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INJECTIVE MODELS OF G -DISCONNECTED SIMPLICIAL SETS

by Marek GOLASIŃSKI

The purpose of this paper is to extend the rational homotopy theory on G -disconnected simplicial sets being not necessary of finite G -type.

Sullivan [12] introduced the rational de Rham theory for connected simplicial complexes and applied it to show that the de Rham algebra A_X^* of \mathbb{Q} -differential forms on a simply connected complex X of finite type determines its rational homotopy type. The central results of Sullivan's theory has been generalized by Triantafillou (see [13], [15]) to equivariant context but under the assumption that a simplicial set X of finite type with a finite group G action is G -connected and nilpotent, *i.e.* the fixed point simplicial subsets X^H are nonempty, connected, and nilpotent for all subgroups $H \subseteq G$. In this case not only A_X^* with the induced G -action are considered but also the de Rham algebras $A_{X^H}^*$ of X^H for all subgroups $H \subseteq G$. It means that a functor \mathcal{A}_X^* on the category $\mathcal{O}(G)$ of canonical orbits is studied and its componentwise injectivity is the key observation for the existence of an equivariant analog of Sullivan's minimal models. Unfortunately, G -connectedness is a much more severe restriction on a G -simplicial set than connectedness is in the nonequivariant context, since it is impossible to break up a G -simplicial set into "connected components", as one would do nonequivariantly. Therefore, instead of the orbit category $\mathcal{O}(G)$, we have to work in this paper over the category $\mathcal{O}(G, X)$ with one object for each component of each fixed point simplicial subset X^H of a G -simplicial set X for all subgroups $H \subseteq G$.

This paper grew out of our attempt to understand and generalize

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the Ph. D. thesis by B.L. Fine (see [4], the Chicago University, 1992). It is not clearly written there, but the used methods and obtained results can be only applied for G -CW-spaces of finite G -type and finitely G -connected (*i.e.* all its fixed point subspaces own a finite number of components). Namely, the category $\mathcal{O}(G, X)$ is infinite in general, even for a G -CW-space X of finite G -type. Put $k\text{-Mod}^f$ (resp. $k\text{-Mod}$) for the category of k -modules (resp. finitely generated) over a field k . It does not arise from this thesis if the category of functors $\mathcal{O}(G, X) \rightarrow k\text{-Mod}^f$ or $\mathcal{O}(G, X) \rightarrow k\text{-Mod}$ is considered. However, none of them is appropriate to be applied for any (even of finite G -type) G -CW-space. In the category of functors $\mathcal{O}(G, X) \rightarrow k\text{-Mod}^f$ sufficiently many injective objects do not exist. Moreover, in the category of functors $\mathcal{O}(G, X) \rightarrow k\text{-Mod}$ the tensor product of two injective objects is not injective. These two properties are crucial to make further steps in the Ph. D. thesis by B.L. Fine for studies of (even of finite G -type) G -CW-spaces.

Here is a brief summary of the paper. In Section 1 we investigate the category $k\mathbb{I}\text{-Mod}$ of covariant functors on a small category \mathbb{I} to the category of k -modules over a field k . This approach is inspired by a category of functors on categories related to the orbit category $\mathcal{O}(G)$ determined by a finite group G . For simplicity we replace these categories by an EI -category \mathbb{I} (*i.e.* a small category such that all endomorphisms are isomorphisms). We introduce basic notions and present some prerequisites about injective objects in the category $k\mathbb{I}\text{-Mod}$.

Unfortunately, injective $k\mathbb{I}$ -modules are not preserved by tensor product. Therefore, we move to the category of functors from an EI -category \mathbb{I} to the very useful but rather neglected category $k\text{-Mod}^c$ of linearly compact k -modules considered already by Lefschetz in [10]. We recall the basic terminology in the category $k\text{-Mod}^c$, define complete tensor product and prove Proposition 1.4 on its behaviour on linearly compact k -modules. Then complete tensor and symmetric powers are defined in the category of graded linearly compact $k\mathbb{I}$ -modules and some of their properties are stated in Remark 1.6.

In Section 2 we extend our previous investigations on the category $\mathbb{I}\text{-DGA}_k$ of functors from an EI -category \mathbb{I} to the category DGA_k of differential graded algebras over a field k . For a complete injective (as a $k\mathbb{I}$ -module) $k\mathbb{I}$ -algebra \mathcal{A} and a complete $k\mathbb{I}$ -module M we consider its cohomologies $H^*(\mathcal{A})$, $H^*(\mathcal{A}, M)$ and a convergent spectral sequence

$$E_2^{pq} = \text{Ext}^p(M, H^q(\mathcal{A})) \implies H^{p+q}(\mathcal{A}, M)$$

which is a crucial tool in the sequel.

Then, we generalize the results of [13] and present an existence of an injective minimal model for a complete injective $k\mathbb{I}$ -algebra \mathcal{A} , for an EI -category \mathbb{I} . The above spectral sequence plays a key role in a construction of this model and for this reason the injectivity of \mathcal{A} (as a $k\mathbb{I}$ -module) is necessary. On the other hand, by [8] for any complete $k\mathbb{I}$ -algebra \mathcal{A} there exists a complete injective $k\mathbb{I}$ -algebra $\Omega(\mathcal{A})$ and a natural cohomology isomorphism $\mathcal{A} \rightarrow \Omega(\mathcal{A})$. First, we show in Propositions 2.3 and 2.4 that injective minimal $k\mathbb{I}$ -algebras behave (up to homotopy) as cofibrant ones. Then we prove in Theorem 2.8 an existence and uniqueness of an injective minimal model $\mathcal{M}_{\mathcal{A}}^{(i)}$ of a complete injective $k\mathbb{I}$ -algebra \mathcal{A} .

Our object in Section 3 is to apply the results assembled in the previous sections to the category $G\text{-}\mathbb{S}\mathbb{S}$ of G -simplicial sets, where G is a finite group. We show in Proposition 3.3 that on the de Rham algebra A_X^* of rational polynomial forms on a simplicial set X there is a natural complete linear topology. Next we observe that much of algebraic-topological information on a G -simplicial set X is encoded in the cofinite EI -category $\mathcal{O}(G, X)$ with one object for each component of each fixed point simplicial subset X^H , for all subgroups $H \subseteq G$.

With any G -simplicial set X , we associate the de Rham $\mathbb{Q}\mathcal{O}(G, X)$ -algebra \mathcal{A}_X^* , where \mathbb{Q} is the field of rationals; its injectivity was presented in [6]. We show in Lemma 3.6 that the category of injective linearly compact $k\mathcal{O}(G, X)$ -modules is closed with respect to the complete tensor product and deduce in Theorem 3.7 an existence of an injective minimal model for the de Rham $\mathbb{Q}\mathcal{O}(G, X)$ -algebra \mathcal{A}_X^* of a G -simplicial set X . Finally, we state Theorem 3.11 as the main result and describe the rational homotopy type of a nilpotent G -simplicial set X by means of injective minimal model of the de Rham algebra \mathcal{A}_X^* .

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1. Injective modules over a category.

Let k be a field. The category of (left) k -modules is denoted by $k\text{-Mod}$. If \mathbb{I} is a small category then a covariant functor $\mathbb{I} \rightarrow k\text{-Mod}$ is called a *left $k\mathbb{I}$ -module* and the functor category of left $k\mathbb{I}$ -modules is denoted by $k\mathbb{I}\text{-Mod}$, and called the *category of left $k\mathbb{I}$ -modules*. We also have the

category of contravariant functors $\mathbb{I} \rightarrow k\text{-Mod}$, alias *right $k\mathbb{I}$ -modules* and denoted by $\text{Mod-}k\mathbb{I}$.

The notions *submodule*, *quotient module*, *kernel*, *image* and *cokernel* for $k\mathbb{I}$ -modules are defined object-wise. For each object $I \in \text{Ob}(\mathbb{I})$ we have the right $k\mathbb{I}$ -module

$$k\mathbb{I}(-, I) : \mathbb{I} \longrightarrow k\text{-Mod}$$

determined by the Yoneda functor $\mathbb{I}(-, I)$ and similarly, the left $k\mathbb{I}$ -module $k\mathbb{I}(I, -)$. *Projective* and *injective* $k\mathbb{I}$ -modules are defined by usual lifting properties. Observe that the category of projective right $k\mathbb{I}$ -modules is isomorphic to the category of all injectives in the category of all covariant functors from \mathbb{I} to the category $k\text{-Mod}^{\text{op}}$ dual to $k\text{-Mod}$.

In various categories considered in algebraic topology endomorphisms are isomorphisms. Therefore, let \mathbb{I} be an *EI-category*, which by definition, is a small category in which each endomorphism is an isomorphism. Following [11], we define a partial order (which is crucial for the sequel) on the set $\text{Is}(\mathbb{I})$ of isomorphism classes \bar{I} of objects $I \in \text{Ob}(\mathbb{I})$ by

$$\bar{I} \leq \bar{J} \quad \text{if} \quad \mathbb{I}(I, J) \neq \emptyset.$$

This induces a partial ordering on the set $\text{Is}(\mathbb{I})$ of isomorphism classes of objects, since the *EI*-property ensures that $\bar{I} \leq \bar{J}$ and $\bar{J} \leq \bar{I}$ implies $\bar{I} = \bar{J}$. We write that $\bar{I} < \bar{J}$ if $\bar{I} \leq \bar{J}$ and $\bar{I} \neq \bar{J}$. If $I \in \text{Ob}(\mathbb{I})$ with the automorphism group $\text{Aut}(I)$, we let $k[I] = k\text{Aut}(I)$ be the group ring of $\text{Aut}(I)$ and write $k[I]\text{-Mod}$ for the category of left $k[I]$ -modules.

For a fixed $I \in \text{Ob}(\mathbb{I})$ we introduce the following covariant functors needed in the sequel.

Cosplitting functor $S_I : k\mathbb{I}\text{-Mod} \rightarrow k[I]\text{-Mod}$.

If M is a $k\mathbb{I}$ -module, let $S_I(M)$ be the $k[I]$ -submodule of $M(I)$ equal to the intersection of kernels of all k -homomorphisms $M(f) : M(I) \rightarrow M(J)$ induced by all non isomorphisms $f : I \rightarrow J$ with I as source. Each automorphism $g \in \text{Aut}(I)$ induces a map $M(g) : M(I) \rightarrow M(I)$ which maps $S_I(M)$ into itself. Thus $S_I(M)$ becomes a left $k[I]$ -module. It is clear how S_I is defined on morphisms.

Coextension functor $E_I : k[I]\text{-Mod} \rightarrow k\mathbb{I}\text{-Mod}$.

This functor sends N to $\text{Hom}_{k[I]}(k\mathbb{I}(-, I), N)$.

It is easy to observe that the category $k\mathbb{I}\text{-Mod}$ has sufficiently many injectives. Hereafter, we assume that \mathbb{I} is an EI -category with the filtration $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_m = \text{Is}(\mathbb{I})$ such that

$$(\star) \quad \bar{I} \in T_k, \quad \bar{J} \in T_\ell, \quad \bar{I} < \bar{J} \text{ implies } k > \ell.$$

Injective $k\mathbb{I}$ -modules for such a category \mathbb{I} have been analysed in [6]. It turns out that they can be constructed from injective modules over group rings.

The dual category $k\text{-Mod}^{\text{op}}$ is isomorphic to the category $k\text{-Mod}^c$ of linearly compact k -modules considered in [10]. For our further purpose we briefly present some results on the category $k\text{-Mod}^c$. A topological k -module M is said to be *linearly topological* if it is Hausdorff and there is a fundamental system $\mathcal{N}(M)$ of neighborhoods of zero consisting of k -submodules. A linearly topological k -module M is called *linearly compact* if for every collection of its closed affine subsets $\{F_i\}_{i \in I}$ (i.e. $F_i = m_i + M_i$ for some closed k -submodule $M_i \subseteq M$) with the finite intersection property it holds $\bigcap_{i \in I} F_i \neq \emptyset$. For linearly topological k -modules M and N let $\text{Hom}_k^t(M, N)$ be the set of all continuous k -linear maps. We topologize this k -module by requiring that for any linearly compact k -submodule $K \subseteq M$ and an open k -submodule $V \subseteq N$ the k -submodules $\{f \in \text{Hom}_k^t(M, N); f(K) \subseteq V\}$ form a subbasis of a linear topology on $\text{Hom}_k^t(M, N)$. For a k -module M , let $M^* = \text{Hom}_k^t(M, k)$ be its topological dual.

