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DYNAMICS OF QUADRATIC POLYNOMIALS: COMPLEX BOUNDS FOR REAL MAPS

by M. LYUBICH and M. YAMPOLSKY

1. INTRODUCTION

Complex *a priori* bounds proved to be a key analytic issue of the Renormalization Theory. They lead to rigidity results, local connectivity of Julia sets and the Mandelbrot set, and convergence of the renormalized maps (see [HJ], [L2], [McM1], [McM2], [MS], [R], [S]). By definition, an infinitely renormalizable map f has complex bounds if all its renormalizations $R^n f$ extend to quadratic-like maps with definite moduli of the fundamental annuli. Sullivan established this property for real infinitely renormalizable maps with bounded combinatorics (see [S] and [MS]). On the other hand, it was shown in [L1], [L2] that the map $R^n f$ has a big modulus provided the "essential period" $p_e(R^{n-1} f)$ (see §3 for the precise definition) is big. Thus the gap between [S] and [L2] consists of quadratics of "essentially bounded but unbounded type". Loosely speaking for such maps the high renormalization periods are due to saddle-node behavior of the return maps. The goal of this paper is to analyze this specific phenomenon.

Given a quadratic-like map f, denote by $\mathrm{mod}(f)$ the supremum of the moduli of various fundamental annuli of f. We say that a real quadratic-like map f is *close to the cusp* if it has an attracting fixed point with the multiplier greater than $\frac{1}{2}$ (one can replace $\frac{1}{2}$ with $1-\epsilon$ for a fixed but otherwise arbitrary $\epsilon > 0$). Note that a renormalizable map has no attracting fixed points and therefore is not close to the cusp.

Key words: One-dimensional dynamics – Renormalization – Quadratic polynomials – Complex bounds – Local connectivity.
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THEOREM 1.1. — *Let $f : z \mapsto z^2 + c$, $c \in \mathbb{R}$, be any n times renormalizable real quadratic polynomial. Let*

$$\max_{1\le k\le n-1} p_e(R^k f) \le \bar{p}_e.$$

Then

$$\operatorname{mod}(R^n f) \ge \mu(\bar{p}_e) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp.

This fills the above mentioned gap:

COMPLEX BOUNDS THEOREM. — *There exists a universal constant $\mu > 0$ with the following property. Let f be any n times renormalizable real quadratic. Then*

$$\operatorname{mod}(R^n f) \ge \mu,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp. In particular, infinitely renormalizable real quadratics have universal complex a priori bounds.

Let us mention here only one consequence of this result. By work of Hu and Jiang [HJ], [J] and McMullen [McM2], "unbranched" complex *a priori* bounds imply local connectivity of the Julia set $J(f)$. (In §3 we will give the definition of "unbranched" bounds and will show that this property is indeed satisfied for real maps.) On the other hand, the Yoccoz Theorem gives local connectivity of $J(f)$ for at most finitely renormalizable quadratic maps (see [H], [M1]). Thus we have:

LOCAL CONNECTIVITY THEOREM. — *The Julia set of any real quadratic map $z \mapsto z^2 + c$, $c \in \left[-2, \frac{1}{4}\right]$, is locally connected.*

The proof of Theorem 1.1 is closer to [S] rather than to [L2]. It turns out, however, that Sullivan's Sector Lemma (see [MS]) is not valid for essentially bounded (but unbounded) combinatorics; the pullback of the plane with two slits is not necessarily contained in a definite sector. What turns out to be true instead is that the *little Julia sets* $J(R^n f)$ are contained in a definite sector.

We will derive Theorem 1.1 from the following quadratic estimate for the renormalizations (appropriately normalized):

$$\left|R^n f(z)\right| \ge c|z|^2, \tag{1.1}$$

with some $c > 0$ depending on the bound on the essential period. The main technical point of this work is to prove (1.1). In particular, this estimate implies that the diameters of the little Julia sets $J(R^n f)$ shrink to zero (see the discussion in §4), which already yields local connectivity of $J(f)$ at the critical point.

A quadratic-like map with a big modulus is close to a quadratic polynomial which is one of the reasons why it is important to analyze when the renormalizations have big moduli. It was proven in [L2] that $\operatorname{mod}(Rf)$ is big if and only if f has a big essential period, which together with Theorem 1.1 implies:

BIG SPACE CRITERION. — *There is a universal constant $\gamma > 0$ and two functions $\mu(p) > \nu(p) > \gamma > 0$ tending to ∞ as $p \to \infty$ with the following property. For an n times renormalizable quadratic polynomial f,*

$$\nu(p_e(R^{n-1}f)) \leq \operatorname{mod}(R^n f) \leq \mu(p_e(R^{n-1}f)),$$

unless the n-th renormalization is of doubling type and $R^n f$ is close to the cusp.

Let us briefly outline the structure of the paper. The next section, §2, contains some background and technical preliminaries. In §3 and §6 we describe the essentially bounded combinatorics and the related saddle-node phenomenon. In §4 we state the main technical lemmas, and derive from them our results. In §5 we give a quite simple proof of complex bounds in the case of bounded combinatorics, which will model the following argument. The proofs of the main lemmas are given in the final section, §7.

Remarks:

1. The key estimates for the moduli for maps with essentially high periods appeared in [L1], §4, while in [L2], §8, they were appropriately refined and interpreted.

2. When Theorem 1.1 was proven the authors received a manuscript by Levin and van Strien [LS] with an independent proof of the Complex Bounds Theorem. The method of [LS] is quite different; instead of a detailed combinatorial analysis it is based on specific numerical estimates for the real geometry. It does not address the phenomenon of big space.

Also, the gap between [S] and [L2] was independently filled by Graczyk and Swiatek [GS2]. The method of the latter work is specifically adopted to essentially bounded but unbounded combinatorics. Note also that a related analysis of the big space phenomenon for real quadratics was independently carried out in [GS1].

3. We were primarily concerned with the dynamics of quadratic maps, so we will only briefly dwell on the higher degree case. By replacing 2 to d, Theorem 1.1 extends to higher degree unimodal polynomials $z \mapsto z^d + c$. On the other hand, by an appropriate adjustment of logic (see §8 of [L2]), the results of [L2] concerning *a priori* bounds also extend to the higher degree case (the first author noticed this after receiving a manuscript [LS] where the complex bounds for higher degree maps were proven in an essentially different manner). Thus the whole above discussion except growing of $\nu(p)$ is still valid for higher degrees.

4. All the above results will actually be proven for maps of Epstein class $\mathcal{E}_\lambda$ (see §4). In this case the quadratic-like extension with a definite modulus (independent of λ) appears after skipping first $N = N(\lambda)$ renormalization levels.

5. Unimodal maps with essentially bounded combinatorics studied in this paper are closely related to critical circle maps. Indeed, high combinatorics for circle maps is always associated to saddle-node behavior (see [He]). Our method is well suited for the circle dynamics, and it was transferred to that setting by the second author [Y], who proved complex bounds for all critical circle maps. This complements the work of de Faria [F] where complex bounds were established for circle maps with bounded combinatorics.

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2. PRELIMINARIES

2.1. General notations and terminology.

Let $\mathbb{D}_r = \{z : |z| < r\}$.

We use $|J|$ for the length of an interval J, dist and diam for the Euclidean distance and diameter in $\mathbb{C}$. Notation $[a, b]$ stands for the (closed) interval with endpoints a and b without specifying their order.

Two sets X in Y in $\mathbb{C}$ are called *K-commensurable* or simply *commensurable* if

$$K^{-1} \leq \operatorname{diam} X / \operatorname{diam} Y \leq K$$

with a constant $K > 0$ which may depend only on the specified combinatorial bounds.

Given a univalent function ϕ in a domain $U \subset \mathbb{C}$, the *distortion* of ϕ is defined as $\sup_{z,\zeta \in U} \log |\phi'(z)/\phi'(\zeta)|$.

We say that an annulus A has a *definite modulus* if $\operatorname{mod} A \geq \delta > 0$, where δ may also depend only on the specified combinatorial bounds.

For a pair of intervals $I \subset J$ we say that I is contained *well inside* of J if for any of the components L of $J \setminus I$, $|L| \geq K|I|$ where the constant $K > 0$ may depend only on the specified quantifiers.

A smooth interval map $f : I \to I$ is called *unimodal* if it has a single critical point, and this point is an extremum. A C^3 unimodal map is called *quasi-quadratic* if it has negative Schwarzian derivative, and its critical point is non-degenerate.

Given a unimodal map f and a point $x \in I$, x' will denote the dynamically symmetric point, that is, such that $fx' = fx$. Notation $\omega(z) \equiv \omega_f(z)$ means as usual the limit set of the forward orbit $\{f^n z\}_{n=0}^{\infty}$.

Set $Q_c(z) = z^2 + c$.

2.2. Hyperbolic disks.

Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J \equiv \mathbb{C} \backslash (\mathbb{R} \backslash J)$ denote the plane slit along two rays. Let $\bar{\mathbb{C}}_J$ denote the completion of this domain in the path metric in $\mathbb{C}_J$ (which means that we add to $\mathbb{C}_J$ the banks of the slits).

By symmetry, J is a hyperbolic geodesic in $\mathbb{C}_J$. The *geodesic neighborhood* of J of radius r is the set of all points in $\mathbb{C}_J$ whose hyperbolic distance to J is less than r. It is easy to see that such a neighborhood is the union of two $\mathbb{R}$-symmetric segments of Euclidean disks based on J and having angle $\theta = \theta(r)$ with $\mathbb{R}$. We will denote this hyperbolic disk by $D_\theta(J)$ (see Figure 1). A particular example of a geodesic neighborhood of J is the Euclidean disk $D(J) \equiv D_{\pi/2}(J)$.

These hyperbolic neighborhoods were introduced into the subject by Sullivan [S]. They are a key tool for getting complex bounds due to the following version of the Schwarz Lemma:

SCHWARZ LEMMA. — *Let us consider two intervals $J' \subset \mathbb{R}$, $J \subset \mathbb{R}$. Let $\phi : \mathbb{C}_J \to \mathbb{C}_{J'}$ be an analytic map such that $\phi(J) \subset J'$. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

Let $J = [a, b]$, $a < b$. For a point $z \in \overline{\mathbb{C}}_J$, the *angle between z and J*, $(\widehat{z, J})$ is the least of the angles between the intervals $[a, z]$, $[b, z]$ and the corresponding rays $(-\infty, a]$, $[b, +\infty)$ of the real line, measured in the range $0 \leq \theta \leq \pi$.

The following consequence of the Schwarz Lemma will provide us a key to control the inverse branches expansion.

LEMMA 2.1. — *Under the circumstances of the Schwarz Lemma, assume that ϕ admits a univalent extension $(\mathbb{C}_T, T) \to (\mathbb{C}_{T'}, T')$, where both components of $T \setminus J$ have length $2\rho|J|$. Let us consider a point $z \in \mathbb{C}_J$ such that $(\widehat{z, J}) \geq \epsilon$. Then*

$$\frac{\operatorname{dist}(\phi z, J')}{|J'|} \leq C \frac{\operatorname{dist}(z, J)}{|J|}$$

for some constant $C = C(\rho, \epsilon)$.

Proof. — Let us normalize the situation in such a way: $J = J' = [0, 1]$. Since the space of univalent maps normalized at two points is compact (by the Koebe Theorem), the statement is true if $\operatorname{dist}(z, J) \leq \rho$. So assume that $\operatorname{dist}(z, J) \geq \rho$.

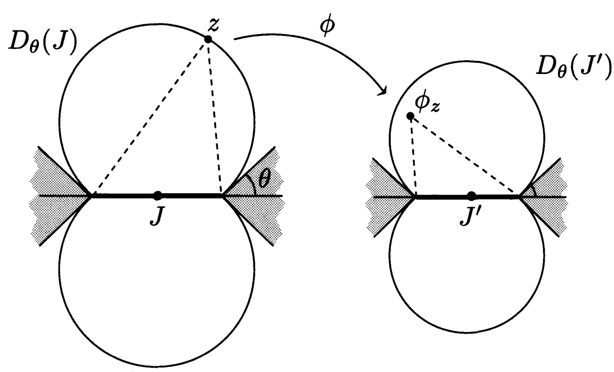

Figure 1

Observe that the smallest (closed) geodesic neighborhood $\operatorname{cl} D_\theta(J)$ enclosing z satisfies: $\operatorname{diam} D_\theta(J) \leq C(\epsilon) \operatorname{dist}(z, J)$ (*cf.* Figure 1). Indeed, if $\theta \geq \frac{1}{2}\epsilon$ then $\operatorname{diam} D_\theta(J) \leq C(\epsilon)$, which is fine since $\operatorname{dist}(z, J) \geq \rho$.

