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TERENCE NAPIER

MOHAN RAMACHANDRAN

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## THE BOCHNER-HARTOGS DICHOTOMY FOR WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

by T. NAPIER (\*) & M. RAMACHANDRAN (\*\*)

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### Introduction.

A complex manifold  $M$  for which  $H_c^1(M, \mathcal{O}) = 0$  is said to have the *Bochner-Hartogs property* (see Hartogs [H], Bochner [B], and Harvey and Lawson [HL]). Equivalently, for every  $C^\infty$  compactly supported form  $\alpha$  of type  $(0, 1)$  with  $\bar{\partial}\alpha = 0$  on  $M$ , there is a  $C^\infty$  compactly supported function  $\beta$  on  $M$  such that  $\bar{\partial}\beta = \alpha$ . Andreotti and Vesentini [AV] proved that a strongly  $(n-1)$ -complete complex manifold of dimension  $n > 1$  has the Bochner-Hartogs property, and Grauert and Riemenschneider [GR], that a strongly hyper- $(n-1)$ -convex Kähler manifold of dimension  $n > 1$  has the Bochner-Hartogs property (see Section 1). In [R], the second author proved that if the universal covering  $\tilde{M}$  of a compact Kähler manifold (or a Galois covering  $M$  with infinite covering group of more than quadratic growth) admits a nonconstant holomorphic function, then  $M$  satisfies the following dichotomy:

(BHD) *Either  $M$  has the Bochner-Hartogs property or there exists a proper holomorphic mapping of  $M$  onto a Riemann surface.*

A complex manifold which admits a continuous plurisubharmonic exhaustion function is said to be *weakly 1-complete*. The main result of this paper (Theorem 2.5) is the following:

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THEOREM. — *If  $(M, g)$  is a connected noncompact weakly 1-complete Kähler manifold which has exactly one end, then  $M$  satisfies (BHD).*

If a complex manifold  $M$  has the Bochner-Hartogs property, then every holomorphic function  $f$  on a neighborhood of infinity with no relatively compact connected components extends to a holomorphic function  $f_0$  on  $M$  as follows. Cutting off away from infinity, one obtains a  $C^\infty$  function  $\lambda$  on  $M$ . For  $\alpha = \bar{\partial}\lambda$ , there is a function  $\beta$  as above and the function  $f_0 = \lambda - \beta$  is the desired extension. In particular, a Riemann surface cannot have the Bochner-Hartogs property and a complex manifold cannot satisfy both of the conditions in (BHD). Moreover, a manifold with the Bochner-Hartogs property has only one end because, on a manifold with more than one end, there exists a function which is locally constant, but not constant, near infinity. A related result due to Arapura, Bressler, and the second author [ABR] is that the universal covering of a compact Kähler manifold has at most one end. In fact, as shown in [NR], a complete noncompact connected Kähler manifold  $M$  which satisfies  $H^1(M, \mathbb{R}) = 0$  and which has bounded geometry or is weakly 1-complete has exactly one end. It was also proved in [NR] that a complete Kähler manifold with at least three ends which has bounded geometry or is weakly 1-complete admits a proper holomorphic mapping onto a Riemann surface.

Facts concerning strictly  $q$ -plurisubharmonic functions and Green's functions are collected in Section 1. Section 2 contains the proof of the theorem. The main step is to prove Proposition 2.3 that (BHD) holds for a weakly 1-complete Kähler manifold on which there exists, outside a compact subset, a pair of pluriharmonic functions with linearly independent differentials (this may be thought of as a generalization of a theorem of Ohsawa [O1]). One may assume that  $M$  is complete and admits a positive Green's function since one can exhaust  $M$  by domains with these properties. Hence, if  $[\alpha] \in H_c^1(M, \mathcal{O})$  is a nonzero element, then one can form the  $L^2$  harmonic projection  $\gamma$  and a function  $\beta$  such that  $\gamma = \alpha - \bar{\partial}\beta$ . In particular,  $\beta$  is pluriharmonic outside a compact set  $K$ . One then forms a pluriharmonic function on a covering space, pushes down to obtain a second pluriharmonic function on  $M \setminus K$ , and applies Proposition 2.3.

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## 1. Preliminaries on $q$ -plurisubharmonic functions.

Most of the facts discussed in this section are known, so the proofs are only sketched. Throughout this section,  $(M, g)$  denotes a Kähler manifold of dimension  $n$  and  $q$  denotes a positive integer.

Let  $\varphi$  be a real-valued continuous function on  $M$ . We will say that  $\varphi$  is *strictly  $q$ -plurisubharmonic* if  $\varphi$  is an element of the class  $\Psi(q)$  defined by Wu [W]. We will call  $\varphi$   *$q$ -plurisubharmonic* if the function  $\varphi + \psi$  is strictly  $q$ -plurisubharmonic for every continuous strictly  $q$ -plurisubharmonic function  $\psi$  on  $M$ .

*Remarks.*

1. If  $\varphi$  of class  $C^2$ , then  $\varphi$  is  $q$ -plurisubharmonic (strictly  $q$ -plurisubharmonic) if and only if, for each point  $x_0 \in M$ , the trace of the restriction of the *Levi form*

$$\mathcal{L}(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

of  $\varphi$  to any complex subspace of  $T_{x_0}^{1,0}M$  of dimension  $q$  is nonnegative (respectively, positive).

2. If  $\varphi$  is  $q$ -plurisubharmonic (strictly  $q$ -plurisubharmonic), then  $\varphi$  is  $(q+1)$ -plurisubharmonic (respectively, strictly  $(q+1)$ -plurisubharmonic).

3. A real-valued function of class  $C^2$  on a complex manifold is said to be *strictly  $q$ -convex* if its Levi form has at most  $q-1$  nonpositive eigenvalues at each point. A function  $\psi$  on a complex space  $X$  is said to be *strictly  $q$ -convex* if, for each point  $x \in X$ , there is a proper embedding of a neighborhood  $U$  of  $x$  into an open subset  $V$  of some complex Euclidean space and an extension of  $\psi|_U$  to a strictly  $q$ -convex function on  $V$ . It follows that if  $\varphi$  is of class  $C^2$  and strictly  $q$ -plurisubharmonic on  $M$ , then  $\varphi$  is strictly  $q$ -convex on  $M$  and on any analytic subset of  $M$ .

4. The set of smooth elements of  $\Psi(q)$  is dense in the following sense:

**PROPOSITION 1.1** (Wu [W], Proposition 1). — *If  $\varphi$  is a continuous strictly  $q$ -plurisubharmonic function on a Kähler manifold  $M$  and  $\alpha$  is a positive continuous function on  $M$ , then there exists a  $C^\infty$  strictly  $q$ -plurisubharmonic function  $\psi$  such that  $|\varphi - \psi| < \alpha$  on  $M$ .*

In particular, it follows that the restriction of a continuous  $q$ -plurisubharmonic (strictly  $q$ -plurisubharmonic) function to a complex submanifold of dimension  $q$  is subharmonic (respectively, strictly subharmonic).

5. The Kähler manifold  $(M, g)$  is said to be *hyper- $q$ -complete* if  $M$  admits a  $C^\infty$  strictly  $q$ -plurisubharmonic exhaustion function. If there exists a  $C^\infty$   $q$ -plurisubharmonic exhaustion function which is strictly  $q$ -plurisubharmonic on the complement of some compact subset of  $M$ , then  $(M, g)$  is said to be *strongly hyper- $q$ -convex*.

6. Standard arguments show that if  $\varphi$  and  $\varphi'$  are continuous  $q$ -plurisubharmonic functions on  $M$ , then  $\varphi + \varphi'$ ,  $\max(\varphi, \varphi')$ , and the composition  $\chi(\varphi)$  of any nondecreasing convex function  $\chi$  with  $\varphi$ , are all  $q$ -plurisubharmonic.

7. Hunt and Murray [HM] and Kalka [K] studied functions which satisfy a condition which they called  $q$ -plurisubharmonicity but which is weaker than the above notion.

