{\infty} x^{p^i} \quad \text{and} \quad Q_\bullet^{\mathrm{Mil}}(x) = 0.$$

(b) *The Atiyah-Hirzebruch spectral sequence for $K(n)^*(L_n)$ collapses for dimensional reasons and we can describe the ring $K(n)^*(L_n)$ as the $K(n)^*$ algebra $\Lambda(\bar{z}) \otimes K(n)^*[\bar{x}]/(\bar{x}^{p^n})$ with total Bockstein operation given by $Q_\bullet(\bar{z}) = \sum_{i=0}^{n-1} \bar{x}^{p^i}$ and $Q_\bullet(\bar{x}) = 0$.* □

THEOREM 3.7. — *The first non-zero differential in the spectral sequence (3.1) for $K(n)^*(D_p(X))$ is d_{p-1}, whose action is as described in (3.2) and is associated to a non-zero multiple of the Bockstein operation Q_{n-1}.*

Proof. — By computing dimensions we see that the first possible candidate for a non-zero differential is d_{p-1} acting via the operation Q_{n-1}.

We shall show that this possibility is realised in the spectral sequence for $K(n)^*(D_p(L_n))$. Let us write $D_p^s(L_n)$ for the inverse image in $D_p(L_n)$ of $BC_p^{(s)}$, the s-skeleton of BC_p and consider the case $s = p-1$. The differential d_{p-1} in (3.1) is as claimed if and only if the differential d_{p-1} in the spectral sequence for the fibration

$$(L_n)^p \longrightarrow D_p^{p-1}(L_n) \longrightarrow BC_p^{(p-1)}$$

is non-zero; moreover, for dimensional reasons, this is the only potentially non-zero differential in this restricted spectral sequence.

Now consider the Atiyah-Hirzebruch spectral sequence for

$$K(n)^*(D_p^{p-1}(L_n)).$$

By (3.5) and (3.6) this spectral sequence fails to collapse; hence, by counting dimensions, neither does the Serre spectral sequence for

$$K(n)^*(D_p^{p-1}(L_n)).$$

□

Remark 3.8. — The technique of playing off the two spectral sequences is a very powerful one and one that can be used to obtain much computational information. In essence it is a tool that systematically uses the geometric information arising from the double filtration on $D_p(X)$ coming from the skeletal filtration on $D_p(X)$ and the inverse image filtration of the skeletal filtration on BC_p.

Higher differentials than the ones considered in (3.7) can also be studied. The second differential in the Atiyah-Hirzebruch spectral sequence is given by the relation $(Q_n^{\mathrm{Mil}})^2 = 0$, *i.e.* , on elements of the form $a^{\otimes p}$ by the relation

$$\begin{aligned} 0 &= Q_n^{\mathrm{Mil}}(Q_n^{\mathrm{Mil}}(a^{\otimes p})) \\ &= \sum_{i \neq j} \mu_{i,j}((Q_{n-i}^{\mathrm{Mil}} Q_{n-j}^{\mathrm{Mil}} + Q_{n-j}^{\mathrm{Mil}} Q_{n-i}^{\mathrm{Mil}})a)^{\otimes p} \cdot x^{l(i)+l(j)} \\ &\qquad + \sum_i \mu_{i,i}((Q_{n-i}^{\mathrm{Mil}} Q_{n-i}^{\mathrm{Mil}})a)^{\otimes p} \cdot x^{2l(i)} \end{aligned}$$

where $\mu_{i,j}$ is some scalar, again derived from the formulæ (3.4). Thus the second differential on an element $a^{\otimes p}$ takes the form

$$\sum_{i,j} \mu_{i,j}(Q_{n-i,n-j}^{\mathrm{Mil}} a)^{\otimes p} \cdot x^{l(i)+l(j)}$$

where $Q_{r,s}^{\mathrm{Mil}}$ is the secondary operation related to $Q_r^{\mathrm{Mil}} Q_s^{\mathrm{Mil}} + Q_s^{\mathrm{Mil}} Q_r^{\mathrm{Mil}} = 0$ if $r \neq s$ and $Q_r^{\mathrm{Mil}} Q_r^{\mathrm{Mil}} = 0$ otherwise. These formulæ are discussed in much more detail in [24].

