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AREA PRESERVING pl HOMEOMORPHISMS AND RELATIONS IN K_2

by Peter GREENBERG

Peter Greenberg (1956–1993) came to Grenoble in 1991. He graduated in M.I.T., next holding positions at Fargo, North Dakota and at I.P.N., Mexico City.
He was appreciated everywhere by his students and colleagues.
The present paper is quite representative of the inspired mixture of geometry, algebra and arithmetic one encounters in his work.
It was completed a few weeks before his bike-accident.
Therefore, we feel that this journal is the right place to publish it.

1. Introduction.

This work is part of an effort to get a geometric understanding of the algebraic K-theory of real fields, using an approach not unrelated to those of Lichtenbaum [Li] (involving scissors congruence) and Goncharov and others [BGSV], [G] (using configurations in geometry). Our aim, more or less accomplished here for K_2, is to reproduce a situation in dimension 1, which we now describe.

1.1. The idea of a "derivative".

Let $\mathrm{PL}_c\,\mathbb{R}$ be the group of all piecewise-linear, compactly supported homeomorphisms of the real line, which are C^1 except at a finite number of points.

Key words: Algebraic K-theory – K_2 – Piecewise linear homeomorphisms – Torsion.
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PROPOSITION 1. — *For every $g \in \mathrm{PL}_c\,\mathbb{R}$ there is a function $Dg = \mathbb{R} \to \mathbb{R}^+$ such that*

(1) *$Dg(x) = 1$ except for a finite set of $x \in \mathbb{R}$;*

(2) *If $g, h \in \mathrm{PL}_c\,\mathbb{R}$, $x \in \mathbb{R}$ then $D(gh)(x) = Dg(hx) \cdot Dh(x)$;*

(3) $\prod_{x \in \mathbb{R}} Dg(x) = 1$.

Proof. — Let

$$Dg(x) = \frac{\lim\limits_{y \to x^+} g'(y)}{\lim\limits_{y \to x^-} g'(y)}. \qquad \square$$

Remark 2. — (a) Let k be a subfield of $\mathbb{R}$, and let $\mathrm{PL}_c\,k$ be the subgroup of $g \in \mathrm{PL}_c\,\mathbb{R}$ such that if $Dg(x) \neq 1$ then $x, g(x) \in k$; thus g' where defined, is an element of k^+. Then we consider $Dg : k \to k^+$ and we note:

(b) $k^+ = \ker\bigl(K_1(k) \to \pi_0(\mathrm{GL}(\mathbb{R}))\bigr)$.

(c) The function D arises from a homomorphism $\delta : \mathrm{pl}_0 \to k^+$, where pl_0 is the group of germs of $\mathrm{PL}_c\,k$-homeomorphisms at 0; indeed pl_0 is just $k^+ \times k^+$ and $\delta(s,t) = s/t$. Thus the log function composed with δ gives an element $\mathrm{Log} \in H^1(\mathrm{pl}_0\,;\mathbb{R})$.

(d) Relations in k^+ arise from Proposition 1, part (3). For example, let $x, y \in (0,1)$, $x, y \in k$. Then there is a unique $g \in \mathrm{PL}_c\,k$ such that $g(0) = 0$, $g(1) = 1$, $g(x) = y$ and such that $Dg(t) = 1$, $t \neq 0, x, 1$. The relation in Proposition 1, part (3) is then:

$$\frac{y}{x} \cdot \frac{x(1-y)}{y(1-x)} \cdot \frac{1-x}{1-y} = 1.$$

Our aim is to mimic Proposition 1, and the accompanying remarks, in higher dimensions. In dimension 2 we have the following:

THEOREM 3. — *Let $\mathrm{SPL}_c\,\mathbb{R}^2$ be the group of compactly supported, area preserving, piecewise-linear homeomorphisms of the real plane. For each $g \in \mathrm{SPL}_c\,\mathbb{R}^2$ there is a function $Dg : \mathbb{R}^2 \to K_2(\mathbb{R})$ such that*

(1) *$Dg(v) \neq 1$ for only a finite subset of $v \in \mathbb{R}^2$;*

(2) *for $g, h \in \mathrm{SPL}_c\,\mathbb{R}^2$, $v \in \mathbb{R}^2$ we have*

$$D(gh)(v) = Dg(hv)Dh(v);$$

(3) *for all* $g \in \mathrm{SPL}_c\,\mathbb{R}^2$ *we have*

$$\prod_{v \in \mathbb{R}^2} Dg(v) = 1.$$

Further, parallel to the remarks above we have:

Remark 4. — (a) If $g \in \mathrm{SPL}_c\,\mathbb{R}^2$, the set $\mathrm{sing}(g)$ of points where g is not affine is a finite graph linearly embedded in the plane. Let $\mathrm{supp}(g)$ denote the set of vertices of $\mathrm{sing}(g)$. Let k be a subfield of $\mathbb{R}$, and let $\mathrm{SPL}_c\,k^2$ be the subgroup of $g \in \mathrm{SPL}_c\,\mathbb{R}^2$ such that $\mathrm{supp}(g)$, $g(\mathrm{supp}(g)) \subset k^2$. Then for $g \in \mathrm{SPL}_c\,k^2$.

