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# ABOUT $G$ -BUNDLES OVER ELLIPTIC CURVES

by Yves LASZLO(\*)

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## 1. Introduction.

In this note, we study principal bundles over a complex elliptic curve  $X$  with reductive structure group  $G$ . As in the vector bundle case, we first show that a non semistable bundle has a canonical semistable  $L$ -structure with  $L$  some Levi subgroup of  $G$  reducing the study of  $G$ -bundles to the study of semistable bundles (Proposition 3.2). We then look at the coarse moduli space  $M_G$  of topologically trivial semistable bundles on  $X$  (there is not any stable topologically trivial  $G$ -bundle) and prove that it is isomorphic to the quotient  $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$  where  $\Gamma(T)$  is the group of one parameter subgroups of a maximal torus  $T$  and  $W = N(G, T)/T$  is the Weyl group (Theorem 4.16). Suppose that  $G$  is simple and simply connected and let  $\theta$  be the longest root (relative to some basis  $(\alpha_1, \dots, \alpha_l)$  of the root system  $\Phi(G, T)$ ). The coroot  $\theta^\vee$  of  $\theta$  has a decomposition  $\theta^\vee = \sum_i g_i \alpha_i^\vee$  with  $g_i$  a positive integer. Using Theorem 4.16 and Looijenga's isomorphism

$$[\Gamma(T) \otimes_{\mathbf{Z}} X]/W \xrightarrow{\sim} \mathbf{P}(1, g_1, \dots, g_l),$$

one gets that  $M_G$  is isomorphic to the weighted projective space  $\mathbf{P}(1, g_1, \dots, g_l)$  (see 4.17), generalizing the well-known isomorphism  $M_{\mathbf{SL}_{l+1}} \xrightarrow{\sim} \mathbf{P}^l$  (see [T] for instance). One recovers for instance the Verlinde formula in this case.

We know that these results are certainly well-known from experts, but we were unable to find any reference in the literature, except of course when  $G$  is either  $\mathbf{SL}$  or  $\mathbf{GL}$ . For another point of view, see [BG].

During the referee process of this paper, an independent paper of Friedman, Morgan and Witten has appeared in Duke's eprints (see [FMW]), where the link between Looijenga's result and bundles on elliptic curves is studied from another point of view.

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*Notations.* — By scheme, we implicitly mean a complex scheme. Let  $G$  be a reductive group with a Borel subgroup (resp. a maximal torus)  $B$  (resp.  $T \subset B$ ). The corresponding Lie algebras will be denoted by  $\mathfrak{t}$ ,  $\mathfrak{b}$  and  $\mathfrak{g}$ .

We denote by  $W = N(G, T)/T$  the Weyl group and by  $\Gamma(T)$  the  $W$ -module  $\text{Hom}(G_m, T)$ .

### 2. Review on the Harder-Narasimhan reduction.

Let  $X$  be an algebraic curve which is smooth, projective and connected and  $E$  be a  $G$ -bundle on  $X$ . Recall that  $E$  is semistable if and only if the adjoint bundle  $\mathcal{E} = \text{Ad}(E)$  is a semistable vector bundle. Following [AB], let me recall how to define the Harder-Narasimhan reduction of  $E$ . Pick a non degenerate invariant quadratic form  $q$  on the Lie algebra of  $G$ . Then  $q$  defines a non degenerate quadratic form on  $\mathcal{E}$ .

LEMMA 2.1. — *The length  $r - 1$  of the Harder-Narasimhan filtration*

$$0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_i \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

of  $\mathcal{E}$  is even.