LEMMA 3.9. — *Suppose that r is an odd number in the range $1 \leq r \leq 2p^n - 1$. Then the image of*

$$K(n)^*(D_p^r(X))/\mathrm{Im}(\mathrm{tr}) \quad \text{in} \quad K(n)^*(D_p^1(X))/\mathrm{Im}(\mathrm{tr})$$

is a free module over $K(n)^*(S^1) = \Lambda(z)$; here tr *denotes the transfer homomorphism.*

Proof. — In the Atiyah-Hirzebruch spectral sequence modulo the transfer the only non-zero differentials are generated by the action of the Milnor operation and its higher operations on elements of the form $a^{\otimes p}$. Moreover, the E_2 page is itself a free module over $K(n)^*(S^1)$. However, formulæ such as in (3.5) and (3.8) show that all the possible differentials will preserve the freeness over this ring. Thus E_∞ and by extension $K(n)^*(D_p^r(X))/\mathrm{Im}(\mathrm{tr})$ has the property claimed. □

We next consider the general differentials d_r in (3.1) for $r < 2p^n$. The following induction shows the form of a differential d_r with $r < 2p^n - 1$ to be comparable with that of the first possibly non-zero differential in this range.

THEOREM 3.10. — *Suppose $r < 2p^n - 1$ and that d_r represents the q^{th} non-zero differential in the spectral sequence (3.1). Then*

(a) *E_r is generated as an algebra by $E_r^{0,*}$ and the elements $z \in E_r^{1,0}$ and $x \in E_r^{2,0}$,*

(b) *the differential d_r is trivial on z, x and elements in $B^* \subset E_r^{0,*}$ and*

(c) *the differential d_r acts on an element $a^{\otimes p} \in E_r^{0,*}$ by sending it to $(\delta_r^q(a))^{\otimes p} \cdot e_r$ where δ_r^q is a q^{th} order $K(n)$ operation, defined on the kernel of the operations associated to the previous differentials. Moreover, δ_r^q is of odd degree and hence r is even.*

Proof. — We proceed by induction on q, the case of $q = 1$ having been settled by (3.2). We suppose the result proved for the first $q-1$ non-zero differentials and suppose d_r with $r < 2p^n - 1$ is potentially the next. Suppose d_s with $s < r$ was the $(q-1)^{\text{th}}$ non-zero differential.

Part (a) follows immediately from the structure of $E_s^{*,*}$ and the fact that δ_s^{q-1} is odd dimensional since this implies that s is even. Note that multiplication by $x\colon E_r^{t,*} \to E_r^{t+2,*}$ is onto for $t \geq 0$ and is an isomorphism for $t \geq s/2$. Part (b) follows as in (3.2).

For part (c) the argument to associate δ_r^q with a q^{th} order operation is identical with that in (3.2). It remains to show that δ_r^q is of odd order. Suppose it is not so r is odd and so for some space X we have an element $a^{\otimes p} \in E_r^{0,*}$ supporting a non-trivial d_r but for which $d_r(a^{\otimes p} \cdot z) = 0$. Consider the spectral sequence for $D_p^r(X)$, the inverse image of the r-skeleton of BC_p; the differentials of (3.1) restrict to this sequence, but d_r is the final differential. This supposed differential d_r now contradicts (3.9). □

We have now dealt with all the differentials d_r for $r < 2p^n - 1$. As noted before, d_{2p^n-1} acts non-trivially on the element z. It is trivial however on all elements of $E_{2p^n-1}^{0,*}$: on the elements B^* the argument of (3.2)(b) applies, while the argument of (3.10)(c) shows that it acts on elements of the form $a^{\otimes p}$ via an *odd* degree higher order operation $\delta_{2p^n-1}^q$. However, $\delta_{2p^n-1}^q$ clearly has to be an even degree operation and hence is zero. When d_{2p^n-1} has acted the element x^{p^n} is killed and there is no room for any further differentials as $E_{2p^n}^{m,*}$ is zero for $m > 2(p^n - 1)$. We thus arrive at the following description of the spectral sequence (3.1).

THEOREM 3.11. — *There are only a finite number of non-trivial differentials in the sequence (3.1), all acting as in (3.10) for Bockstein and higher order operations δ_r^q, $r < 2p^n - 1$, except for the final one, d_{2p^n-1}, which acts trivially on $E_{2p^n-1}^{0,*}$ and as induced on the elements $z \in E_{2p^n-1}^{1,0}$ and $x \in E_{2p^n-1}^{2,0}$ by the splitting (2.2).* □

We finish this section with some examples. We begin by specialising to the case $n = 1$ to obtain the mod p K-theory results of [17] and [3].