Otherwise the intervals $[0, z]$ and $[1, z]$ cut out segments of angle size at least ϵ on the appropriate circle arc of $\partial D_\theta(J)$. Hence the lengths of these intervals are commensurable with $\operatorname{diam} D_\theta(J)$ (with a constant depending on ϵ). On the other hand, these lengths are at most $(1+\rho^{-1})\operatorname{dist}(z, J)$, provided that $\operatorname{dist}(z, J) \geq \rho|J|$.

Together with the Schwarz Lemma this yields:

$$\operatorname{dist}(\phi z, J') \leq \operatorname{diam}\big(D_\theta(J')\big) = \operatorname{diam}\big(D_\theta(J)\big) \leq C(\rho, \epsilon)\operatorname{dist}(z, J),$$

and the claim follows. □

2.3. Square root.

In the next lemma we collect for further reference some elementary properties of the square root map. Let $\phi(z) = \sqrt{z}$ be the branch of the square root mapping the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ into itself.

LEMMA 2.2. — *Let $K > 1$, $\delta > 0$, $K^{-1} \leq a \leq K$, $T = [-a, 1]$, $T' = [0, 1]$. Then:*

- *$\phi(D_\theta(T) \setminus \mathbb{R}_-) \subset D_{\theta'}(T')$, with θ' depending on θ and K only.*
- *If $z' \in \phi D(T) \setminus D([-\delta, 1+\delta])$, then*

$$\widehat{(z', T')} > \epsilon(\delta) > 0 \quad \textit{and} \quad C(K, \delta)^{-1} < \operatorname{dist}(z', T') < C(K, \delta).$$

LEMMA 2.3. — *Let $\zeta \in \mathbb{C}$, $J = [a, b] \subset [0, +\infty)$, $\zeta' = \phi(\zeta)$ and $J' = [a', b'] = \phi J$. Then:*

- *If $\operatorname{dist}(\zeta, J) > \delta|J|$ then*

$$\frac{\operatorname{dist}(J', \zeta')}{|J'|} < C(\delta)\frac{\operatorname{dist}(J, \zeta)}{|J|}.$$

- *Let θ denote the angle between $[\zeta, a]$ and the ray of the real line which does not contain J; η' denote the angle between $[\zeta', b']$ and the corresponding ray of the real line. If $\theta \leq \frac{1}{2}\pi$ then $\eta' \geq \frac{1}{4}\pi$.*

(According to our convention, in the last statement we don't assume that $a < b$.)

2.4. Branched coverings.

Let $0 \in U' \subset U \subset \mathbb{C}$ be two topological disks different from the whole plane, and $f : U' \to U$ be an analytic double branched covering map with critical point at 0. Let $\mathcal{B}$ denote the space of such double branched coverings.

For $f \in \mathcal{B}$, the *filled Julia set* $K(f)$ is naturally defined as the set of non-escaping points of f, $K(f) = \bigcap_{n \geq 0} f^{-n}U$, and the *Julia set* is defined as its boundary, $J(f) = \partial K(f)$. These sets are not necessarily compact and may change as the map is restricted to a smaller topological disk V' (such that this restriction is still a map of class $\mathcal{B}$). The Julia set (and the filled Julia set) are connected if and only if the critical point itself is non-escaping, $0 \in K(f)$.

If additionally $\mathrm{cl}\, U' \subset U$ then the map f is called *quadratic-like.* If the Julia set $J(f)$ of a quadratic-like map is connected then it does not change as the map is restricted to a smaller domain V' (such that this restriction is still quadratic-like), see [McM], Thm 5.11. Moreover, the Julia set of a quadratic-like map is compact, and this is actually the criterion for admitting a quadratic-like restriction:

LEMMA 2.4 (compare [McM2], Prop. 4.10). — *Let $U' \subset U$ be two topological disks, and $f : U' \to U$ be a double branched covering with non-escaping critical point and compact Julia set. Then there are topological discs $U \supset V \supset V' \supset K(f)$ such that the restriction $g : V' \to V$ is quadratic-like. Moreover, if* $\mathrm{mod}(U \setminus K(f)) \geq \epsilon > 0$ *then* $\mathrm{mod}(V \setminus V') \geq \delta(\epsilon) > 0$.

Proof. — Let us consider the topological annulus $A = U \setminus K(g)$. Let $\phi : A \to R = \{z : 1 < |z| < r\}$ be its uniformization by a round annulus. It conjugates g to a map $G : R' \to R$ where R' is a subannulus of R with the same inner boundary, unit circle S^1. As G is proper near the unit circle, it is continuously extended to it, and then can be reflected to the symmetric annulus. We obtain a double covering map $\widehat{G} : \widehat{R}' \to \widehat{R}$ of the symmetric annuli preserving the circle. Moreover $\widehat{R}$ is a round annulus of modulus at least 2ϵ.

Let ℓ denote the hyperbolic metric on $\widehat{R}$, $\widehat{V}$ denote the hyperbolic 1-neighborhood of S^1, and $\widehat{V}' = \widehat{G}^{-1}\widehat{V} \subset \widehat{V}$. As $\widehat{G} : S^1 \to S^1$ is a double covering, we have:

$$2\ell(S^1) = \int_{S^1} \|Df(z)\|\, d\ell \leq \max_{S^1} \|Df(z)\|\, \ell(S^1),$$

so that $\max_{S^1} \|Df(z)\| \geq 2$. As $\operatorname{mod} \widehat{R} \geq 2\epsilon$, $\ell(S^1) \leq L(\epsilon)$. Hence $\|Df(z)\| \geq \rho(\epsilon) > 1$ for all $z \in \widehat{V}$. It follows that $\widehat{V}'$ is contained in $(1/\rho(\epsilon))$-neighborhood of S^1. But then each component of $V \setminus V'$ is an annulus of modulus at least $\delta(\epsilon) > 0$.

We obtain now the desired domains by going back to U: $V = \phi^{-1}\widehat{V} \cup K(f)$, $V' = \phi^{-1}\widehat{V}' \cup K(f)$. □

Let us supply the space $\mathcal{B}$ of double branched coverings with the *Carathéodory topology* (see [McM1]). Convergence of a sequence $f_n : U'_n \to U_n$ in this topology means Carathéodory convergence of $(U_n, 0)$ and $(U'_n, 0)$, and compact-open convergence of f_n.

2.5. Epstein class.

Let us consider a quasi-quadratic interval map $f : I = [\beta, \beta'] \to I$ with $f(\beta') = f(\beta) = \beta$, where β is a non-attracting fixed point: $f'(\beta) \geq 1$. By definition, f belongs to Epstein class (see [E], [S]) if it admits an analytic extension to a double branched covering $f : U' \to U$ such that $U = \mathbb{C}_T$ and U' is an $\mathbb{R}$-symmetric topological disk meeting the real line along an interval T' containing I. (For reasons which will become clear in §2.6 we do not assume that $T' \subset T$.) Any map f in Epstein class admits a representation

$$f(z) = (\phi(z))^2 + c \equiv Q_c \circ \phi, \tag{2.1}$$

where $\phi : U' \to \Delta(\phi)$ is a univalent map onto the complex plane with four slits, which double covers $\mathbb{C}_T$ under the quadratic map $Q_c : z \mapsto z^2 + c$. As the range $\Delta(\phi)$ is determined by T and c, we will also denote it as $\Delta_{T,c}$.

For purely notational convenience we will also assume the maps f of the Epstein class to be even: $f(z) = f(-z)$. Then the map ϕ is odd, and the intervals I, T' and the domain U' are symmetric about 0. Moreover, the interval T and hence the domain $U = \mathbb{C}_T$ can also be assumed symmetric: just shrink T to make it symmetric and adjust T' accordingly.

Remark. — Of course, all the maps of Epstein class associated to a quadratic map (restricted iterates of a quadratic map) are automatically symmetric. To carry the argument through in the non-symmetric case, one should just observe that the dynamical involution $z \mapsto z'$, $f(z) = f(z')$, has bounded distortion on compact subsets of U.

Let $\mathcal{E}$ stand for the Epstein class modulo affine conjugacy (that is, rescaling of I). We will always normalize $f \in \mathcal{E}$ so that 0 is its critical point.

Given a $\rho > 1$, let $\mathcal{E}^\rho \subset \mathcal{E}$ denote the space of maps of Epstein class modulo affine conjugacy such that $|T| > \rho|I|$.

LEMMA 2.5. — *For any $\rho > 0$, the space $\mathcal{E}^\rho$ is Carathéodory compact.*

Proof. — Let us normalize a map f in $\mathcal{E}^\rho$ so that $I = [-1, 1]$, $\beta = 1$. Then $T \supset [-\rho, \rho]$. Moreover, since the modulus of the topological annulus $\mathbb{C}_T \setminus I$ is at most twice the modulus of $\mathbb{C}_{T'} \setminus I$, $T' \supset [-\rho', \rho']$ with $\rho' > 1$ depending only on ρ. Since the critical value c divides T into two intervals of length at least $\rho - 1$, the range $\Delta(\phi) \equiv \Delta_{T,c}$ covers the disk $\mathbb{D}_r$ with $r = \sqrt{\rho - 1}$.

Let us now have a sequence $f_n = Q_{c_n} \circ \phi_n$ of normalized maps of Epstein class (2.1). Clearly we can select a subsequence such that the slit domains $\mathbb{C}_{T_n}$ and $\Delta(\phi_n)$ Carathéodory converge respectively to some $\mathbb{C}_T$ and $\Delta_{T,c}$, where $T \supset [-\rho, \rho]$ and $\Delta_{T,c} \supset \mathbb{D}_r$.

Moreover, since $Q_c(\phi_n I) \subset [-1, 1]$, we have: $|\phi_n I| \leq 2\sqrt{2}$, so that we can make $\phi_n I$ converge to some interval $J = [-a, a]$. This interval is contained in $\Delta_{T,c}$, since the intervals $\phi_n I$ are well inside $\Delta(\phi_n)$.

Since $\phi_n(\beta) \leq \sqrt{2}$ and $f_n'(\beta) \geq 1$, $\phi_n'(\beta)$ stays away from 0. So, the points $\phi_n\beta \to a$ stay definite distance from the boundary of $\Delta_{T,c}$ and $(\phi_n^{-1})'(\phi_n\beta)$ are bounded from above. By the Koebe Theorem, the family of univalent maps ϕ_n^{-1} is normal on $\Delta_{T,c}$.

Let us select a subsequence ϕ_n^{-1} uniformly converging on compact subsets of $\Delta_{T,c}$. Since $\phi_n I$ are intervals of bounded length staying away from the boundary of $\Delta_{T,c}$, the limit of the ϕ_n^{-1} is non-constant, and hence is a univalent function ϕ^{-1}. It follows that the domains U_n' of the maps ϕ_n Carathéodory converge to $U' = \phi^{-1}\Delta_{T,c}$.

Let us now observe that by the Koebe Theorem, the sequence of direct functions ϕ_n is normal on any domain $\Omega \supset I$ compactly contained in U'. Indeed, this is a family of univalent functions bounded on I, with the derivatives $\phi_n'(\beta)$ bounded away from 0. It follows that $\phi_n \to \phi$ uniformly on compact sets of U'.

Since $c_n \to c$, we conclude that $f_n \to Q_c \circ \phi$. □

The above proof also yields:

LEMMA 2.6. — *Given a $\rho > 1$, there is a domain $O_\rho \supset [-1, 1]$ with the following property. For any $f \in \mathcal{E}^\rho$ normalized so that $I = [-1, 1]$, the* *univalent map* ϕ *in* (2.1) *is well-defined and has bounded distortion on* O_ρ. *Moreover, in scale* ϵ *the distortion of* ϕ *is bounded by* $C(\rho)\epsilon$.