The following result is contained implicitly in the work of Greene and Wu [GW], Ohsawa [O2], and Demailly [D2]:

**THEOREM 1.2** (Demailly, Greene-Wu, Ohsawa). — *Let  $X$  be an analytic subset of dimension  $m \leq q$  in the Kähler manifold  $M$  and let  $Y$  be the union of the singular set  $X_{\text{sing}}$  with all irreducible components of  $X$  which are noncompact or which have dimension strictly less than  $q$ . Then there exist neighborhoods  $V$  of  $X$  and  $W$  of  $Y$  in  $M$  and a  $C^\infty$  strictly  $(q+1)$ -plurisubharmonic function  $\varphi$  on  $V$  such that  $\varphi|_X$  exhausts  $X$ ,  $W \subset V$ , and  $\varphi|_W$  is strictly  $q$ -plurisubharmonic.*

The proof is an easy modification of Demailly's [D2] proof of the analogous result for strictly  $q$ -convex functions, but we include a sketch here for completeness. Similarly, as in [D2] and in the work of Coltoiu [C], a hyper- $q$ -complete submanifold admits a hyper- $q$ -complete neighborhood, but we won't use this fact and the proof will not be sketched.

By a theorem of Richberg [Ri], a  $C^\infty$  strictly plurisubharmonic function on an analytic subset of a complex space extends to a  $C^\infty$  strictly plurisubharmonic function on a neighborhood. Demailly proved a version of this theorem for strictly  $q$ -convex functions in which the function is approximated by a strictly  $q$ -convex function on a neighborhood (see also [P]). A natural modification of Richberg's proof shows that if a

function on an analytic subset of a Kähler manifold admits local  $C^\infty$  strictly  $q$ -plurisubharmonic extensions, then it admits a  $C^\infty$  strictly  $q$ -plurisubharmonic extension to a neighborhood. We will only need this fact for submanifolds and the proof is simple in this case.

**PROPOSITION 1.3** (Richberg). — *If  $\varphi$  is a  $C^\infty$  strictly  $q$ -plurisubharmonic function on a complex submanifold  $N$  of  $M$  (relative to the Kähler metric  $g|_N$ ), then there exists a  $C^\infty$  strictly  $q$ -plurisubharmonic function  $\psi$  on a neighborhood of  $N$  in  $M$  such that  $\psi|_N = \varphi$ .*

*Sketch of the proof.* — Let  $m = \dim N$  and suppose  $(U, (z_1, \dots, z_n))$  is a holomorphic coordinate neighborhood in which  $N$  is the zero set of  $w = (z_1, \dots, z_{n-m})$ . If  $\varphi'$  is a function on a relatively compact polydisk  $D$  in  $U$  obtained by composing  $\varphi$  with the associated projection mapping and  $C > 0$  is sufficiently large, then the function  $\varphi' + C|w|^2$  is strictly  $q$ -plurisubharmonic on a neighborhood of  $N \cap D$ . By using a partition of unity, one may patch these local extensions to obtain the desired function  $\psi$ . Q. E. D.

The main point of Demailly's [D2] proof of Ohsawa's theorem [O2] is essentially the following version of the theorem of Greene and Wu [GW] on the existence of subharmonic exhaustion functions (see [D2], Proof of Theorem 2):

**PROPOSITION 1.4** (Demailly, Greene-Wu). — *Suppose  $X$  is a complex space with no compact irreducible components,  $Y$  is an analytic subset which contains the singular set  $X_{\text{sing}}$ ,  $U$  is a neighborhood of  $Y$  in  $X$ , and  $h$  is a Hermitian metric on  $X \setminus Y$ . Then there exists a  $C^\infty$  nonnegative function  $\varphi$  on  $X$  such that*

- (i)  $\varphi$  is positive and exhaustive on  $X \setminus U$ ,
- (ii)  $\varphi$  vanishes on a neighborhood of  $Y$ , and
- (iii)  $\varphi$  is subharmonic (with respect to  $h$ ) on  $X \setminus Y$  and strictly subharmonic on the subset  $\{x \in X \mid \varphi(x) > 0\}$  of  $X \setminus Y$ .

*Proof of Theorem 1.2 ([D2]).* — The statement of the theorem makes sense and is trivial for  $q = 0$ . We proceed by induction on  $q$ . Assume that  $q > 0$  and that the theorem holds for nonnegative integers less than  $q$ . Let  $A$  be the union of  $X_{\text{sing}}$  and all of the irreducible components of  $X$  of dimension less than  $q$ , let  $B$  be the union of all of the noncompact irreducible components of  $X$  of dimension  $q$ , and let  $C$  be the closure

of  $X \setminus (A \cup B)$ .

Applying the induction hypothesis to  $A$  and cutting off away from  $A$ , we obtain a  $C^\infty$  nonnegative function  $\alpha$  on  $M$  and a neighborhood  $U$  of  $A$  in  $M$  such that  $\alpha$  is strictly  $q$ -plurisubharmonic on  $U$  and exhaustive on  $\overline{U}$ .

Next, by Proposition 1.4, there exists a nonnegative  $C^\infty$  function  $\psi$  on  $X$  such that  $\text{supp } \psi \subset B \setminus A = B \setminus (A \cup C)$ ,  $\psi > 2$  on  $B \setminus U$ ,  $\psi$  exhausts  $B \setminus U$ , and  $\psi$  is strictly  $q$ -plurisubharmonic on the subset  $N \equiv \psi^{-1}((0, \infty))$  of  $B \setminus (A \cup C)$ . By Richberg's theorem (Proposition 1.3), there exists a  $C^\infty$  strictly  $q$ -plurisubharmonic function  $\tau$  on a neighborhood  $V_1$  of  $N$  in  $M$  such that  $\tau = \psi$  on  $N$ . Let  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  nondecreasing convex function such that  $\chi(t) = 0$  if  $t < 1$ ,  $\chi(t) > t$  if  $t > 2$ , and  $\chi'(t) > 0$  if  $t > 1$ . Since  $\psi^{-1}([0, 1))$  is a neighborhood of the boundary of  $N$  in  $X$ , we may assume (shrinking  $V_1$  if necessary) that there is a neighborhood  $V_2$  of  $X \setminus N$  in  $M$  such that  $\tau < 1$  on  $V_1 \cap V_2$ . Thus the function  $\chi(\tau)$  may be extended to a nonnegative  $C^\infty$   $q$ -plurisubharmonic function  $\beta$  which is defined on the neighborhood  $V_1 \cup V_2$  of  $X$  and vanishes on a neighborhood of  $A \cup C$ . Moreover, since  $\psi > 2$  on  $B \setminus U$ ,  $\beta$  is strictly  $q$ -plurisubharmonic and positive on a neighborhood of  $B \setminus U$  in  $M$  and  $\beta$  exhausts  $B \setminus U$ .

Since  $\dim C < q+1$ , every  $C^\infty$  function on the subset  $C \setminus Y$  of  $C \setminus X_{\text{sing}}$  is strictly  $(q+1)$ -plurisubharmonic. As in the construction of  $\beta$ , we may form a nonnegative  $C^\infty$   $(q+1)$ -plurisubharmonic function  $\gamma$  on a neighborhood of  $X$  such that  $\gamma$  is positive and strictly  $(q+1)$ -plurisubharmonic on a neighborhood of  $C \setminus U$  in  $M$ ,  $\gamma$  exhausts  $C \setminus U$ , and  $\gamma$  vanishes on a neighborhood of  $A \cup B = Y$ .

It is now easy to check that if  $\lambda$  is a  $C^\infty$  increasing convex function on  $\mathbb{R}$  and  $\lambda'(t) \rightarrow \infty$  sufficiently fast as  $t \rightarrow \infty$ , then the function  $\varphi \equiv \alpha + \lambda(\beta) + \lambda(\gamma)$  has the required properties on some neighborhood  $V$  of  $X$  and some neighborhood  $W$  of  $Y = A \cup B$ . Q. E. D.