COROLLARY 3.12. — *The spectral sequence (3.1) for $K(1)^*(D_p(X))$ has just two non-zero differentials. The first is d_{p-1} acting as described in (3.7) via the operation Q_0, and the second is d_{2p-1} forced by the Atiyah-Hirzebruch spectral sequence differential for $K(1)^*(BC_p)$.*

Proof. — For $K(1)$ the only potential operations are the $Q_0^{(s)}$, $s = 1, 2, \ldots$, where $Q_0^{(s)}$ is the mod p^s Bockstein. As each $Q_0^{(s)}$ is of dimension 1 they can only occur in differentials d_r where $r - 1 + p \equiv 0 \pmod{2p-2}$. As the differential corresponding to $Q_0^{(1)}$ occurs potentially non-trivially in d_{p-1}, a differential corresponding to $Q_0^{(2)}$ can occur at the earliest at $r = 3p - 3$. However, as noted in (3.11) all differentials after d_{2p-1} are zero. □

Note that our calculations are easier than those in [17] as we have the advantage of $2(p-1)$ sparseness in $K(1)$.

The case $K(1)$ is reasonably well behaved. However, calculations for lens spaces show there to be many higher order operations coming into play for $K(n)$, $n > 1$. The following gives an example of non-trivial differentials d_s occuring with $p-1 < s < 2p^n - 1$. Let us write M_p for the mod p Moore space $S^1 \cup_p E^2$, the 2-skeleton of L_n. Then $K(n)^*(M_p)$ is free of rank 2 over $K(n)_*$ on generators y and $Q_0(y)$ in dimensions 1 and 2 respectively. We also write y and $Q_0^{\mathrm{Mil}}(y)$ for the 1 and 2 cycles in the E_2 term of the Atiyah-Hirzebruch spectral sequence for $K(n)^*(D_p(M_p))$.

PROPOSITION 3.13. — *The spectral sequence (3.1) for*

$$K(2)^*(D_p(M_p))$$

has two non-zero differentials. The first is $d_{(2p+1)(p-1)}$ acting as described in (3.10) via the secondary operation related to Q_0, and the second is d_{2p^2-1} acting as usual.

Proof. — Computation of Milnor's Q_2^{Mil} in the Atiyah-Hirzebruch spectral sequence for $K(2)^*(D_p(M_p))$ shows it to have the form

$$Q_2^{\mathrm{Mil}}(y^{\otimes p}) = \lambda_1 (Q_1^{\mathrm{Mil}} y)^{\otimes p} \cdot x^{(p-1)/2} + \lambda_2 (Q_0^{\mathrm{Mil}} y)^{\otimes p} \cdot x^{(2p+1)(p-1)/2}$$

where each λ_i is non-zero. Of course $Q_1^{\mathrm{Mil}}(y) = 0$ but as $Q_0^{\mathrm{Mil}}(y) \neq 0$ we see that the Atiyah-Hirzebruch spectral sequence for $K(n)^*(D_p^{(2p+1)(p-1)}(M_p))$ does not collapse. As before we conclude that we get a differential

$$d_{(2p+1)(p-1)} \colon y^{\otimes p} \mapsto (\lambda Q_0(y))^{\otimes p} \cdot x^{(2p+1)(p-1)/2} \qquad 0 \neq \lambda \in \mathbb{F}_p$$

in the Serre spectral sequence for $K(2)^*(D_p(M_p))$ before the final d_{2p^2-1}. □

Remark 3.14. — The investigation of differentials related to the primary Bockstein operations Q_i, $i = n-1, \ldots, 1, 0$, of [2] can be carried out for given values of n using the space L_n and its subcomplexes. For possible differentials related to more complex Bockstein operations there is a whole family of lens spaces, various subcomplexes of BC_{p^k}, that can be brought into play. These carry non-trivial higher order $K(n)$ operations and are discussed in more detail in [2]. However, we understand that using the machinary of [4] McClure has shown that the only operations that can arise non-trivially in (3.1) are those associated to the primary operations Q_i.

We conclude this section by observing a simple reproof (and in fact an extension) of the results of [1], [9] and [10].

COROLLARY 3.15. — *The spectral sequence (3.1) for $K(n)^*(D_p(X))$ is simple for any space X with Morava K-theory concentrated either entirely in even dimensions or entirely in odd dimensions.*

Proof. — The operations $\delta_r^q(-)$ are all odd dimensional so they are all zero on a space X with Morava K-theory concentrated as supposed. □

4. Landweber exact theories.

In this section we prove results which show that the spectral sequence (2.1) is simple for a variety of cohomology theories $G^*(-)$ in various circumstances. We need to restrict to spaces X for which the E_2 page of (2.1) is similar to that given in §2; the following definition encapsulates our requirements.