We will refer to the above property by saying that f is a *quadratic map up to bounded distortion.* The last statement (which certainly follows from the Koebe Distortion Theorem) shows, in particular, that in some scale ϵ depending only on ρ the distortion of ϕ is bounded by 2.

We will mostly be concerned with a subset of Epstein maps specified by a stronger condition. Given a $\lambda \in (0,1)$, let $\mathcal{E}_\lambda \subset \mathcal{E}$ be space of maps of Epstein class modulo affine conjugacy such that $T' \subset T$ and each component J of $T \setminus T'$ is λ^{-1}-commensurable with T'. Note that there exists $\lambda \in (0,1)$, such that all real quadratics Q_c, $c \in \left[-2, \frac{1}{4}\right]$, belong to the Epstein class $\mathcal{E}_\lambda$ (with T selected as a fixed large 0-symmetric interval).

LEMMA 2.7. — *Given a* $\lambda \in (0,1)$, *let* $f \in \mathcal{E}_\lambda$ *and* $[-1,1] = I \subset T' \subset T$ *be as above:*

- *the space* $\mathcal{E}_\lambda$ *is Carathéodory compact*;

- *both* T *and* T' *are* $K(\lambda)$*−commensurable with* I, *and* I *is contained well inside* T';

- *denote by* J_i^n, $i = 1, 2$, *the components of* $(f|_{\mathbb{R}})^{-n}(T \setminus T')$. *If* f *is not close to the cusp then* $|J_i^n|$ *is* $K(\lambda)$*-commensurable with* $\mathrm{dist}(J_i^n, \partial I)$.

Proof. — As $\mathcal{E}_\lambda$ is a closed subset of some $\mathcal{E}^{\rho(\lambda)}$, the first property follows from Lemma 2.5.

Furthermore, there exists $\mu = \mu(\lambda) > 0$ such that the annulus $A = D(T) \setminus D(T')$ has modulus at least μ. Since $\mathrm{mod}(f^{-n}A) = 2^{-n}\mu(\lambda)$, there exist $K_n = K_n(\lambda)$ such that $|J_i^n| \geq K_n|T \setminus T'|$. Using the fact that both components of $T \setminus T'$ are λ^{-1}-commensurable with $|T'|$ we have $|J_i^n| > L_n|T'|$. As J_i^1 are contained in $T' \setminus I$, $|I|/|T'|$ is bounded from above.

Set $T'' = [\gamma', \gamma] = (f|_{\mathbb{R}})^{-1}T'$ where γ lies on the same side of 0 as the fixed point $\beta = 1$. Commensurability of the J_i^0 with T' and Koebe Distortion Theorem imply that ϕ in (2.1) has a $C(\lambda)$-bounded distortion on this interval. Hence $|T'| \geq |fT''| \asymp |T''|^2$. It follows that $|T''|/|T'| \to 0$ as $|T'| \to \infty$. Since J_i^2 is commensurable with T', the length of T' must be bounded. Thus I is commensurable with T' (and T).

To prove the last statement, let us consider the interval $S = [\frac{1}{2}\beta, \gamma]$. Bounded distortion of $\phi|S$ and elementary distortion properties of the quadratic map imply that f has bounded distortion on S.

By compactness of $\mathcal{E}_\lambda$, for $f \in \mathcal{E}_\lambda$ which is not close to the cusp the multiplier of β is bounded away from 1. Moreover, we can take a point $a \in (\beta, \gamma)$ which divides (β, γ) into $K(\lambda)$-commensurable parts and such that $f'(x) \geq q = q(\lambda) > 1$ for $x \in (\beta, a)$. Let $J^n \equiv J_1^n$ stand for the intervals lying on the side of the fixed point $\beta \in \partial I$. Then only bounded number of intervals J^n may be outside (β, a). For the rest of them, $|J^n| \geq (q-1)\,\mathrm{dist}(J^n, \beta)$ which finishes the proof of the lemma. $\square$

All maps in this paper will be assumed to belong to the Epstein class $\mathcal{E}$.

2.6. Renormalization.

We assume that the reader is familiar with the notion of renormalization in one-dimensional dynamics (see *e.g.*, [MS]).

Let f be k times renormalizable quasi-quadratic map, $0 \leq k \leq \infty$. For $\ell \leq k$, let the closed interval $P^\ell \ni 0$ be a central periodic interval corresponding to the ℓ-fold renormalization $R^\ell f$ of f, n_ℓ be its period: $f_\ell \equiv f^{n_\ell} : P^\ell \to P^\ell$. Let P_i^ℓ be the component of $f^{-(n_\ell - i)} P^\ell$ containing $f^i 0$. These intervals always have disjoint interiors. We say that the intervals P_i^ℓ, $i = 0, 1, \ldots, n_\ell - 1$, form the *cycle of level ℓ*. Note that the periodic interval P^ℓ is not canonically defined. Possible choices are $P^\ell = B^\ell = [\beta_\ell, \beta'_\ell]$ where β_ℓ is the an appropriate fixed point of f_ℓ; and $P^\ell = [f_\ell 0, f_\ell^2 0]$.

By definition, the ℓ-fold renormalization $R^\ell f$ is equal to $f_\ell | P^\ell$ up to the choice of P^ℓ and rescaling. To be definite, we will assume that it is normalized so that B^ℓ is rescaled to $[-1, 1]$:

$$R^\ell f(z) = q^{-1} f_\ell(qz), \quad \text{where} \quad q = \tfrac{1}{2}|B^\ell|.$$

Let $p_\ell = n_\ell / n_{\ell-1}$ be the relative periods, $\bar{p}_k(f) = \max_{1 \leq \ell \leq k} p_\ell$. We say that an infinitely renormalizable map f has *bounded combinatorics* if the sequence of relative periods is bounded.

If $n_1 = 2$ then f is called *immediately renormalizable*, and the corresponding renormalization is called *doubling*. In this case the maximal periodic intervals $P^1 = [\beta_1, \beta'_1]$ and P_1^1 touch at their common fixed point β_1 (which coincides with the fixed point α of f with negative multiplier). In all other cases the periodic intervals P_i^1 are disjoint.

Besides β_ℓ, the quasi-quadratic map f_ℓ has one more fixed point on B^ℓ which will be denoted by α_ℓ. At the cusp (*i.e.*, when $f'_\ell(\beta_\ell) = 1$) these two points coincide. Note that if $\ell < k$ (so that $R^\ell f$ is renormalizable), then $f_\ell'(\alpha_\ell) < -1$.

Let $S_1^\ell \supset P_1^\ell$ be the maximal interval such that the restriction of $f^{n_\ell - 1}$ to it is monotone. Set $T^\ell = f^{n_\ell - 1}(S_1^\ell)$ and $S^\ell = f^{-1}(S_1^\ell)$. Then $f_\ell : S^\ell \to T^\ell$ is a unimodal map.

Let us now state some basic geometric properties of infinitely renormalizable maps usually referred to as *real bounds* (see [G], [BL1], [BL2], [S], [MS] for the proofs). Below we assume that f is a k times renormalizable quasi-quadratic map of Epstein class $\mathcal{E}_\lambda$, $0 \le k \le \infty$.

LEMMA 2.8. — *For a quasi-quadratic map $f \in \mathcal{E}_\lambda$ as above:*

- *The interval P^k is well inside T^k and S^k. Moreover, after skipping initial $N(\lambda)$ levels, the space in between these intervals becomes absolute (i.e., independent of λ).*

- *The renormalizations $R^k f$ belong to some class $\mathcal{E}^r$ with $r = r_k(\lambda) \le r(\lambda) < 1$ which becomes absolute after skipping the initial $N(\lambda)$ levels.*

- *If α_k has negative multiplier then $S^k \subset T^k$.*

- *If $f'(\alpha_k) \le -\epsilon < 0$ then S^k is well inside T^k (with the space depending on ϵ).*

Proof. — The first statement is proven in the above quoted works (see *e.g.*, [MS], Lemma VI.2.1). The second statement is the consequence of the first one.

Let us consider the component J of $S^k \setminus (\beta, \beta')$ containing β. If $S^k \supset T^k$ then f_k monotonically maps J into itself. Hence it has an attracting fixed point $\gamma \in J$ with positive multiplier. Since the critical point is attracted by the cycle of γ, γ belongs to some interval P_j^k. It follows that $R^k f$ also has an attracting fixed point with positive multiplier contradicting the assumption. This proves the third statement.

The last statement follows by compactness of $\mathcal{E}^\rho$. □

LEMMA 2.9. — *The map $f^{n_k - i} : P_i^k \to P^k$, $0 < i < n_k$, of a non-central interval onto the central one is a diffeomorphism whose distortion is bounded by an absolute constant.*

Let G_s^k be the *gaps* of level k, that is the components of $P_j^{k-1} \setminus \cup P_i^k$. *Geometry* of f is said to be *δ-bounded* (up to level n) if there is a choice of periodic intervals P_i^k, such that for any intervals P_i^k, $G_s^k \subset P_j^{k-1}$, we have: $|P_i^k|/|P_j^{k-1}| \geq \delta$ and $|G_s^k|/|P_j^{k-1}| \geq \delta$, $k = 1, \ldots, n$. In other words, all the intervals and the gaps of level k contained in some interval of level $k-1$ are commensurable with the latter.

Let $\bar{p}$ be an upper bound on the essential periods of the first k renormalizations of f: $\bar{p}_i(f) \leq \bar{p}$, $i = 0, 1, \ldots, k$.

THEOREM 2.10. — *Any map f as above has a δ-bounded geometry, where δ depends only on $\bar{p}$. In particular, infinitely renormalizable maps with bounded combinatorics have bounded geometry.*

COROLLARY 2.11. — *Let P_i^k be a non-central interval which belongs to the central interval P^{k-1}. Then the map $f^{n_k - i} : P_i^k \to P^k$ has a derivative bounded away from 0 and ∞ by constants depending on $\bar{p}$ only.*

Proof. — Indeed, by Theorem 2.10, the intervals P_i^k and P^k are commensurable, while by Lemma 2.9, the map between them has a bounded distortion. □

COROLLARY 2.12. — *Let $f'(\alpha_k) \leq -\epsilon < 0$. Then S^k is $K(\bar{p})$-commensurable with T^k, and the renormalization $R^k f$ belongs to some class $\mathcal{E}_\mu$, with μ depending only on λ, $\bar{p}$ and ϵ.*

Proof. — Given the last statement of Lemma 2.8, we only need to show that $|S^k|/|T^k|$ is bounded from below. But $S^k \subset P^{k-1}$, since the map $f_k = f_{k-1}^2$ is 0-symmetric and at least 3-modal on P^{k-1}. As P^{k-1} is f_k-invariant, $T^k \subset P^{k-1}$ as well. As by Theorem 2.10 P^k and P^{k-1} are $K(\bar{p})$-commensurable, we are done. □

2.7. Bounds and unbranching.

Let us state a result which gives an estimate of the modulus of a quadratic-like map after one renormalization:

THEOREM 2.13 (see [L2], Cor. 5.6). — *Let f be a renormalizable quadratic-like map with $\operatorname{mod} f \geq \rho > 0$. Then Rf is also quadratic-like, and*

$$\operatorname{mod} Rf \geq \delta(\rho) > 0$$

unless the renormalization is a doubling and Rf is close to the cusp.

A fundamental annulus A of a renormalization $R^k f$ is called *unbranched* if $A \cap \omega_f(0) = \emptyset$.

LEMMA 2.14 (see [L2], Lemma 9.3). — *Let f be an infinitely renormalizable $\mathbb{R}$-symmetric quadratic-like map with a priori bounds. Then every other renormalization $R^n f$ has an unbranched fundamental annulus with a definite modulus (depending on a priori bounds only).*

3. ESSENTIALLY BOUNDED COMBINATORICS AND GEOMETRY

Let f be a renormalizable quasi-quadratic map.