A  $C^\infty$  strictly  $q$ -convex function  $\varphi$  on a complex space  $X$  of pure dimension  $q$  has no local maximum points. Wu's approximation theorem (Proposition 1.1) implies that the same is true of the restriction of a continuous strictly  $q$ -plurisubharmonic function on  $M$  to an analytic subset  $X$  of pure dimension  $q$ . Similarly, we have the following:

**PROPOSITION 1.5 (Maximum principle).** — *If the restriction  $\varphi|_X$  of a continuous  $q$ -plurisubharmonic function  $\varphi$  on  $M$  to a connected analytic subset  $X$  of pure dimension  $q$  assumes its maximum value  $m = \varphi(x_0)$  at some point  $x_0 \in X$ , then  $\varphi|_X$  is constant.*

*Proof.* — Since  $\varphi|_{X_{\text{reg}}}$  is subharmonic, it suffices to show that  $\varphi(x) = m$  at some point  $x \in X_{\text{reg}}$ . Assume that this is not the case. Taking successive singular sets  $X_{\text{sing}}, (X_{\text{sing}})_{\text{sing}}, \dots$ , we may assume that, for some nowhere dense analytic subset  $Y$  of  $X$ , we have  $x_0 \in Y_{\text{reg}}$  and  $\varphi < m$  on  $X \setminus Y$ . Every  $C^\infty$  function on  $Y_{\text{reg}}$  is strictly  $q$ -plurisubharmonic since  $\dim Y < q$ . So, by applying Proposition 1.3, we may form a relatively compact neighborhood  $U$  of  $x_0$  in  $M$  and a  $C^\infty$  strictly  $q$ -plurisubharmonic function  $\psi$  on a neighborhood of  $\bar{U}$  such that  $\psi(x_0) = 0$  and  $\psi < 0$  on  $(\bar{U} \setminus \{x_0\}) \cap Y$ . In particular, on some neighborhood  $V$  of  $Y \cap \partial U$ , we have  $\psi < 0$  and hence  $\varphi + \epsilon\psi < m$  for every  $\epsilon > 0$ . We also have  $\varphi + \epsilon\psi < m$  on  $(X \cap \partial U) \setminus V$  provided  $\epsilon$  is sufficiently small. Therefore  $\varphi + \epsilon\psi < m = (\varphi + \epsilon\psi)(x_0)$  on  $X \cap \partial U$ . This contradicts the maximum principle for continuous strictly  $q$ -plurisubharmonic functions, so the proposition follows. Q. E. D.

In this paper, the main tool for obtaining the Bochner-Hartogs property is the following result of Grauert and Riemenschneider [GR] (who proved a version of the vanishing theorem for higher cohomology groups) and of Siu [S], Lemma 5.10 (who proved a version in the more general setting of harmonic maps into manifolds satisfying a certain curvature condition).

**THEOREM 1.6** (Grauert-Riemenschneider, Siu). — *Suppose  $\Omega$  is a relatively compact domain in a Kähler manifold  $M$  and  $\Omega$  has a  $C^\infty$   $(n-1)$ -plurisubharmonic defining function  $\varphi$  whose differential is nonzero at every point in  $\partial\Omega$ .*

(a) *If  $\beta$  is a  $C^\infty$  function on  $\bar{\Omega}$  which is harmonic on  $\Omega$  and which satisfies the tangential Cauchy-Riemann equations  $\bar{\partial}_b\beta = 0$  on  $\partial\Omega$ , then  $\beta$  is pluriharmonic on  $\Omega$ .*

(b) *If  $\varphi$  is strictly  $(n-1)$ -plurisubharmonic on a neighborhood of some point in  $\partial\Omega$ , then  $H_c^1(\Omega, \mathcal{O}) = 0$ .*

*Sketch of the proof.* — Suppose first that  $\beta$  is a function as in (a). Let  $\gamma = \bar{\partial}\beta$  and let  $\eta = *\bar{\gamma}$ , where  $*$  denotes the Hodge star operator. Then  $\eta$  may be thought of as a  $C^\infty$  form of type  $(n, n-1)$  or as a form of type  $(0, n-1)$  with values in the canonical bundle  $K_M$  on  $\bar{\Omega}$ . A computation in normal coordinates then shows that  $\eta$  lies in the domain of the adjoint operator  $\bar{\partial}^*$ . The Bochner-Kodaira formula is then (see [GR])



or [S], formula 2.1.4)

$$\|\bar{\partial}\eta\|_{L^2(\Omega)}^2 + \|\bar{\partial}^*\eta\|_{L^2(\Omega)}^2 = \|\bar{\nabla}\eta\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} |\gamma|^2 \cdot \tau \, d\sigma,$$

where  $d\sigma$  is the volume element on  $\partial\Omega$  and  $\tau$  is the trace of the restriction of the Levi form  $\mathcal{L}(\varphi)$  of  $\varphi$  to  $T^{1,0}(\partial\Omega)$ . Here, the curvature terms drop out because  $\eta$  is of type  $(n, n-1)$  and we have used the fact that  $\bar{\partial}\beta \wedge \bar{\partial}\varphi$  vanishes at each point of  $\partial\Omega$ . Since  $\beta$  is harmonic, the left-hand side is equal to zero. Since  $\varphi$  is  $(n-1)$ -plurisubharmonic, we have  $\tau \geq 0$  and hence  $\bar{\nabla}\eta = 0$ . A computation in normal coordinates now shows that  $\beta$  is pluriharmonic (see [S], Proof of Lemma 5.6(d)) and (a) is proved.

Suppose now that  $\alpha$  is a  $C^\infty$  compactly supported form of type  $(0, 1)$  with  $\bar{\partial}\alpha = 0$ . Let  $\beta$  be the  $C^\infty$  function on  $\bar{\Omega}$  which vanishes on  $\partial\Omega$  and which satisfies  $-\frac{1}{2}\Delta\beta = \bar{\partial}^*\bar{\partial}\beta = \bar{\partial}^*\alpha$  on  $\Omega$  (where  $\Delta = -(d^*d + dd^*)$  is the Laplacian), let  $\gamma = \alpha - \bar{\partial}\beta$ , and let  $\eta = *\bar{\gamma}$ . Then

$$\bar{\partial}^*\eta = -*\partial*(*\bar{\gamma}) = *\bar{\partial}\bar{\gamma} = 0 \quad \text{and} \quad \bar{\partial}\eta = -*[\bar{\partial}^*\alpha - \bar{\partial}^*\bar{\partial}\beta] = 0.$$

Moreover, since  $\alpha$  has compact support,  $\gamma = -\bar{\partial}\beta$  near  $\partial\Omega$ ; and, since  $\beta$  vanishes on  $\partial\Omega$ ,  $\bar{\partial}_b\beta \equiv 0$ . Applying the Bochner-Kodaira formula to  $\eta$  as in the proof of (a), we get  $\bar{\nabla}\eta = 0$  and it follows that  $\partial\gamma = 0$  (and hence  $d\gamma = 0$ ). Thus  $\bar{\gamma}$  is a holomorphic 1-form on  $\Omega$ .

On the other hand, since  $\tau > 0$  at some point, the Bochner-Kodaira formula implies that  $|\gamma| = |\bar{\partial}\beta| = 0$  on some nonempty open subset in  $\partial\Omega$ . If  $U$  is a connected neighborhood of a boundary point and  $f$  is a  $C^\infty$  function on  $U \cap \bar{\Omega}$  which is holomorphic on  $U \cap \Omega$  and which vanishes on  $U \cap \partial\Omega$ , then one may extend  $f$  to a continuous function  $h$  on  $U$  which vanishes outside  $\Omega$ . But then  $\bar{\partial}h = 0$  in the weak sense, so  $h$  is holomorphic. It follows that  $h$ , and therefore  $f$ , must vanish identically. Letting  $f$  be a coefficient of the holomorphic 1-form  $\bar{\gamma}$  with respect to some local holomorphic frame, we see that  $\gamma$  vanishes on a nonempty open subset of  $\Omega$  and hence on all of  $\Omega$ . Thus  $\beta$  is holomorphic outside the support  $K$  of  $\alpha$ . Since  $\beta$  vanishes on  $\partial\Omega$ , the above discussion (with  $f = \beta$ ) implies that  $\beta$  vanishes on each connected component of  $\Omega \setminus K$  which is not relatively compact in  $\Omega$ . In other words,  $\beta$  has compact support and  $\alpha = \bar{\partial}\beta$ . Thus (b) is proved. Q. E. D.

Since the proof of the main theorem involves related arguments on a complete Kähler manifold, we close this section with a discussion of Green's functions on Riemannian manifolds. A connected noncompact Riemannian manifold  $N$  which admits a positive symmetric Green's function  $G(x, y)$  is said to be *hyperbolic* (otherwise,  $N$  is called *parabolic*). We normalize  $G$  so

that, for each point  $x_0 \in N$ ,

$$\Delta_{\text{distr}} G(\cdot, x_0) = -\delta_{x_0}$$

where  $\delta_{x_0}$  is the Dirac function at  $x_0$  and  $\Delta = -(d^*d + dd^*)$  is the Laplacian. We will use the same notation for the corresponding integral operator  $G$  given by

$$(G\alpha)(x) = \int_N G(x, y)\alpha(y) dV(y) \quad \forall x \in N$$

for each suitable function  $\alpha$  on  $N$ . If  $\alpha$  is a  $C^\infty$  compactly supported function, then  $\beta \equiv -G\alpha$  is a  $C^\infty$  bounded function with finite energy (i.e.  $\int_N |\nabla\beta|^2 dV < \infty$ ) and  $\Delta\beta = \alpha$ . Moreover,  $\beta(x_\nu) \rightarrow 0$  if  $\{x_\nu\}$  is a sequence in  $N$  with  $x_\nu \rightarrow \infty$  and  $G(\cdot, x_\nu) \rightarrow 0$ . Such a sequence  $\{x_\nu\}$  always exists and will be called a *regular sequence*.