THEOREM 1.1 (see [10]).

- (1) A linearly topological k -module M is linearly compact if and only if M^* is discrete.
- (2) If M is linearly compact or discrete then the canonical map $M \rightarrow M^{**}$ is a topological isomorphism.
- (3) If M and N are linearly compact or discrete k -modules then the canonical map $\text{Hom}_k^t(M, N) \rightarrow \text{Hom}_k^t(N^*, M^*)$ is a topological isomorphism.

There is another interesting link between linearly compact k -modules and discrete k -modules.

Remark 1.2 (see [10]). — For a k -module M any two of the following properties imply the third:

- (1) M is discrete;

- (2) M is linearly compact;
- (3) M is finitely generated.

For a linearly topological k -module M and its closed k -submodule M' the quotient topology on M/M' is linear. In particular, if M' is an open submodule then this topology on M/M' is discrete. Let $\omega_{M'}: M \rightarrow M/M'$ be the canonical map. For $V_1, V_2 \in \mathcal{N}(M)$ such that $V_1 \subseteq V_2$, let $\omega_{V_2}^{V_1}: M/V_1 \rightarrow M/V_2$ be the canonical map and

$$M^\wedge = \varprojlim_{V \in \mathcal{N}(M)} M/V.$$

Write $\pi_V: M^\wedge \rightarrow M/V$ for the canonical projection. Then the collection of k -submodules $\{\ker \pi_V; V \in \mathcal{N}(M)\}$ forms a subbasis of a linear topology on M^\wedge . The k -module M^\wedge with this topology is called the *completion* of M . The collection of maps $\omega_V: M \rightarrow M/V$, for $V \in \mathcal{N}(M)$ determines a continuous monomorphism $\omega: M \rightarrow M^\wedge$ and $\omega(M)$ is dense in M^\wedge . A topological k -module M is said to be *complete* if the map ω is a topological isomorphism. Of course, if M is linearly compact or discrete then $\omega(M)$ is closed in M^\wedge and thus M is complete as well.

For two linearly topological k -modules M and N let $M \otimes N$ be their tensor product over k . If $V \subseteq M$ and $W \subseteq N$ are two open k -submodules, we write

$$[V, W] = V \otimes N + M \otimes W.$$

Then, the following lemma holds.

LEMMA 1.3. — *If M and N are linearly topological k -modules then the collection of k -submodules $[V, W]$ of $M \otimes N$ with open k -submodules $V \subseteq M$ and $W \subseteq N$ forms a linear topology on $M \otimes N$ and such that the canonical bilinear map $M \times N \rightarrow M \otimes N$ is universal with respect to uniformly continuous k -bilinear maps to linearly topological k -modules.*

Write $M \widehat{\otimes} N$ for the completion $(M \otimes N)^\wedge$ and call it the *complete tensor product* of M and N . Then the canonical map $M \times N \rightarrow M \widehat{\otimes} N$ is universal with respect to uniformly continuous k -bilinear maps to complete k -modules. Now we are in position to show

PROPOSITION 1.4. — *If M and N are linearly compact (resp. discrete) k -modules then $M \widehat{\otimes} N$ is linearly compact (resp. discrete) and there is a topological isomorphism $M^* \widehat{\otimes} N^* \rightarrow (M \widehat{\otimes} N)^*$.*

Proof. — Observe that

$$M \widehat{\otimes} N = \varprojlim_{\substack{V \in \mathcal{N}(M) \\ W \in \mathcal{N}(N)}} M \otimes N/[V, W] \approx \varprojlim_{\substack{V \in \mathcal{N}(M) \\ W \in \mathcal{N}(N)}} M/V \otimes N/W.$$

If M and N are linearly compact then the quotient k -modules M/V and N/W are linearly compact and discrete. Hence by Remark 1.2 both of them are finitely generated. Thus $M \widehat{\otimes} N$ is linearly compact as an inverse limit of finitely generated k -modules and $(M \widehat{\otimes} N)^*$ is discrete. Of course, if M and N are discrete then the induced topology on $M \otimes N$ is discrete, so $M \widehat{\otimes} N = M \otimes N$ and $(M \widehat{\otimes} N)^*$ is linearly compact.

Define a map

$$\hat{f}: M^* \widehat{\otimes} N^* \longrightarrow (M \widehat{\otimes} N)^*$$

as follows. For $\phi \in M^*$ and $\psi \in N^*$ there is a k -bilinear uniformly continuous map $\phi\psi: M \times N \rightarrow k$ such that $(\phi\psi)(m, n) = \phi(m)\psi(n)$ for $m \in M$ and $n \in N$. Thus there is a unique map $\phi \widehat{\otimes} \psi: M \widehat{\otimes} N \rightarrow k$ since k is a complete k -module. Therefore, we can write $f(\phi \otimes \psi) = \phi \widehat{\otimes} \psi$ to get a continuous k -map $\hat{f}: M^* \widehat{\otimes} N^* \rightarrow (M \widehat{\otimes} N)^*$. It is not difficult to see that \hat{f} is a continuous monomorphism.

Let now $\gamma \in (M \widehat{\otimes} N)^*$ and M, N be linearly compact. Then there are $V \in \mathcal{N}(M)$ and $W \in \mathcal{N}(N)$ such that $\gamma([V, W]) = 0$. Let $\bar{\gamma}: M/V \otimes N/W \rightarrow k$ be the induced map. But M/V and N/W are finitely generated k -modules, hence

$$(M/V \otimes N/W)^* \approx (M/V)^* \otimes (N/W)^*.$$

Let $\alpha_1, \dots, \alpha_r \in (M/V)^*$ and $\beta_1, \dots, \beta_r \in (N/W)^*$ be such that

$$\bar{\gamma} = \alpha_1 \otimes \beta_1 + \dots + \alpha_r \otimes \beta_r.$$

Define $\phi_i \in N^*$ and $\psi_i \in M^*$ by $\phi_i = \alpha_i \pi_V$ and $\psi_i = \beta_i \pi_W$ for $i = 1, \dots, r$. Then $\gamma = \hat{f}(\phi_1 \otimes \psi_1 + \dots + \phi_r \otimes \psi_r)$ and \hat{f} is an epimorphism.

Let now M and N be discrete and $\gamma \in (M \otimes N)^*$. For finitely generated k -submodules $\widetilde{V} \subseteq M$ and $\widetilde{W} \subseteq N$ let $V \subseteq M^*$ and $W \subseteq N^*$ be the corresponding open k -submodules. Then the restriction $\gamma|_{\widetilde{V} \otimes \widetilde{W}}$ determines an element in $M^*/V \otimes N^*/W$. Thus we get

$$\phi \in M^* \widehat{\otimes} N^* = \varprojlim_{\substack{V \in \mathcal{N}(M^*) \\ W \in \mathcal{N}(N^*)}} M^*/V \otimes N^*/W$$

such that $\hat{f}(\phi) = \gamma$. □

Theorem 1.1 and Proposition 1.4 yield immediately

COROLLARY 1.5.

(1) Any linearly compact k -module is topologically isomorphic to a product of one-dimensional k -modules.

(2) If $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$ are collections of linearly compact k -modules then there exists a topological isomorphism

$$\prod_{i \in I} V_i \hat{\otimes} \prod_{j \in J} W_j \approx \prod_{\substack{i \in I \\ j \in J}} V_i \hat{\otimes} W_j.$$

Let now \mathbb{I} be an $E\mathbb{I}$ -category. A covariant functor from \mathbb{I} to $k\text{-Mod}^c$ is said to be a *linearly compact left $k\mathbb{I}$ -module*. Then [6] yields a full description of all injective $k\mathbb{I}$ -modules. For two linearly compact left $k\mathbb{I}$ -modules M, N we define their *complete tensor product* $M \hat{\otimes} N$ as a linearly compact left $k\mathbb{I}$ -modules such that

$$(M \hat{\otimes} N)(I) = M(I) \hat{\otimes} N(I) \text{ for all } I \in \text{Ob}(\mathbb{I}).$$

Put $Q = \{Q_i\}_{i \geq 0}$ for a graded linearly compact left $k\mathbb{I}$ -module. Then for any $I \in \text{Ob}(\mathbb{I})$ we get a graded linearly compact left k -module $Q(I) = \{Q_i(I)\}_{i \geq 0}$ and let $|q| = i$ for $q \in Q_i(I)$. For $n > 0$, by means of associativity of the complete tensor product $\hat{\otimes}$, we can define graded linearly compact left $k\mathbb{I}$ -modules $\hat{T}^n Q$ and $\hat{S}^n Q$ (called the *n th tensor* and *symmetric power*, respectively) as follows:

$$(\hat{T}^n Q)_i(I) = \bigoplus_{i_1 + \dots + i_n = i} Q_{i_1}(I) \hat{\otimes} \dots \hat{\otimes} Q_{i_n}(I)$$

and

$$(\hat{S}^n Q)_i(I) = ((S^n Q)_i(I))^\wedge$$

for $i \geq 0$, where $((S^n Q)_i(I))^\wedge = (T^n Q_i(I)/(R^n Q)_i(I))^\wedge$ and $(R^n Q)_i(I)$ is the homogeneous k -submodule of $(T^n Q)_i(I)$ generated by elements

$$q_1 \otimes \dots \otimes q_n - (-1)^{|q_k| \cdot |q_{k+1}|} q_1 \otimes \dots \otimes q_{k+1} \otimes q_k \otimes \dots \otimes q_n$$

for $q_k \in Q_{|q_k|}(I)$ and $k = 1, \dots, n$. Then the natural canonical map $\pi_I: T^n Q(I) \rightarrow T^n Q(I)/R^n Q(I)$ determines $\hat{\pi}_I: \hat{T}^n Q(I) \rightarrow \hat{S}^n Q(I)$ for $I \in \text{Ob}(\mathbb{I})$.

Moreover, if characteristic of k is zero then there is a natural map

$$\sigma_I : S^n Q(I) \longrightarrow T^n Q(I)$$

such that

$$\sigma_I(q_1 \otimes \cdots \otimes q_n + R^n Q(I)) = \frac{1}{n!} \sum_{\tau \in S_n} \varepsilon(\tau) q_{\tau(1)} \otimes \cdots \otimes q_{\tau(n)}$$

for $q_k \in Q_{|q_k|}(I)$, $I \in \text{Ob}(\mathbb{I})$ and $k = 1, \dots, n$, where S_n is the n th symmetric group and $\varepsilon : S_n \rightarrow \{+1, -1\}$ the sign map. Then we get the induced natural map

$$\hat{\sigma}_I : \hat{S}^n Q(I) \longrightarrow \hat{T}^n Q(I)$$

such that $\hat{\pi}_I \hat{\sigma}_I = \text{id}_{\hat{S}^n Q(I)}$ for $I \in \text{Ob}(\mathbb{I})$ and $\hat{S}^n Q$ is a direct summand of $\hat{T}^n Q$. Moreover, we define $\hat{T}Q$ and $\hat{S}Q$, the graded linearly compact *tensor* and *symmetric* left $k\mathbb{I}$ -algebra, where for $i \geq 0$

$$(\hat{T}Q)_i = \bigoplus_{n \geq 0} (\hat{T}^n Q)_i \quad \text{and} \quad (\hat{S}Q)_i = \bigoplus_{n \geq 0} (\hat{S}^n Q)_i.$$

Observe that $\hat{S}Q = \hat{T}Q / \hat{R}Q$, where $\hat{R}Q$ is the closed homogeneous ideal of $\hat{T}Q$ generated by elements

$$x \hat{\otimes} y - (-1)^{|x| \cdot |y|} y \hat{\otimes} x \quad \text{for} \quad x, y \in \hat{T}Q.$$

Remark 1.6. — If $Q = \{Q_i\}_{i \geq 0}$ is a graded injective linearly compact left $k\mathbb{I}$ -module and the complete tensor product preserves injective linearly compact left $k\mathbb{I}$ -modules then the graded linearly compact left $k\mathbb{I}$ -modules $\hat{T}^n Q$, $\hat{S}^n Q$ for $n > 0$ and $\hat{T}Q = \{(\hat{T}Q)_i\}_{i > 0}$, $\hat{S}Q = \{(\hat{S}Q)_i\}_{i > 0}$ are injective linearly compact left $k\mathbb{I}$ -modules.