Recall that $\beta \equiv \beta_0$ and α stand for the fixed points of f with positive and negative multipliers correspondingly. Let $B \equiv B(f) = [\beta, \beta']$, $A \equiv A(f) = [\alpha, \alpha'] \subset B$.

If f is immediately renormalizable then A is a periodic interval with period 2. Otherwise let us consider the principal nest $A \equiv I^0 \equiv I^0(f) \supset I^1 \equiv I^1(f) \supset \ldots$ of intervals of f (see [L1]). It is defined in the following way. Let $t(m)$ be the first return time of the orbit of 0 back to I^{m-1}. Then I^m is defined as the component of $f^{-t(m)} I^{m-1}$ containing 0. Moreover $\bigcap_m I^m = B(Rf)$.

For $m > 1$, let

$$g_m : \bigcup_i I_i^m \to I^{m-1}$$

be the *generalized renormalization* of f on the interval I^{m-1}, that is, the first return map restricted onto the intervals intersecting the postcritical set (here $I^m \equiv I_0^m$). Note that $g_m \equiv f^{t(m)} : I^m \to I^{m-1}$ is unimodal with $g_m(\partial I^m) \subset \partial I^{m-1}$, while $g_m : I_i^m \to I^{m-1}$ is a diffeomorphism for all $i \neq 0$.

Let us consider the following set of levels:

$$\begin{aligned} X \equiv X(f) &= \{m : t(m+1) > t(m)\} \cup \{0\} \\ &= \{0 = m(0) < m(1) < m(2) < \cdots < m(\chi)\}. \end{aligned}$$

A level $m = m(k)$ belongs to X iff the return to level $(m-1)$ is *non-central*, that is $g_m 0 \in I^{m-1} \backslash I^m$. For such a moment the map $g_{m+1|I^{m+1}}$ is essentially different from $g_{m|I^m}$ (that is not just the restriction of the latter to a smaller domain). Let us use the notation $h_k \equiv g_{m(k)+1}$, $k = 1, \dots \chi$. The number $\chi = \chi(f)$ is called the *height* of f. (In the immediately renormalizable case set $\chi = -1$.)

The nest of intervals

$$I^{m(k)+1} \supset I^{m(k)+2} \supset \cdots \supset I^{m(k+1)} \tag{3.1}$$

is called a *central cascade.* The *length* ℓ_k of the cascade is defined as $m(k+1) - m(k)$. Note that a cascade of length 1 corresponds to a non-central return to level $m(k)$.

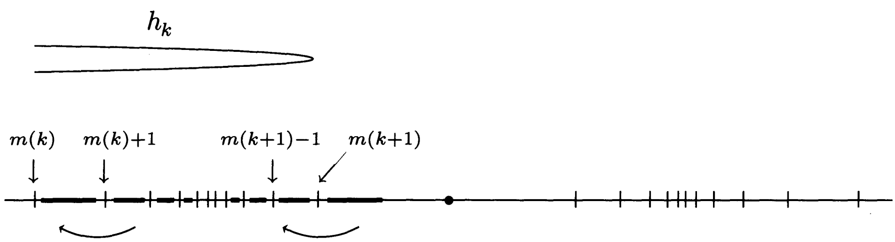

Figure 2. A long saddle-node cascade

A cascade (3.1) is called *saddle-node* if $h_k I^{m(k)+1} \not\ni 0$ (see Figure 2). Otherwise it is called *Ulam-Neumann.* For a long saddle-node cascade the map h_k is combinatorially close to $z \mapsto z^2 + \frac{1}{4}$. For a long Ulam-Neumann cascade it is close to $z \mapsto z^2 - 2$.

Given a cascade (3.1), let

$$K_j^{m(k)+i} \subset I^{m(k)+i-1} \setminus I^{m(k)+i}, \quad i = 1, \dots, m(k+1) - m(k), \tag{3.2}$$

denote the pullbacks of $I_r^{m(k)+1}$ under h_k^{i-1} (*i.e.*, the connected components of the preimage of $I_r^{m(k)+1}$ under the corresponding inverse map). Clearly, $K_j^{m(k)+i+1}$ are mapped by h_k onto $K_r^{m(k)+i}$, $i = 1, \dots, m(k+1) - m(k) - 1$, while $K_j^{m(k)+1} \equiv I_j^{m(k)+1}$ are mapped onto the whole $I^{m(k)}$. This family of intervals is called the *Markov family* associated with the central cascade.

For $x \in \omega(0) \cap (I^{m(k)} \setminus I^{m(k)+1})$ set

- $d(x) = \min\{j - m(k), m(k+1) - j\}$,

  if $h_k x \in I^j \setminus I^{j+1}$ for $m(k) \le j \le m(k+1) - 1$ and
- $d(x) = 0$ otherwise (*i.e.*, when $h_k x \in I^{m(k+1)}$).

This parameter shows how deep the orbit of x lands inside the cascade. Let us now define d_k as the maximum of $d(x)$ over all $x \in \omega(0) \cap (I^{m(k)} \setminus I^{m(k)+1})$.

Given a saddle-node cascade (3.1), let us call *neglectable* all levels $m(k) + d_k < \ell < m(k+1) - d_k$.

Let us now define *the essential period* $p_e = p_e(f)$. Let p be the period of the periodic interval $J = B(Rf)$. Let us remove from the orbit $\{f^k J\}_{k=0}^{p-1}$ all the intervals whose first return to some $I^{m(k)}$ belongs to a neglectable level. The essential period is the number of the intervals which are left.

We say that an infinitely renormalizable map f has *essentially bounded combinatorics* if $\sup_n p_e(R^n f) < \infty$.

Remark. — Bounded essential period is equivalent to a bound on the following combinatorial factors: the height, the return times of the I_k^m to I^{m-1} under iterates of g_{m-1}, the lengths of the Ulam-Neumann cascades, and the depths d_k of landing at the saddle-node cascades.

THEOREM 3.1 (see [L2], Thm V). — *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map of Epstein class. There is $\bar{p}_\lambda > 0$ and a function $\nu_\lambda(p) \to \infty$ as $p \to \infty$ with the following property. If $p_e(f) \geq \bar{p}_e \geq \bar{p}_\lambda$ then Rf has an unbranched fundamental annulus A such that* $\mathrm{mod}(A) \geq \nu_\lambda(\bar{p}_e)$.

Let $\sigma(f) = |B(Rf)|/|B(f)|$. Let us say that f has *essentially bounded geometry* if $\inf_n \sigma(R^n f) > 0$.

By the *gaps* G_j^m of level m we mean the components of $I^{m-1} \setminus \cup I_j^m$. We say that a level m is *deep inside the cascade* if $m(k)+\bar{p}_e \leq m \leq m(k+1)-\bar{p}_e$. The following lemma says that the maps with essentially bounded combinatorics have essentially bounded geometry (the inverse is true by Theorem 3.1).

LEMMA 3.2 (see [L2], Lemma 8.8). — *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map with $p_e(f) \leq \bar{p}_e$. Then all the intervals I^m in the principal nest of f are $C(\bar{p}_e, \lambda)$-commensurable. Moreover, the non-central intervals I_i^m, $i \neq 0$, and the gaps G_j^m of level m are $C(\bar{p}_e, \lambda)$-commensurable with $I^{m-1} \setminus I^m$. This is also true for the central interval I^m, provided m is not deep inside the cascade.*

Note that the last statement of the lemma is definitely false when m is deep inside a cascade: then I^m occupies almost the whole of I^{m-1}. So we observe commensurable intervals in the beginning and in the end of the cascade, but not in the middle. This is the saddle-node phenomenon which is in the focus of this work.

COROLLARY 3.3. — *Let $f \in \mathcal{E}_\lambda$ be a renormalizable quasi-quadratic map with $p_e(f) \le \bar{p}_e$ such that Rf is not close to the cusp. Then $Rf \in \mathcal{E}^\rho$ with $\rho = \rho(\lambda, \bar{p}_e)$. If Rf has no attracting fixed points then $Rf \in \mathcal{E}_\mu$ with $\mu = \mu(\lambda, \bar{p}_e)$.*

Proof. — In view of Lemmas 2.8 and 3.2 it is enough to notice that $I^{m(\chi-1)} \supset T^1 \supset S^1 \supset I^{m(\chi)+1}$ where $\chi = \chi(f)$ is the height of f (compare Corollary 2.12). □

The following important distortion result will replace Lemma 2.9 in the case of unbounded combinatorics:

THEOREM 3.4 (see [GJ], [Ma]). — *For any quasi-quadratic map f, the return map $g_m : I_j^{m+1} \to I^m$ is a composition of the quadratic map $z \mapsto z^2$ and a map h with bounded distortion. Moreover, h^{-1} has a definite Koebe extension around I^m.*

The following two statements extend Corollary 2.11 to the case of essentially bounded combinatorics.

COROLLARY 3.5. — *Let f be a quasi-quadratic map with $p_e(f) \le \bar{p}_e$.*

- *For a non-central interval $I_j^{m(k)+1} \subset I^{m(k)} \setminus I^{m(k)+1}$ the derivative of the restriction $h_k|_{I_j^{m(k)+1}}$ is bounded away from 0 and ∞.*

- *For any $m(k) \le \ell < s < m(k+1)$, which are not deep inside the cascade, the derivative of the transition map*

$$h_k^{s-\ell} : I^s \setminus I^{s+1} \to I^\ell \setminus I^{\ell+1}$$

is bounded away from 0 and ∞.

The constants depend only on $\bar{p}_e$.

Proof. — By Lemma 3.2, any non-central interval $I_j^{m(k)+1}$ is commensurable to its distance to 0. Hence the quadratic map has bounded distortion on $I_j^{m(k)+1}$. By Theorem 3.4, the return map $g_m : I_j^{m+1} \to I^m$ has bounded distortion as well. Since its domain and range are commensurable (by Lemma 3.2 again), we see that its derivative is bounded away from 0 and ∞.

Furthermore, the Koebe Principle easily implies that the transition map along the cascade has bounded distortion. Hence by essentially bounded geometry, it must have bounded derivative. □

COROLLARY 3.6. — *Under the assumptions of the previous corollary, let f be renormalizable. Let $P_i \subset I^{m(k+1)} \backslash I^{m(k+2)}$ be a non-central periodic interval. Consider its first return $f^s P_i \equiv P_{i+s}$ back to $I^{m(k+1)}$. Then P_{i+s} is $K(\bar{p}_e)$-commensurable with P_i.*

This is also true for the intermediate returns to $I^{m(k)}$, that is the intervals P_{i+j} satisfying $0 < j < s$ and $P_{i+j} \subset I^m \setminus I^{m+1}$ with $m(k) \leq m \leq m(k+1)+1$, provided m is not deep inside the cascade.

Proof. — The first statement follows from the previous lemma.

The second statement follows in a similar way from Theorem 3.4 and the second part of Corollary 3.5. □

4. REDUCTIONS TO THE MAIN LEMMAS

In this section we will state the Main Lemmas and will derive all the results from them. The lemmas will be proven in the following sections. As everything will be done in the setting of the Epstein class, let us start with the corresponding version of Theorem 1.1.

THEOREM 4.1. — *For any $\lambda \in (0,1)$ there exists $N = N(\lambda)$ with the following property. Let $f \in \mathcal{E}_\lambda$ be an $n-$times renormalizable map, $N \leq n \leq \infty$. Let*

$$\max_{1 \leq k \leq n-1} p_e(R^k f) \leq \bar{p}_e.$$

Then $R^n f$ has a quadratic-like extension with

$$\operatorname{mod}(R^n f) \geq \mu(\bar{p}_e) > 0,$$

unless the last renormalization is of doubling type and $R^n f$ is close to the cusp.

4.1. Main Lemmas.

Let P^k, $f_k \equiv f^{n_k}$, *etc.* be as in §2.6. Set $S \equiv S^0$, $T \equiv T^0$, so that $f: S \to T$ is unimodal. Let us consider the decomposition:

$$f_k = \psi_k \circ f, \tag{4.1}$$

where ψ_k is a univalent map from a neighborhood U^k of P_1^k onto $\mathbb{C}_{T^k}$.