*Proof.* — Let

$$0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_i \subset \dots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder-Narasimhan filtration of  $\mathcal{E}$ . The Harder-Narasimhan filtration of  $\mathcal{E}^*$  is

$$0 \subset \dots \subset (\mathcal{E}/\mathcal{E}_{r-i})^* \subset \dots \subset \mathcal{E}^*.$$

Because  $\mathcal{E}$  is self-dual, the Harder-Narasimhan filtration is self-dual and one has an isomorphism

$$(2.1) \quad \mathcal{E}_i \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{r-i})^*$$

inducing isomorphisms

$$(2.2) \quad gr_i \xrightarrow{\sim} (gr_{r+1-i})^*, i = 1, \dots, r$$

where  $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ . Assume that the length  $r - 1$  is odd. Let us consider the morphisms

$$m_k : \mathcal{E}_{r/2} \otimes \mathcal{E}_k \rightarrow \mathcal{E}/\mathcal{E}_{k-1}, 0 \leq k \leq r - 1$$

deduced from the Lie bracket of  $\mathcal{E}$ . The equality (2.2) gives the inequalities

$$(2.3) \quad \mu_1 > \dots > \mu_{r/2} > -\mu_{r/2} > \dots > -\mu_1$$

where  $\mu_i$  is the slope of the semistable vector bundle  $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ . In particular, the slopes  $\mu_i + \mu_j$  of the subquotients  $gr_i \otimes gr_j, i \leq r/2$  and  $j \leq k$  which appear in  $\mathcal{E}_{r/2} \otimes \mathcal{E}_k$  are not less than  $\mu_{r/2} + \mu_k > \mu_k$  though the slopes of the subquotients  $gr_i, i \geq k$  which appear  $\mathcal{E}/\mathcal{E}_{k-1}$  are  $\leq \mu_k$ . This shows that  $m_k$  is zero for all  $k$  and that all elements of  $\mathcal{E}_{r/2}$  are nilpotent. By (2.1), this algebra is also lagrangian. Suppose that the center of  $G$  is of positive dimension. Then  $\mathcal{E}$  contains a trivial sub-bundle (of positive rank) as a direct summand which implies that some of the  $\mu_i$ 's is zero, contradicting (2.3). The Lie algebra bundle  $\mathcal{E}$  is therefore semisimple and therefore has no non trivial lagrangian sub-Lie algebra (with respect of the Killing form form) consisting of nilpotent elements, just by a dimension argument.  $\square$

It follows that one can index the Harder-Narasimhan filtration of  $\mathcal{E}$  such that

$$(2.4) \quad 0 = \mathcal{E}_{-r} \subset \mathcal{E}_{-r+1} \subset \dots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \dots \subset \mathcal{E}_{r-1} = \mathcal{E}$$

where  $\mathcal{E}_{-j}$  is the orthogonal of  $\mathcal{E}_{j-1}$ . One checks that  $\mathcal{E}_0$  is a subalgebra of  $\mathcal{E}$ . Notice that  $\mathcal{E}_0/\mathcal{E}_{-1}$  is self-dual and therefore has slope zero. In particular, the slope of  $gr_{-j}, j > 0$  is  $> 0$ . As in the proof of the preceding lemma, this immediately implies the sequence of inclusions

$$(2.5) \quad [\mathcal{E}_{-j}, \mathcal{E}_{-1}] \subset \mathcal{E}_{-j-1} \text{ for all } j.$$

In particular, all elements of  $\mathcal{E}_{-1}$  are nilpotent. For sake of completeness, let me prove this easy lemma (cf. Th VIII.10.1 of [Bo]).

LEMMA 2.2. — *Let  $\mathfrak{g}$  be a reductive algebra endowed with an invariant non degenerate bilinear form. Let  $\mathfrak{n}'$  be a subalgebra of  $\mathfrak{g}$  whose elements are nilpotent. Then, if the orthogonal of  $\mathfrak{n}'$  is a sub-Lie algebra of  $\mathfrak{g}$ , it is parabolic.*

*Proof.* — The Lie algebra  $\mathfrak{n}'$  is nilpotent. Let  $\mathfrak{b}$  be a maximal solvable Lie subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{n}'$  and  $\mathfrak{n}$  its nilpotent ideal. By [Bo], Definition VIII.3.3.1,  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{g}$ . Because all elements of  $\mathfrak{n}'$  are nilpotent,  $\mathfrak{n}'$  is contained in  $\mathfrak{n}$ . By [Bo], proposition VII.1.3.10 (iii), the orthogonal of  $\mathfrak{n}$  is  $\mathfrak{b}$  and the lemma follows.  $\square$