## 2. Proof of the main result.

We begin with two lemmas. The first is a special case of a result of Nishino [Ni] who proved it without the assumption that  $M$  is Kähler. In the Kähler case, one may prove it using arguments contained in the proof of [NR], Theorem 4.6.

LEMMA 2.1. — *Suppose  $(M, g)$  is a connected weakly 1-complete Kähler manifold and there exists a proper holomorphic mapping of some nonempty open subset of  $M$  onto a Riemann surface. Then  $M$  also admits a proper holomorphic mapping onto a Riemann surface.*

The second lemma often helps one obtain a holomorphic mapping to a Riemann surface. An elementary proof is given here. One may also prove this fact by using holomorphic equivalence relations (see, for example, [Ka]).

LEMMA 2.2. — *If  $\omega_1$  and  $\omega_2$  are two linearly independent closed holomorphic 1-forms satisfying  $\omega_1 \wedge \omega_2 \equiv 0$  on a connected complex manifold  $M$ , then the meromorphic function  $h \equiv \omega_1/\omega_2$  has no points of indeterminacy in  $M$  and is locally constant on the analytic set  $S \equiv \{x \in M \mid (\omega_1)_x = 0\} \cup \{x \in M \mid (\omega_2)_x = 0\}$ .*

*Proof.* — Let  $I$  be the set of points of indeterminacy of  $h$  and, for each  $\zeta \in \mathbb{P}^1$ , let

$$F_\zeta = \overline{(h|_{(M \setminus I)})^{-1}(\zeta)} \supset I$$

be the fiber over  $\zeta$  (see [Gu]). Since the problem is local, we may assume that there exist holomorphic functions  $f_1$  and  $f_2$  on  $M$  such that  $df_i = \omega_i$  for  $i = 1, 2$ . One may see that  $f_1$  and  $f_2$  are locally constant on  $F_\zeta$  for each  $\zeta \in \mathbb{P}^1$  as follows. Near each smooth point  $x$  of  $F_\zeta$  at which  $df_2 \neq 0$ , we may choose holomorphic coordinates  $z = (z_1, \dots, z_n)$  in which  $z_1 = f_2$  on a connected neighborhood of  $x$ . Since  $df_1 \wedge dz_1 = 0$ , we have then  $f_1 = f_1(z_1)$  and hence  $h = f'_1(z_1)$ . In particular,  $h$  is constant along each fiber of  $z_1$  and hence each of these fibers (being of codimension 1) must be an open set in  $F_\zeta$ . Therefore, if  $v$  is a vector tangent to  $F_\zeta$  at  $x$ , then  $\omega_2(v) = df_2(v) = 0$ . This is also the case if  $df_2 = 0$  at  $x$ . Thus  $f_2$  is locally constant on  $F_\zeta$  for each  $\zeta \in \mathbb{P}^1$ , and, by symmetry, the same is true of  $f_1$ . It follows that, if  $x_0 \in I$ , then  $f_2^{-1}(f_2(x_0))$  contains the connected component  $H_\zeta$  of  $F_\zeta$  containing  $x_0$  for each  $\zeta \in \mathbb{P}^1$ . But  $\{H_\zeta\}_{\zeta \in \mathbb{P}^1}$  is an infinite collection of distinct analytic sets of pure dimension  $n-1$ , so this is impossible. Therefore  $I = \emptyset$ .

It remains to show that  $h$  is locally constant on the analytic set  $S$ , which we may assume to be irreducible. Since  $f_1$  or  $f_2$  is then constant on  $S$ , we may assume that  $S$  lies in some irreducible component  $T$  of the zero set of  $f_1 \cdot f_2$  and it suffices to show that  $h$  is constant on  $T$ . By working near a generic point of  $T$ , we may assume that there exist holomorphic coordinates  $z = (z_1, \dots, z_n)$  on  $M$  in which  $T$  is the zero set of  $z_1$  and, for  $j = 1, 2$ ,  $f_j(z) = z_1^{m_j} g_j(z)$  where  $g_j$  is a nonvanishing holomorphic function and  $m_j \geq 0$ . By replacing  $z_1$  by  $z_1 u$  for some (local)  $m_2^{\text{th}}$  root  $u$  of  $g_2$ , we may also assume that  $g_2 \equiv 1$ . Since  $df_1 \wedge df_2 \equiv 0$ , we then get  $f_1 = f_1(z_1)$  and  $g_1 = g_1(z_1)$ . Hence the function  $h(z) = f'_1(z_1)/f'_2(z_1)$  depends only on  $z_1$  and is therefore constant on  $T$ . Thus the lemma is proved.

The main step in the proof of the theorem stated in the introduction is the following proposition, which is, in a sense, a generalization of a theorem of Ohsawa [O1] and a result in [NR].

**PROPOSITION 2.3.** — *Let  $(M, g)$  be a connected noncompact Kähler manifold of dimension  $n$  on which there exists a continuous  $(n-1)$ -plurisubharmonic exhaustion function  $\varphi$ . Suppose there is a compact subset  $K$  of  $M$  such that, on each connected component  $E$  of  $M \setminus K$ , there exists a pair of real-valued pluriharmonic functions  $\rho_1$  and  $\rho_2$  with linearly independent differentials  $d\rho_1$  and  $d\rho_2$ . Then  $M$  satisfies (BHD).*

*Remarks.*

1. The above condition on  $E$  holds if, for example, there exists a

nonconstant holomorphic function on  $E$ .

2. If  $M$  satisfies the above conditions and has more than one end, then there exists a proper holomorphic mapping of  $M$  onto a Riemann surface.

*Proof.* — Choosing  $a \in \mathbb{R}$  sufficiently large, we may assume without loss of generality that

$$K = \{x \in M \mid \varphi(x) \leq a\} \neq \emptyset.$$

Let  $E$  be a connected component of  $M \setminus K$  with noncompact closure and let  $\rho_1$  and  $\rho_2$  be pluriharmonic functions on  $E$  as in the statement of the proposition. We may assume that  $\rho_1$  and  $\rho_2$  extend to pluriharmonic functions on a neighborhood of  $\overline{E}$ .

There are three possibilities:

(a) The analytic subset  $X = \{x \in E \mid (\partial\rho_1 \wedge \partial\rho_2)_x = 0\}$  of  $E$  is nowhere dense,

(b)  $X = E$  and the set  $Q$  of points in  $E$  which lie in a compact level of the holomorphic mapping

$$h = \frac{\partial\rho_1}{\partial\rho_2} : E \rightarrow \mathbb{P}^1$$

(see Lemma 2.2) is nonempty, or

(c)  $X = E$  and  $Q = \emptyset$ .

We will show that, if (a) or (c) holds, then  $M$  admits an exhaustion by  $C^\infty$  domains which are strongly hyper- $(n-1)$ -convex at each boundary point in  $E$ ; and if (b) holds, then  $E$  admits a proper holomorphic mapping onto a Riemann surface. Briefly, in the case (a), we will work on a relatively compact (weakly) hyper- $(n-1)$ -convex domain  $\Omega$  in  $M$ . Combining  $\varphi$ ,  $\rho_1^2 + \rho_2^2$ , and a strictly  $(n-1)$ -plurisubharmonic function near  $X$  (obtained from Theorem 1.2), we will obtain a strictly  $(n-1)$ -plurisubharmonic function near  $E \cap \partial\Omega$ . In the case (b), we will show that  $Q = E$  and hence, by Stein factorization, one obtains the desired mapping. Finally, in the case (c), we will again work on a hyper- $(n-1)$ -convex domain  $\Omega$  and we will apply a standard patching argument to strictly  $(n-1)$ -plurisubharmonic functions on neighborhoods of fibers of  $h$  (or of a suitable holomorphic function if  $h$  is constant) to obtain a strictly  $(n-1)$ -plurisubharmonic function as in the case (a).