2. Algebras over a category and their injective minimal models.

Let DGA_k be the category of homologically connected commutative differential graded k -algebras (or simply k -algebras). We briefly recall some constructions presented in [9]. For a given map $\gamma : B \rightarrow E$ in DGA_k , where B is augmented, Halperin [9] considers its “minimal factorization”. Namely, he generalizes the notion of a minimal k -algebra (*cf.* [12]) to a *minimal KS-extension* given by a sequence of augmented k -algebras

$$\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A,$$

where A is free as a graded commutative k -algebra generated by some graded k -module $M = \{M_i\}_{i \geq 0}$. If $M_0 = 0$ then the extension \mathbb{E} is called *positive*. Next in [9], it is shown that for any map $\gamma: B \rightarrow E$ of connected k -algebras, where B is augmented there is a unique (up to isomorphism) minimal KS -extension

$$\mathbb{E}: B \xrightarrow{i} C \xrightarrow{\pi} A$$

and a homology isomorphism $\rho: C \rightarrow E$ such that $\rho \circ i = \gamma$.

The extension \mathbb{E} together with the map $\rho: C \rightarrow E$ is called a *KS-minimal model* for γ . In particular, for a k -algebra A and the canonical map $k \rightarrow A$ one gets a *minimal algebra* M_A together with a homology isomorphism $\rho_A: M_A \rightarrow A$ called the *minimal model* for A .

An object $A = \{A^n\}_{n \geq 0}$ in DGA_k is called *complete* if

(1) A^n is a complete linearly topological k -module and the differential $d: A^n \rightarrow A^{n+1}$ is continuous for all $n \geq 0$;

(2) multiplication $A^n \times A^m \rightarrow A^{n+m}$ is uniformly continuous for all $n, m \geq 0$ (with respect to the linear product topology on $A^n \times A^m$).

Write DGA_k^\wedge for the subcategory of DGA_k determined by complete differential graded k -algebras.

Let A be a complete k -algebra with the differential d , M a (non-graded) linearly topological k -module and $\tau: M \rightarrow Z^{n+1}A$ a k -map to the $(n+1)$ -cocycles of A for a fixed $n \geq 0$. Denote by (M, n) the graded k -module with M in degree n and 0 otherwise. Define a differential d_τ on $A \hat{\otimes} \hat{S}(M, n)$ by

$$d_\tau|_A = d \quad \text{and} \quad d_\tau|_M = \tau,$$

where $\hat{S}(M, n)$ is the completion of the symmetric algebra $S(M, n)$ on the graded k -module (M, n) . Then the k -algebra $(A \hat{\otimes} \hat{S}(M, n), d_\tau)$, denoted by $A \hat{\otimes}_\tau \hat{S}(M, n)$, is called an *elementary extension* of M and the class

$$[\tau] \in H^{n+1}(\text{Hom}_k(M, A)) = \text{Hom}_k(M, H^{n+1}A)$$

the *structure class*. For a minimal k -algebra M let $M(n)$ be its subalgebra generated by elements of degree less or equal n . Then M is said to be *nilpotent* if each $M(n)$ is constructed from $M(n-1)$ by a finite number of elementary extensions. A homologically connected k -algebra A is said to be *nilpotent* if its minimal model M_A is nilpotent. If X is a (connected)

nilpotent simplicial set then the de Rham \mathbb{Q} -algebra A_X^* of differential form is nilpotent as it was shown in [1]. If a k -algebra A is augmented let $\tilde{A} = \ker(A \rightarrow k)$ be its augmentation ideal. Recall that *decomposability* of the differential d of A means that $d(A) \subseteq \tilde{A} \cdot \tilde{A}$.

Let \mathbb{I} be an EI -category and $k\mathbb{I}\text{-DGA}_k$ the category of all covariant functors from \mathbb{I} to DGA_k called *$k\mathbb{I}$ -algebras* (or simply *systems of k -algebras*). We say that a $k\mathbb{I}$ -algebra \mathcal{A} is *complete* (resp. *linearly compact*) if the algebras $\mathcal{A}(I)$ are complete (resp. linearly compact) for all $I \in \text{Ob}(\mathbb{I})$ and \mathcal{A} is *injective* if the left $k\mathbb{I}$ -modules \mathcal{A}^n are injective for $n \geq 0$, where $\mathcal{A}^n(I) = (\mathcal{A}(I))^n$ for all $I \in \text{Ob}(\mathbb{I})$.

For any complete injective (as a $k\mathbb{I}$ -module) $k\mathbb{I}$ -algebra \mathcal{A} and a complete left $k\mathbb{I}$ -module M we consider two types of cohomology of \mathcal{A} .

(1) The $k\mathbb{I}$ -module $\mathbf{H}^n(\mathcal{A})$ such that $\mathbf{H}^n(\mathcal{A})(I) = H^n(\mathcal{A}(I))$ for $I \in \text{Ob}(\mathbb{I})$ and $n \geq 0$.

(2) The cohomology $H^n(\mathcal{A}, M) = H^n(\text{Hom}(M, \mathcal{A}))$ with coefficients in M for $n \geq 0$, where $\{\text{Hom}(M, \mathcal{A}^n)\}_{n \geq 0}$ is a cochain complex in the category of complete left $k\mathbb{I}$ -modules. For a projective resolution $M^{(*)}$ of M in the category of complete $k\mathbb{I}$ -modules we form the double complex $\text{Hom}(M^{(*)}, \mathcal{A})$. The standard homological algebra arguments yield a spectral sequence

$$E_2^{pq} = \text{Ext}^p(M, \mathbf{H}^q(\mathcal{A})) \implies H^{p+q}(\mathcal{A}, M).$$

Notice that the injectivity of \mathcal{A} (as a $k\mathbb{I}$ -module) implies the convergence of this sequence and $H^n(\mathcal{A}, M) = \text{Hom}(M, \mathbf{H}^n(\mathcal{A}))$ if M is projective. This spectral sequence is an essential tool in our further investigations.

Remark 2.1. — If \mathcal{A} and \mathcal{B} are injective (as $k\mathbb{I}$ -modules) $k\mathbb{I}$ -algebras then a map $f: \mathcal{A} \rightarrow \mathcal{B}$ induces an isomorphism $\mathbf{H}^n(f): \mathbf{H}^n(\mathcal{A}) \rightarrow \mathbf{H}^n(\mathcal{B})$ for $n \geq 0$ if and only if for any $k\mathbb{I}$ -module M the induced map $H^n(f, M): H^n(\mathcal{A}, M) \rightarrow H^n(\mathcal{B}, M)$ is an isomorphism for $n \geq 0$.

Proof. — If $\mathbf{H}^n(f)$ is an isomorphism for $n \geq 0$ then from the above spectral sequence it follows that $H^n(f, M)$ is an isomorphism as well for $n \geq 0$.

Let now $H^n(f, M)$ be an isomorphism for any $k\mathbb{I}$ -module M and $n \geq 0$. For a fixed object $I \in \text{Ob}(\mathbb{I})$ consider a $k\mathbb{I}$ -module M_I such that $M_I(J) = k\mathbb{I}(J, I)$. Then $H^n(M_I, \mathcal{A}) = H^n(\mathcal{A}(I)) = \mathbf{H}^n(\mathcal{A})(I)$ and $\mathbf{H}^n(f)$ is an isomorphism for any $n \geq 0$. \square

We also consider relative cohomology

$$\begin{aligned} H^n(\mathcal{A}_2, \mathcal{A}_1) &= H^n(\mathcal{A}_2^* \times \mathcal{A}_1^{*+1}) \quad \text{and} \\ H^n(\mathcal{A}_2, \mathcal{A}_1; M) &= H^n(\text{Hom}(M, \mathcal{A}_2^* \times \mathcal{A}_1^{*+1})) \end{aligned}$$

for a map $f: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ of $k\mathbb{I}$ -algebras with differentials d_1 and d_2 , where $d: \mathcal{A}_2^n \times \mathcal{A}_1^{n+1} \rightarrow \mathcal{A}_2^{n+1} \times \mathcal{A}_1^{n+2}$ is given by

$$d(a_2, a_1) = (d_2(a_2) + f(a_1), -d_1(a_1))$$

for $a_1 \in \mathcal{A}_1^{n+1}$, $a_2 \in \mathcal{A}_2^n$ and $n \geq 0$.

Throughout, \mathbb{I} is a *cofinite EI*-category (each isomorphism class \bar{I} has only finitely many predecessors). For any $I \in \text{Ob}(\mathbb{I})$ we define its *height* as the number of its predecessors.

Hereafter, we assume that all $k\mathbb{I}$ -algebras \mathcal{A} are homologically connected, i.e. satisfy $H^0(\mathcal{A}) = \underline{k}$, where \underline{k} is the constant $k\mathbb{I}$ -module determined by a field k of characteristic zero. To define an injective minimal model of \mathcal{A} we need to remind the following result presented in [8] and generalizing [5].

THEOREM 2.2. — *If \mathbb{I} is an EI-category such that $k[I]$ is a semisimple ring for all $I \in \text{ob}(\mathbb{I})$ and there is a filtration*

$$\emptyset = T_0 \subset T_1 \subset \cdots \subset T_m = \text{Is}(\mathbb{I})$$

satisfying (\star) then for any complete $k\mathbb{I}$ -algebra \mathcal{A} there is a complete and injective (as a $k\mathbb{I}$ -module) $k\mathbb{I}$ -algebra $\Omega(\mathcal{A})$ and a natural inclusion $i_{\mathcal{A}}: \mathcal{A} \rightarrow \Omega(\mathcal{A})$ which is a cohomology isomorphism.

If \underline{k} is the constant $k\mathbb{I}$ -algebra determined by a field k then \underline{k} is not in general injective as $k\mathbb{I}$ -module. But for any $k\mathbb{I}$ -algebra \mathcal{A} (injective as a $k\mathbb{I}$ -module) there is a map $\Omega(\underline{k}) \rightarrow \mathcal{A}$ of \mathbb{I} -algebras extending the canonical inclusion $\underline{k} \rightarrow \mathcal{A}$ as it follows from a more general fact.

LEMMA 2.3. — *Let \mathbb{I} be an EI-category satisfying (\star) . If $f: \mathcal{A} \rightarrow \mathcal{B}$ is a map of $k\mathbb{I}$ -algebras and \mathcal{B} is injective as a $k\mathbb{I}$ -module then there is an extension map $\tilde{f}: \Omega(\mathcal{A}) \rightarrow \mathcal{B}$ of $k\mathbb{I}$ -algebras.*

Proof. — We construct by induction over the filtration of $\text{Is}(\mathbb{I})$ a sequence of maps $f_{\ell}: \Omega_{\ell}(\mathcal{A}) \rightarrow \mathcal{B}$ for $\ell = 0, 1, \dots, n$.