Because the orthogonal of  $\mathcal{E}_{-1}$  is  $\mathcal{E}_0$ , the Lie subalgebra  $\mathcal{E}_0$  is therefore parabolic with radical  $\mathcal{E}_{-1}$  (see [AB], p. 589). Let  $P$  be the unique standard

parabolic subgroup defined by  $\mathcal{E}_0$ . If  $F$  is the bundle of local trivialization  $G_S \xrightarrow{\sim} E_S$  ( $S \rightarrow X$  étale) whose differential sends  $\text{Lie}(P)_S$  to  $\mathcal{E}_0$ , then  $F$  is a  $P$ -structure of  $E$ . Let us denote by  $U$  the unipotent radical of  $P$  and by  $\bar{P}$  (resp.  $\bar{F}$ ) the quotient  $P/U$  (resp.  $F/U$ ). By construction,  $\bar{F}$  is semistable (because  $\text{Ad}(\bar{F}) = \mathcal{E}_0/\mathcal{E}_{-1}$ ).

**DEFINITION 2.3.** — *With the notation above, the  $P$ -bundle  $F$  is the Harder-Narasimhan reduction of  $E$ .*

**Remark 2.4.** — It is easy to check that the filtration and therefore the corresponding reduction does not depend on the particular choice of the invariant non degenerate quadratic form on  $\text{Lie}(G)$ .

**Example 2.5.** — Suppose that  $E$  is the  $\mathbf{GL}_n$ -bundle of local frames of a vector bundle  $\mathcal{E}$  on  $X$  with Harder-Narasimhan filtration  $0 = \mathcal{E}_0 \dots \subset \mathcal{E}_i \dots \subset \mathcal{E}_k = \mathcal{E}$ . Let  $P$  be the quasi-triangular subgroup of  $\mathbf{GL}_n$ -defined by the partition  $[r_i = \text{rk}(\mathcal{E}_{i+1}) - \text{rk}(\mathcal{E}_i)]_{0 \leq i < k}$  of  $\text{rk}(\mathcal{E})$ . Then  $F$  is the  $P$ -bundle of local frames compatible with the filtration and  $\bar{F}$  is the  $\times_i \mathbf{GL}_{r_i}$ -bundle of local frames of  $\bigoplus_i \mathcal{E}_{i+1}/\mathcal{E}_i$ .

**We suppose once for all that  $X$  is an elliptic curve.**

### 3. Non semistable $G$ -bundles.

Let  $E$  be a  $G$ -bundle on  $X$  and let  $F$  be the Harder-Narasimhan reduction of  $E$ . Let us consider a Levi factor of  $P$  thought as a section  $\sigma : \bar{P} \rightarrow P$  of the canonical projection  $P \twoheadrightarrow \bar{P} = P/U$ .

**Remark 3.1.** — Following Humphreys (see [Hu], 30.2) a Levi factor is a factor of the unipotent radical and not of the radical itself (as in Bourbaki for instance).

**PROPOSITION 3.2.** — *With the notations above, the  $P$ -bundle  $\sigma_*(\bar{F})$  is isomorphic to  $F$ .*

**Remark 3.3.** — This is the generalization of the well-known (and easy) fact that any vector bundle on  $X$  is a direct sum of semistable vector bundles.

*Proof.* — Let us denote by

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{P} \rightarrow \bar{\mathcal{P}} \rightarrow 1$$

be the twist of

$$1 \rightarrow U \rightarrow P \rightarrow \bar{P} \rightarrow 1$$

by  $F$  (see [S], chap. I & 5). Geometrically,  $\mathcal{P}$  (resp.  $\bar{\mathcal{P}}$ ) is the group scheme  $\mathcal{A}ut_P(F)$  (resp.  $\mathcal{A}ut_{\bar{P}}(\bar{F})$ ). The twisted group  $\mathcal{U}$  is the unipotent radical of  $\mathcal{P}$  and is isomorphic  $F \times_P U$  ( $P$  acts on the normal subgroup  $U$  by conjugation). As usual, the map