(b) D_g takes values in $\ker(K_2(\mathbb{R}) \to \pi_1(\mathrm{GL}(\mathbb{R})))$ if $k = \mathbb{R}$, more generally in the subgroup of $K_2(k)$ generated by Steinberg symbols $\{s, t\}$ where at least one of s, t is positive.

(c) The definition of D arises from a homomorphism $w\colon \mathrm{spl}_0\,k \to K_2(k)$, where spl_0 is the group of germs of $\mathrm{SPL}_c\,k^2$ homeomorphisms at the origin. This was described in joint work with Vlad Sergiescu [GS]; its definition and properties will be described below. After Dupont and Parry-Sah [Du], [PS] the dilogarithm appears as a class in $H^2(\widetilde{\mathrm{SL}_2\mathbb{R}};\mathbb{R})$ which by [G1] is isomorphic to $H_1(\mathrm{spl}_0\,;\mathbb{R})$.

(d) Relations in $K_2(k)$ arise from Theorem 3, (3); we now consider some examples. It seems reasonable to conjecture that the reciprocity relation for triangles (Corollary 5 below) provides enough relations to define $K_2(k)$.

1.2. Examples.

Applications of Theorem 3, part (3) to various families of SPL transformations give relations in K_2 of various fields of functions, as will be explained later in the paper. For now we give some sample results.

The first example concerns triangles in the plane.

So let r^i, s^i, $i = 0, 1, 2$ be six points in the real plane, and let r^i_j (resp. s^i_j) be the barycentric coordinates of the points r^i with respect to the triple s^j (resp. s^i with respect to the r^j). Let

$$\chi_i = r^j_i r^k_j + r^k_i r^j_k + r^j_i r^k_i, \quad i \neq j \neq k.$$

Then we have:

Corollary 5. — *In K_2 of the field of rational functions in the r^i_j, the following relation holds :*

$$\prod_i \left\{1 - \frac{r_i^{i+1} r_{i+1}^i}{r_{i+1}^{i+1} r_i^i} \cdot \frac{r_i^{i+1} \chi_{i+1}}{r_{i+1}^i \chi_i}\right\} = (\textit{same expression with the } s^i_j)^{-1}.$$

(We note that $\chi_i = R(s_i^j s_j^k + s_i^k s_k^j + s_i^j s_i^k)$, where R is the determinant of the matrix of r^i_j.)

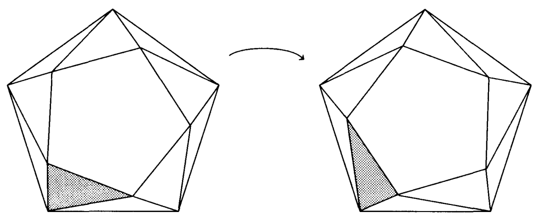

Figure 1: Partial twist of a pentagon in another

Here, and in the following example, the expressions in brackets are Steinberg symbols. SPL maps with symmetries produce, *via* Theorem 3, part (3), elements of torsion in K_2. For example a "partial twist" of an N-gon inside another N-gon (see figure 1) yields the:

Corollary 6. — *Let $N > 2$ and let $T_N = 2\pi/N$. Let*

$$x = \frac{-A\sin(T_N) - R\sin(\phi) + R\sin(T_N + \phi)}{A\sin(T_N) - R\sin(\phi) + R\sin(\phi - T_N)},$$
$$y = \frac{-A\sin(T_N) + R\sin(\phi) - R\sin(\phi - T_N)}{A\sin(T_N) - R\sin(\phi - T_N) + R\sin(\phi - 2T_N)},$$

$$a = \frac{R\sin(T_N) + A\sin(\phi) - A\sin(T_N + \phi)}{R\sin(T_N) - A\sin(\phi) + A\sin(\phi - T_N)},$$
$$b = \frac{R\sin(T_N) - A\sin(2T_N - \phi) - A\sin(\phi - T_N)}{R\sin(T_N) - A\sin(\phi) + A\sin(\phi - T_N)}.$$

Then the expression

$$\left\{1 - \frac{x}{y}, y\right\} \cdot \{1 - ab, a\}$$

is an element of order N, if N is even, $2N$ if N is odd, in K_2 of the field of rational functions in $A, R, \sin(T_N), \cos(T_N), \sin(\phi), \cos(\phi)$. (Note that all of the sin *functions in the expressions for x, y, a, b are elements of this field.)*