Let $\tilde{f}_0 = f$. Given $\tilde{f}_\ell : \Omega_\ell(\mathcal{A}) \rightarrow \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \Omega_{\ell-1}(\mathcal{A}) & \xrightarrow{i_{\ell-1}} & \Omega_\ell(\mathcal{A}) \\ & \searrow \tilde{f}_{\ell-1} & \downarrow \tilde{f}_\ell \\ & & \mathcal{B} \end{array}$$

commutes we construct a map $\tilde{f}_{\ell+1} : \Omega_{\ell+1}(\mathcal{A}) \rightarrow \mathcal{B}$ as follows. The $k\mathbb{I}$ -algebra \mathcal{B} is injective, so by [6] there is an isomorphism of $k\mathbb{I}$ -modules

$$\mathcal{B} \approx \prod_{I \in \text{Is}(\mathbb{I})} E_I S_I \mathcal{B}$$

and \tilde{f}_ℓ induces maps $E_I S_I \tilde{f}_\ell : E_I S_I \Omega_\ell(\mathcal{A}) \rightarrow E_I S_I \mathcal{B}$ for $I \in \text{Ob}(\mathbb{I})$, where E_I and S_I are the coextension and cosplitting functors considered in Section 1. Then \tilde{f}_ℓ together with these maps determines a map

$$\tilde{f}_{\ell+1} : \Omega_{\ell+1}(\mathcal{A}) \longrightarrow \mathcal{B}.$$

The map $\tilde{f} = \tilde{f}_m$ satisfies the required property. □

A $k\mathbb{I}$ -algebra \mathcal{A} with a map $\Omega(\underline{k}) \rightarrow \mathcal{A}$ is called a $k\mathbb{I}$ -algebra under $\Omega(\underline{k})$ or a *based $k\mathbb{I}$ -algebra*. A homotopy between maps of based $k\mathbb{I}$ -algebras is called a *based homotopy*. A $k\mathbb{I}$ -algebra \mathcal{A} is called *nilpotent* if $\mathcal{A}(I)$ is nilpotent for all $I \in \text{Ob}(\mathbb{I})$.

A based injective, nilpotent and complete $k\mathbb{I}$ -algebra \mathcal{M} is said to be *i-minimal* if it satisfies the following:

- (1) there is an inclusion $\Omega(\underline{k}) \hookrightarrow \mathcal{M}$;
- (2) $\mathcal{M}(I)$ is a positive KS -extension of $\Omega(\underline{k})(I)$ for all $I \in \text{Ob}(\mathbb{I})$;
- (3) $\mathcal{M}(I)$ is a minimal KS -extension of $\Omega(\underline{k})(I)$ for all terminal $I \in \text{Ob}(\mathbb{I})$;
- (4) if d is the differential of \mathcal{M} then $d|_{S_I \mathcal{M}}$ is decomposable for all $I \in \text{Ob}(\mathbb{I})$, where S_I is the cosplitting functor considered in Section 1.

We shall show that *i-minimal $k\mathbb{I}$ -algebras* play the same role in the category of nilpotent complete $k\mathbb{I}$ -algebras as minimal algebras in the category of nilpotent k -algebras.

PROPOSITION 2.4. — *Let \mathbb{I} be an EI -category satisfying (\star) . If a map $f : \mathcal{M} \rightarrow \mathcal{N}$ of *i-minimal $k\mathbb{I}$ -algebras* induces an isomorphism on cohomology then f is an isomorphism.*

For the proof, we proceed by induction over the filtration of $\text{Is}(\mathbb{I})$ and mimic the proof of Theorem 5.2 in [13]. For the surjectivity of f , we make use of the structure of injective $k\mathbb{I}$ -modules considered in [6]. \square

The following properties of i -minimal $k\mathbb{I}$ -algebras are completely analogous to the nonequivariant ones (see *e.g.*, [1] and *cf.* [13]).

PROPOSITION 2.5. — *Let \mathbb{I} be an EI-category satisfying (\star) and $f : \mathcal{A} \rightarrow \mathcal{B}$ a cohomology isomorphism of nilpotent based $k\mathbb{I}$ -algebras and $g : \mathcal{M} \rightarrow \mathcal{B}$ a map of $k\mathbb{I}$ -algebras, where \mathcal{M} is i -minimal. Then there is a map $\tilde{f} : \mathcal{M} \rightarrow \mathcal{A}$, unique up to based homotopy, such that the diagram*

$$\begin{array}{ccc} & & \mathcal{A} \\ & \nearrow \tilde{f} & \downarrow f \\ \mathcal{M} & \xrightarrow{g} & \mathcal{B} \end{array}$$

commutes up to based homotopy.

Then, we can state

COROLLARY 2.6. — *Let \mathbb{I} be an EI-category satisfying (\star) .*

(1) *If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a cohomology isomorphism of nilpotent based $k\mathbb{I}$ -algebras and \mathcal{M} is an i -minimal $k\mathbb{I}$ -algebra then the induced map $f_* : [\mathcal{M}, \mathcal{A}] \rightarrow [\mathcal{M}, \mathcal{B}]$ of the sets of based homotopy classes is a bijection.*

(2) *If \mathcal{A} is a nilpotent $k\mathbb{I}$ -algebra, $\rho : \mathcal{M} \rightarrow \mathcal{A}$, $\rho' : \mathcal{M}' \rightarrow \mathcal{A}$ are two cohomology isomorphisms and $\mathcal{M}, \mathcal{M}'$ are i -minimal $k\mathbb{I}$ -algebras then there is a (unique up to based homotopy) isomorphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ such that the diagram*

$$\begin{array}{ccc} \mathcal{M} & & \\ \uparrow f & \searrow \rho & \mathcal{A} \\ \mathcal{M}' & \nearrow \rho' & \end{array}$$

commutes up to based homotopy.

Let now \mathcal{A} be a nilpotent $k\mathbb{I}$ -algebra and $\rho : \mathcal{M}^{(i)} \rightarrow \mathcal{A}$ a cohomology isomorphism, where $\mathcal{M}^{(i)}$ is i -minimal. Then $\mathcal{M}^{(i)}$ is called the *i -minimal model* of \mathcal{A} . The above uniqueness results make this definition meaningful.

To show the existence of an i -minimal model for a $k\mathbb{I}$ -algebra we first make the following important construction. Let \mathcal{A} be a complete $k\mathbb{I}$ -algebra, M a complete $k\mathbb{I}$ -module and $\tau: M \rightarrow Z^{n+1}\mathcal{A}$ a $k\mathbb{I}$ -map, where $Z^{n+1}\mathcal{A}$ is the $k\mathbb{I}$ -module of cocycles of degree $n+1$ of the $k\mathbb{I}$ -algebra \mathcal{A} for some $n \geq 0$. We construct a linearly topological $k\mathbb{I}$ -algebra $\mathcal{A}(M)$ called the *elementary extension* of \mathcal{A} .

Let

$$0 \rightarrow M \xrightarrow{\omega} M_0 \xrightarrow{\omega_0} M_1 \xrightarrow{\omega_1} \dots$$

be the minimal injective resolution of M in the category of linearly topological $k\mathbb{I}$ -modules, *i.e.* M_ℓ is the injective envelope of $\text{Im } \omega_{\ell-1}$ for $\ell = 0, 1, \dots$. Then, the map $\tau: M \rightarrow Z^{n+1}\mathcal{A}$ induces $k\mathbb{I}$ -maps $\tau_\ell: M_\ell \rightarrow \mathcal{A}^{n+\ell+1}$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\omega} & M_0 & \xrightarrow{\omega_0} & M_1 \xrightarrow{\omega_1} \dots \\ & & \downarrow \tau & & \downarrow \tau_0 & & \downarrow \tau_1 \\ & & Z^{n+1}\mathcal{A} & \longrightarrow & \mathcal{A}^{n+1} & \xrightarrow{d} & \mathcal{A}^{n+2} \xrightarrow{d} \dots \end{array}$$

commutes. Let $M_* = \{M_\ell\}_{\ell \geq 0}$ be the graded $k\mathbb{I}$ -module, where M_ℓ has degree $n + \ell$. Define the complete $k\mathbb{I}$ -algebra

$$\mathcal{A}(M) = \mathcal{A} \hat{\otimes} \hat{S}M_*,$$

where the differential on $\mathcal{A}(M)$ restricts on \mathcal{A} to the given on \mathcal{A} and on M_ℓ to $\omega_\ell + (-1)^\ell \tau_\ell$ for $\ell = 0, 1, \dots$. Observe that by Proposition 1.4 and Remark 1.6 the $k\mathbb{I}$ -algebra $\mathcal{A}(M)$ is injective and linearly compact if \mathcal{A} and M are so.

LEMMA 2.7. — *Let \mathbb{I} be an EI -category satisfying (\star) and \mathcal{A} and \mathcal{B} complete $k\mathbb{I}$ -algebras and $\tau: M \rightarrow Z^{n+1}\mathcal{A}$ a $k\mathbb{I}$ -map. If \mathcal{B} is injective (as a $k\mathbb{I}$ -module) and complete and $\phi: M \rightarrow \mathcal{B}^n$ a map of $k\mathbb{I}$ -modules then for a map $f: \mathcal{A} \rightarrow \mathcal{B}$ of $k\mathbb{I}$ -algebras such that $d\phi(m) = f\tau(m)$ for $m \in M$, where d is the differential on \mathcal{B} then there exists a map $\tilde{f}: \mathcal{A}(M) \rightarrow \mathcal{B}$ extending f and ϕ .*

Proof. — Let

$$0 \rightarrow M \xrightarrow{\omega} M_0 \xrightarrow{\omega_0} M_1 \xrightarrow{\omega_1} \dots$$

be the minimal injective resolution of M in the category of linearly

topological $k\mathbb{I}$ -modules. By the injectivity of \mathcal{B}^n there is a map $\phi_0: M_0 \rightarrow \mathcal{B}^n$ of $k\mathbb{I}$ -algebras and such that $\phi_0\omega = \phi$. By assumption

$$d\phi_0 - f\tau_0|_M = 0,$$

hence it defines a map $M/M_0 \rightarrow \mathcal{B}^{n+1}$ which, by the injectivity of \mathcal{B}^{n+1} , extends to a map $\phi_1: M_1 \rightarrow \mathcal{B}^{n+1}$ such that $\phi_1\omega_0 = d\phi_0 - f\tau_0$.

Assume inductively that there is a map $\phi_\ell: M_\ell \rightarrow \mathcal{B}^{n+\ell}$ such that

$$d\phi_\ell - (-1)^\ell f\tau_\ell|_{\text{Im } \omega_{\ell-1}} = 0.$$

Hence it determines a map $M_\ell/\text{Im } \omega_{\ell-1} \rightarrow \mathcal{B}^{n+\ell+1}$ and by the injectivity of $\mathcal{B}^{n+\ell+1}$ it extends to a map $\phi_{\ell+1}: M_{\ell+1} \rightarrow \mathcal{B}^{n+\ell+1}$ such that $\phi_{\ell+1}\omega_\ell = d\phi_\ell - (-1)^\ell f\tau_\ell$. Then

$$d\phi_{\ell+1} - (-1)^{\ell+1} f\tau_{\ell+1}|_{\text{Im } \omega_\ell} = 0.$$

The $k\mathbb{I}$ -algebra \mathcal{B} is complete, therefore the maps $f: \mathcal{A} \rightarrow \mathcal{B}$ and $\phi_\ell: M_\ell \rightarrow \mathcal{B}^{n+\ell}$ for $\ell = 0, 1, \dots$ determine a map $\tilde{f}: \tilde{\mathcal{A}}(M) \rightarrow \mathcal{B}$ of $k\mathbb{I}$ -algebras extending f and ϕ . \square

THEOREM 2.8. — *Let \mathbb{I} be an EI -category satisfying (\star) . If \mathcal{A} is a nilpotent, injective (as a $k\mathbb{I}$ -module) and complete $k\mathbb{I}$ -algebra with linearly compact cohomology and the category of injective linearly compact $k\mathbb{I}$ -modules is closed with respect to the complete tensor product $\hat{\otimes}$ then there exist an i -minimal $k\mathbb{I}$ -algebra $\mathcal{M}^{(i)}$ and a cohomology isomorphism $\rho: \mathcal{M}^{(i)} \rightarrow \mathcal{A}$.*

Proof. — We proceed inductively on degree and construct $\mathcal{M}^{(i)}$ as the increasing union of i -minimal $k\mathbb{I}$ -algebras $\mathcal{M}^{(i)}(n, \ell)$ for $n \geq 0$ and $\ell \geq -1$, where

$$\mathcal{M}^{(i)}(n) = \mathcal{M}^{(i)}(n, -1) = \bigcup_{\ell \geq 0} \mathcal{M}^{(i)}(n-1, \ell)$$

with a map $\rho(n): \mathcal{M}^{(i)}(n) \rightarrow \mathcal{A}$ which is a cohomology isomorphism in degree $\leq n$ and a cohomology monomorphism in degree $n+1$.