$$\left\{ \begin{array}{ccc} H^1(X, P) & \rightarrow & H^1(X, \mathcal{P}) \\ F'' & \mapsto & \mathcal{I}som_P(F, F'') \end{array} \right. \text{ resp. } \left\{ \begin{array}{ccc} H^1(X, \bar{P}) & \rightarrow & H^1(X, \bar{\mathcal{P}}) \\ \bar{F}'' & \mapsto & \mathcal{I}som_{\bar{P}}(\bar{F}, \bar{F}'') \end{array} \right.$$

are bijective. The image of  $\mathcal{I}som_P(F, \sigma_*\bar{F})$  in  $H^1(X, \bar{\mathcal{P}})$  is the trivial torsor  $\mathcal{I}som_{\bar{P}}(\bar{F}, \bar{F})$  and it is enough to show the equality  $H^1(X, \mathcal{U}) = \{[\mathcal{U}]\}$  to prove the isomorphism  $F \xrightarrow{\sim} \sigma_*(\bar{F})$ . With the notations of (2.4), the Lie algebra of  $\mathcal{U}$  is  $\mathcal{E}_{-1}$ . By (2.5), the Lie algebra  $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}$  is abelian for any  $j \geq 1$ . This induces a filtration

$$1 = \mathcal{U}_{-r} \subset \mathcal{U}_{-r+1} \subset \dots \subset \mathcal{U}_{-2} \subset \mathcal{U}_{-1} = \mathcal{U}$$

by unipotent group schemes where the exponential defines isomorphisms

$$\mathcal{U}_{-j}/\mathcal{U}_{-j-1} \xrightarrow{\sim} \mathcal{E}_{-j}/\mathcal{E}_{-j-1} \quad j \geq 1$$

of abelian group schemes. By construction,  $\mathcal{E}_{-j}/\mathcal{E}_{-j-1}, j \geq 1$  is semistable of positive slope and therefore

$$H^1(X, \mathcal{E}_{-j}/\mathcal{E}_{-j-1}) = 0, j \geq 1$$

because  $g(X) = 1$ . This implies the equality

$$H^1(X, \mathcal{U}) = \{[\mathcal{U}]\}.$$

□

#### 4. The coarse moduli space $M_G$ .

Let  $M_G$  be the coarse moduli space of semistable  $G$ -bundle of trivial topological type (what is the same, the component containing the trivial torsor  $G_X$ ). Recall that the (closed) points of  $M_G$  are  $S$ -equivalence classes of semistable  $G$ -bundles. The only thing which will be needed about this equivalence relation is the following (cf. [Ra1], Corollary 3.12.1):

4.1. Every class  $\xi$  defines a Levi subgroup  $L$  such that there exists a stable  $L$ -bundle  $F$  with  $F(G) \in \xi$ . Moreover,  $F(G)$  is well defined up to isomorphism.

*Remark 4.2.* — Ramanathan’s construction of  $M_G$  is written for a curve of genus  $\geq 2$ , but the construction can be made in general (see for instance [LeP] in the case of  $G = GL_n$  from which the general case follows).

4.3. We denote by  $a \otimes b$  the product of two  $T$ -bundles  $a$  and  $b$  (for the natural structure of abelian group of  $H^1(X, T)$ ). Let  $\underline{\psi} = (\psi)_{i \in I}$  be a finite family of one parameter subgroups and  $\underline{L} = (L_i \in I)$  a family of line bundles of degree 0 on  $X$  (thought as  $G_m$ -torsors). Then,  $\otimes_{i \in I} L_i(\psi_i)$  is a  $T$ -structure of a  $G$ -bundle  $\underline{L}_{\underline{\psi}}$  on  $X$  which is semistable. This defines a morphism of abelian groups

$$p : \Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow H^1(X, T).$$

Chose a (closed) point  $x$  of  $X$  which defines an isomorphism  $\text{Pic}^0(X) \xrightarrow{\sim} X$  and a Poincaré line bundle  $\mathcal{P}$  on  $X \times \text{Pic}^0(X)$ . This allows to construct a universal semistable  $T$ -bundle  $\mathbf{L}$  on  $X \times \Gamma(T) \otimes_{\mathbf{Z}} X$ .