LEMMA 4.2. — *Let $f:[-1,1]\to[-1,1]$ be a k times renormalizable quasi-quadratic map of Epstein class $\mathcal{E}_\lambda$. Assume that $p_e(R^\ell f)\le \bar{p}_e$ for $\ell=0,1,\dots,k-1$. Then there exist $C=C(\bar{p}_e)>0$ and $t=t(\lambda,\bar{p}_e)\in\mathbb{N}$, such that for all $z\in D(T^t)\cap\mathbb{C}_{T^k}$ the following estimate holds:*

$$\frac{\operatorname{dist}(\psi_k^{-1}z,P_1^k)}{|P_1^k|}\le C\frac{\operatorname{dist}(z,P^k)}{|P^k|}. \tag{4.2}$$

Thus the maps ψ_k^{-1} after appropriate rescaling (that is normalizing $|P^k|=|P_1^k|=1$) have at most linear growth depending on λ and $\bar{p}$ only. This implies, in particular, that for sufficiently big ℓ (depending on λ and $\bar{p}_e$ only), $\psi_k^{-1}(D(T^\ell))$ is contained in the range where f^{-1} is the square root map up to bounded distortion (see Lemma 2.6). This yields the quadratic estimate (1.1) stated in the Introduction:

$$\frac{\operatorname{dist}(f^{n_k}(z),P^k)}{|P^k|}>c\Big(\frac{\operatorname{dist}(z,P^k)}{|P^k|}\Big)^2,\quad z\in\Omega^k\equiv f^{-1}\psi_k^{-1}(D(T^\ell)), \tag{4.3}$$

where c and ℓ depend only on $\bar{p}$ and λ.

COROLLARY 4.3. — *Under the circumstances of Lemma 4.2, there exists $N=N(\lambda,\bar{p}_e)$ with the following property. For any $k>N$, $f_k:P^k\to P^k$ admits a quadratic-like extension whose little Julia set is $K(\bar{p}_e)$-commensurable with the interval P^k.*

Proof. — The above estimate (4.3) implies that for a sufficiently large r we have $|f^{n_k}(z)|>2|z|$, provided $\operatorname{dist}(f^{n_k}(z),P^k)>r|P^k|$. By real bounds there exists s depending only on λ and $\bar{p}_e$, such that $\operatorname{dist}(\zeta,P^k)>r|P^k|$ for $k>s+\ell$ and any $\zeta\in\partial D(T^{k-s})$.

Set $V^k=D(T^{k-s})\cap\mathbb{C}_{T^k}$, $\Delta^k=(f_k|\Omega^k)^{-1}V^k$. Then by the above estimate, $\partial\Delta^k$ cannot touch $\partial D(T^{k-s})$. Neither can it touch $T^{k-s}\setminus T^k$ since $\partial\Delta^k\cap\mathbb{R}=S^k$. Hence Δ^k is compactly contained in V^k, so that the restriction $f^{n_k}:\Delta^k\to V^k$ is quadratic-like. Its little Julia set is contained in $D(T^{k-s})$ which is commensurable with P^k. □

Carrying the argument for Lemma 4.2 further, we will prove the following result:

LEMMA 4.4. — *Under the circumstances of the previous lemma, the little Julia set $J(f_k)$ (for the quadratic-like extension of $f_k : P^k \to P^k$) is contained in the hyperbolic disk $D_\epsilon(B^k)$, where $\epsilon > 0$ depends only on $\bar{p}_e$, unless the k-th renormalization is of doubling type and $R^k f$ is close to the cusp.*

4.2. Proof of the main results.

Proof of Theorem 4.1. — Choose N as in Corollary 4.3. Let us assume first that $R^k f$ does not have an attracting fixed point. Then by Lemma 2.8, B^k is well inside of T^k. Hence the hyperbolic disk $D_\epsilon(B^k)$ is well inside the slit plane $\mathbb{C}_{T^k}$. By Lemma 4.4, the Julia set $J(R^n f)$ is also well inside $\mathbb{C}_{T^k}$, and the desired follows from Lemma 2.4.

If $R^k f$ has an attracting cycle, let us go one level up. As $R^{k-1} f$ does not have attracting points, it has a definite modulus. By Theorem 2.13 its first renormalization, $R^k f$, also has a definite modulus, unless it is of doubling type and close to the cusp. □

Proof of Theorem 1.1. — For $n > N$ the claim follows from Theorem 4.1. As $\operatorname{mod} f = \infty$ for any quadratic polynomial f, for all preceding levels $n \leq N$ we have bounds by Theorem 2.13. □

The statement of the Complex Bounds Theorem needs an obvious adjustment for maps of Epstein class (where one should skip first $N(\lambda)$ levels), or for quadratic-like maps (where the bounds depend on $\operatorname{mod}(f)$). Note also that due to the Straightening Theorem (see [DH], [McM1]), the latter case follows from the quadratic one.

Proof of the Complex Bounds Theorem. — By Lemma 2.8, all the renormalizations $R^m f$, $N(\lambda) \leq m < k$, belong to a class $\mathcal{E}_\theta$ with an absolute θ. Without loss of generality we can assume that $N(\lambda) = 0$ (taking into account Theorem 2.13 in the quadratic-like case).

Take a $\mu > 0$, *e.g.* $\mu = 1$. By Theorem 3.1, there is a $\tilde{p} = \tilde{p}(\mu)$ such that $\operatorname{mod}(Rf) \geq \mu$ for all renormalizable maps f of Epstein class $\mathcal{E}_\theta$ with $p_e(f) \geq \tilde{p}$. So we have complex bounds for all renormalizations $R^{n+1} f$ such that $p_e(R^n f) \geq \tilde{p}$. For all intermediate levels we have bounds by Theorem 2.13 and Theorem 4.1 (except perhaps for the first N levels with an absolute $N = N(\theta)$).

The latter bounds depend on $\tilde{p}$. But with the choice $\mu = 1$, $\tilde{p}$ and hence the bounds are absolute. □

Now the Complex Bounds Theorem and Lemma 2.14 yield:

LEMMA 4.5. — *Let $f \in \mathcal{E}_\lambda$. Then for every other level $k \geq N(\lambda)$, the renormalization $R^k f$ has an unbranched fundamental annulus with a definite modulus.*

By a *puzzle piece* we mean a topological disk bounded by rational external rays and equipotentials (compare [H], [L2], [M1]).

Proof of the Local Connectivity Theorem. — By [HJ], [J], [McM2], unbranched *a priori* bounds imply local connectivity of the Julia set. For the sake of completeness we will supply the argument below.

For now, f is an infinitely renormalizable map of class $\mathcal{E}_\lambda$. Since quadratic-like maps (considered up to rescaling) with *a priori* bounds form a compact family, the Julia set $K(g)$ depends upper semi-continuously on g, and the β-fixed point depends continuously on the map (see [McM1], §4, for all these properties), the little Julia sets $J(f_k)$ are commensurable with the intervals B^k. Hence the $J(f_k)$ shrink to the critical point. By the Douady and Hubbard renormalization construction (see [D], [L2], [McM2]), each little Julia set is the intersection of a nest of puzzle pieces. As each of these pieces contains a connected part of the Julia set, $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$ (by a standard "spreading around" argument). Take a puzzle piece $V \ni 0$. The set of points which never visit V, $Y_V = \{\zeta : f^n\zeta \notin V,\ n = 0, 1, \ldots\}$, is expanding. (Cover this set by finitely many non-critical puzzle pieces, thicken them a bit, and use the fact the branches of the inverse map are contracting with respect to the Poincaré metric in these pieces.) It follows that if $z \in Y_V$ then there is a nest of puzzle pieces shrinking to z, and we are done.

Let now $z \notin Y_V$, for any critical puzzle piece V. Take an unbranched level k. Then there is a puzzle piece $V^k \supset J(f_k)$ with a definite space in between it and the rest of the postcritical set. Take the first moment ℓ_k such that $f^{\ell_k} z \in V^k$. Then there exists a single-valued inverse branch $f^{-\ell_k} : \mathbb{C}_{(1+\epsilon)B^k} \to \mathbb{C}$ whose image contains z (where ϵ depends on λ only).

Furthermore, there is an $r \in (0,1)$ depending on λ only such that the hyperbolic disk Ω^k in $\mathbb{C}_{(1+\epsilon)B^k}$ of radius r (centered at 0) contains V^k. Moreover, by the Koebe Theorem this disk has a bounded shape.

Using the Koebe Theorem once more, we see that the $f^{-\ell_k}$ have a bounded distortion on Ω^k. Hence the pullbacks $U^k = f^{-\ell_k}\Omega^k$ have a bounded distortion as well. As they cannot contain a disk of a definite radius (as any disk $B(z,\epsilon)$ must cover the whole Julia set under some iterate of f), we conclude that $\operatorname{diam} U^k \to 0$. All the more, the pullbacks of the V^k under $f^{-\ell_k}$ shrink. This is the desired nest of puzzle pieces about z. □

Proof of the Big Space Criterion. — It follows from Theorem 3.1 and Theorem 1.1. □

5. BOUNDED COMBINATORICS

We first prove the complex bounds in the case when the map f has bounded combinatorics. The result is well-known in this case [MS], [S], but we give a quite simple proof which will be then generalized onto the case of essentially bounded combinatorics.

5.1. The ε-jumping points.

Given an interval $T \in \mathbb{R}$ let $f: U' \to \mathbb{C}_T$ be a map of Epstein class.

For a point $x \in \mathbb{R} \cap U'$ which is not critical for f^n, let $V_n(x) \equiv V_n(x,f)$ denote the maximal domain containing x which is univalently mapped by f^n onto $\mathbb{C}_T$. Its intersection with the real line is the monotonicity interval $H_n(x) \equiv H_n(x,f)$ of f^n containing x. Let $f_x^{-n}: \mathbb{C}_T \to V_n(x)$ denote the corresponding inverse branch of f^{-n} (continuous up to the boundary of the slits, with different values on the different banks). If J is an interval on which f^n is monotone, then the notations $V_n(J)$ and $H_n(J)$ and f_J^{-n} make an obvious sense.

Take an $x \in \mathbb{R}$ and a $z \in \mathbb{C}_T$. If we have a backward orbit of $x \equiv x_0, x_{-1}, \dots, x_{-\ell}$ of x which does not contain 0, the *corresponding* backward orbit $z \equiv z_0, z_{-1}, \dots, z_{-\ell}$ is obtained by applying the appropriate branches of the inverse functions: $z_{-n} = f_{x_{-n}}^{-n} z$. The same terminology is applied when we have a monotone pullback $J \equiv J_0, \dots, J_{-\ell}$ of an interval J.

Let $H \supset J$ be two intervals. Let $S_{\theta,\epsilon}(H,J)$ denote the union of two 2ϵ-wedges with vertices at ∂J (symmetric with respect to the real line) cut off by the neighborhood $D_\theta(H)$ (see Figure 3).

Let $C_\epsilon(J)$ denote the complement of the above two wedges (that is, the set of points looking at J at an angle at least ϵ).