DEFINITION 4.1. — *For a space X and multiplicative cohomology theory $G^*(-)$, say the ring $G^*(X)$ is* regular *as a G^* module if the E_2 page of (2.1) is generated by the image of the transfer, the image of $H^*(BC_p; G^*)$ under the splitting map j of (2.2) and by elements of the form $a^{\otimes p} \in E_2^{0,*}$ where $a \in G^*(X)$.*

Of course, if $G_*(X)$ is a free G_* module then $G^*(X)$ is certainly regular, but we shall see below that other $G^*(X)$ will also satisfy (4.1).

Note that, as in the proofs of (3.2) and (3.10), elements in the image of the transfer are always permanent cycles in the spectral sequence. Thus the first potentially non-zero differential in the spectral sequence (2.1) for a space X with $G^*(X)$ regular is either one arising from a differential in the Atiyah-Hirzebruch spectral sequence for $G^*(BC_p)$ or else one which is non-zero on some element $a^{\otimes p} \in E_2^{0,*}$.

The case of theories $G^*(-)$ with torsion free coefficients is particularly interesting in light of the following result.

LEMMA 4.2. — *Suppose $G^*(-)$ is a multiplicative complex oriented cohomology theory with torsion free coefficients concentrated in even dimensions. Then the Atiyah-Hirzebruch spectral sequence for $G^*(BC_p)$*

collapses and hence the spectral sequence (2.1) for spaces X with $G^*(X)$ regular is simple if and only if it collapses.

Proof. — The assumptions on the coefficients G^* mean that the Atiyah-Hirzebruch spectral sequence for $G^*(BC_p)$ has E_2 term concentrated in even dimensions and hence collapses. Under these conditions the elements in the image of the homomorphism

$$j^*: H^*(BC_p; G^*) \longrightarrow H^*(BC_p; G^*(X^p))$$

are thus permanent cycles in (2.1) and so to say that (2.1) is simple is to say that none of the ring generators of E_2 support non-zero differentials.

□

LEMMA 4.3. — *If $G^*(-)$ is a multiplicative complex oriented cohomology theory with torsion free coefficients then $G^*(X)$ is regular if and only if $H^1(BC_p; G^*(X^p)) = 0$, or, equivalently, if and only if the E_2-term of (2.1) is concentrated in $E_2^{\text{even},*}$.*

Proof. — This follows as $H^*(BC_p; M)$ is periodic of degree 2 for any coefficients M. □

A case of particular interest is that of p complete Brown-Peterson cohomology $BP_p^\wedge$; see [22] for a discussion of the role of this theory. The following result gives us a useful criterion for a space X to have $(BP_p^\wedge)^*(X)$ regular.

PROPOSITION 4.4. — *Suppose $\widehat{E(n)}^*(X)$ is a completed free $\widehat{E(n)}^*$ module and there is a completed Künneth isomorphism $\widehat{E(n)}^*(X^p) = \widehat{E(n)}^*(X)^{\widehat{\otimes} p}$. Then $(BP_p^\wedge)^*(X)$ is regular.*

Proof. — By [6], the natural inclusion of $BP_p^\wedge$ into the product of the $K(n)$ localisations $\prod_{n>0} L_{K(n)}BP$ is a split monomorphism. Hence by (4.3) it suffices to show that $H^1(C_p; (L_{K(n)}BP)^*(X^p)) = 0$. However, by [7] and [8] this will follow by showing that each $H^1(C_p; \widehat{E(n)}^*(X^p)) = 0$, which follows by the usual calculations from the hypotheses of the proposition.

□

Remark 4.5. — There are many examples of spaces X satisfying the hypotheses of (4.4). One particularly interesting class of examples is that of spaces X with $K(n)^{\mathrm{odd}}(X) = 0$ for all n since in this case the spectral sequence leading from $K(n)^*(X)$ to $\widehat{E(n)}^*(X)$ collapses and we can appeal to [8]. This class includes the classifying spaces of many finite groups, Eilenberg MacLane spaces, QS^{2n}, $BO(n)$, $MO(n)$, BO, ImJ, and so on; see [22] for a detailed discussion of such spaces. There are however other classes of spaces satisfying (4.4); for example the types of space QX considered in [12].

The next theorem considers the Landweber exact cohomology theories [13]. Recall that a complex oriented theory $G^*(-)$ is said to be Landweber exact if on finite CW complexes X its relation to complex cobordism is given by the tensor product

$$G^*(X) = G^* \underset{MU^*}{\otimes} MU^*(X)$$

where the MU^* module action on G^* comes from the complex orientation. Examples of such theories include complex K-theory, elliptic cohomology, the Johnson-Wilson theories $E(n)$ and of course MU, BP and their p completions themselves. These are all multiplicative with torsion free coefficient rings, concentrated in even degrees.