Begin with $\mathcal{M}^{(i)}(0) = \Omega(k)$ and the based map $\rho(0): \mathcal{M}^{(i)}(0) \rightarrow \mathcal{A}$. Assume we have an i -minimal $k\mathbb{I}$ -algebra $\mathcal{M}^{(i)}(n-1)$ and a map

$$\rho(n-1): \mathcal{M}^{(i)}(n-1) \longrightarrow \mathcal{A}$$

which is a cohomology isomorphism in degree $\leq n-1$ and a monomorphism in degree n . We consider

$$M'_n = \operatorname{coker} \mathbf{H}^n(\rho(n-1))$$

and show (as in [13] by comparing some spectral sequences) an existence of maps $\tau'(n-1): M'_n \rightarrow Z^{n+1}\mathcal{M}^{(i)}(n-1)$ and $\rho'(n-1): M'_n \rightarrow \mathcal{A}^n$. Then, by Lemma 2.7 there exists a map

$$\rho(\widetilde{n-1}): \mathcal{M}^{(i)}(n-1)(M'_n) \longrightarrow \mathcal{A}$$

extending $\rho(n-1)$ and $\rho'(n-1)$ and inducing a cohomology isomorphism in degree $\leq n$.

Now let

$$M''_n = \ker \mathbf{H}^{n+1}(\rho(\widetilde{n-1})).$$

We study again a relation between some spectral sequences to find maps

$$\tau''(n-1): M''_n \rightarrow Z^{n+1}\mathcal{M}(n-1)(M'_n) \quad \text{and}$$

$$\rho''(n-1): M''_n \rightarrow \mathcal{A}^n.$$

By Lemma 2.7 there is a map

$$\rho(n-1, 0): \mathcal{M}^{(i)}(n-1, 0) = \mathcal{M}^{(i)}(n-1)(M'_n)(M''_n) \longrightarrow \mathcal{A}$$

extending $\rho(\widetilde{n-1})$ and $\rho''(n-1)$. Of course, the extension

$$\mathcal{M}^{(i)}(n-1)(M'_n)(M''_n)$$

may cause $\rho(n-1, 0)$ to be not monomorphic in degree $n+1$. If now one repeats the construction countably many times and takes the increasing union of these extensions then the resulting $k\mathbb{I}$ -algebra $\mathcal{M}^{(i)}(n)$ will be i -minimal and by nilpotency of \mathcal{A} the map $\rho(n): \mathcal{M}^{(i)}(n) \rightarrow \mathcal{A}$ induced by the maps $\rho(n-1, \ell): \mathcal{M}^{(i)}(n-1, \ell) \rightarrow \mathcal{A}$ at each stage will be a cohomology isomorphism in degree $\leq n$ and a monomorphism in degree $n+1$, as desired. \square

Remark 2.9. — Let \mathcal{A} be any complete $k\mathbb{I}$ -algebra with linearly compact cohomology and $\Omega(\mathcal{A})$ the associated componentwise injective $k\mathbb{I}$ -algebra determined by Theorem 2.2. Then, by the above theorem there is an i -minimal $k\mathbb{I}$ -algebra $\mathcal{M}^{(i)}$ and a cohomology isomorphism $\rho: \mathcal{M}^{(i)} \rightarrow \Omega(\mathcal{A})$ and $\mathcal{M}^{(i)}$ is called the i -minimal model of \mathcal{A} .

3. Applications to rational homotopy theory.

For the field of rationals \mathbb{Q} let $DGA_{\mathbb{Q}}$ be the category of commutative graded differential \mathbb{Q} -algebras. Given a simplicial set X one can form a \mathbb{Q} -algebra A_X^* by taking collections of \mathbb{Q} -polynomial forms on each simplex (sums of terms of type $\omega(t_0, \dots, t_n) dt_{i_1} \wedge \dots \wedge dt_{i_\ell}$, where $\omega(t_0, \dots, t_n)$ is a \mathbb{Q} -polynomial) that agree when restricted to common faces (see [1] for more details). We prove below that A_X^* admits a natural linearly complete topology.

Let $F: \mathbb{SS}^{\text{op}} \rightarrow k\text{-Mod}$ be a contravariant functor from the category \mathbb{SS} of simplicial sets to that of k -modules, where k is a field. For a simplicial set X we define a *natural topology* on $F(X)$ as follows: for any map $\tilde{x}: \Delta(\ell) \rightarrow X$ k -submodules $\ker(F(\tilde{x}): F(X) \rightarrow F(\Delta(\ell)))$, where $\Delta(\ell)$ is the ℓ -simplex form a fundamental system of neighborhoods of zero in $F(X)$. From the definition easily follows

LEMMA 3.1.

(1) For any simplicial map $\tilde{x}: \Delta(\ell) \rightarrow X$ the induced map $F(\tilde{x}): F(X) \rightarrow F(\Delta(\ell))$ is continuous.

(2) If M is linearly topological k -module then a map $f: M \rightarrow F(X)$ of k -modules is continuous if and only if for any simplicial map $\tilde{x}: \Delta(\ell) \rightarrow X$ the composition $F(\tilde{x})f: M \rightarrow F(\Delta(\ell))$ is continuous.

From this one can deduce

COROLLARY 3.2.

(1) If $f: X \rightarrow Y$ is a simplicial map then the induced map $F(f): F(Y) \rightarrow F(X)$ is continuous.

(2) If $F, G: \mathbb{SS}^{\text{op}} \rightarrow k\text{-Mod}$ are contravariant functors and $\Phi: F \rightarrow G$ is a natural transformation then for any simplicial set X the map $\Phi(X): F(X) \rightarrow G(X)$ is continuous (with respect to the natural topology).

In particular, for $k = \mathbb{Q}$ we get a natural topology on the \mathbb{Q} -module A_X^n of n -forms on a simplicial set X for $n \geq 0$.

PROPOSITION 3.3. — Let X be a simplicial set. Then:

(1) the natural topology on A_X^n is complete for all $n \geq 0$;

- (2) the multiplication $A_X^n \times A_X^m \rightarrow A_X^{n+m}$ of differential forms is uniformly continuous (with respect to the product topology on $A_X^n \times A_X^m$);
- (3) the differential $d_X^n : A_X^n \rightarrow A_X^{n+1}$ is continuous.

Proof.

(1) First observe that for a simplicial map $\tilde{x} : \Delta(\ell) \rightarrow X$ there is an isomorphism $A_X^n / \ker A_{\tilde{x}}^n \approx A_{\Delta(\ell)}^n$ of discrete \mathbb{Q} -modules. Then, the map

$$\phi : A_X^n \longrightarrow \varprojlim_{\tilde{x} : \Delta(\ell) \rightarrow X} A_X^n / \ker A_{\tilde{x}}^n \approx \varprojlim_{\tilde{x} : \Delta(\ell) \rightarrow X} A_{\Delta(\ell)}^n$$

such that $\phi(\omega) = (A_{\tilde{x}}^n(\omega))_{\tilde{x} : \Delta(\ell) \rightarrow X}$, for $\omega \in A_X^n$ is a required topological isomorphism.

(2) For a simplicial map $\tilde{x} : \Delta(\ell) \rightarrow X$ and the corresponding open k -submodule $V = \ker(A^{n+m}(\tilde{x}) : A_X^{n+m} \rightarrow A_{\Delta(\ell)}^{n+m})$, consider the subspaces

$$U_1 = \ker(A^n(\tilde{x}) : A_X^n \rightarrow A_{\Delta(\ell)}^n),$$

$$U_2 = \ker(A^m(\tilde{x}) : A_X^m \rightarrow A_{\Delta(\ell)}^m)$$

of A_X^n and A_X^m , respectively. Then, the image of $U_1 \times A_X^m$ and $A_X^n \times U_2$ by the multiplication map of differential forms is contained in V , so it is uniformly continuous.

(3) The differential d_X^n is natural with respect to X , hence it is continuous by Lemma 3.1. \square

Let now $C_n(X, \mathbb{Q})$ be the discrete \mathbb{Q} -module of n -chains on a simplicial set X with the coefficients in the field of rationals \mathbb{Q} for $n \geq 0$. Then on the \mathbb{Q} -module $C^n(X, \mathbb{Q})$ of n -cochains, for $n \geq 0$ there is a linearly compact topology considered in Section 1 which is the same as the natural one. In particular, it follows that the induced topology on the cohomology groups $H^n(X, \mathbb{Q})$ is linearly compact for $n \geq 0$ and the dual \mathbb{Q} -module $(H^n(X, \mathbb{Q}))^*$ is isomorphic to the n th homology group $H_n(X, \mathbb{Q})$ for $n \geq 0$. On the other hand the map

$$i_X^* : A_X^* \longrightarrow C^*(X, \mathbb{Q})$$

given by the integration of forms is a natural transformation, so it is continuous by Corollary 3.2. Therefore, by the de Rham theorem (see [1]) the induced map on cohomology

$$i_X^n : H^n(A_X^*) \longrightarrow H^n(X, \mathbb{Q})$$

is a continuous isomorphism for all $n \geq 0$.

Let now X be a G -simplicial set with G a finite group. Much of the algebraic-topological information about X is encoded in the form of functors from the category $\mathcal{O}(G, X)$ which may be strictly described as follows:

- The set $\text{Ob}\mathcal{O}(G, X)$ of objects consists of pairs $(G/H, \alpha)$, with H a subgroup of G and α a connected component of the fixed point subset X^H (i.e. $\alpha \in \pi_0(X^H)$).
- Morphisms $(G/K, \beta) \rightarrow (G/H, \alpha)$ are G -maps $\phi: G/K \rightarrow G/H$ such that $\pi_0(\bar{\phi})(\alpha) = \beta$, with $\bar{\phi}: X^H \rightarrow X^K$ the induced map of the fixed point simplicial subsets.

In the sequel we will identify an object $(G/H, \alpha)$ of $\mathcal{O}(G, X)$ with α and denote by X_α^H the connected component of X^H corresponding to $\alpha \in \pi_0(X^H)$. Observe that the category $\mathcal{O}(G, X)$ results also as Grothendieck construction, namely $\mathcal{O}(G) \int \pi_0(X)$, where $\mathcal{O}(G)$ is the category of canonical orbits of the group G and $\pi_0(X): \mathcal{O}(G)^{\text{op}} \rightarrow \text{Set}$ is a contravariant functor such that $\pi_0(X)(G/H) = \pi_0(X^H)$ for $G/H \in \text{Ob}\mathcal{O}(G)$.

Let $H = H_0, H_1, \dots, H_m$ be all distinct subgroups of G conjugate to H . For $\alpha \in \pi_0(X^H)$ let $\alpha_k \in \pi_0(X^{H_k})$ be one of the corresponding component of X^{H_k} . Then the set of morphisms $\mathcal{O}(G, X)((G/K, \beta), (G/H, \alpha))$ is in one to one correspondence with the disjoint union

$$\bigcup_{\substack{K \subseteq H_k \\ \alpha_k \subseteq \beta}} \mathcal{O}(G, X)((G/H_k, \alpha_k), (G/H, \alpha))$$

of morphisms, hence we may identify both of them in our further investigations. Moreover $\mathcal{O}(G, X)$ is a cofinite EI -category and for the isomorphism class $\bar{\alpha}$ of its object α there is the largest number $d(\alpha) = n$ such that there is a sequence $\bar{\alpha} = \bar{\alpha}_1 < \dots < \bar{\alpha}_n$. The group G is finite, therefore for the set of isomorphism classes $\text{Is}\mathcal{O}(G, X)$ we can define a filtration

$$\emptyset = T_0 \subset T_1 \subset \dots \subset T_m = T = \text{Is}\mathcal{O}(G, X)$$

satisfying $(*)$, where $T_\ell = \{\bar{\alpha}; d(\alpha) \leq \ell\}$ for $\ell = 0, 1, \dots, m$.