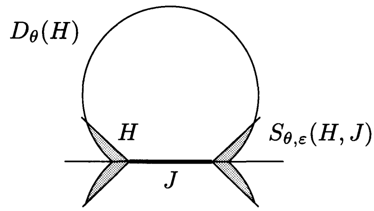

Figure 3

LEMMA 5.1. — *Let f be a quadratic map. Let $J \equiv J_0, J_{-1}, \dots, J_{-\ell} \equiv J'$ be a monotone pullback of an interval J, $z \equiv z_0, z_{-1}, \dots, z_{-\ell} \equiv z'$ be the corresponding backward orbit of a point $z \in \mathbb{C}_T$. Then for all sufficiently small $\epsilon > 0$ (independent of f), either $z_{-k} \in C_\epsilon(J_{-k})$ at some moment $k \le \ell$, or $z' \in S_{\theta,\epsilon}(H_\ell(J'), J')$ with $\theta = \frac{1}{2}\pi - 0(\epsilon)$.*

If the first possibility of the lemma occurs we say that the backward orbit of z *ϵ-jumps.*

Proof. — Assume that the backward orbit of z does not "ϵ-jump", that is, z_{-k} belongs to an $\mathbb{R}$-symmetric 2ϵ-wedge centered at $a_{-k} \in \partial J_{-k}$, $k = 0, 1 \dots, \ell$. By the second statement of Lemma 2.3, $fa_{-(k+1)} = a_{-k}$. Let $M_{-k} = f^{\ell-k} H_\ell(J')$, and b_{-k} be the boundary point of M_{-k} on the same side of J_{-k} as a_{-k}. Let us take the moment k when $b_{-k} = 0$. At this moment the point z_{-k} belongs to a right triangle based upon $[a_{-k}, b_{-k}]$ with the ϵ-angle at a_{-k} and the right angle at b_{-k}. Hence $z_{-k} \in D_\theta(M_{-k})$ with $\theta = \frac{1}{2}\pi - 0(\epsilon)$. It follows by Schwarz Lemma that $z' \in D_\theta(H_\ell(J'))$, and we are done. □

In view of Lemma 2.6, the above lemma admits the following straightforward extension onto the Epstein class:

LEMMA 5.2. — *The conclusion of Lemma* 5.1 *still holds, provided f is a map of Epstein class $\mathcal{E}_\lambda$, and the backward orbit of z stays sufficiently close to the real line* (*depending on* λ).

5.2. Proof of Lemma 4.2 (for bounded combinatorics).

For technical reasons we consider a new family of intervals $\widetilde{S}^k$ and $\widetilde{T}^k$, for which $P^k \subset \widetilde{S}^k \subset S^k \subset \widetilde{T}^k \subset T^k$, each of the intervals is commensurable with the others and contained well inside the next one, and $f_k(\widetilde{S}^k) = \widetilde{T}^k$.

Let us fix a level k, and set $n \equiv n_k$,

$$J_0 \equiv P_0^k, J_{-1} \equiv P_{n-1}^k, \ldots, J_{-(n-1)} \equiv P_1^k. \tag{5.1}$$

Take now any point $z_0 \in D(T^t) \cap \mathbb{C}_{T^k}$ with sufficiently big $t = t(\lambda)$. Let $z_{-1}, \ldots, z_{-(n-1)}$ be its backward orbit corresponding to the above backward orbit of J_0. Our goal is to prove that

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \le C(\bar{p}) \frac{\operatorname{dist}(z_0, J_0)}{|J_0|}. \tag{5.2}$$

Take a big quantifier $\bar{K} > 0$. Let us say that s is a "good" moment of time if J_{-s} is $\bar{K}$-commensurable with J_0. For example, let $J_{-s} \subset P^\ell$ and $s < n_{\ell+1}$, that is s is a moment of backward return to P^ℓ preceding the first return to $P^{\ell+1}$. Thus J_{-s} is contained in one of the non-central intervals $P_i^{\ell+1} \subset P^\ell$. By Corollary 2.11 we see that the moment s is good, provided $\bar{K}$ is selected sufficiently big.

We proceed inductively:

LEMMA 5.3. — *Let $J = J_{-s}$ and $J' = J_{-(s+n_\ell)}$ be two consecutive returns of the backward orbit (5.1) to a periodic interval P^ℓ, $\ell < k$. Let z and z' be the corresponding points of the backward orbit of z_0. If $z \in D(\widetilde{T}^\ell)$ then $\operatorname{dist}(z', J') \le C(\bar{p})|\widetilde{T}^\ell|$. Moreover, either $z' \in D(\widetilde{T}^\ell)$, or $\widehat{(z', J')} > \epsilon(\bar{p}) > 0$.*

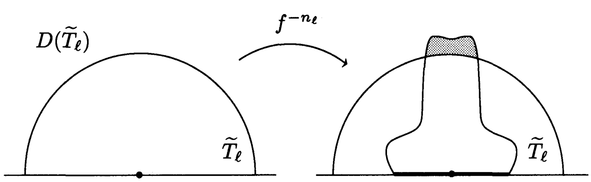

Figure 4

Proof. — Let us consider the decomposition (4.1). By Lemma 2.8 the space between the intervals T and $\widetilde{T}$ depends only on $\bar{p}$. Applying the Koebe Distortion Theorem to the map ψ_ℓ^{-1} we see that its distortion on $\widetilde{T}^\ell$ is $C(\bar{p})$-bounded. Set $\widetilde{Z}^\ell = \psi_\ell^{-1}\widetilde{T}^\ell$. By bounded geometry, the point $f_\ell 0$ divides $\widetilde{T}^\ell$ into commensurable parts. Hence the critical value

$f0 = \psi_\ell^{-1}(f_\ell 0)$ divides $\widetilde{Z}^\ell$ into commensurable parts; let $A = A(\bar{p})$ stand for a bound of the ratio of these parts.

By the Schwarz Lemma, domain $V = \psi_\ell^{-1}(D(\widetilde{T}^\ell))$ is contained in $D(\widetilde{Z}^\ell)$. By Lemma 2.6 and Lemma 2.2 its pullback, $f^{-1}V$ is contained in a domain $W = f^{-1}D(\widetilde{Z}^\ell)$ intersecting the real line by $\widetilde{S}^\ell$, with $\operatorname{diam} W \leq K(\bar{p})|\widetilde{S}^\ell|$; moreover, $W \setminus D(\widetilde{T}^\ell)$ is contained in a sector $\mathbf{C}_\epsilon(\widetilde{S}^\ell)$ with ϵ depending only on A (see Figure 3), and thus the proof is completed. □

Let us now give a more precise statement:

LEMMA 5.4. — *Let $J = J_{-s}$ and $J' = J_{-s'}$ be two returns of the backward orbit (5.1) to P^ℓ, where $s' = s + tn_\ell$. Let z and z' be the corresponding points of the backward orbit of z_0. Assume $z \in D(\widetilde{T}^\ell)$. Then either for some $0 \leq i \leq t$, a point $z_{-(s+in_\ell)}$ ϵ-jumps and $|z_{-(s+in_\ell)}| \leq C|T^\ell|$, or $z_{-s'} \in D_{\theta'}(H')$, where H' is the monotonicity interval of f^{tn_ℓ} containing J', and $\theta' = \frac{1}{2}\pi - O(\epsilon)$.*

Proof. — Assume that the above points do not ϵ-jump. Then by Lemma 5.3 they belong to the disk $D(\widetilde{T}^\ell)$. As the map ψ_ℓ^{-1} from (4.1) has bounded distortion, none of the points z_{-m} δ-jumps for $s \leq m \leq s'$, where $\delta = O(\epsilon)$ as $\epsilon \to 0$. Now the claim follows from Lemma 5.1. □

The following lemma will allow us to make an inductive step:

COROLLARY 5.5. — *Let $J = J_{-n_\ell}$, $J' = J_{-n_{\ell+1}}$, and z, z' be the corresponding points of the backward orbit of z_0. Assume $z \in D(\widetilde{T}^{\ell-1})$. Then either there is a good moment $-m \in (-n_\ell, -n_{\ell+1})$ when the point z_{-m} ϵ-jumps and $|z_{-m}| \leq C|T^\ell|$, or $z' \in D(\widetilde{T}^\ell)$.*

Proof. — Note that by bounded geometry (Corollary 2.11) all the moments

$$-n_\ell, -(n_\ell + n_{\ell-1}), -(n_\ell + 2n_{\ell-1}), \ldots, -n_{\ell+1},$$

when the intervals of (5.1) return to $P^{\ell-1}$ before the first return to $P^{\ell+1}$, are good (provided the quantifier $\bar{K}$ is selected sufficiently big). Hence by Lemma 5.4 either the first possibility of the claim occurs, or $z' \in D_{\theta'}(L')$, where L' is the monotonicity interval of $f^{n_{\ell+1}-n_\ell}$ containing J', and $\theta' = \frac{1}{2}\pi - O(\epsilon)$. As $n_{\ell+1} - n_\ell \geq n_\ell$, L' is contained in S^ℓ, which is well inside $\widetilde{T}^\ell$. Thus $D_{\theta'}(L') \subset D(\widetilde{T}^\ell)$, provided ϵ is sufficiently small. □

We are ready to carry out the inductive proof of (5.2). Let j be the smallest level for which

$$z_0 \in D(\widetilde{T}^j). \tag{5.3}$$

By Lemma 5.3, either $z_{-n_j} \in D(\widetilde{T}^j)$, or z_{-n_j} ϵ-jumps. Moreover, in the latter case $|z_{-n_j}| \le C|z_0|$, and the map ψ_j^{-1} from (4.1) admits a univalent extension to $\mathbb{C}_{T^j}$. So Lemma 2.1 yields (5.2).

In the former case we will proceed inductively. Assume that either $z_{-n_\ell} \in D(\widetilde{T}^{\ell-1})$, or z_{-t} ϵ-jumps at some good moment $-t \ge -n_\ell$. If the latter happens, we are done. If the former happens, we pass to $\ell + 1$ by Corollary 5.5. Lemma 4.2 is proved (for bounded combinatorics). □

5.3. Proof of Lemma 4.4 (for bounded combinatorics).

We will show here that $J(f_k) \subset D_\varepsilon(\widetilde{T}^k)$ which is sufficient for applications (see also §7.2).

By Corollary 4.3, $\operatorname{diam} J(f_k) \le C|T^k|$, with a $C = C(\bar{p})$. Hence $J(f_k) \subset D(\widetilde{T}^\ell)$, where $\ell \ge k - N(\bar{p})$. Let $\zeta' \in J(f_k)$, $\zeta = f_k\zeta'$, and $\zeta = \zeta_0, \zeta_{-1}, \dots, \zeta_{-n} = \zeta'$ be the corresponding backward orbit under iterates of f_ℓ.

By Lemma 5.3, either ζ_{-j} ϵ-jumps at some moment, or $\zeta' \in D(\widetilde{T}^k)$. If the former happens then $\zeta_{-j} \in D_\theta(J_{-jn_\ell})$, where $\theta = \theta(\epsilon) > 0$, and J_{-m} are the intervals from (5.1). But then by the Schwarz Lemma $\zeta' \in D_{\theta'}(P^k)$ with some θ' depending on λ and $\bar{p}$ only. Thus $J(f_k) \subset D_{\theta'}(P^k) \cup D(\widetilde{T}^k)$, and we are done.

6. SADDLE-NODE CASCADES

Let $f \in \mathcal{E}_\lambda$ be a map of Epstein class.

Let us note first that for a long saddle-node cascade (3.1), the map $h_k : I^{m(k)+1} \to I^{m(k)}$ is a small perturbation of a map with a parabolic fixed point.

LEMMA 6.1 (see [L2]). — *Let h_k be a sequence of maps of Epstein class $\mathcal{E}_\lambda$ having saddle-node cascades of length $\ell_k \to \infty$. Then any limit point $f : I' \to I$ of this sequence (in the Carathéodory topology) has on the real line topological type of $z \mapsto z^2 + \frac{1}{4}$, and thus has a parabolic fixed point.*

Proof. — It takes ℓ_k iterates for the critical point to escape $I^{m(k)+1}$ under iterates of h_k. Hence the critical point does not escape I' under iterates of f. By the kneading theory [MT], f has on the real line topological type of z^2+c with $-2 \le c \le \frac{1}{4}$. Since small perturbations of f have escaping critical point, the choice for c boils down to only two boundary parameter values, $\frac{1}{4}$ and -2. Since the cascades of h_k are of saddle-node type, $fI' \not\ni 0$, which rules out $c = -2$. □

Remark 6.1. — Thus the plane dynamics of h_k with a long saddle node cascade resembles the dynamics of a map with a parabolic fixed point: the orbits follow horocycles (see Figure 5).