*Remark 4.4.* — The theta line bundle  $\Theta$  on  $X = \text{Pic}^1(X)$  becomes through the isomorphism  $X \xrightarrow{\sim} \text{Pic}^0(X)$  the determinant bundle  $\det(R\Gamma\mathcal{P})^*$ .

The family of semistable bundles  $\mathbf{L}(G)$  defines a morphism of (reduced) schemes

$$\Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow M_G.$$

The action of the Weyl group  $W$  on  $\Gamma(T)$  defines an action  $\Gamma(T) \otimes_{\mathbf{Z}} X$  such that  $w.L_{\psi} \xrightarrow{\sim} L_{\psi}$  for all  $w \in W$ . Let

$$\pi : [\Gamma(T) \otimes_{\mathbf{Z}} X] / W \rightarrow M_G$$

be the induced morphism. We want to prove that  $\pi$  is an isomorphism.

4.5. Let us prove that  $\pi$  is finite. Let  $G \rightarrow \mathbf{SL}_N$  be a faithful representation of  $G$  inducing a morphism  $M_G \rightarrow M_{\mathbf{SL}_N}$ . Let  $L$  be the inverse of the determinant bundle on  $M_{\mathbf{SL}_N}$ .

*Remark 4.6.* — Notice that in this case,  $M_{\mathbf{SL}_N} = \mathbf{P}^{N-1}$  and that the determinant bundle is just  $\mathcal{O}(1)$  (see [Tu], Theorem 7 for instance).

LEMMA 4.7. — *The line bundle  $\pi^*(L)$  is ample.*

*Proof.* — One can assume that  $G$  is semisimple. Let  $q$  be the natural morphism

$$q : \Gamma(T) \otimes_{\mathbf{Z}} X \rightarrow M_G.$$

It is enough to prove that  $q^*(L)$  is ample. Let us choose a basis of  $\Gamma(L)$  identifying  $\Gamma(T) \otimes_{\mathbf{Z}} X$  with  $X^l$  ( $l$  is the rank of  $G$ ). Let  $\gamma : G_m \rightarrow T$  be a non trivial element in  $\Gamma(T)$ . Let  $q_\gamma : X \rightarrow M_G$  be the morphism defined by  $\gamma$ . One can assume that  $\gamma(z) = \text{diag}(z^{\gamma_1}, \dots, z^{\gamma_l})$  for  $z \in \mathbf{C}^*$  (with  $\sum \gamma_i = 0$ ). Then (see Remark 4.4),

$$q_\gamma^*(L) = \Theta^{\sum_i \gamma_i^2}$$

which is ample because  $\sum_i \gamma_i^2 > 0$  (recall that  $\gamma$  is non trivial). The rank- $N$  vector bundle parameterized by  $(x_1, \dots, x_l)$  is

$$\bigoplus_i \mathcal{O} \left( \sum_{\gamma \in \gamma} \gamma_i(x_\gamma - x) \right).$$

By additivity of the determinant bundle,  $q^*(L)$  is of the form

$$\boxtimes_{1 \leq i \leq l} \Theta^{b_i} \text{ with } b_i > 0$$

and therefore is ample. □

The fibers of  $\pi$  are therefore finite, and the proper morphism  $\pi$  is finite.

4.8. Let  $\pi^{-1}(0)$  be the fiber of  $\pi$  at the trivial bundle  $G_X$ . Let us first prove that  $\pi^{-1}(0)$  is set-theoretically reduced to  $[0]$ , the class  $W.0$ . Let us first prove the following general result.

4.9. Let us consider the following situation: let  $p : \mathcal{X} \rightarrow S$  be a proper morphism such that  $\mathcal{O}_S \rightarrow p_* \mathcal{O}_{\mathcal{X}}$  is an isomorphism. Assume that  $p$  has a section  $\sigma : S \rightarrow \mathcal{X}$ . Let  $A \subset B$  be a reductive subgroup of a linear group  $B$ .