For a G -simplicial set X let \mathcal{A}_X^* be a $\mathbb{Q}\mathcal{O}(G, X)$ -algebra defined by

$$\mathcal{A}_X^*(G/H, \alpha) = A_{X_\alpha^H}^*$$

and the maps on forms are those induced by the action of G on the connected components of the fixed point simplicial subsets. In [6] we proved that \mathcal{A}_X^* is an injective and complete as a $\mathbb{Q}\mathcal{O}(G, X)$ -module.

Denote by $\mathcal{C}_*(X, \mathbb{Q})$ the right $\mathbb{Q}\mathcal{O}(G, X)$ -module defined by

$$\mathcal{C}_*(X, \mathbb{Q})(G/H, \alpha) = C_*(X_\alpha^H, \mathbb{Q})$$

for $(G/H, \alpha) \in \text{Ob}(\mathcal{O}(G, X))$, where the latter stands for the ordinary chain complex of X_α^H with coefficients in \mathbb{Q} . Then we get the induced right $\mathbb{Q}\mathcal{O}(G, X)$ -module $\mathbf{H}_n(X, \mathbb{Q})$ such that

$$\mathbf{H}_n(X, \mathbb{Q})(G/H, \alpha) = H_n(X_\alpha^H, \mathbb{Q})$$

for $(G/H, \alpha) \in \text{Ob}(\mathcal{O}(G, X))$ and $n \geq 0$.

PROPOSITION 3.4. — *The right $\mathbb{Q}\mathcal{O}(G, X)$ -module $\mathcal{C}_*(X, \mathbb{Q})$ is projective.*

Proof. — Let

$$\begin{array}{ccccc} & & \mathcal{C}_*(X, \mathbb{Q}) & & \\ & \swarrow \theta & \downarrow \gamma & & \\ M & \xrightarrow{\delta} & N & \longrightarrow & 0 \end{array}$$

be a diagram of right $\mathbb{Q}\mathcal{O}(G, X)$ -modules with an exact row and θ to be constructed. Let $X' \subseteq X$ be a subset containing exactly one element from each orbit of G in X . Given $x' \in X'$, consider x' as an element of $C_*(X_\alpha^{G_{x'}}, \mathbb{Q})$ for some component $\alpha \in \pi_0(X^{G_x})$, where $G_{x'}$ is the isotropy subgroup of x' . Define $\theta(G/G_{x'}, \alpha)(x') \in M(G/G_{x'}, \alpha)$ to be any element with

$$\delta(G/G_{x'}, \alpha)\theta(G/G_{x'}, \alpha)(x') = \gamma(G/G_{x'}, \alpha)(x').$$

Extend θ to the whole orbit by $\theta(gx') = g^*\theta(x')$ for $g \in G$, where $g^* = M(\hat{g}: (G/G_{gx'}, g\alpha) \rightarrow (G/G_{x'}, \alpha))$.

If now $x \in C_*(X_\beta^H, \mathbb{Q})$ represents one of generators then x is an element of $X_\alpha^{G_{x'}} \subseteq X_\beta^H$ and $H \subseteq G_x$. Hence we get a map $\phi: (G/H, \beta) \rightarrow (G/G_x, \alpha)$. Thus $x = C_*(X, \mathbb{Q})(\phi)(x)$ and we define

$$\theta(G/H, \beta)(x) = M(\phi)\theta(G/G_x, \alpha)(x).$$

□

For any right $\mathbb{Q}\mathcal{O}(G, X)$ -module M , we write

$$\mathcal{C}_G^*(X, M) = \text{Hom}(\mathcal{C}_*(X, \mathbb{Q}), M)$$

and we consider two types of cohomology of X (resp. \mathcal{A}_X^*).

- (1) The $\mathbb{Q}\mathcal{O}(G, X)$ -module $\mathbf{H}^n(X, \mathbb{Q})$ (resp. $\mathbf{H}^n(\mathcal{A}_X^*)$) such that

$$\mathbf{H}^n(X, \mathbb{Q})(G/H, \alpha) = H^n(X_\alpha^H, \mathbb{Q})$$

$$(\text{resp. } (\mathbf{H}^n(\mathcal{A}_X^*))(G/H, \alpha) = H^n(\mathcal{A}_{X_\alpha^H}^*))$$

for $(G/H, \alpha) \in \text{Ob } \mathcal{O}(G, X)$ and $n \geq 0$.

- (2) The cohomology $H_G^n(X, M) = H^n(\mathcal{C}_G^*(X, M))$ with coefficients in a right $\mathbb{Q}\mathcal{O}(G, X)$ -module M (resp. $H^n(\mathcal{A}_X^*, N)$ with coefficients in a linearly complete left $\mathbb{Q}\mathcal{O}(G, X)$ -module N) for $n \geq 0$.

Standard homological algebra arguments yield a spectral sequence (see [2])

$$E_2^{p,q} = \text{Ext}^p(\mathbf{H}_q(X, \mathbb{Q}), M) \implies H_G^{p+q}(X, M)$$

$$(\text{resp. } E_2^{p,q} = \text{Ext}^p(N, \mathbf{H}^q(\mathcal{A}_X^*)) \implies H_G^{p+q}(\mathcal{A}_X^*, N)).$$

Notice that the projectivity of $\mathcal{C}^*(X, \mathbb{Q})$ (resp. injectivity of \mathcal{A}_X^* as an $\mathbb{Q}\mathcal{O}(G, X)$ -module) implies the convergence of the spectral sequence.

For a right $\mathbb{Q}\mathcal{O}(G, X)$ -module M , let M^* denote its dual left $\mathbb{Q}\mathcal{O}(G, X)$ -module defined by

$$M^*(G/H, \alpha) = \text{Hom}(M(G/H, \alpha), \mathbb{Q})$$

for $(G/H, \alpha) \in \text{Ob } \mathcal{O}(G, X)$ with the linearly compact topology on each $M^*(G/H, \alpha)$. Taking these facts into account we got in [7] the following equivariant de Rham Theorem (*cf.* [13]).

THEOREM 3.5. — *If X is a G -simplicial set and M a right $\mathbb{Q}\mathcal{O}(G, X)$ -module then there is an isomorphism*

$$H_G^n(X, M) \approx H^n(\mathcal{A}_X^*, M^*) \quad \text{for all } n \geq 0.$$

A G -simplicial set X is called *nilpotent* if simplicial subsets X_α^H are nilpotent for all subgroups $H \subseteq G$ and $\alpha \in \pi_0(X^H)$. Therefore, the associated \mathbb{Q} -algebra \mathcal{A}_X^* is nilpotent for a nilpotent G -simplicial set X .

Let $f: X \rightarrow Y$ be map of G -simplicial sets and $\mathcal{O}(G, f): \mathcal{O}(G, X) \rightarrow \mathcal{O}(G, Y)$ the induced functor. Then there is a map of $\mathbb{Q}\mathcal{O}(G) \int$ -algebras

$$(\pi_0(f), f^*): \mathcal{A}_Y^* \circ \mathcal{O}(G, f) \longrightarrow \mathcal{A}_X^*$$

such that

$$f^*(G/H, \alpha) = A_{f_\alpha^H}^*: A_{Y_{\pi_0(f^H)(\alpha)}^H}^* \longrightarrow A_{X_\alpha^H}^*.$$

Therefore, we get a functor

$$\mathcal{A}^*: G\text{-}\mathbb{S}\mathbb{S} \longrightarrow \mathcal{O}(G) \int \text{-} DGA_{\mathbb{Q}}^\wedge,$$

where $\mathcal{O}(G) \int \text{-} DGA_{\mathbb{Q}}^\wedge$ is the category of functors from $\mathcal{O}(G, X)$ to the category $DGA_{\mathbb{Q}}^\wedge$ for X running over all G -simplicial sets.

To show an existence of an i -minimal model of \mathcal{A}_X^* we need the following

LEMMA 3.6. — *The category of injective linearly compact left $k\mathcal{O}(G, X)$ -modules is closed with respect to the complete tensor product $\widehat{\otimes}$ for any field k .*

Proof. — First observe that any injective linearly compact left $k\mathcal{O}(G, X)$ -module is a direct summand of a product of co-Yoneda $k\mathcal{O}(G, X)$ -modules of the form

$$(k\mathcal{O}(G, X)(-, (G/H, \alpha)))^*$$

for some $(G/H, \alpha) \in \text{Ob } \mathcal{O}(G, X)$. Therefore, by Corollary 1.5 for the proof it is sufficient to show that the complete tensor product

$$\begin{aligned} & (k\mathcal{O}(G, X)(-, (G/H_1, \alpha_1)))^* \widehat{\otimes} (k\mathcal{O}(G, X)(-, (G/H_2, \alpha_2)))^* \\ &= (k\mathcal{O}(G, X)(-, (G/H_1, \alpha_1)) \otimes k\mathcal{O}(G, X)(-, (G/H_2, \alpha_2)))^* \end{aligned}$$

of two co-Yoneda $k\mathcal{O}(G, X)$ -modules is a product of some co-Yoneda $k\mathcal{O}(G, X)$ -modules. But for an object $(G/K, \beta)$ in the category $\mathcal{O}(G, X)$ the free k -module

$$k\mathcal{O}(G, X)((G/K, \beta), (G/H_1, \alpha_1)) \otimes k\mathcal{O}(G, X)((G/K, \beta), (G/H_2, \alpha_2))$$

is freely generated by the set

$$\mathcal{O}(G, X)((G/K, \beta), (G/H_1, \alpha_1)) \times \mathcal{O}(G, X)((G/K, \beta), (G/H_2, \alpha_2)).$$

On the other hand the G -set $G/H_1 \times G/H_2$ is in one to one correspondence with the disjoint union $\bigcup_{i=1}^m G/K_i$, where K_i is the isotropy group of some point

$$x_i = (g'_i H_1, g''_i H_2) \in G/H_1 \times G/H_2 \quad \text{for } i = 1, \dots, m.$$

Then $g'^{-1}_i K_i g'_i \subseteq H_1$, $g''^{-1}_i K_i g''_i \subseteq H_2$ and the set $\mathcal{O}(G, X)((G/K, \beta), (G/H_1, \alpha_1)) \times \mathcal{O}(G, X)((G/K, \beta), (G/H_2, \alpha_2))$ is in one to one correspondence with the disjoint union

$$\bigcup_{i=1}^m \mathcal{O}(G, X)((G/K, \beta), (G/K_i, g'_i \alpha_1)) \times \mathcal{O}(G, X)((G/K, \beta), (G/K_i, g''_i \alpha_2)).$$

Thus, by Proposition 1.4 there is a topological isomorphism of left $k\mathcal{O}(G, X)$ -modules

$$\begin{aligned} & (k\mathcal{O}(G, X)(-, (G/H_1, \alpha_1)))^* \widehat{\otimes} (k\mathcal{O}(G, X)(-, (G/H_2, \alpha_2)))^* \\ & \approx \prod_{i=1}^m (k\mathcal{O}(G, X)(-, (G/K_i, g'_i \alpha_1)))^* \sqcap \prod_{i=1}^m (k\mathcal{O}(G, X)(-, (G/K_i, g''_i \alpha_2)))^* \end{aligned}$$

and this ends the proof. \square

For a nilpotent G -simplicial set X the $\mathbb{Q}\mathcal{O}(G, X)$ -algebra \mathcal{A}_X^* is complete by Proposition 3.3 and injective by [6]. Hence, Theorem 2.8 yields

THEOREM 3.7. — *If X is a nilpotent G -simplicial set then there is an i -minimal $\mathbb{Q}\mathcal{O}(G, X)$ -algebra $\mathcal{M}_X^{(i)}$ (called the i -minimal model of X) and a homology isomorphism $\rho_X : \mathcal{M}_X^{(i)} \rightarrow \mathcal{A}_X^*$.*

We present now examples of i -minimal models of some G -simplicial sets. Let M_n be a left $\mathbb{Q}\mathcal{O}(G)$ -module and $n \geq 2$. Then by [2], [3] there is an Eilenberg-Mac Lane G -simplicial set $K(M_n, n)$ of type (M_n, n) . Let

$$0 \rightarrow M_n^* \xrightarrow{\omega} M_{n,0} \xrightarrow{\omega_0} M_{n,1} \xrightarrow{\omega_1} \dots$$

be the minimal injective resolution in the category $\mathbb{Q}\mathcal{O}(G)\text{-Mod}^c$ of the dual linearly compact $\mathbb{Q}\mathcal{O}(G)$ -module M_n^* and let $M_{n,*} = \{M_{n,\ell}\}_{\ell \geq 0}$ be the

graded $\mathbb{Q}\mathcal{O}(G)$ -module, where $M_{n,\ell}$ has degree $n + \ell$. Then, the symmetric left $\mathbb{Q}\mathcal{O}(G)$ -algebra $\widehat{S}(M_{n,*})$ is the i -minimal injective model of $K(M_n, n)$ with the differential restricting on M_ℓ to ω_ℓ .