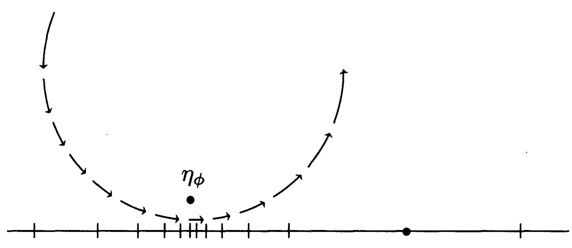

Figure 5. The backward trajectory of a point corresponding to a saddle-node cascade

LEMMA 6.2. — *Let us consider a saddle-node cascade (3.1) generated by a return map h_k. Let us also consider a backward orbit of an interval $E \subset I^{m(k)} \setminus I^{m(k)+1}$ under iterates of h_k:*

$$E \equiv E_0,\ E_{-1} \subset I^{m(k)+1} \setminus I^{m(k)+2}, \dots, E_{-j} \equiv E' \subset I^{m(k)+j} \setminus I^{m(k)+j+1},$$

where $m(k)+j+1 \le m(k+1)$. Let $z = z_0, z_{-1}, z_{-2}, \dots, z_{-j} = z'$ be the corresponding backward orbit of a point $z \in D(I^{m(k)})$. If the length of the cascade is sufficiently big, then either $z' \in D(I^{m(k)})$, or $\widehat{(z', J')} > \epsilon$ and $\operatorname{dist}(z', J') \le C(\bar{p})|I^{m(k)}|$.

Proof. — To be definite, let us assume that the intervals E_{-i} lie on the left of 0 (see Figure 2). Without loss of generality, we can assume that $z \in \mathbb{H}$. Let $\phi = h_k^{-1}$ be the inverse branch of h_k for which $\phi E_{-i} = E_{-(i+1)}$. As ϕ is orientation preserving on $(-\infty, h_k 0]$, it maps the upper half-plane $\mathbb{H}$ into itself: $\phi(\mathbb{H}) \subset \{z = r\,e^{i\theta} \mid r > 0,\ \pi > \theta > \frac{1}{2}\pi\}$.

By Lemma 6.1, if the cascade (3.1) is sufficiently long, the map ϕ has an attracting fixed point $\eta_\phi \in \mathbb{H} \cap D(I^{m(k)+2})$ (which is a perturbation of the parabolic point for some map of type $z^2 + \frac{1}{4}$). By the Denjoy-Wolf Theorem, $\phi^n(\zeta) \xrightarrow[n\to\infty]{} \eta_\phi$ for any $\zeta \in \mathbb{H}$, uniformly on compact subsets of $\mathbb{H}$. Thus for a given compact set $K \Subset \mathbb{H}$, there exists $N = N(K, \phi)$ such that $\phi^N(K) \subset D(I^{m(k)+1})$. By a normality argument, the choice of N is actually independent of a particular ϕ under consideration.

By Lemma 2.2 the set $K = \phi(D(I^{m(k)})) \setminus D(I^{m(k)}) \cap \mathbb{H}$ is compactly contained in $\mathbb{H}$, and $\operatorname{diam} K \leq C|I^{m(k)}|$. For N as above we have $z' \in \bigcup_{i=0}^{N-1} \phi^i(K) \cup D(I^{m(k)})$, and the lemma is proved. □

7. PROOFS OF THE MAIN LEMMAS

The case of essentially bounded combinatorics is more involved than the bounded case treated above (§5). Above we needed only quite rough combinatorial information in between two consecutive renormalization levels. Below we will need to pull the point more carefully through the principal nest waiting until it jumps. A difficulty arises if the jump occurs at a "bad" moment. Then the corresponding iterate of the periodic interval is deep inside of a cascade and hence is not commensurable with its original size. The analysis of saddle-node behavior given in §6 will allow us to handle this problem.

7.1. Proof of Lemma 4.2.

In view of Lemma 2.8, we can assume without loss of generality that all the renormalizations $R^\ell f$, $\ell = 0, \ldots, k-1$, belong to a class $\mathcal{E}_\lambda$ with an absolute λ. Let us start with a little lemma:

LEMMA 7.1. — *Let $f \in \mathcal{E}_\lambda$ be a map of Epstein class which is not close to the cusp. Then both components of $B \setminus A$ contain an f-preimage of* 0 *which divides them into commensurable parts.*

Proof. — The interval $[\alpha, \beta']$ is mapped by f onto $[\beta, \alpha] \ni 0$. Denote by $\eta = f^{-1}(0) \cap [\alpha, \beta']$. Under our assumption this point is clearly different from α and β'.

As the space of maps of Epstein class $\mathcal{E}_\lambda$ which are not close to the cusp is compact, η divides $[\alpha, \beta']$ into commensurable parts. The analogous statement is certainly true for the symmetric point $\eta' \in [\beta, \alpha']$. □

As in §5, let us fix a level τ, let $n = n_\tau$, and set

$$J_0 \equiv P^\tau, J_1 \equiv P^\tau_{n-1}, \dots, J_{-(n-1)} \equiv P^\tau_1. \tag{7.1}$$

Let $z \in D(T^t) \cap \mathbb{C}_{T^\tau}$ with sufficiently big $t = t(\lambda)$. Let

$$z \equiv z_0, z_{-1}, z_{-2}, \dots, z_{-(n-1)} \tag{7.2}$$

be the backward orbit of z corresponding to the orbit (7.1). We should prove that

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \le C(\bar{p}_e)\frac{\operatorname{dist}(z_0, J_0)}{|J_0|}. \tag{7.3}$$

We will proceed inductively along the principal nest. Namely, we will show below that the backward z-orbit (7.2) either ϵ-jumps at some good moment, or follows the backward J-orbit (7.1) with at most one level delay.

In what follows we work with a fixed renormalization level l and skip index l in the notations: $f \equiv f_\ell \equiv R^\ell(f_0)$, $S = S^\ell$, $A \equiv A^\ell, B \equiv B^\ell$. We will use notations of §3 for different combinatorial objects. Let $H_s(x)$ be the monotonicity intervals as defined in §5.1.

LEMMA 7.2 (Return to A). — *Let $E = E_0, E_{-1}, \dots, E_{-s} = E'$ be consecutive returns of the backward orbit* (7.1) *to B, between two consecutive returns to A. Let $\zeta = \zeta_0, \zeta_{-1}, \dots \zeta_{-s} = \zeta'$ be the corresponding points of the backward orbit* (7.2). *Assume $\zeta \in D(S)$. Then either $\zeta' \in D(B)$, or there is a moment when $-i \in [-s, 0]$ when the point ζ_{-i} ϵ-jumps : $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and moreover*

$$\frac{\operatorname{dist}(\zeta_{-i}, E_{-i})}{|E_{-i}|} \le C(\bar{p}_e)\frac{\operatorname{dist}(\zeta_0, E_0)}{|E_0|}. \tag{7.4}$$

Proof. — By definition of the essential period, $s \le \bar{p}_e$. Note that the interval $f^{-1}(S)$ is contained well inside S. By the Schwarz Lemma and Lemma 2.2, if a point $\zeta_{-i} \notin D(S)$, then it ϵ-jumps. Combining Lemma 2.3 and Lemma 2.6 we see that (7.4) holds up to the first moment $-i$ when ζ_{-i} ϵ-jumps.

By Lemma 7.1 each component of $B \setminus A$ contains an f-preimage of 0 which divides B into K-commensurable intervals, with $K = K(\bar{p}_e)$. Hence the monotonicity interval of f, $H = H_s(E_{-s})$, is well inside of B. As $f : B \to B$ has an extension of Epstein class $\mathcal{E}_{\mu(\lambda)}$ (Corollary 2.12), we can apply Lemma 5.2. It follows that if none of the points ζ_{-i} ϵ-jumps, then $\zeta_{-i} \in D_\theta(H)$, $0 \ge -i \ge -s$, with $\theta = \frac{1}{2}\pi - O(\epsilon)$. Thus $\zeta_{-s} \in D(B)$ for sufficiently small $\epsilon < \epsilon(\bar{p}_e)$, and the proof is completed. $\square$

We say that a point/interval is deep inside of the cascade (3.1) if it belongs to $I^{m(k)+\bar{p}_e} \backslash I^{m(k+1)-\bar{p}_e}$. (In the case of essentially bounded combinatorics this cascade must be of saddle-node type). Recall that a moment $-i$ is called *good* if the interval J_{-i} is commensurable with J_0. By Lemma 3.6, this happens, *e.g.*, when for some k, the interval J_{-i} lies in $I^{m(k)} \setminus I^{m(k+1)}$ before the first entering to $I^{m(k+1)}$ but is not deep inside the corresponding cascade.

LEMMA 7.3 (First return to $I^{m(1)}$). — *Assume that f is not immediately renormalizable. Let $E \equiv E_0, E_{-1}, \ldots, E_{-s} \equiv E'$ be the consecutive returns of the backward orbit (7.1) to A until the first return to $I^{m(1)}$. Let $\zeta \in \mathbb{C}_A \cap D(B)$, and let $\zeta \equiv \zeta_0, \zeta_{-1} \ldots \zeta_{-s} \equiv \zeta'$ be the corresponding points in the backward orbit of ζ_0. Then either $\zeta' \in D(A)$, or $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and $\operatorname{dist}(\zeta_{-i}, E_{-i}) \le C(\bar{p}_e)|B|$ at some good moment $0 \ge -i \ge -s$.*

Proof. — Let $H = H_s(E_{-s})$. As f is not immediately renormalizable, we have the interval $I^1 = [p, p']$, which is contained well inside of A by Lemma 3.2. If p is chosen on the same side of 0 as α, then $f^2[\alpha, p] \supset [\alpha, \alpha']$. Denote by η the f^2-preimage of 0 in $[\alpha, p]$. Since f is quadratic up to bounded distortion (Lemma 2.6), the map $f^2|_{[\alpha,p]}$ is quasi-symmetric (that is, maps commensurable adjacent intervals onto commensurable ones). It follows that η divides $[\alpha, p]$, and hence A, into $K = K(\bar{p}_e, \lambda)$-commensurable parts. Hence $H \subset [\eta, \eta']$ is well inside A.

By Lemma 5.2 and Lemma 7.2, either $\zeta' \in D_\theta(H)$ with $\theta = \frac{1}{2}\pi - O(\epsilon)$, or there is a moment $i \le s$ such that

$$(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon \quad \text{and} \quad \operatorname{dist}(\zeta_{-i}, E_{-i}) \le C(\bar{p}_e)|B|. \tag{7.5}$$

In the former case we are done as $D_\theta(H) \subset D(A)$ for sufficiently small ϵ.

Let the latter case occur. Then we are done if the moment $-i$ is good. Otherwise E_{-i} is deep inside the cascade $A = I^0 \supset I^1 \supset \cdots \supset I^{m(1)}$.

Consider the largest r such that $E_{-(i+q)} \subset I^{t+q-1} \setminus I^{t+q}$ for all $0 \le q \le r$. Note that by essentially bounded combinatorics (Corollary 3.6), the moment $-j = -(i+r)$ has to be good. By Lemma 6.2, either (7.5) occurs for ζ_{-j}, and we are done, or $\zeta_{-j} \in D(A)$.

In the latter case let $\widetilde{K} \subset I^{m(1)-1} \setminus I^{m(1)}$ be the interval containing $E_{-(s-1)}$ which is homeomorphically mapped under h_1^{s-1-j} onto A (to see that such an interval exists, consider the Markov scheme described in §3). By the the Schwarz Lemma $\zeta_{-(s-1)} \in D(\widetilde{K}) \subset D(A)$. Now the claim follows from Lemmas 2.2 and 2.6. □

Now we are in a position to proceed inductively along the principal nest: Note that the assumption of the following lemma is checked for $k = 1$ in Lemma 7.3.

LEMMA 7.4 (Further returns to $I^{m(k)}$). — *Let E and E' be two consecutive returns of the backward orbit (7.1) to the interval $I^{m(k)}$. Let ζ and ζ' be the corresponding points of the backward orbit of z_0. Assume that $\zeta \in D(I^{m(k-1)})$. Then, either $\zeta' \in D(I^{m(k)})$, or $(\widehat{\zeta', E'}) > \epsilon(\bar{p}_e) > 0$, and $\operatorname{dist}(\zeta', E') < C(\bar{p}_e)|I^{m(k-1)}|$.*

Proof. — Denote by $\widetilde{E}$ the last interval in the backward orbit (7.1) between E and E', which visits $I^{m(k-1)}$ before returning to $I^{m(k)}$. Then $h_{k-1}E' = \widetilde{E}$ and $h^j_{k-1}\widetilde{E} = E$ for an appropriate j.