LEMMA 4.10. — *Let  $\alpha$  be an  $A$ -bundle trivial along  $\sigma$ . Then, if the associated  $B$ -bundle  $\beta = \alpha(B)$  is trivial, the  $A$ -bundle  $\alpha$  is so.*

*Proof.* — Let  $s$  be the section of  $\beta/A$  defined by  $\alpha$ . Because  $\beta/A$  is affine over  $S$ , the section  $s$  factors through  $p$  in a section  $\tilde{s}$ . Because  $\alpha$  is trivial along  $\sigma$ , the section  $\tilde{s}$  comes from a section of the restriction to  $\sigma$  of the trivial bundle  $\beta$  and can be lifted to a section  $s'$  of  $\beta$ . The section  $s' \bmod A$  of  $\beta/A$  is equal to  $s$  and defines a trivialization of  $\alpha = s^* \beta$ . □

4.11. Choose an embedding  $G$  in a product  $G' = \prod_i \mathbf{GL}_{n_i}$  of linear groups such that  $Z_0(G) \subset Z_0(G')$  ( $Z_0$  denotes the neutral component). Let  $T'$  be a maximal torus of  $G'$  containing  $T$ . Let  $f : M_G \rightarrow M_{G'}$  be a natural morphism (see [Ra2], Corollary of Theorem 7.1). Let  $E$  be a  $T$ -bundle such that  $E \in \pi^{-1}(0)$  and let  $E'$  be the corresponding  $T'$ -bundle.



Because  $f(E(G)) = [E'(G')]$ , the semistable bundle is equivalent to the trivial bundle and is therefore trivial (a direct sum of line bundles of degree 0 is equivalent to the trivial bundle if and only if all summands are trivial). Applying the preceding lemma with  $\alpha = E, A = T, B = G'$  and  $\mathcal{X} = X$  for instance, one gets that  $E$  is trivial.

4.12. It remains to show that  $\pi$  is étale at the origin: this will follow from the fact that the completion of  $\pi$  at the origin can be identified to the completion at the origin of the Chevalley isomorphism  $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/G$ .

LEMMA 4.13. — *The morphism  $\pi : (\Gamma(T) \otimes \text{Pic}^0(X))/W \rightarrow M_G$  is étale at the origin.*

*Proof.* — Let's briefly recall how to construct the moduli space  $M_G$  (see [Ra1], [BLS]), or better of an affine neighborhood  $M$  of the trivial bundle  $X \times G$  as a GIT quotient  $Y/H$  of a smooth affine scheme  $Y$  by some reductive group  $H$  (with Lie algebra  $\mathfrak{h}$ ). One choose first a faithful representation  $G \hookrightarrow \mathbf{GL}_n$  inducing an embedding  $\Gamma(T) \otimes \text{Pic}^0(X) \hookrightarrow (\text{Pic}^0(X))^n$ . For  $m$  big enough, one knows that the canonical morphism

$$\iota_P : H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \otimes \mathcal{O} \rightarrow P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$$

is surjective for all semistable bundles  $P$  and that  $H^0(\iota_P)$  is bijective. Let  $\chi$  be the Euler characteristic of some  $P(\mathbf{C}^n) \otimes \mathcal{O}(mx)$ . By the theory of Hilbert schemes, the pairs  $(P, \iota)$  where  $P$  is a semistable  $G$ -bundle and  $\iota$  an isomorphism

$$H^0(P(\mathbf{C}^n) \otimes \mathcal{O}(mx)) \xrightarrow{\sim} \mathbf{C}^\chi$$

are parameterized by a smooth scheme  $Y$  and  $M_G$  is a GIT quotient of this scheme by  $H = \mathbf{GL}_\chi$  (see [BLS]). Notice that the stabilizer of the “trivial pair” is  $G$  itself.