Let X and Y be G -nilpotent simplicial sets with $\mathcal{O}(G, X) = \mathcal{O}(G, Y)$, and $\rho_X: \mathcal{M}_X^{(i)} \rightarrow \mathcal{A}_X^*$, $\rho_Y: \mathcal{M}_Y^{(i)} \rightarrow \mathcal{A}_Y^*$ cohomology isomorphisms, where $\mathcal{M}_X^{(i)}$ and $\mathcal{M}_Y^{(i)}$ are i -minimal models of X and Y , respectively. Then by the Künneth formula, the canonical map $f: \mathcal{A}_X^* \widehat{\otimes} \mathcal{A}_Y^* \rightarrow \mathcal{A}_{X \times Y}^*$ and the induced map $\rho_X \widehat{\otimes} \rho_Y: \mathcal{M}_X^{(i)} \widehat{\otimes} \mathcal{M}_Y^{(i)} \rightarrow \mathcal{A}_X^* \widehat{\otimes} \mathcal{A}_Y^*$ are cohomology isomorphisms. From Proposition 2.5, we deduce that $\mathcal{M}_X^{(i)} \widehat{\otimes} \mathcal{M}_Y^{(i)}$ is the i -minimal model of $X \times Y$ with the cohomology isomorphism $\rho_{X \times Y} = f(\rho_X \widehat{\otimes} \rho_Y): \mathcal{M}_X^{(i)} \widehat{\otimes} \mathcal{M}_Y^{(i)} \rightarrow \mathcal{A}_{X \times Y}^*$.

Thus, we may generalize the above example as follows. Let M_{n_1}, \dots, M_{n_k} be $\mathbb{Q}\mathcal{O}(G)$ -modules and $n_1, \dots, n_k \geq 2$. Then, the left $\mathbb{Q}\mathcal{O}(G)$ -algebra $\widehat{S}(M_{n_1,*}) \widehat{\otimes} \dots \widehat{\otimes} \widehat{S}(M_{n_k,*})$ is the i -minimal model of the G -simplicial set $K(M_{n_1}, n_1) \times \dots \times K(M_{n_k}, n_k)$ with the respective differential.

To relate the homotopy groups of a nilpotent G -simplicial set X and its i -minimal model \mathcal{M}_X we need another index category $\mathcal{O}'(G, X)$ defined as follows:

- (1) the set $\text{Ob}(\mathcal{O}'(G, X))$ of objects consists of pairs $(G/H, x)$ with H a subgroup of G and $x \in X^H$;
- (2) morphisms $(G/H, x) \rightarrow (G/K, y)$ are given by G -maps $\phi: G/H \rightarrow G/K$ such that $\bar{\phi}(y) = x$ with $\bar{\phi}: X^K \rightarrow X^H$.

Equivalently, the category $\mathcal{O}'(G, X)$ is given by the Grothendieck construction $\mathcal{O}(G) \int \mathcal{X}_0(X)$, where $\mathcal{X}_0(X): \mathcal{O}(G) \rightarrow \text{Set}$ is a contravariant functor such that $\mathcal{X}_0(X)(G/H) = X_0^H$ for $G/H \in \text{Ob}\mathcal{O}(G)$, where X_0^H is the set of 0-simplexes of the fixed point simplicial subset X^H .

The category $\mathcal{O}(G, X)$ is a quotient of $\mathcal{O}'(G, X)$ and let $\eta_X: \mathcal{O}'(G, X) \rightarrow \mathcal{O}(G, X)$ be the quotient map induced by the natural transformation of functors $\mathcal{X}_0(X) \rightarrow \pi_0(X)$. If X and Y are G -simplicial sets and $\phi: \pi_0(X) \rightarrow \pi_0(Y)$ a map of $\mathcal{O}(G)$ -sets then using methods presented in the proof of Proposition 3.4 we can prove that there exists a lifting $\phi': \mathcal{X}_0(X) \rightarrow \mathcal{X}_0(Y)$ of ϕ . But $\phi': \mathcal{X}_0(X) \rightarrow \mathcal{X}_0(Y)$ determines a G -map $\psi: X_0 \rightarrow Y_0$ of 0-simplexes defined as follows: let $x \in X_0$ and let G_x be isotropy group of x then we write $\psi(x) = \phi'(G/G_x)(x)$. Therefore, we may state

Remark 3.8. — If X and Y are G -simplicial sets and $\phi: \pi_0(X) \rightarrow \pi_0(Y)$ is a map of $\mathcal{O}(G)$ -sets then there exist a lifting $\phi': \mathcal{X}_0(X) \rightarrow \mathcal{X}_0(Y)$ of ϕ such that the diagram

$$\begin{array}{ccc} \mathcal{X}_0(X) & \xrightarrow{\phi'} & \mathcal{X}_0(Y) \\ \downarrow & & \downarrow \\ \pi_0(X) & \xrightarrow{\phi} & \pi_0(Y) \end{array}$$

commutes and a G -map $\psi: X_0 \rightarrow Y_0$ of 0-simplexes such that $\psi^H = \phi'(G/H)$ for any subgroup $H \subseteq G$.

Let X be a G -simplicial set and $\pi_n(X): \mathcal{O}'(G, X) \rightarrow \mathbb{G}p$ the functor to the category $\mathbb{G}p$ of groups such that $\pi_n(X)(G/H, x) = \pi_n(X^H, x)$ is the n th homotopy group of the based simplicial set (X^H, x) for $(G/H, x)$ in $\text{Ob}(\mathcal{O}'(G, X))$ and $n \geq 1$. Then there is a natural transformation

$$\pi_1(X) \times \pi_n(X) \longrightarrow \pi_n(X)$$

defined by the action

$$\pi_1(X^H, x) \times \pi_n(X^H, x) \longrightarrow \pi_n(X^H, x)$$

for $n \geq 1$, a subgroup $H \subseteq G$ and $x \in X^H$. Because of the functoriality of the lower central series of a group we can define inductively functors

$$\Gamma_\ell \pi_n(X): \mathcal{O}'(G, X) \longrightarrow \mathbb{G}p$$

to the category of groups $\mathbb{G}p$ for all $\ell \geq 0$ such that $\Gamma_0 \pi_n(X) = \pi_n(X)$ and for a given $\Gamma_\ell \pi_n(X)$ let

$$\Gamma_{\ell+1} \pi_n(X)(G/H, x) = \{x - ax; x \in \Gamma_\ell \pi_n(X)(G/H, x), a \in \pi_1(X^H, x)\}.$$

Then we obtain a decreasing filtration

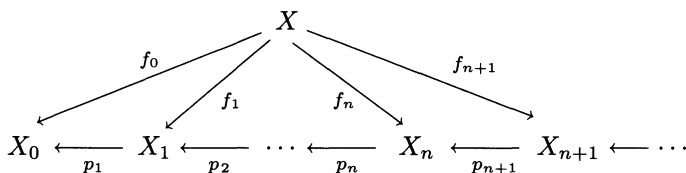
$$\pi_n(X) = \Gamma_0 \pi_n(X) \supseteq \Gamma_1 \pi_n(X) \supseteq \Gamma_2 \pi_n(X) \supseteq \cdots$$

of $\pi_n(X)$ for all $n \geq 1$ and a short exact sequence

$$\begin{aligned} 0 \rightarrow \Gamma_\ell \pi_n(X) / \Gamma_{\ell+1} \pi_n(X) &\longrightarrow \pi_n(X) / \Gamma_{\ell+1} \pi_n(X) \\ &\longrightarrow \pi_n(X) / \Gamma_\ell \pi_n(X) \rightarrow 0 \end{aligned}$$

for $n \geq 1$ and $\ell \geq 0$ with the trivial action of $\pi_1(X)$ on $\Gamma_\ell \pi_n(X) / \Gamma_{\ell+1} \pi_n(X)$. Of course, if X is a nilpotent G -simplicial set then $\Gamma_\ell \pi_n(X) = 0$ for some $\ell \geq 0$. The Postnikov tower of a nilpotent G -simplicial set X has the following properties (*cf.* [15] for a connected case).

PROPOSITION 3.9. — *If X is a nilpotent G -simplicial set and*



its Postnikov tower then:

- (1) $\pi_n(X_m) = 0$ for $n > m$;
- (2) $\pi_n(f_m)$ is an isomorphism for $n \leq m$;
- (3) the fibration $p_n : X_{n+1} \rightarrow X_n$ admits a refinement

$$X_{n+1} = X_{n,N} \longrightarrow X_{n,N-1} \longrightarrow \cdots \longrightarrow X_{n,1} \longrightarrow X_{n,0} = X_n,$$

where $X_{n,\ell+1} \rightarrow X_{n,\ell}$ is a principal G -fibration with the fibre given by the G -Eilenberg-MacLane simplicial set $K(\Gamma_\ell \pi_n(X)/\Gamma_{\ell+1} \pi_n(X), n+1)$.

We say that a G -simplicial set X is *rational* if the homotopy groups $\pi_n(X^H, x)$ are \emptyset -local for any subgroup $H \subseteq G$, $x \in X^H$ and $n \geq 1$. Of course, for $n \geq 2$ this means that $\pi_n(X^H, x)$ are \mathbb{Q} -modules, where \mathbb{Q} is the field of rationals. From [13] it follows that for any G -simplicial set X there is a rational G -simplicial set X_0 and a G -map $f : X \rightarrow X_0$ with some universal property. But for nilpotent G -simplicial sets holds (cf. [15] for a connected case)

THEOREM 3.10. — *Let $f : X \rightarrow X_0$ be a map of G -simplicial sets with X_0 rational. Then the following conditions are equivalent:*

- (1) $H_G^n(f, M) : H_G^n(X_0, M) \rightarrow H_G^n(X, M)$ is an isomorphism for all left $\mathbb{Q}\mathcal{O}(G, X)$ -modules M and $n \geq 0$;
- (2) $H_n^G(f, M) : H_n^G(X, M) \rightarrow H_n^G(X_0, M)$ is an isomorphism for all right $\mathbb{Q}\mathcal{O}(G, X)$ -modules M and $n \geq 0$;
- (3) $\pi_n(f) : \pi_n(X) \otimes \mathbb{Q} \rightarrow \pi_n(X_0) \otimes \mathbb{Q} = \pi_n(X_0)$ is an isomorphism for $n \geq 0$;
- (4) $H_n(f, \mathbb{Q}) : H_n(X, \mathbb{Q}) \rightarrow H_n(X_0, \mathbb{Q})$ is a isomorphism for $n \geq 0$;
- (5) for any G -map $g : X \rightarrow Y$, where Y is rational there is a (unique up

to G -homotopy) map $\tilde{g} : X_0 \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ f \downarrow & \nearrow \tilde{g} & \\ X_0 & & \end{array}$$

commutes (up to G -homotopy).