We can see that the result is nontrivial as follows: set $N = 4$, $A = R = 1$, and $t = \tan(\phi)$. Then in the field of rational functions in t, the expression

$$\left\{\frac{4t-t^2-1}{4}, \frac{(t-1)^2}{2}\right\} \cdot \left\{\frac{4(t-1)^2}{(4t-t^2-1)(3t^2-1)}, \frac{3t^2-1}{4t-t^2-1}\right\}$$

is an element of order 4. Now set $t = 8/15$ and take the same symbol at the prime 17; one finds -4, which is indeed an element of order 4, but not 2, in $(\mathbb{Z}/17)^*$. It is also not hard to see that neither Steinberg symbol is trivial in K_2.

The organization of this paper is as follows: in the following section, the map w and the function D are defined, after a brief review on $K_2(2,k)$. In Section 3, the three unit formula of Jun Morita [Mor] is applied to evaluate w, in the simplest nontrivial cases, in terms of Steinberg symbols. This leads to the calculations of Corollaries 5 and 6 in Section 4. Finally, Section 5 contains a conjecture in piecewise $\mathrm{SL}_2(\mathbb{Z})$ geometry. This work has profited from discussions with J. Helmstetter, R. Kenyon, A. Martin, N. Moser, G. Robert and V. Sergiescu, to whom I am very grateful.

2. Definition of w and D.

The definition of w given here is joint work with Vlad Sergiescu [GS].

2.1. Background of K-theory.

Let k be a subfield of the real numbers. The group $\mathrm{SL}_2(k)$ acts transitively on the circle

$$S_k = \left(k^2 - (0,0)\right)/k^+$$

of rays from the origin. The stabilizer of the ray through $(s,1)$ (as well as that through $(-s,-1)$ is called P_s and is isomorphic to k; explicitly

$$P_s = \left\{\begin{pmatrix} 1+t & -ts \\ t/s & 1-t \end{pmatrix};\ t \in k\right\}.$$

Further, the stabilizer of the rays through $(1,0)$ and $(-1,0)$ is denoted and given by

$$P_0 = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix};\ t \in k\right\}$$

and the stabilizer of the rays through $(0,1)$ and $(0,-1)$ is denoted and given by

$$P_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} ;\ t \in k \right\}.$$

In fact P_0 and P_∞ generated $\mathrm{SL}_2\, k$ and, setting for $r \in k$, $t \in k^*$,

$$e_a^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad e_{-a}^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad v_{\pm a}^t = \begin{pmatrix} 0 & \pm t^{\pm 1} \\ \mp t^{\mp 1} & 0 \end{pmatrix},$$

where, (see [Mor]), a and $-a$ are the roots in the Weyl system for $\mathrm{SL}_2\, k$, we have

$$e_{\pm a}^t e_{\pm a}^s = e_{\pm a}^{t+s} \quad \text{and} \quad v_{\pm a}^u e_{\pm a}^r v_{\pm a}^{-u} = e_{\mp a}^{-u^{-2} r}.$$

The *Steinberg group* $\mathrm{St}_2\, k$ is defined to be the group generated by $x_{\pm a}(r)$, $r \in k$, with the relations

$$x_{\pm a}(s)\, x_{\pm a}(t) = x_{\pm a}(s+t), \quad w_{\pm a}^u\, x_{\pm a}(r)\, w_{\pm a}^{-u} = x_{\mp a}(-u^{-2} r),$$

where $w_{\pm a}^u = x_{\pm a}(u)\, x_{\mp a}(-u^{-1})\, x_{\pm a}(u)$; there is an evident surjection

$$\mathrm{St}_2\, k \longrightarrow \mathrm{SL}_2\, k,$$

whose kernel is called $K_2(2,k)$.

Now set $h_a(u) = w_a^u\, w_a^{-1} \in \mathrm{St}_2(k)$, and set

$$c(s,t) = h_a(uv)\, h_a(u)^{-1} h_a(v)^{-1};$$

one verifies that $c(s,t) \in K_2(2,k)$.

As it happens [Mil], [Mor], for k a field, $K_2(2,k)$ is central in $\mathrm{St}_2\, k$ and surjects onto the group $K_2(k)$, defined *via* a stable version of the above; by Matsumoto's theorem we can take $K_2(k)$ to be the abelian group generated by the images of the $c(s,t)$. These are the antisymmetric, bilinear *Steinberg symbols* $\{s,t\}, s,t \in k^*$, with the relations $\{t, 1-t\} = 1$, $t \neq 0,1$ see [Mor], [Mil].

The following lemma is crucial to the definition of w.