Let  $\mathcal{U}$  be the universal  $T$ -bundle on  $X \times (\Gamma(T) \otimes \text{Pic}^0(X))$ . Let us chose a trivialization of the vector bundle  $R\Gamma(\mathcal{L} \boxtimes \mathcal{O}(mx))$  on some symmetric affine neighborhood  $S^0$  of 0 in  $\text{Pic}^0(X)$ . Therefore, the direct image of  $\mathcal{U}(\mathbf{C}^n) \boxtimes \mathcal{O}(m)$  is trivial on  $S = (S^0)^n \cap \Gamma(T) \otimes \text{Pic}^0(X)$  and the trivialization is  $W$ -equivariant. The induced morphism  $\pi : S \rightarrow M_G = Y/H$  is therefore induced by a  $W$ -equivariant morphism  $S \rightarrow Y$  mapping 0 to  $y$ . Notice that the orbit  $H.y$  is closed. By considering some  $H$ -invariant affine open neighborhood of  $y$ , one can assume that  $Y$  is affine (the quotient  $Y/H$  is now a neighborhood of  $H.y$  in  $M_G$ ).

Let's consider the following commutative diagram:

$$(4.1) \quad \begin{array}{ccc} \mathbf{C}[Y]_+ & \longrightarrow & \mathbf{C}[S]_+ \\ \downarrow & & \downarrow \\ V = (T_y^*Y)/\mathfrak{h} & \xrightarrow{k} & T_0^*S \end{array}$$

where  $\mathbf{C}[Y]_+$  (resp.  $\mathbf{C}[S]_+$ ) denotes the maximal ideal of  $y$  (resp.  $0$ ). The transpose of  $k$  is the tangent map of  $S \rightarrow \mathcal{M}_G$  from  $S$  to the stack of  $G$ -bundles on  $X$ , namely the Kodaira-Spencer map

$$k : \mathfrak{t} = \mathfrak{t} \otimes H^1(X, \mathcal{O}_X) = T_0S \rightarrow (T_yY)/\mathfrak{h} = \mathfrak{g} \otimes H^1(X, \mathcal{O}) = \mathfrak{g}.$$

LEMMA 4.14. — *The Kodaira-Spencer map  $k$  is the canonical inclusion  $\mathfrak{t} \hookrightarrow \mathfrak{g}$ .*

*Proof.* — By functoriality, one is reduced to the case where  $G = \mathbf{GL}_n$  and  $T$  is the torus of invertible diagonal matrices. Consider the one parameter subgroup of differential  $aE_{i,i}$  for some integer  $a$  ( $E_{i,i}$  is the standard diagonal rank 1 matrix). If  $(\lambda_{\alpha,\beta})$  is a Cech-cocycle representing  $\lambda \in H^1(\mathcal{O})$ , the derivative

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0)$$

is defined by the vector bundle on  $X[\epsilon]/(\epsilon^2 = 0)$  with cocycle  $1 + a\epsilon\lambda_{\alpha,\beta}E_{i,i}$ . In other words,

$$\frac{\partial \pi}{\partial(\gamma \otimes \lambda)}(0) = \lambda d\gamma,$$

which proves the lemma. □

Notice that  $k$  is  $N(G, T)$ -equivariant. By Luna's results ([Lu]), one obtains an étale slice of  $Y \rightarrow Y/H$  as follows.

One choose an  $H_y = \text{Aut}_G(X \times G) = G$ -invariant section  $\sigma$  of  $\mathbf{C}[Y]_+ \rightarrow (T_y^*Y)/\mathfrak{h}$  and the induced morphism  $Y \rightarrow V$  identifies étale locally  $Y/H$  and  $V/H_y$ . The group  $N(G, T)$  being reductive and  ${}^t k$  being surjective, one pick an invariant section  $\tau$  of  ${}^t k : \mathfrak{g}^* \rightarrow \mathfrak{t}^*$  which defines a morphism (still denoted by  $\tau$ )

$$\mathfrak{t}^* \rightarrow \mathbf{C}[Y]_+ \rightarrow \mathbf{C}[S]_+$$

which is  $W$ -equivariant. This is a  $W$ -equivariant section of  $\mathbf{C}[S]_+ \rightarrow T_0^*S$  and therefore defines an étale slice of  $S \rightarrow S/W$ . Shrinking  $S$  and  $Y$  if necessary, one obtains from the diagram (4.1) the commutative diagram

$$\begin{array}{ccc}
 \mathbf{C}[Y]_+^H & \xrightarrow{\pi} & \mathbf{C}[S]_+^W \\
 \mathbf{S}(\sigma) \uparrow & & \uparrow \mathbf{S}(\tau) \\
 (\mathbf{S} \mathfrak{g}^*)^G & \xrightarrow{\mathbf{S}(\iota k)} & (\mathbf{S} \mathfrak{t}^*)^W
 \end{array}$$

where  $\mathbf{S}(\sigma)$  and  $\mathbf{S}(\tau)$  are étale. By Chevalley’s theorem,  $\pi$  is therefore étale at the origin. □