To formulate the next result we need some notations. By Theorem 3.7 there exist the i -minimal model $\rho_X : \mathcal{M}_X^{(i)} \rightarrow \mathcal{A}_X^*$ of the $\mathbb{Q}\mathcal{O}(G, X)$ -algebra and a map $\mathcal{M}_X^{(i)} \rightarrow \underline{\mathbb{Q}}$ of $\mathbb{Q}\mathcal{O}(G, X)$ -algebras for a nilpotent G -simplicial set X , where $\underline{\mathbb{Q}}$ is the constant $\mathbb{Q}\mathcal{O}(G, X)$ -algebra determined by the field \mathbb{Q} . Hence we may define a left $\mathbb{Q}\mathcal{O}(G, X)$ -module $\pi^n(\mathcal{M}_X^{(i)})$ such that

$$\pi^n(\mathcal{M}_X^{(i)})(G/H, \alpha) = \pi^n(\mathcal{M}_X^{(i)}(G/H, \alpha))$$

and $\Gamma^\ell \pi^n(\mathcal{M}_X^{(i)}) \subseteq \pi^n \mathcal{M}_X^{(i)}$ to be the image of the obvious map $\pi^n \mathcal{M}_X^{(i)}(n, \ell - 1) \rightarrow \pi^n \mathcal{M}_X^{(i)}$ for all $n \geq 1$ and $\ell \geq 0$. This gives a natural increasing filtration

$$0 = \Gamma^0 \pi^n \mathcal{M}_X^{(i)} \subseteq \Gamma^1 \pi^n \mathcal{M}_X^{(i)} \subseteq \Gamma^2 \pi^n \mathcal{M}_X^{(i)} \subseteq \dots$$

of $\pi^n \mathcal{M}_X^{(i)}$ for all $n \geq 1$.

Now we may state the main result.

THEOREM 3.11. — *If X is a nilpotent G -simplicial set and $\mathcal{M}_X^{(i)}$ its i -minimal model then there are natural isomorphisms :*

(1) $H_G^n(X, M) \approx H^n(\mathcal{A}_X^*, M^*) \approx H^n(\mathcal{M}_X^{(i)}, M^*)$ for all $n \geq 0$, where M is a right $\mathbb{Q}\mathcal{O}(G, X)$ -module and M^* its dual linearly compact left $\mathbb{Q}\mathcal{O}(G, X)$ -module;

(2) $H^n(X, \mathbb{Q}) \approx H^n(\mathcal{A}_X^*) = H^n(\mathcal{M}_X^{(i)})$ for all $n \geq 0$;

(3) the correspondence $X \mapsto \mathcal{M}_X^{(i)}$ is a bijection from rational G -homotopy types to isomorphisms classes of i -minimal $\mathbb{Q}\mathcal{O}(G, X)$ -algebras;

(4) $(\pi_n(X)/\Gamma_\ell \pi_n(X))^* \approx \Gamma^\ell \pi^n(\mathcal{M}_X^{(i)} \eta_X)$ for $n \geq 2$, $\ell \geq 1$ and $(\Gamma_\ell \pi_n(X)/\Gamma_{\ell+1} \pi_n(X))^* \approx \Gamma^{\ell+1} \pi^n(\mathcal{M}_X^{(i)} \eta_X)/\Gamma^\ell \pi^n(\mathcal{M}_X^{(i)} \eta_X)$ for $n, \ell \geq 1$, where $\eta_X : \mathcal{O}'(G, X) \rightarrow \mathcal{O}(G, X)$ is the quotient functor;

(5) $\mathcal{M}_X^{(i)}(n, \ell)$ is the i -minimal model for the (n, ℓ) -stage $X_{n, \ell}$ of the Postnikov tower of X ;

(6) the map $\tau : M_{n+1,\ell} \rightarrow Z^{n+2}(\mathcal{M}_X^{(i)}(n, \ell - 1))$ determining the elementary extension $\mathcal{M}_X^{(i)}(n, \ell) = \mathcal{M}^{(i)}(n, \ell - 1)(M_{n+1,\ell})$ yields the k -invariant

$$\begin{aligned} k_\ell^{n+2} &\in H_G^{n+2}(X_{n,\ell-1}, \Gamma_\ell \pi_{n+1}(X) \otimes \mathbb{Q} / \Gamma_{\ell+1} \pi_{n+1}(X) \otimes \mathbb{Q}) \\ &\approx [X_{n,\ell-1}, K(\Gamma_\ell \pi_{n+1}(X) \otimes \mathbb{Q} / \Gamma_{\ell+1} \pi_{n+1}(X) \otimes \mathbb{Q}, n+2)]_G. \end{aligned}$$

Proof. — The assertions of (1) and (2) follow from the equivariant de Rham Theorem 3.5. In virtue of (4), (5) and Theorem 3.10 there are isomorphisms

$$\begin{aligned} H_G^{n+2}(X_{n,\ell-1}, \Gamma_\ell \pi_{n+1}(X) \otimes \mathbb{Q} / \Gamma_{\ell+1} \pi_{n+1}(X) \otimes \mathbb{Q}) \\ \approx H^{n+2}(\mathcal{M}_{X_{n,\ell-1}}^{(i)}, M_{n+1,\ell}) \\ \approx \text{Hom}(M_{n+1,\ell}, \mathbf{H}^{n+2}(\mathcal{M}_{X_{n,\ell-1}}^{(i)})). \end{aligned}$$

Thus (6) is straightforward.

We mimic [13] to sketch a proof of the remaining parts for G -simplicial sets X such that $\pi_1(X) = 0$, for the sake of ease of notation and clarity of exposition. The modifications required for the nilpotent case should at this point be clear. We use the notations X_n and $\mathcal{M}_X(n)$ to denote the n th stage of the Postnikov tower of X and of the i -minimal model $\mathcal{M}_X^{(i)}$ of X , respectively for all $n \geq 0$. These assertions are obvious for $\mathcal{M}_X(0) = \Omega(\mathbb{Q})$. Assume by induction that they are true for $\mathcal{M}_X(n-1)$ and X_{n-1} . By Theorem 2.8 there are isomorphisms

$$\mathcal{M}_{X_n} \approx \mathcal{M}_X(n-1)(\pi_n(X)^*), \quad \mathcal{M}_X(n) \approx \mathcal{M}_X(n-1)(\pi_n(X)^*)$$

and maps $\rho_{X_n} : \mathcal{M}_{X_n} \rightarrow \mathcal{A}_{X_n}^*$, $\rho_X(n) : \mathcal{M}_X(n) \rightarrow \mathcal{A}_X^*$. If $f_n : X \rightarrow X_n$ is the G -map from X to its n th stage X_n in the Postnikov tower then we get a map $g_n : \mathcal{M}_{X_n} \rightarrow \mathcal{M}_X(n)$ and a commutative (up to based homotopy) diagram

$$\begin{array}{ccc} \mathcal{M}_{X_n} & \xrightarrow{g_n} & \mathcal{M}_X(n) \\ \rho_{X_n} \downarrow & & \downarrow \rho_X(n) \\ \mathcal{A}_{X_n}^* & \xrightarrow{\mathcal{A}_{f_n}} & \mathcal{A}_X. \end{array}$$

The resulting map g_n is a (based) isomorphism on cohomology up to degree n , hence in all degrees because of the structure of $\mathbb{Q}\mathcal{O}(G, X)$ -algebras in question. Therefore, g_n is an isomorphism by Proposition 2.4, since both $\mathbb{Q}\mathcal{O}(G, X)$ -algebras \mathcal{M}_{X_n} and $\mathcal{M}_X(n)$ are i -minimal. \square

The last result contains the following proposition. We refer to Remark 3.8, for the zero step in the inductive proof of its second part.

PROPOSITION 3.12.

(1) Let $f : X \rightarrow Y$ be a G -map of simplicial sets and $\rho_X : \mathcal{M}_X^{(i)} \rightarrow \mathcal{A}_X^*$, $\rho_Y : \mathcal{M}_Y^{(i)} \rightarrow \mathcal{A}_Y^*$ i -minimal models of X and Y , respectively. Then there is a (unique up to based homotopy) map $\tilde{f} : \mathcal{M}_Y^{(i)} \circ \mathcal{O}(G, f) \rightarrow \mathcal{M}_X^{(i)}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}_Y^* \circ \mathcal{O}(G, f) & \xrightarrow{(\pi_0(f), f^*)} & \mathcal{A}_X^* \\ \rho_Y \circ \mathcal{O}(G, f) \uparrow & & \uparrow \rho_X \\ \mathcal{M}_Y^{(i)} \circ \mathcal{O}(G, f) & \xrightarrow{(\pi_0(f), \tilde{f})} & \mathcal{M}_X^{(i)} \end{array}$$

commutes (up to based homotopy).

(2) If $\phi : \pi_0(X) \rightarrow \pi_0(Y)$ is a map of $\mathcal{O}(G)$ -sets, $\mathcal{O}(G) \int \phi : \mathcal{O}(G, X) \rightarrow \mathcal{O}(G, Y)$ the induced functor and $\tilde{f} : \mathcal{M}_Y^{(i)} \circ (\mathcal{O}(G) \int \phi) \rightarrow \mathcal{M}_X^{(i)}$ a map of $\mathbb{Q}\mathcal{O}(G, X)$ -algebras then there is a (unique up to G -homotopy) G -map $f : X \rightarrow Y$ such that $\pi_0(f) = \phi$ and the diagram

$$\begin{array}{ccc} \mathcal{A}_Y^* \circ (\mathcal{O}(G) \int \phi) & \xrightarrow{(\phi, f^*)} & \mathcal{A}_X^* \\ \rho_Y \circ (\mathcal{O}(G) \int \phi) \uparrow & & \uparrow \rho_X \\ \mathcal{M}_Y^{(i)} \circ (\mathcal{O}(G) \int \phi) & \xrightarrow{(\phi, \tilde{f})} & \mathcal{M}_X^{(i)} \end{array}$$

commutes (up to based homotopy).

In particular, for nilpotent G -simplicial sets X and Y (with Y rational), there is a bijection

$$[X, Y]_G \approx [\mathcal{M}_Y^{(i)}, \mathcal{M}_X^{(i)}]_{\mathcal{O}(G)_f},$$

where $[X, Y]_G$ is the set of G -homotopy classes of G -maps from X to Y and $[\mathcal{M}_Y^{(i)}, \mathcal{M}_X^{(i)}]_{\mathcal{O}(G)_f}$ the set of homotopy classes of maps in the category $\mathcal{O}(G) \int DGA_{\mathbb{Q}}^\wedge$.

At the end, we mention some applications of the constructed i -minimal models. First, observe that with the i -minimal model $\mathcal{M}_X^{(i)}$ of a G -nilpotent

simplicial set X we can associate an $\mathcal{O}(G, X)$ -simplicial set $F\mathcal{M}_X^{(i)}$ such that

$$(F\mathcal{M}_X^{(i)})(G/H, \alpha) = F\mathcal{M}_X^{(i)}(G/H, \alpha),$$

where $F: DGA_{\mathbb{Q}} \rightarrow \mathbb{S}\mathbb{S}$ is the adjoint functor to the de Rham functor $A^*: \mathbb{S}\mathbb{S} \rightarrow DGA_{\mathbb{Q}}$. By [1], p. 64, the adjunction maps

$$X_{\alpha}^H \longrightarrow F\mathcal{M}_X^{(i)}(G/H, \alpha)$$

are rationalizations of X_{α}^H , for $(G/H, \alpha) \in \text{Ob } \mathcal{O}(G, X)$. If now $\mathcal{FM}_X^{(i)}$ is the G -simplicial set associated with the $\mathcal{O}(G, X)$ -simplicial set $F\mathcal{M}_X^{(i)}$ (see [3]) then the canonical G -map $X \rightarrow \mathcal{FM}_X^{(i)}$ is the rationalization of X .

Moreover, let \mathbb{Z}_n be the cyclic group of order n and X a Hopf \mathbb{Z}_{p^k} -simplicial set. Then, mimic the methods in presented [14], one may deduce that the \mathbb{Q} -localization X_0 of the simplicial set X is \mathbb{Z}_{p^k} -equivalent to a product of Eilenberg-Mac Lane \mathbb{Z}_{p^k} -simplicial sets, without any restrictions on the G -connectivity as well as G -finite type of X .

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