LEMMA 7. — *Let $\overline{P}_0 \subset \mathrm{St}_2\, k$ be the subgroup isomorphic to k, consisting of the $x_a(t)$. Then for each $s \in k$ there is a unique conjugate $\overline{P}_s$ of $\overline{P}_0$ lying over P_s, which consists of the $x_{-a}(1/s)\, x_a(t)\, x_{-a}(-1/s)$.*

Proof. — This is a consequence of the definitions, and of the centrality of $K_2(2,k)$. □

Remark 8. — It is perhaps helpful, as regards the lemma, to consider the well-defined lifts of the P_s to the topological cover of $\mathrm{SL}_2(k)$ (that is, the set of periodic lifts of elements of $\mathrm{SL}_2(k)$ to $\mathbb{R} = \widetilde{S}_{\mathbb{R}}$). These are just the lifts whose elements have fixed points.

With Lemma 7 in place, we now define, following joint work with Sergiescu [GS] the homomorphism

$$w\colon \mathrm{spl}_0\, k \longrightarrow K_2(k),$$

and then the functions Dg. Let $\gamma \in \mathrm{spl}_0$. Then (see figure 2), γ is determined by a finite set of ordered pairs (r_i, g_i), $i = 1, \dots, n$ where the $r_i \in S_k$ are in counterclockwise order and where $\gamma \equiv g_i \in \mathrm{SL}_2\, k$ in the sector from r_i to r_{i+1}.

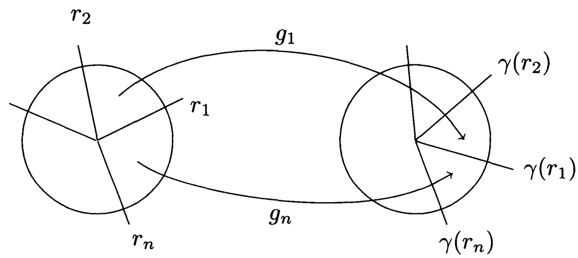

Figure 2. A germ

Now $p_i = g_i^{-1} g_{i-1} \in P_{r_i}$, and of course $p_n p_{n-1} \cdots p_1 = 1$ in $\mathrm{SL}_2\, k$. We define

$$\bar{w}(\gamma) = \bar{p}_n \cdots \bar{p}_1,$$

where $\bar{p}_i$ is the lift by Lemma 7; evidently $\bar{w}(\gamma) \in K_2(2,k)$ and we set $w(\gamma)$ to be its image in $K_2(k)$.

As it happens (see [GS], [G]), $\bar{w}$ is an isomorphism

$$H_1(\mathrm{spl}_0\, k) \longrightarrow H_2(\widetilde{\mathrm{SL}_2 k}) \cong K_2(k,2).$$

We define then, for $g \in \mathrm{SPL}_c\, k, v = (x,y) \in \mathbb{R}^2$, $Dg(v) = w(\gamma)$, where

$$\gamma(s,t) = g\big((s,t) + (x,y)\big) - g(x,y),$$

and now proceed to the:

Proof of Theorem 3. — We first show part (1), that Dg has support in the set $\mathrm{supp}(g)$ of vertices of the singular graph $\mathrm{sing}(g)$ of g; in fact if $v \in \mathbb{R}^2 - \mathrm{sing}(g)$, $Dg(v)$ is evidently trivial, and if $v \in \mathrm{sing}(g) - \mathrm{supp}(g)$, then $\bar{w}$ has the form $\bar{p}^{-1}\bar{p}$ and so is trivial.

The part (2), $Dgh(v) = Dg(hv)Dh(v)$, follows from the fact (see [GS]) that w is a homomorphism.

The proof of part (3), that

$$\prod_{v\in\mathbb{R}^2} Dg(v) = 1,$$

turns on the fact that $K(2,k)$ is *central* in $\mathrm{St}_2\,k$. That is if $\gamma \in \mathrm{spl}_0$ then $w(\gamma) = \bar{p}_n \cdots \bar{p}_1 = \bar{p}_{n-1} \cdots \bar{p}_1 \bar{p}_n$, and so on. We formulate the proof as follows:

Let $g \in \mathrm{SPL}_c$, and let $* \in \mathbb{R}^2$ be a point far from $\mathrm{sing}(g)$. We define a homomorphism

$$\mathcal{D}\colon \pi_1(\mathbb{R}^2 - \mathrm{supp}(g), *) \longrightarrow K_2(2,k).$$

Let $\alpha \in \pi_1(\mathbb{R}^2 - \mathrm{supp}(g))$ be represented by a path a transverse to $\mathrm{sing}(g)$. Let $c_1, \dots, c_m$ denote the connected components of $\mathbb{R}^2 - \mathrm{sing}(g)$ which are traversed by the curve a, in the order in which a crosses them (thus, a component, such as the unbounded one, may occur twice in the list). Let $j_1, \dots, j_m \in \mathrm{SL}_2\,k$ be the linear parts of the restriction of g to the c_i, let $p_i = j_i^{-1} j_{i-1}$, let $\bar{p}_i \in \mathrm{St}_2\,k$ be its lift, and set $\mathcal{D}(\alpha) = \bar{p}_m \cdots \bar{p}_1$. Then $\mathcal{D}$ is a homomorphism, and if a encloses just one vertex $v \in \mathrm{supp}(g)$, we have $\mathcal{D}(\alpha) = D(g)$ by the centrality remarked above. Then if A is a curve contained in the unbounded component of $\mathbb{R}^2 - \mathrm{sing}(g)$, the fact that $\mathcal{D}(A) = 1$ concludes the proof.