4.15. The morphism  $\pi$  is therefore a finite morphism between normal varieties and is of degree 1. We have proved the

**THEOREM 4.16.** — *The morphism*

$$\pi : [\Gamma(T) \otimes_{\mathbf{Z}} X]/W \rightarrow M_G$$

*is an isomorphism.*

4.17. Assume that  $G$  is simple and simply connected. Let  $\theta$  be the longest root and  $\alpha_i, i = 1, \dots, l$  the basis of the root system  $\Phi(B, G)$ . The coroot  $\theta^\vee$  is a sum

$$\theta^\vee = \sum_i g_i \alpha_i^\vee$$

where  $\alpha_i^\vee$  is the coroot of  $\alpha$ . Then Looijenga [Lo] has proved that  $[\Gamma(T) \otimes_{\mathbf{Z}} X]/W$  is the weighted projective space  $\mathbf{P}(1, g_1, \dots, g_l)$ .

*Remark 4.18.* — The proof in [Lo] is not correct. See [BS] for a more general result and hints for a complete proof.

4.19. It is interesting to remark ([D], remarques 1.8) that  $\mathcal{O}(l)$  is locally free if and only if  $l$  is a multiple of  $\text{lcm}(g_i)$  although it is reflexive (Lemme 4.1 of *loc. cit.*). In particular,  $M_G$  is locally factorial if and only if  $\text{lcm}(g_i) = 1$ , condition which is equivalent to  $G$  special in the sense of Serre (look at the table of [Bo]). If one notice that  $\text{lcm}(g_i)$  is also the minimal Dynkin index of the representations of  $G$  (see [LS]), this funny characterization of special groups in terms of  $M_G$  is the version in the genus one case of Proposition 13.2 of [BLS] (which deals with the genus  $> 1$ ). In all the cases, one has the formula

$$(4.2) \quad \dim H^0(M_G, \mathcal{O}(l)) = \text{card}(P_l)$$

where  $P_l$  is the number of dominant weights  $w$  such that  $\langle \theta^\vee, w \rangle \leq l$ , as predicted by the Verlinde formula (see [Be]).

4.20. Let us explain briefly the link between the theorem of Narasimhan and Seshadri and our description of  $M_G$ . Suppose that  $G$  is semisimple with maximal compact subgroup  $K$ . The theorem of Narasimhan and Seshadri says that the complex points of  $M_G$  are parameterized by equivalence classes of pairs of elements of  $K$  which commutes ( $K$  acting on these pairs diagonally through the adjoint action). Suppose further that  $G$  is simply connected. Then such a class has a representative in  $T_{\mathbf{R}} \times T_{\mathbf{R}}$  (where  $T_{\mathbf{R}}$  is the maximal torus of  $K$ ). Suppose that  $X(\mathbf{C})$  is a complex torus  $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$  of period  $\tau$  in the Poincaré upper half plane. The complex structure  $(a, b) \rightarrow a - \tau b$  on  $\mathbf{R} \times \mathbf{R}$  induces a complex structure on  $T_{\mathbf{R}} \times T_{\mathbf{R}}$  which is naturally the maximal torus  $T$  of  $G$ . We get a diagram

$$\begin{array}{ccc}
 \Gamma(T) \otimes \mathbf{C} & \xrightarrow{\sim} & T_{\mathbf{R}} \times T_{\mathbf{R}} \\
 \downarrow & & \downarrow \\
 \Gamma(T) \otimes X(\mathbf{C}) & \xrightarrow{\pi} & M_G(\mathbf{C})
 \end{array}$$

One checks easily that this diagram commutes.

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