The Markov scheme (3.2) provides us with an interval $\widetilde{K} \subset I^{m(k)-1} \setminus I^{m(k)}$ containing $\widetilde{E}$ which is homeomorphically mapped under h^j_{k-1} onto $I^{m(k-1)}$. By essentially bounded geometry $\widetilde{K}$ is well inside $I^{m(k)-1} \setminus I^{m(k)}$. Note that the critical value $h_{k-1}(0)$ is contained in one of the intervals $I^{m(k)}_j$, which is well inside $\widetilde{K}$ by essentially bounded geometry. Thus this critical value divides $\widetilde{K}$ into commensurable parts.

Let $K' \supset E'$ be the pull-back of $\widetilde{K}$ by $h_k \mid I^{m(k)}$. It follows that K' is contained well inside $I^{m(k)}$.

Let $\tilde{\zeta} = h_{k-1}\zeta'$ be the point of the orbit (7.2) corresponding to $\widetilde{E}$. By the Schwarz Lemma, $\tilde{\zeta} \in D(\widetilde{K})$. By the previous remarks and Lemma 2.2, $\zeta' \in D(I^{m(k)})$, or $(\widehat{\zeta', E'}) > \epsilon(\bar{p}_e)$ and $\operatorname{dist}(\zeta', E') < C(\bar{p}_e)|I^{m(k-1)}|$. □

Lemma 7.4 is not enough for making inductive step since the jump can occur at a bad moment. The following lemma takes care of this possibility in the way similar to Lemma 7.3.

LEMMA 7.5 (First return to $I^{m(k+1)}$, $k \geq 1$). — *Let $E \equiv E_0, E_{-1}, \ldots, E_{-s} \equiv E'$ be the consecutive returns of the orbit (7.1) to $I^{m(k)}$ until the first return to $I^{m(k+1)}$. Let $\zeta \equiv \zeta_0, \zeta_{-1} \ldots, \zeta_{-s} \equiv \zeta'$ be the corresponding points in the backward orbit of ζ. Assume that $\zeta_{-1} \in \mathbb{C}_{I^{m(k+1)}} \cap D(I^{m(k)})$. Then either $\zeta' \in D(I^{m(k)})$, or $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and $\operatorname{dist}(\zeta_{-i}, E_{-i}) < C(\bar{p}_e)|I^{m(k)}|$ at some good moment $-1 \geq -i \geq -s$.*

Proof. — Consider the largest moment $r \leq s-1$ such that $E_{-q} \subset I^{m(k)+q-1} \setminus I^{m(k)+q}$ for all $q \leq r$. By essentially bounded combinatorics and geometry, this moment is good.

By Lemma 6.2, either

$$(\widehat{\zeta_{-r}, E_{-r}}) > \varepsilon \quad \text{and} \quad \operatorname{dist}(\zeta_{-r}, E_{-r}) \leq C(\bar{p}_e)\big|I^{m(k)}\big|,$$

or $\zeta_{-r} \in D(I^{m(k)})$. In the former case we are done by Lemma 2.1. So let us assume the latter. If $r = s-1$ the claim follows from Lemmas 2.2 and 2.6. Otherwise, the Markov scheme (3.2) provides us with an interval $K \subset I^{m(k+1)-1} \setminus I^{m(k+1)}$ containing $E_{-(s-1)}$ which is mapped homeomorphically onto $I^{m(k)}$ by h_k^{s-1-r}. By the Schwarz Lemma, $\zeta_{-(s-1)} \in D(K) \subset D(I^{m(k)})$, and the desired conclusion follows again from Lemmas 2.2 and 2.6. □

The following lemma will allow us to pass to the next renormalization level. Note that the statement is almost identical to that of Lemma 7.2. Let us now restore the label ℓ for the renormalization level.

LEMMA 7.6 (To the next renormalization level: period > 2 case). — *Assume that f_ℓ is not immediately renormalizable.*

Let $E = E_{-1}, \ldots, E_{-r} = E', \ldots, E_{-(r+s)} = E''$ be the returns of the backward orbit (7.1) to $B^{\ell+1}$, and let E', E'' be two consecutive returns to $A^{\ell+1}$. Let $\zeta = \zeta_{-1}, \ldots, \zeta', \ldots, \zeta_{-(r+s)} = \zeta''$ be the corresponding points of the backward orbit (7.2), and suppose $\zeta \in D(I^{m(\chi-1)})$, where $\chi = \chi(f_\ell)$ is the height of f_ℓ. Then either $\zeta'' \in D(B^{\ell+1})$, or $(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon(\bar{p}_e) > 0$ and $\operatorname{dist}(\zeta_{-i}, E_{-i}) \leq C(\bar{p})|B^{\ell+1}|$ for some $1 \leq i \leq r+s$. Moreover, all these moments are good.

Proof. — First, $r + s \leq 2\bar{p}_e$ by definition of the essential period $\bar{p}_e$, and the last statement follows from Lemma 2.6.

By Lemma 7.4, either $(\widehat{\zeta_{-2}, E_{-2}}) > \epsilon$, $\operatorname{dist}(\zeta_{-2}, E_{-2}) \leq C(\bar{p}_e)|B^{\ell+1}|$, or $\zeta_{-2} \in D(I^{m(\chi)})$.

By the Schwarz Lemma and Lemma 2.2, if $\zeta_{-i} \in D(I^{m(\chi)})$, then either $\zeta_{-(i+1)} \in D(I^{m(\chi)})$, or $\operatorname{dist}(\zeta_{-(i+1)}, E_{-(i+1)}) \leq C(\bar{p}_e)|B^{\ell+1}|$ and $(\widehat{\zeta_{-(i+1)}, E_{-(i+1)}}) > \epsilon(\bar{p}_e) > 0$. In the latter case we are done.

If the former case occurs for all $i < r+s$ then by Lemma 5.2, $\zeta'' \in D_\theta(H)$, where $H = H_{r+s-1}(E'', f_{\ell+1})$ and $\theta = \frac{1}{2}\pi - O(\epsilon)$. By Lemma 7.1, H is well inside $B^{\ell+1}$, and hence $D_\theta(H) \subset D(B^{\ell+1})$ for sufficiently small $\epsilon > 0$. □

Our last lemma takes care of the case when the map f_ℓ is immediately renormalizable.

LEMMA 7.7 (To the next renormalization level: period 2 case). — *Assume that f_ℓ is immediately renormalizable, so that $A^\ell = B^{\ell+1}$. Let $E \subset B^{\ell+1}$, $E \equiv E_0, E_{-1}, \ldots, E_{-s} \equiv E'$ be the consecutive returns of the backward orbit* (7.1) *to B^ℓ, until the first return to $A^{\ell+1}$.*

Let $\zeta \equiv \zeta_0, \ldots, \zeta_{-s} \equiv \zeta'$ be the corresponding points of the backward orbit (7.2). *Assume also that $\zeta \in \mathbb{C}_{A^\ell} \cap D(B^\ell)$. Then either $\zeta' \in D(B^{\ell+1})$, or*

$$(\widehat{\zeta_{-i}, E_{-i}}) > \epsilon \quad \textit{and} \quad \operatorname{dist}(\zeta_{-i}, E_{-i}) < C(\bar{p}_e)|B^\ell|$$

for some $0 \geq -i \geq -s$. Moreover, all these moments are good.

Proof. — By essentially bounded combinatorics, $s \leq 2\bar{p}_e$ which yields the last statement.

Further, by Lemma 7.1, the monotonicity interval $H_s(E_{-s}, f_\ell)$ is contained well inside of $B^{\ell+1}$, and the claim follows from Lemma 5.2. □

Let us now summarize the above information. When $f_{\tau-1}$ is immediately renormalizable, set $V_\tau = B^{\tau-1}$. Otherwise let $V_\tau = I^{m(\chi-1)}(f_{\tau-1})$ where $\chi = \chi(f_{\tau-1})$ is the height of $f_{\tau-1}$.

LEMMA 7.8. — *Let $f_\tau = R^\tau f$. Let us consider the backward orbit* (7.1) *of an interval J and the corresponding orbit* (7.2) *of a point z. Then there exist $\epsilon = \epsilon(\bar{p}_e) > 0$ such that either one of the points z_{-s} ϵ-jumps at some good moment, or $z_{-(n-1)} \in D(V_\tau)$.*

Proof of Lemma 4.2. — If the former possibility of Lemma 7.8 occurs than Lemma 2.1 yields (7.3) (note that the assumptions of Lemma 2.1 are satisfied due to Theorem 3.4). In the latter possibility happens then

$$\frac{\operatorname{dist}(z_{-(n-1)}, J_{-(n-1)})}{|J_{-(n-1)}|} \leq C(\bar{p}_e)$$

by essentially bounded geometry, and we are done again.

The lemma is proved. □

7.2. Proof of Lemma 4.4.

Let us first show that $J(f_k) \subset D_\theta(S^k)$ with a $\theta = \theta(\bar{p}_e)$ (recall that $S^k \ni 0$ is the maximal interval on which f_k is unimodal).

By Corollary 4.3, $\operatorname{diam} J(f_\tau) \leq C(\bar{p}_e)|B^\tau|$.

Take $\zeta'' \in J(f_\tau)$. Let $\zeta' = f_\tau(\zeta'')$, $\zeta = f_\tau(\zeta')$, and

$$\zeta = \zeta_0, \zeta_{-1}, \dots, \zeta_{-n} = \zeta', \dots, \zeta_{-2n} = \zeta''$$

be the corresponding backward orbit.

Let the first possibility of Lemma 7.8 occur and ζ_{-s} ϵ-jumps at a good moment $s \leq n-1$. Then $\zeta_{-s} \in D_\delta(J_{-s})$ with $\delta = \delta(\bar{p}_e) > 0$, since $\operatorname{dist}(\zeta_{-s}, J_{-s})$ is commensurable with $|J_{-s}|$. But then by the Schwarz lemma and Lemma 2.2, $\zeta'' \in Q_\theta(S_\tau)$ with a $\theta = \theta(\bar{p}_e) > 0$.

Let the second possibility of Lemma 7.8 occur.

Let us first consider the case when $f_{\tau-1}$ is not immediately renormalizable. Then $\zeta' \in D(I^{\tau-1,m(\chi-1)})$. By Lemma 7.4, $\zeta'' \in D(I^{\tau-1,m(\chi)}) \subset D(S^\tau)$. Thus $J(f_\tau) \subset Q_\epsilon(S_\tau)$, and we are done.

In the case when $f_{\tau-1}$ is immediately renormalizable $\zeta' \in D(B^{\tau-1})$. Consider the interval of monotonicity of $f_{\tau-1}$, $H = H_2(\zeta'') \subset S_\tau$. By Lemma 5.2, $\zeta'' \in D_\theta(H)$ with $\theta = \frac{1}{2}\pi - O(\epsilon)$, and the claim follows.

Let us now show how to replace S^τ by B^τ. By essentially bounded geometry, the space $S^\tau \setminus B^\tau$ is commensurable with $|B^\tau|$ (see Corollary 3.3 and the second statement of Lemma 2.7).

By the last statement of Lemma 2.7, for any $\delta > 0$, there is an $N = N(\bar{p}_e, \delta)$ such that the N-fold pull-back of S^τ by f_τ is contained in $(1+\delta)B^\tau$. By the Schwarz Lemma and Lemma 2.2, $J(f_\tau) \subset D_\rho((1+\delta)B^\tau)$, with a $\rho = \rho(\delta, \bar{p}_e)$.

By the compactess Lemma 2.5, for some $\delta > 0$ (independent of τ) the map f_τ is linearizable in the $\delta|B^\tau|$-neighborhood of the fixed point β_τ. In the corresponding local chart the Julia set $J(f_\tau)$ is invariant with respect to $f'_\tau(\beta_\tau)$-dilation. Hence further pull-backs will keep it within a definite sector. $\square$

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