3. Calculations with Jun Morita's formula.

In this section, the "three unit formula" of J. Morita [Mor] is applied to calculate $w(\gamma)$, in terms of Steinberg symbols, in the simplest nontrivial cases.

We begin by recalling the notation of the previous section; thus a germ $\gamma \in \mathrm{spl}_0$ is described by a finite set of pairs $(r_i, g_i) \in S_k \times \mathrm{SL}_2\,k$, $i = 1, \dots, n$ with the r_i in counterclockwise order, and with $g_i \neq g_{i+1}$. As remarked in the proof of Theorem 3, part (1), $w(\gamma) = 1$ if $n = 1$ or if $n = 2$. It is not hard to see that there are no examples with $n = 3$, so we will calculate $w(\gamma)$ in the cases where $n = 4$.

We distinguish two cases (see figure 3). Indeed, after composition with elements of $\mathrm{SL}_2\, k$ (on which w is trivial) we can assume that $g_1 \equiv \mathrm{id}$ and that γ is supported (case (a)) in the lower-right quarter circle from $(0,-1)$ to $(1,0)$ (in rectilinear coordinates), or (case (b)) in the exterior of the upper-right quarter circle from $(1,0)$ to $(0,1)$. In case (b), we also assume that the angle from r_i to r_{i+1} is less than π. In the calculations, we will refer to the notations in the figure.

Recalling Lemma 7 and the definition of w in the previous section, it is evident that our task is to translate strings of $x_{\pm a}s$ into Steinberg symbols. The three unit formula of J. Morita [Mor], applied to fields, is exactly what is needed for the cases we consider. We recall the statement (in the case of a field):

THEOREM 9 (see Morita [Mor], Theorem 3). — *Let $t, u, v \in k^*$ and set $A = u + v - t$. Then we have*

$$x_{\mp a}(A)\, x_{\pm a}(-1/u)\, x_{\mp a}(Au/v)\, x_{\pm a}(v/tu)\, x_{\mp a}(-At/v)\, x_{\pm a}(1/t) = \{t/v, u\} \cdot \{t, -v\}^{-1} \in K_2(2, k).$$

We now apply the three unit formula to calculate:

COROLLARY 10.

(a) *Let $\gamma \in \mathrm{spl}_0$ as in case (a). Then $xt = yz$ and*

$$w(\gamma) = \Big\{1 - \frac{x}{y}, \frac{y}{z}\Big\}.$$

(b) *Let $\gamma \in \mathrm{spl}_0$ as in case (b). Then $ab = \alpha\beta < 1$ and*

$$w(\gamma) = \big\{1 - ab, a\beta\big\}.$$

Consequently, the image of w is the subgroup of $K_2(k)$ generated by symbols $\{s, t\}$ where at least one of s, t, is positive.

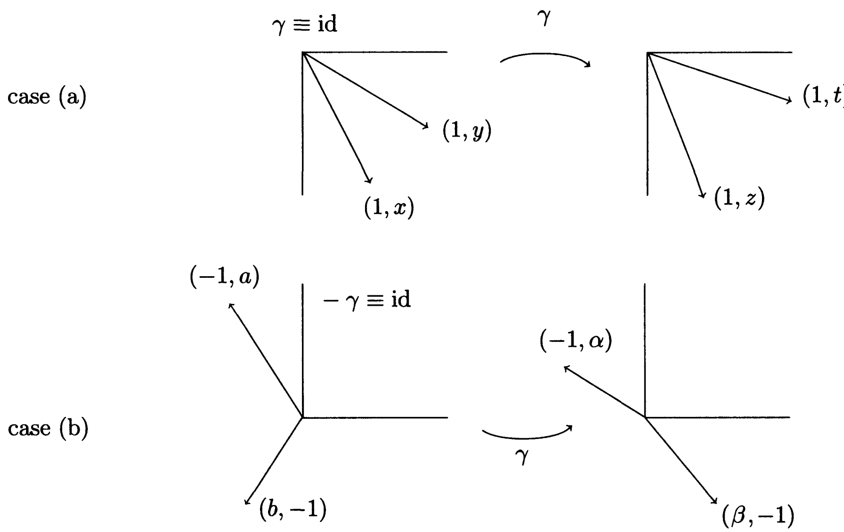

Figure 3. Notation for Corollary 10 — application of the Morita formula — cases (a) *and* (b)