Suppose first that  $E$  satisfies the condition (a) and let  $Y$  be the union of all of the compact irreducible components of  $X$  of dimension  $n-1$ . The

maximum principle (Proposition 1.5) implies that  $\varphi(Y)$  is a countable set, so we may choose  $b \in (a, \infty) \setminus \varphi(Y)$  so large that  $K$  is contained in some connected component  $\Omega$  of  $\{x \in M \mid \varphi(x) < b\}$ . By Theorem 1.2 (Demailly, Greene-Wu, Ohsawa), there exists a positive  $C^\infty$  function  $\psi$  on  $E \setminus Y$  which is strictly  $(n-1)$ -plurisubharmonic on a neighborhood of  $X \setminus Y$  in  $E$ . Therefore, since  $\rho_1^2 + \rho_2^2$  is strictly  $(n-1)$ -plurisubharmonic on  $E \setminus X$  and since  $\varphi^{-1}(b) \cap Y = \emptyset$ , the function  $\gamma \equiv r \cdot (\rho_1^2 + \rho_2^2) + \psi$ , where  $r$  is a sufficiently large positive constant, is strictly  $(n-1)$ -plurisubharmonic on a relatively compact neighborhood  $U$  of  $\varphi^{-1}(b) \cap E$  in  $E \setminus Y$ . Applying Wu's approximation theorem [W] (Proposition 1.1) to the function  $\gamma - \log(b - \varphi)$  on  $\Omega \cap U$ , we get a  $C^\infty$  strictly  $(n-1)$ -plurisubharmonic function  $\lambda$  on  $U \cap \Omega$  which approaches infinity at  $\partial\Omega$ . Finally, let  $c > 0$  and let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  nondecreasing convex function such that  $\chi(t) = 0$  for  $t \leq c$ ,  $\chi'(t) > 0$  for  $t > c$ , and  $\chi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . If  $c$  is sufficiently large, then the function  $\chi(\lambda)$  extends to a  $C^\infty$  nonnegative  $(n-1)$ -plurisubharmonic function on  $\Omega \cup (M \setminus E)$  which vanishes on  $M \setminus E$ , which is strictly  $(n-1)$ -plurisubharmonic near  $E \cap \partial\Omega$ , and which exhausts  $\Omega \cap \overline{E}$ .

Assuming now that (b) holds, we show that  $Q = E$ . We first observe that  $Q$  is open for point-set topological reasons. For if  $x_0 \in Q$  and  $L_0$  is the (compact) level of  $h$  through  $x_0$ , then there is a relatively compact neighborhood  $U$  of  $L_0$  in  $E \setminus (h^{-1}(h(x_0)) \setminus L_0)$ . Since  $\partial U$  is compact, there is a neighborhood  $V$  of  $h(x_0)$  in  $\mathbb{P}^1$  such that  $h^{-1}(V) \cap \partial U = \emptyset$ . Hence the level through each point in the neighborhood  $h^{-1}(V) \cap U$  does not meet  $\partial U$  and must, therefore, be a compact subset of  $U$ .

Next, let  $b > a$  be so large that there is a connected component  $\Omega$  of the set  $\{x \in M \mid \varphi(x) < b\}$  such that  $Q \cap \Omega \neq \emptyset$ ,  $K \subset \Omega$ , and  $\Omega \cap E$  is connected. If  $x_0 \in \overline{Q} \cap \Omega$ , then the irreducible component  $A$  of  $h^{-1}(h(x_0))$  containing  $x_0$  is a compact subset of  $\Omega \cap E$ . For if  $\{x_\nu\}$  is a sequence in  $Q \cap \Omega \setminus \{x_0\}$  converging to  $x_0$  and, for each  $\nu$ ,  $L_\nu$  is the (compact) level of  $h$  through  $x_\nu$ , then, by the Remmert-Stein-Thullen theorem (see [Gu]),  $A$  lies in the closure of  $\bigcup_\nu L_\nu$ . On the other hand,  $\varphi|_{L_\nu}$  is constant for each  $\nu$  and  $\varphi(x_\nu) \rightarrow \varphi(x_0)$  where  $a < \varphi(x_0) < b$ . Therefore, since  $\varphi = a$  or  $b$  at each boundary point of  $\Omega \cap E$ ,  $\bigcup_\nu L_\nu$  must lie in some compact subset of  $\Omega \cap E$  and the claim follows.

Since  $h$  extends to a holomorphic mapping on a neighborhood of  $\overline{E}$  ( $\rho_1$  and  $\rho_2$  extend by the choice of  $a$ ), the set of critical values of  $h|_{(\Omega \cap E)}$  is finite and the inverse image of this finite set is an analytic subset  $B$  of  $\Omega \cap E$ . The above discussion implies that, if  $L_0$  is the level of  $h$  through

a point  $x_0 \in \overline{Q} \cap \Omega$  and  $L_0 \cap \Omega$  is smooth, then  $L_0 \subset Q \cap \Omega$ . Therefore  $Q \cap \Omega \setminus B$  is a closed subset of the connected set  $\Omega \cap E \setminus B$  and hence, since  $Q$  is open, we must have equality. In particular,  $Q \cap \Omega$  is dense in  $\Omega \cap E$ . Applying the above again, we see that if  $L_0$  is any level of  $h$  which meets  $\Omega$ , then every irreducible component of  $L_0$  that meets  $\Omega$  must be a compact subset of  $\Omega \cap E$ . It follows that  $L_0$  is compact. Thus  $\Omega \cap E \subset Q$  and, since the choice of  $b$  was arbitrary, we get  $Q = E$ . Therefore every level of  $h$  is compact and, by Stein factorization [St], we obtain a proper holomorphic mapping of  $E$  onto a Riemann surface.

Finally, assuming that  $X = E$  and  $Q = \emptyset$  (i.e. that  $E$  satisfies the condition (c)), we apply a modification of a construction due to Ohsawa [O1] for the case of a weakly 1-complete surface. We also assume for now that the mapping  $h : E \rightarrow \mathbb{P}^1$  is nonconstant.

We first show that the union  $C$  of the collection  $\mathcal{C}$  of all compact irreducible components of fibers of  $h$  is a nowhere dense analytic subset of  $E$ . For if  $K_0$  is a compact subset of  $E$  and  $L_0$  is a level of  $h$ , then any compact irreducible component  $C_0$  of  $L_0$  which meets  $K_0$  must lie in the compact subset  $\varphi^{-1}(\varphi(K_0))$ . Moreover, since  $C_0 \neq L_0$ ,  $C_0$  must meet some irreducible component  $H_0$  of the analytic set  $H \equiv \{x \in E \mid (h_*)_x = 0\}$  and, since  $h$  is locally constant on  $H$ , we have  $H_0 \subset L_0$ . Only finitely many irreducible components of  $H$  meet  $\varphi^{-1}(\varphi(K_0))$ , so the collection of all levels  $L_0$  with such an irreducible component  $C_0$  is finite. It follows that  $\mathcal{C}$  is locally finite in  $E$  and hence that  $C$  is an analytic set.

The set  $\varphi(C)$  is discrete, so we may choose a number  $b \in (a, \infty) \setminus \varphi(C)$  so large that there is a connected component  $\Omega$  of  $\{x \in M \mid \varphi(x) < b\}$  such that  $K \subset \Omega$  and  $\Omega \cap E$  is connected. We may also choose a relatively compact neighborhood  $W$  of  $\varphi^{-1}(b) \cap E$  in  $E \setminus C$ . For each point  $x \in \varphi^{-1}(b) \cap E$ , the analytic set  $h^{-1}(h(x)) \setminus C$  has no compact irreducible components and hence, by Theorem 1.2, there is a  $C^\infty$  strictly  $(n-1)$ -plurisubharmonic function  $\gamma_1$  on a neighborhood  $V_1$  of  $h^{-1}(h(x)) \setminus C$  in  $E$ . Moreover, we have  $h^{-1}(D'_1) \cap W \subset V_1$  for any sufficiently small neighborhood  $D'_1$  of  $h(x)$  in  $\mathbb{P}^1$ . There is also a nonnegative  $C^\infty$  function  $\lambda_1$  with  $\text{supp } \lambda_1 \subset D'_1$  and  $\lambda_1 \equiv 1$  on some neighborhood  $D_1$  of  $h(x)$ . We may, therefore, choose  $C^\infty$  strictly  $(n-1)$ -plurisubharmonic functions  $\gamma_1, \dots, \gamma_m$  on open sets  $V_1, \dots, V_m$  in  $E$ , respectively; open sets  $D_1, \dots, D_m$  and  $D'_1, \dots, D'_m$  in  $\mathbb{P}^1$ ; and nonnegative  $C^\infty$  functions  $\lambda_1, \dots, \lambda_m$  on  $\mathbb{P}^1$  such that, for each  $j = 1, \dots, m$ , we have  $\lambda_j \equiv 1$  on  $D_j$ ,  $\text{supp } \lambda_j \subset D'_j$ ,  $h^{-1}(D'_j) \cap W \subset V_j$ , and  $\varphi^{-1}(b) \cap E \subset h^{-1}(D_1) \cup \dots \cup h^{-1}(D_m)$ . Moreover,

by Lemma 2.2,  $h$  is locally constant on the analytic set  $S \equiv \{x \in E \mid (\partial\rho_2)_x = 0\}$ , so  $h(S \cap \phi^{-1}(b))$  is a finite set and we may assume that, for each  $j$ ,  $\lambda_j(h)$  is constant near each point of  $W \cap S$ .