THEOREM 4.6. — *Suppose $G = MU$, BP or any other Landweber exact spectrum with coefficients concentrated in even degrees. Suppose also that X is a space for which $G^*(X)$ is regular. Then the spectral sequence (2.1) collapses.*

Proof. — We begin by considering a universal example. Let us write $\mathbf{G}_{2m}$ for the infinite loop space representing the cohomology group $G^{2m}(-)$. Now the spectral sequence (2.1) for $G^*(D_p(\mathbf{G}_{2m}))$ will certainly collapse for dimensional reasons and by (4.2) since $G_*(\mathbf{G}_{2m})$ is free over G_* and concentrated in even dimensions – this is proved for $G = MU$ and BP in [20], Landweber exact theories with coefficients G^* in even dimensions and countably generated over some subring of the rationals in [5] and for the general case in [11].

Now consider a general X satisfying the hypotheses of the theorem. It will suffice by (4.2) to show that the elements in $E_2^{0,*}$ are all permanent cycles. Suppose not and the first actually non-trivial differential in this spectral sequence was d_r, taking a non-zero value on some $y^{\otimes p} \in E_2^{0,*}$ (note that, once again, the elements in the image of the transfer are all permanent cycles). Suppose for the moment that y is an even dimensional cohomology class, say $y \in G^{2m}(X)$. Then y is represented by a map $X \to \mathbf{G}_{2m}$ and we have an induced map of spectral sequences

$$E_*^{*,*}(G^*(D_p(\mathbf{G}_{2m}))) \longrightarrow E_*^{*,*}(G^*(D_p(X)))$$

the image of which contains the element $y^{\otimes p}$. As the left hand spectral sequence collapses, this contradicts $d_r(y^{\otimes p}) \neq 0$.

Finally, if y was odd dimensional we can argue instead with the space ΣX using the isomorphism of spectral sequences as set up in the proof of (3.3). □

Remark 4.7. — One way to view this result is that these spectral sequences collapse because such cohomology theories have no Bockstein operations. Complex cobordism and the Landweber exact theories may be among the few with so trivial a spectral sequence for the extended power construction. The Bockstein related differentials exhibited for $K(n)$ suggest similar non-trivial differentials for any BP module theory with Bocksteins Q_i, for some $i < n$, but a non-trivial v_n action in coefficients, for example $P(n)$. A borderline case is that of the theory $BP\langle n\rangle$. Here $BP\langle -1\rangle$ is $H\mathbb{F}_p$ and so we have a collapsing sequence by [19], while $BP\langle\infty\rangle = BP$ and so collapses by (4.6). Certainly inverting v_n in $BP\langle n\rangle$ yields $E(n)$, a Landweber exact theory and hence covered by (4.6) as well. Nakaoka's theorem also implies that the Serre spectral sequence for $BP\langle 0\rangle^*(D_p(X)) = H\mathbb{Z}^*_{(p)}(D_p(X))$ has no non-trivial differentials provided $H\mathbb{Z}^*_{(p)}(X)$ is torsion free (the case where $H\mathbb{Z}^*_{(p)}(X)$ has torsion is considerably more complicated and is discussed in detail in [15]).

Theorem (4.6) can be used to deduce the simplicity of the spectral sequence for $G^*(D_p(X))$ in various circumstances for other cohomology theories. The following is such an example and one that finds application in the work of Kashiwabara [12] on $BP^*(QX)$.

COROLLARY 4.8. — *Suppose the space X has $BP^*(X)$ regular and that the reduction homomorphism*

$$K(n)^* \underset{BP^*}{\otimes} BP^*(X) \longrightarrow K(n)^*(X)$$

is onto. Then the spectral sequence (3.1) for $K(n)^*(D_p(X))$ is simple.

Proof. — If the reduction homomorphism as stated is onto then so is the induced homomorphism $E(n)^*(X) \longrightarrow K(n)^*(X)$. The result now follows from (4.6) by naturality of the spectral sequence (2.1). □

A similar argument can be used to prove

COROLLARY 4.9. — *Suppose the space X has $BP^*(X)$ and $G^*(X)$ regular (as BP^* and G^* modules respectively) for some BP module theory $G^*(-)$ with torsion free coefficients concentrated in even dimensions. Suppose further that the reduction homomorphism*

$$G^* \underset{BP^*}{\otimes} BP^*(X) \longrightarrow G^*(X)$$

is onto. Then the spectral sequence (2.1) for $G^*(D_p(X))$ collapses. □

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