Proof. — We prove the corollary for case (a), a similar computation establishes case (b). Note first that

$$g_2 = e_{-a}^{z-x}, \quad g_4 = e_a^{t^{-1}-y^{-1}}.$$

Thus

$$g_3(1,x) = (1,z), \quad g_3(1,y) = (y/t, y)$$

and so since g_3 preserves area, $xt = yz$. We now calculate

$$p_1 = \begin{pmatrix} 1 & t^{-1} - y^{-1} \\ 0 & 1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 & 0 \\ x-z & 1 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 1-\sigma & \sigma/x \\ -\sigma x & 1+\sigma \end{pmatrix}, \quad p_4 = \begin{pmatrix} 1-\tau & \tau/y \\ -\tau y & 1+\tau \end{pmatrix}$$

with $\sigma, \tau \in k$. Recalling now that

$$\begin{pmatrix} 1-q & q/s \\ -qs & 1+q \end{pmatrix} = e_{-a}^{s}\, e_a^{q/s}\, e_{-a}^{-s},$$

we see that

$$\begin{aligned} w(\gamma) &= x_a(t^{-1} - y^{-1})\, x_{-a}(y)\, x_a(\tau/y)\, x_{-a}(-y)\, x_{-a}(x) \\ &\qquad\qquad x_a(\sigma/x)\, x_{-a}(-x)\, x_{-a}(x-z) \\ &= x_{-a}(-z)\, x_a(t^{-1} - y^{-1})\, x_{-a}(y)\, x_a(\tau/y)\, x_{-a}(x-y)\, x_a(\sigma/x) \end{aligned}$$

and, since we know *a priori* that $w(\gamma) \in K_2(k)$, we can apply the three unit formula. One calculates, in the notation above, that

$$A = -z, \quad -1/u = t^{-1} - y^{-1}, \quad Au/v = y, \quad v/tu = \tau/y$$

and then finds

$$A = -z, \quad u = ty/(t-y), \quad v = -z^2/(z-x), \quad t = z(y-x)/(z-x).$$

Now the three unit formula gives

$$w(\gamma) = \left\{\frac{x-y}{z}, \frac{yz}{z-x}\right\} \cdot \left\{\frac{z(y-x)}{z-x}, \frac{z^2}{z-x}\right\}^{-1}$$

and it suffices to apply bilinearity and the identity $\{-s, s\} = 1$ several times to arrive at the stated result.

In the examples below, we need to calculate the value of w for germs with only four singular rays, but not, necessarily in the form of case (a) or (b). This is done (see (b) of figure below) by pre- and post-composing the germ in question with affine transformations, on which w vanishes.

4. Examples for function fields.

The purpose of this section is to give some of the details in the calculations in Corollaries 5 and 6, and to show why the relations obtained hold in K_2 of fields of functions. In fact, for each corollary we will provide a family of examples which contains a *generic* point (in the sense that the values of the relevant variables are algebraically independent); thus the result at this point lifts to a result for the field of functions of the family.

It is then of interest to look at how the geometry of singular points of the family is reflected in the value of the tame symbol at such points.

4.1. Corollary 6; twists of N-gons and torsion.

Consider figure 4 next page, showing a regular N-gon, inscribed in a circle of radius R, twisting by an angle of $2\pi/N - 2\phi$ in a regular N-gon inscribed in a circle of radius A; the transformation leaves the exterior of the larger N-gon fixed. We note that such examples exist for an *open set* of values of A, R, ϕ and thus the relation of Theorem 3, part (3) holds in the function field of rational functions in variables A, $R\cos(\phi)$, $\sin(\phi)$ with coefficients in $\mathbb{Q}$ with $\cos(2\pi/N)$, $\sin(2\pi/N)$ adjoined.

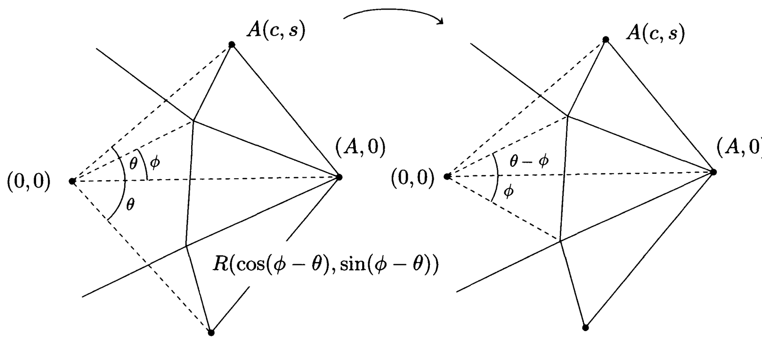

Figure 4. Twisting N-gon

Note that the figure has N-fold rotational symmetry; thus the value of Dg at points $A(\cos(2\pi k/N), \sin(2\pi k/N))$ is independent of k, and likewise at points $R(\cos(\phi + 2\pi k/N), \sin(\phi + 2\pi k/N))$. In fact, the calculations show that