We now show that, for a sufficiently large positive constant  $s$ , the  $C^\infty$  function

$$\gamma \equiv s \cdot (\rho_2)^2 + \sum_{j=1}^m \lambda_j(h) \gamma_j$$

is strictly  $(n-1)$ -plurisubharmonic on some neighborhood of  $\varphi^{-1}(b) \cap E$ . It is easy to see that  $\gamma$  is strictly  $(n-1)$ -plurisubharmonic on a neighborhood of  $S \cap \varphi^{-1}(b) \cap E$ , since, near each point of this set, each of the nonnegative functions  $\lambda_1(h), \dots, \lambda_m(h)$  is constant and at least one of the functions is positive. Given a point  $x$  near  $\varphi^{-1}(b) \cap E$  and a tangent vector  $v \in T_x^{1,0}M$ , we have

$$\begin{aligned} \mathcal{L}(\gamma)(v, v) &= 2s|\partial\rho_2(v)|^2 + \sum_{j=1}^m \gamma_j(x) \mathcal{L}(\lambda_j)(h_*v, h_*v) \\ &\quad + \sum_{j=1}^m 2 \operatorname{Re} [\partial\lambda_j(h_*v) \overline{\partial\gamma_j(v)}] + \sum_{j=1}^m \lambda_j(h(x)) \mathcal{L}(\gamma_j)(v, v). \end{aligned}$$

If  $x$  is not near  $S$ , then, since  $(\partial\rho_2)_x \neq 0$  and  $\partial\rho_1 \wedge \partial\rho_2 \equiv 0$ , we may choose holomorphic coordinates  $(z_1, \dots, z_n)$  near  $x$  in which  $\rho_2 = 2 \operatorname{Re} z_1$ ,  $\rho_1 = \rho_1(z_1)$ , and

$$h = \frac{\partial\rho_1}{\partial\rho_2} = \frac{\partial\rho_1}{\partial z_1} = \frac{\partial\rho_1}{\partial z_1}(z_1) \in \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}.$$

Thus  $\partial h = \frac{\partial h}{\partial z_1} dz_1$  and hence  $|\partial\rho_2(v)|^2 = |dz_1(v)|^2 \geq q_0 |\partial h(v)|^2$  for some positive constant  $q_0$ . Since  $\lambda_1, \dots, \lambda_m \geq 0$ , since  $\max(\lambda_1(h), \dots, \lambda_m(h)) > 0$  near  $\varphi^{-1}(b) \cap E$ , and since, for each  $j$ ,  $\gamma_j$  is strictly  $(n-1)$ -plurisubharmonic on the neighborhood  $V_j$  of  $\operatorname{supp} \lambda_j(h) \cap \varphi^{-1}(b)$ , there exist positive constants  $q_1$  and  $q_2$  (independent of  $s$ ) such that, for every  $\epsilon > 0$ , for every point  $x \in \varphi^{-1}(b) \cap E$ , and for every collection of orthonormal tangent vectors  $e_1, \dots, e_{n-1} \in T_x^{1,0}M$ , we have

$$\sum_{i=1}^{n-1} \mathcal{L}(\gamma)(e_i, e_i) \geq (2sq_0 - q_1 \cdot (1 + (2\epsilon)^{-1})) \cdot \sum_{i=1}^{n-1} |\partial h(e_i)|^2 - \frac{1}{2} \epsilon q_1 + q_2.$$

Thus if we choose  $\epsilon < 2q_2/q_1$  and  $s > (2q_0)^{-1} q_1 \cdot (1 + (2\epsilon)^{-1})$ , then  $\gamma$  will be strictly  $(n-1)$ -plurisubharmonic near points in  $\varphi^{-1}(b) \cap E$  which lie outside an (arbitrarily small) neighborhood of  $S$  as well as those which lie near  $S$ .

Proceeding now as in the case (a), one gets a  $C^\infty$  nonnegative  $(n-1)$ -plurisubharmonic function on  $\Omega \cup (M \setminus E)$  which vanishes on  $M \setminus E$ , which exhausts  $\overline{E} \cap \Omega$ , and which is strictly  $(n-1)$ -plurisubharmonic near  $E \cap \partial\Omega$ .

If  $h$  is constant, then  $\partial(c_1\rho_1 + c_2\rho_2) \equiv 0$  for some pair of constants  $c_1, c_2 \in \mathbb{C}$  which are not both zero. The function  $f = \bar{c}_1\rho_1 + \bar{c}_2\rho_2$  on  $E$  is holomorphic and nonconstant; because the functions 1,  $\rho_1$ , and  $\rho_2$  are linearly independent. One may now proceed as above by using  $f$  in place of the mapping  $h$ .

Thus for each of the (finitely many) connected components  $E$  of  $M \setminus K$  which have noncompact closure, either  $E$  admits a proper holomorphic mapping onto a Riemann surface (and hence a  $C^\infty$  plurisubharmonic exhaustion function), or there exists an arbitrarily large relatively compact domain  $\Omega$  in  $M$  and a  $C^\infty$  nonnegative  $(n-1)$ -plurisubharmonic function on  $\Omega \cup (M \setminus E)$  which vanishes on  $M \setminus E$ , which exhausts  $\overline{E} \cap \Omega$ , and which is strictly  $(n-1)$ -plurisubharmonic near  $E \cap \partial\Omega$ . If all of these connected components of  $M \setminus K$  have the former property, then  $M$  admits a  $C^\infty$  plurisubharmonic exhaustion function and a complete Kähler metric and Lemma 2.1 implies that there is a proper holomorphic mapping of  $M$  onto a Riemann surface. If at least one of these connected components has the latter property, then  $H_c^1(M, \mathcal{O}) = 0$ . For if  $\alpha$  is a  $C^\infty$  compactly supported form of type  $(0, 1)$  and  $\bar{\partial}\alpha = 0$ , then we may choose a  $C^\infty$  relatively compact domain  $\Omega$  which contains the support of  $\alpha$  and which admits a  $C^\infty$   $(n-1)$ -plurisubharmonic defining function which is strictly  $(n-1)$ -plurisubharmonic near at least one connected component of  $\partial\Omega$ . The theorem of Grauert and Riemenschneider [GR] and Siu [S] (Theorem 1.6) then implies that  $H_c^1(\Omega, \mathcal{O}) = 0$  and hence that there is a  $C^\infty$  compactly supported function  $\beta$  on  $\Omega$ , and therefore on  $M$ , such that  $\bar{\partial}\beta = \alpha$ . Thus  $M$  has the property (BHD). Q. E. D.

To apply Proposition 2.3, it will be convenient to have the following lemma, which is essentially a special case of [NR], Theorem 2.6.

LEMMA 2.4. — *Let  $(M, g)$  be a connected noncompact complete hyperbolic Kähler manifold, let  $M_0$  be a  $C^\infty$  relatively compact domain in  $M$ , and let  $E$  be a connected component of  $M \setminus \overline{M}_0$  with noncompact closure. Then there exists a pluriharmonic function  $\tau$  on  $M$  such that  $0 \leq \tau \leq 1$ ,  $\tau$  has finite energy, and, for every regular sequence  $\{x_\nu\}$  in  $M$  approaching  $\infty$  (see Section 1),  $\tau(x_\nu) \rightarrow 1$  if  $x_\nu \in E$  for  $\nu \gg 0$  and  $\tau(x_\nu) \rightarrow 0$  if  $x_\nu \in M \setminus E$  for  $\nu \gg 0$ .*



The construction of  $\tau$  is contained within the proof of [NR], Theorem 2.6. That  $\tau \leq 1$  is not proved explicitly, but one can easily verify this property by forming an exhaustion of  $M$  by  $C^\infty$  relatively compact domains and writing the Green's function on  $M$  as the limit of the corresponding sequence of Green's functions.