$$Dg(A,0) = \{1 - x/y, y\}^2, \quad Dg(R\cos\phi, R\sin\phi) = \{1 - ab, a\}^2$$

where x, y, a, b are as given in the statement of Corollary 6 which explains the exponent $2N$ in the statement; we only get N for N even because of considerations involving the fact that $-\mathrm{Id}$ is in the centre of $\mathrm{SL}_2\,\mathbb{R}$.

The "twists" of Corollary 6 can be considered as involutions defined on an open subset of the variety of pairs of regular N-gons centered at the origin. An involution on the variety of pairs of triangles in the plane provides the transformation from which Corollary 6 is calculated.

4.2. Corollary 5; triangle reciprocity.

Richard Kenyon encouraged the author to consider this family of examples, and Michel Brion and Jacques Helmstetter made very helpful suggestions which lead to the (more-or-less) reasonable statement of the corollary.

We consider, as in the statement of the corollary triangles r^i, s^i, $i = 0, 1, 2$ where we impose the circular order $0 < 1$, $1 < 2$, $2 < 0$ on the i. Let r^i_j (resp. s^i_j) be the j-th barycentric coordinate of r^i with respect to (s^0, s^1, s^2) (resp. for the s^i_j). We recall that the matrices $(s^i_j), (r^i_j)$ are inverse.

Now (see figure 5) there is an open set of s^i, r^i so that the indicated areas are positive, and we search for an SPL$_c$ transformation g, fixing the outside of triangle s^i. Such a transformation is completely described by a new triangle ρ^i so that (see figure 5)

$$\text{area}(\Delta\rho) = \text{area}(\Delta r), \ \text{area}(I) = \text{area}(I'), \ \ldots, \ \text{area}(VI) = \text{area}(VI').$$

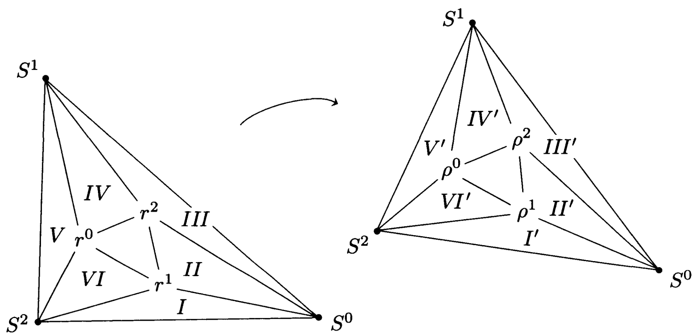

Figure 5

The areas of the regions in question are (up to a multiple of the area of triangle s^i or r^i) the barycentric coordinates r^i_i, s^i_i and with a little algebra one arrives at the following necessary and sufficient conditions on the ρ^i_j:

$$\rho^i_i = r^i_i, \quad \rho^i_j \rho^j_i = r^i_j r^j_i, \quad \rho^i_j + \rho^i_k = r^i_j + r^i_k.$$

This leads a quadratic problem, for which the r^i_j is one (trivial) solution (that is $g = \text{id}$ is a SPL$_c$ transformation as required), the other being

$$\rho^i_j = r^i_j - \frac{\Delta}{\chi_i}, \quad \rho^j_i = r^j_i + \frac{\Delta}{\chi_j}, \quad i < j,$$

where

$$\Delta = r^1_2 r^2_0 r^0_1 - r^1_0 r^0_2 r^2_1,$$

$$\chi_i = r^j_i r^k_j + r^k_i r^j_k + r^k_i r^j_i, \quad i \neq j \neq k,$$

and where we note as well the helpful relation (recall the circular order $0 < 1$, $1 < 2$, $2 < 0$):

$$\Delta = r^i_j \chi_i - r^j_i \chi_i, \quad i < j.$$

Using coordinates with respect to the points s^k, we proceed to calculate the $Dg(s^k)$. In the notation of case (a) of the corollary to Morita's formula we find (for s^2 for example) $x = -r^0_0/r^0_1$, $y = -r^1_0/r^1_1$, $z = -\rho^0_0/\rho^0_1$ and finally:

$$Dg(s^k) = \left\{1 - \frac{r^i_j r^j_i}{r^i_i r^j_j}, \frac{\chi_j r^j_i}{\chi_i r^i_j}\right\};$$

the product of these gives the left hand side of the relation of the corollary.