We are now ready to prove the main result.

**THEOREM 2.5.** — *Let  $(M, g)$  be a connected Kähler manifold of dimension  $n$  which has exactly one end and which admits a continuous  $(n-1)$ -plurisubharmonic exhaustion function  $\varphi$ . Assume that at least one of the following conditions is satisfied:*

- (i)  $\varphi$  plurisubharmonic,
- (ii)  $(M, g)$  is complete and hyperbolic, or
- (iii)  $\varphi$  is of class  $C^\infty$ .

*Then  $M$  satisfies (BHD).*

*Remark.* — An example of Cousin [Co] shows that one cannot remove the requirement that  $M$  have only one end (see, for example, [NR], Example 3.9).

*Proof.* — Assuming that there exists a  $C^\infty$  compactly supported form  $\alpha$  of type  $(0, 1)$  such that  $\bar{\partial}\alpha = 0$  and  $[\alpha] \neq 0$  in  $H_c^1(M, \mathcal{O})$ , we will show that there is a proper holomorphic mapping of  $M$  onto a Riemann surface.

We first assume that (i) or (ii) holds. Fix a  $C^\infty$  relatively compact domain  $M_0$  in  $M$  such that  $\text{supp } \alpha \subset M_0$  and  $M \setminus \bar{M}_0$  is connected. If  $\varphi$  is plurisubharmonic (i.e. (i) holds), let  $a > \sup_{M_0} \varphi$  and let  $\Omega$  be the connected component of  $\{x \in M \mid \varphi(x) < a\}$  containing  $\bar{M}_0$ . Then, by a theorem of Nakano [N] and Demailly [D1], the (weakly 1-complete) domain  $\Omega$  admits a complete Kähler metric  $g'$ . Moreover,  $(\Omega, g')$  is hyperbolic (in fact, the Green's function vanishes at the boundary) and if  $a$  is sufficiently large, then  $\Omega \setminus \bar{M}_0$  is connected. It suffices to show that  $\Omega$  admits a proper holomorphic mapping onto a Riemann surface (for a suitable choice of  $a$ ). For we may then form an exhaustion of  $M$  by such domains. Applying a theorem of Narasimhan [Na], Corollary 1 to the Cartan-Remmert reductions, we get the required mapping on  $M$ . If  $\varphi$  is not plurisubharmonic, we set  $(\Omega, g') = (M, g)$ .

Thus, in either case,  $(\Omega, g')$  admits a continuous  $(n-1)$ -plurisubharmonic exhaustion function and a positive symmetric Green's function  $G(x, y)$ , and the Kähler metric  $g'$  is complete. Therefore, since  $\bar{\partial}^* \alpha$  is a  $C^\infty$  function with compact support on  $\Omega$ , the function  $\beta \equiv -2G(\bar{\partial}^* \alpha)$  is a  $C^\infty$  bounded function with finite energy and  $-\frac{1}{2} \Delta \beta = \bar{\partial}^* \bar{\partial} \beta = \bar{\partial}^* \alpha$ . Moreover,  $\beta(x_\nu) \rightarrow 0$  for any regular sequence  $\{x_\nu\}$  in  $\Omega$  approaching  $\infty$  (see Section 1). The form  $\gamma \equiv \alpha - \bar{\partial} \beta$  is a harmonic form in  $L^2_{0,1}(\Omega, g')$ ; the harmonic projection of  $\alpha$ . By the Gaffney theorem [G],  $\gamma$  is closed (and coclosed). In particular,  $\bar{\gamma}$  is a holomorphic 1-form on  $\Omega$  (since  $\partial \gamma = 0$ ) and  $\beta$  is pluriharmonic on the connected set  $E \equiv \Omega \setminus \overline{M}_0$ . Since  $[\alpha] \neq 0$  in  $H^1_c(M, \mathcal{O})$  and  $\text{supp } \alpha \subset M_0 \subset \subset \Omega$ , we have  $[\alpha] \neq 0$  in  $H^1_c(\Omega, \mathcal{O})$ .

It follows that  $\beta$  is not constant on  $E$ . For, if  $\beta$  were constant on  $E$ , then, since  $\beta$  vanishes at infinity along any regular sequence, we would have  $\beta \equiv 0$  on  $\Omega \setminus \overline{M}_0$ . Hence  $\gamma$  would vanish on  $E$  and, therefore, on  $\Omega$ , since  $\gamma$  is harmonic. But we would then get  $\alpha = \bar{\partial} \beta$ , where  $\beta$  is a  $C^\infty$  function with compact support in  $\Omega$ ; which contradicts the nonvanishing of  $[\alpha]$ . Thus  $\beta|_E$  is not constant (and hence is nowhere locally constant).

If  $\gamma = 0$ , then  $\beta|_E$  is a nonconstant holomorphic function on the end  $E$ . Proposition 2.3 then implies that  $\Omega$  satisfies (BHD) and hence, since  $H^1_c(\Omega, \mathcal{O}) \neq 0$ , there is a proper holomorphic mapping onto a Riemann surface. Thus we may assume that  $\gamma$  is not everywhere zero. Let  $\rho = \text{Re } \beta|_E$  and  $\rho' = \text{Im } \beta|_E$ . If the functions 1,  $\rho$ , and  $\rho'$  are linearly independent on  $E$ , then, again, Proposition 2.3 gives one the required mapping. If 1,  $\rho$ , and  $\rho'$  are linearly dependent on  $E$ , then, since  $\rho$  and  $\rho'$  vanish at infinity along any regular sequence, the functions  $\rho$  and  $\rho'$  must be linearly dependent. Therefore, after multiplying  $\alpha$  by a suitable nonzero complex constant, we may assume that  $\beta|_E = \rho$  (i.e.  $\beta|_E$  is real-valued) and hence that  $\gamma|_E = -\bar{\partial} \rho$ . To obtain a second pluriharmonic function (i.e. to apply Proposition 2.3) we will use Lemma 2.4 to obtain a pluriharmonic function on a covering space of  $\Omega$  with two ends and then push down to  $E$ .

We first observe that, for any point  $x_0 \in E$ , the mapping  $\pi_1(E, x_0) \rightarrow \pi_1(\Omega, x_0)$  is *not surjective*. For if the mapping is surjective, then, since the  $C^\infty$  closed real 1-form  $\theta = -\gamma - \bar{\gamma}$  on  $\Omega$  is equal to  $d\rho$  on  $E$ , it follows that the function  $\rho_0$  given by

$$\rho_0(x) = \rho(x_0) + \int_{x_0}^x \theta \quad \forall x \in \Omega$$

is a well-defined  $C^\infty$  function with  $d\rho_0 = \theta = d\rho$  on  $E$  and  $\rho_0(x_0) = \rho(x_0)$ . Hence  $\rho_0 = \rho$  on  $E$ , so  $\beta - \rho_0$  is a  $C^\infty$  function on  $\Omega$  which vanishes on

$E = \Omega \setminus \overline{M}_0$  and which satisfies

$$\bar{\partial}(\beta - \rho_0) = \bar{\partial}\beta - \bar{\partial}\rho_0 = \alpha - \gamma + \gamma = \alpha.$$

But this contradicts the assumption that  $[\alpha] \neq 0$  in  $H_c^1(\Omega, \mathcal{O})$ , so the mapping cannot be surjective.