In order to calculate the value of Dg at the points r^i, which is easier in coordinates with respect to these points, let (σ^i_j) be the inverse of the matrix ρ^i_j. One finds that

$$\begin{aligned}
\sigma^i_j &= s^i_j + \det(s^i_j)\,\frac{\Delta}{\chi_i}, \quad \sigma^j_i = s^j_i - \det(s^i_j)\,\frac{\Delta}{\chi_i}, && i < j;\\
\chi_k &= \frac{1}{\det(s^i_j)}\,(s^i_k s^j_i + s^j_k s^i_j + s^i_k s^j_k), && i \neq j \neq k;\\
\delta &= \frac{1}{\det(s^i_j)}\,(s^i_j \chi_i - s^j_i \chi_j), && i < j;
\end{aligned}$$

and so that, in the notation of case (b) of the corollary, at the point r^2 (for example) we have $a = -s^0_1/s^0_0$, $b = -s^1_0/s^1_1$, $\beta = -\sigma^1_0/\sigma^1_1$; one calculates:

$$Dg(r_k) = \left\{1 - \frac{s^i_j s^j_i}{s^i_i s^j_j}, \frac{s^i_j \chi_i}{s^j_i \chi_j}\right\}.$$

The product of these terms gives the right-hand side of the equation in Corollary 5.

5. Conjecture in $P\mathbb{Z}$ geometry.

The investigations in this paper are partially motivated by one of a collection of "rigidity" conjectures in $P\mathbb{Z}$ geometry. To explain the conjecture, we recall some aspects of this geometry (see [G2]; especially conjecture after Theorem 2.4), which is, in effect, area-preserving pl geometry over $\mathbb{Z}$.

We fix attention on $\mathbb{Z}^2 \subset \mathbb{R}^2$, and define $\mathbb{A}$ to be the group of affine transformations of the form

$$v \longmapsto Av + b, \quad v \in \mathbb{R}^2, \quad A \in \mathrm{SL}_2\,\mathbb{Z}, \quad b \in \mathbb{Z}^2.$$

An *integral line* is a line passing through at least two points in $\mathbb{Z}^2$, a *polygon* is a polygon whose vertices are in $\mathbb{Z}^2$; these notions are preserved by the group $\mathbb{A}$.

A homeomorphism between (the closures of the interiors of) two polygons $g: P \to Q$ is $P\mathbb{Z}$ if there is a finite set of integral line $\ell_1, \ldots, \ell_n$ such that the restriction of g to any connected component of $P - \coprod \ell_i$ is in $\mathbb{A}$. We denote by $P\mathbb{Z}(P)$ (resp. $P\mathbb{Z}(P, \partial)$ the set of $P\mathbb{Z}$ homeomorphisms of the polygon P (resp. the set of $P\mathbb{Z}$ homeomorphisms of P which fix the boundary pointwise).

A weak form of the conjecture in [G2] is:

CONJECTURE 1. — *Let T be the triangle with vertices* $(0,0)$, $(1,0)$, $(0,1)$. *Then* $P\mathbb{Z}(T, \partial) = \{\mathrm{id}\}$.

(On the other hand, if P is the polygon with vertices $(0,0)$, $(2,0)$, $(1,1)$, $(0,1)$ then $P\mathbb{Z}(P\partial)$ contains a copy of $\mathbb{Z}$.)

Some other conjectures are that $P\mathbb{Z}(P)$ is finitely presented for any polygon P, and that given a genus g, and a positive integer N, the number of $P\mathbb{Z}$ surfaces (that is, 2-manifolds whose charts are $P\mathbb{Z}$) of genus g with area less than N is finite. Richard Kenyon has constructed a $P\mathbb{Z}$ sphere of area 1 — is there more than one with this area?

To state the conjecture related to this paper a little more notation is required. Let

$$L_N = \big\{(a/N, b/N)\,;\ ((a,b), N) = 1\big\}.$$

It is not hard to show that the L_N, $n \geq 1$, are the orbits of the action of $\mathbb{A}$ on $\mathbb{Q}^2$. Let G_N be the group of germs of $P\mathbb{Z}$ homeomorphisms at the point $(1, 1/N)$, and let $\mathbb{A}_N$ be its abelianization. Then if g is a $P\mathbb{Z}$ homeomorphism, and $v \in L_N$ there is a well defined "derivative" $Dg(v) \in \mathbb{A}_N$.

Let $P\mathbb{Z}_c$ be the union, over all polygons P, of the groups $PZ(P, \partial)$.

CONJECTURE 2. — *Let $g \in P\mathbb{Z}_c$. Then $\sum Dg(v) = 0$, where the sum is over all $v \in L_N$ and the equation is in the group $\mathbb{A}_N$.*

As it happens this conjecture is a consequence of the conjecture of [G2].

Notice that the proof of Theorem 3 does not apply.

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