Fix a point  $x_0 \in E$ , let  $\Gamma$  be the image of  $\pi_1(E, x_0)$  in  $\pi_1(\Omega, x_0)$ , and let  $\pi : \tilde{\Omega} \rightarrow \Omega$  be a connected covering space with  $\pi_*(\pi_1(\tilde{\Omega}, x_1)) = \Gamma$  for some point  $x_1 \in \pi^{-1}(x_0)$ . Since  $\Gamma$  is a proper subgroup and  $E$  is a  $C^\infty$  domain,  $\pi$  maps a neighborhood of the closure  $\overline{E}_1$  of the connected component  $E_1$  of  $\tilde{E} = \pi^{-1}(E)$  containing  $x_1$  isomorphically onto a neighborhood of  $\overline{E}$  and  $\tilde{\Omega} \setminus \overline{E}_1$  is not relatively compact in  $\tilde{\Omega}$  (i.e.  $\tilde{\Omega}$  has at least two ends). In particular,  $E_1$  is a hyperbolic end of  $\tilde{\Omega}$  with respect to the complete Kähler metric  $\pi^*g'$ . Therefore, by Lemma 2.4, there exists a pluriharmonic function  $\tau$  on  $\tilde{\Omega}$  such that  $0 \leq \tau \leq 1$ ,  $\tau$  has finite energy, and, for every regular sequence  $\{x_\nu\}$  in  $\tilde{\Omega}$  approaching  $\infty$ ,

$$\lim_{\nu \rightarrow \infty} \tau(x_\nu) = \begin{cases} 1 & \text{if } x_\nu \in E_1 \text{ for } \nu \gg 0 \\ 0 & \text{if } x_\nu \in \tilde{\Omega} \setminus E_1 \text{ for } \nu \gg 0. \end{cases}$$

Since  $\pi$  maps  $E_1$  isomorphically onto  $E$ , the restriction  $\tau|_{E_1}$  determines a pluriharmonic function on  $E$ . If the differential of this function and the differential of the pluriharmonic function  $\rho$  are linearly independent on  $E$ , then, by Proposition 2.3,  $\Omega$  satisfies (BHD). Therefore, since  $\rho$  is nonconstant, it now suffices to assume that there exist real constants  $r$  and  $s$  such that  $\rho \circ \pi = r\tau + s$  on  $E_1$  and to obtain a contradiction.

The first observation is that if  $\tilde{\gamma} = \pi^*\gamma$ , then  $\tilde{\gamma}$  is a closed form of type  $(0, 1)$  on  $\tilde{\Omega}$  which is equal to the form  $-\bar{\partial}(r\tau + s)$  on the nonempty open set  $E_1$  and hence on the entire set  $\tilde{\Omega}$ . Therefore, on the nonempty open set  $\tilde{E} \setminus \overline{E}_1$ , we have

$$-\bar{\partial}(\rho \circ \pi) = \tilde{\gamma} = -\bar{\partial}(r\tau + s).$$

Hence the restriction of the function  $(\rho \circ \pi) - (r\tau + s)$  to  $\tilde{E} \setminus \overline{E}_1$  is real-valued and holomorphic and is therefore locally constant. Thus if  $E_2$  is a connected component of  $\tilde{E}$  which is not equal to  $E_1$ , then, for some real constant  $s'$ , we have  $\rho \circ \pi = r\tau + s'$  on  $E_2$ . Now since  $\pi(E_1) = \pi(E_2) = E$ , we may choose a regular sequence  $\{x_\nu\}$  in  $E$  and sequences  $\{y_\nu\}$  and  $\{z_\nu\}$  in  $E_1$  and  $E_2$ , respectively, such that  $\pi(y_\nu) = \pi(z_\nu) = x_\nu$  for each  $\nu$ . The sequences  $\{y_\nu\}$  and  $\{z_\nu\}$  are then regular sequences in  $\tilde{\Omega}$ , because the lifting of the function  $v = -G(x_0, \cdot)$  to  $\tilde{\Omega}$  is a negative subharmonic function and  $v(y_\nu), v(z_\nu) \rightarrow 0$ . Therefore  $\tau(y_\nu) \rightarrow 1$  and  $\tau(z_\nu) \rightarrow 0$ . Since  $\rho$  vanishes at infinity along any regular sequence, we get

$$0 = \lim \rho(x_\nu) = \lim(r\tau(y_\nu) + s) = r + s$$

and

$$0 = \lim \rho(x_\nu) = \lim(r\tau(z_\nu) + s') = s'.$$

Therefore, for each point  $x \in E$  and each pair of points  $y \in E_1 \cap \pi^{-1}(x)$  and  $z \in E_2 \cap \pi^{-1}(x)$ , we have  $\tau(y) - 1 = r^{-1}\rho(x) = \tau(z)$ . Since  $0 < \tau < 1$  on  $\tilde{\Omega}$ , this is not possible. Thus the proof is complete for the cases (i) and (ii).

Finally, suppose  $\varphi$  is of class  $C^\infty$  (i.e. the condition (iii) holds). As in the case (i), we fix  $M_0$ ,  $a$ , and  $\Omega$ . Here, we choose  $a$  to be a regular value of  $\varphi$  and we let  $g'$  be the restriction of  $g$  to  $\Omega$ . It suffices to show that there exists an arbitrarily large choice of  $a$  for which  $\Omega$  admits a proper holomorphic mapping  $\Psi$  onto a Riemann surface. For if  $\Psi$  is such a mapping, then  $\varphi$  is constant on each level of  $\Psi$ . Hence, near a generic point of  $\Omega$ , there exist holomorphic coordinates  $(z_1, \dots, z_n)$  in which  $\varphi$  is a function of  $z_1$  and we get

$$\mathcal{L}(\varphi) = \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_1} dz_1 d\bar{z}_1 = (2g^{1\bar{1}})^{-1} \Delta \varphi dz_1 d\bar{z}_1 \geq 0.$$

Thus it will follow that  $\varphi$  is plurisubharmonic on each of the domains  $\Omega$ . Letting  $a \rightarrow \infty$ , it will then follow immediately that  $\varphi$  is plurisubharmonic on  $M$  as in the case (i) and the proof in the case (iii) will be complete.

As in the proof of Theorem 1.6 (Grauert-Riemenschneider and Siu), since the boundary of  $\Omega$  is regular, we have  $\gamma = \alpha - \bar{\partial}\beta$  on  $\Omega$  where  $\gamma$  is harmonic and  $\beta$  vanishes on  $\partial\Omega$ . The metric  $g'$  is not complete, but the proof of Theorem 1.6 shows that  $\gamma$  is closed. In particular,  $\beta$  is pluriharmonic on the connected set  $E = \Omega \setminus \overline{M_0}$ . Moreover, since  $[\alpha] \neq 0$ ,  $\beta$  is nonconstant on  $E$ .

If  $\partial\Omega$  is not connected, then we may form a  $C^\infty$  function  $\tau$  on  $\overline{\Omega}$  which is harmonic on  $\Omega$  and locally constant, but not constant, on  $\partial\Omega$ . By Theorem 1.6 (a),  $\tau$  is then pluriharmonic on  $\Omega$  and, since  $\beta$  vanishes on  $\partial\Omega$ ,  $d\tau$  and  $d\beta$  are linearly independent on  $E$ . Therefore, by Proposition 2.3,  $\Omega$  admits a proper holomorphic mapping onto a Riemann surface. Thus we may assume that  $\partial\Omega$  is connected.

If  $a'$  is a regular value in the interval  $(\sup_{M_0} \varphi, a)$  which is sufficiently close to  $a$  and  $\Omega'$  is the connected component of  $\{x \in M \mid \varphi(x) < a'\}$  containing  $\overline{M_0}$ , then  $\Omega' \subset\subset \Omega$  and the set  $E' = \Omega' \setminus \overline{M_0}$  is connected. As above, we may write  $\gamma' = \alpha - \bar{\partial}\beta'$  on  $\Omega'$  where  $\gamma'$  harmonic and  $\beta'$  vanishes on  $\partial\Omega'$ . If  $d\beta$  and  $d\beta'$  are linearly independent on  $E'$ , then Proposition 2.3 implies that  $\Omega'$  satisfies (BHD). If  $d\beta$  and  $d\beta'$  are linearly dependent, then  $\beta$  is constant on  $\partial\Omega'$ . Since  $\beta$  is nowhere locally constant in  $\Omega \setminus \overline{\Omega'}$  and  $\beta \equiv 0$  on  $\partial\Omega$ , the maximum principle implies then that  $\beta$  is equal to a nonzero

constant on  $\partial\Omega'$ . Hence the restriction of the real part or the imaginary part of  $\beta$  to  $\Omega \setminus \overline{\Omega'}$  is a positive or negative pluriharmonic function which vanishes at points in  $\partial\Omega$ . It follows that  $\Omega$  admits a plurisubharmonic exhaustion function and hence, a proper holomorphic mapping onto a Riemann surface. Therefore, by the above remarks,  $M$  satisfies (BHD). Q. E. D.

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T. NAPIER,  
Lehigh University  
Department of Mathematics  
Bethlehem, PA 18015 (USA)  
&  
M. RAMACHANDRAN,  
SUNY at Buffalo  
Department of Mathematics  
Buffalo, NY 14214 (USA).