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SOME REMARKS ON POSITIVE SCALAR AND GAUSS-KRONECKER CURVATURE HYPERSURFACES OF $\mathbb{R}^{n+1}$ AND $\mathbb{H}^{n+1}$

by B. NELLI and H. ROSENBERG

1. Introduction.

We shall discuss isolated singularities of hypersurfaces of $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ with constant positive scalar curvature and hypersurfaces of $\mathbb{R}^{n+1}$ with bounded positive Gauss-Kronecker curvature.

In $\mathbb{R}^3$ the scalar curvature is the Gauss curvature and in this case we are able to prove that a graph of positive Gauss curvature over a punctured disk extends continuously to the puncture. The same holds when the ambient space is $\mathbb{R}^{n+1}$ and the graph has positive Gauss-Kronecker curvature and a strictly convex point.

We will prove that a graph over a punctured disk, with constant positive scalar curvature in $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ (in hyperbolic space for $n \geq 3$) is bounded.

There are examples of rotational surfaces in $\mathbb{R}^3$ with positive constant Gauss curvature that are not C^1 (*cf.* [Sp]), so the regularity of our extension result is optimal.

An interesting problem is to decide if a graph over the punctured disk in $\mathbb{R}^{n+1}$, $n \geq 3$, of constant positive scalar curvature extends to the puncture.

Key words: Scalar and Gauss-Kronecker curvature – Isolated singularity – Maximum principle.
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2. Statement of the theorems.

THEOREM 2.1. — *Let Ω be a domain in $\mathbb{R}^n$, $p \in \Omega$. Let $f : \Omega \setminus \{p\} \to \mathbb{R}$ be a C^2 function such that the graph of f in $\mathbb{R}^{n+1}$ has positive Gauss-Kronecker curvature and has a strictly convex point. Then f extends to a continuous function on Ω. When $n = 2$, Gauss-Kronecker curvature bigger than 0 means every point of the graph is strictly convex.*

THEOREM 2.2. — *Let Ω be a domain in $\mathbb{R}^n$ ($\mathbb{H}^n$, $n \geq 3$), $p \in \Omega$, S a positive constant. Let $f : \Omega \setminus \{p\} \to \mathbb{R}$ be a C^2 function such that the graph of f has scalar curvature equal to S in $\mathbb{R}^{n+1}$ (respectively in $\mathbb{H}^{n+1}$), then f is bounded.*

This is a first step in order to prove that f extends to p (see the case of constant mean curvature in [RSa] and [NSa]).

We must precise what we mean by a graph in hyperbolic space. In fact there are several notions of graph (*cf.* [NSa], [NSe], [NSp]) and here, the opportune one is the following.

Killing Graph with respect to a geodesic γ_p. — Let Ω be a domain in a geodesic hyperplane P and let $p \in \Omega$; let q be any point of Ω and $\eta_p(q)$ the orbit through q of the hyperbolic translation along γ_p (*i.e.* the integral curve of the Killing vector field associated to the hyperbolic translation). Let u be a real function that at each point $q \in \Omega$ associates the point on $\eta_p(q)$ at hyperbolic distance $u(q)$ from P.

We call *Killing cylinder over Ω with respect to γ_p* the set

$$K(\Omega, \gamma_p) = \bigcup_{q \in \Omega} \eta_p(q).$$

3. Isolated singularities of hypersurfaces of $\mathbb{R}^{n+1}$ with $S_n > 0$.

Proof of Theorem 2.1.

We use an elementary geometric method. We can assume that $\Omega \setminus \{p\}$ is the disk of radius one, punctured at the origin, $D^* \subset \{x_{n+1} = 0\}$.

Denote by $x_i(P)$, $i = 1, \ldots, n+1$, the coordinates of a point $P \in \mathbb{R}^{n+1}$ and set $p = (x_1(P), \ldots, x_n(P))$; denote by S^* the graph of f and set $F(p) = (p, f(p))$.

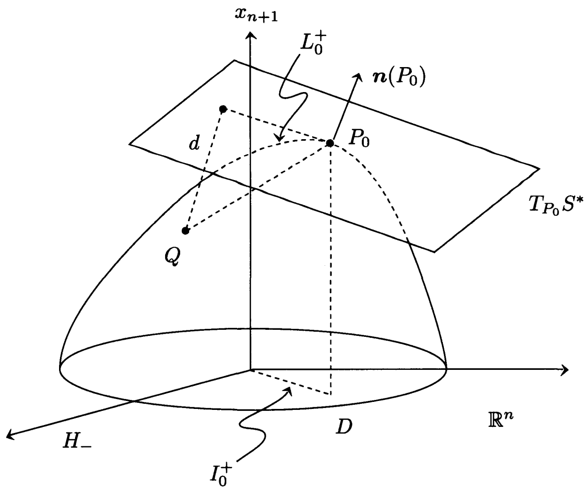

Figure 1

We achieve the proof in four steps.

1. Let $p_0 \in D^*$, $P_0 = (p_0, f(p_0))$ and $T_{P_0}S^*$ be the tangent space to S^* at P_0; then S^* lies in one of the two halfspaces determined by $T_{P_0}S^*$.

2. $|\nabla f|^2$ is bounded in a neighbourhood of 0.

3. The function f is bounded on D^*.

4. There exists $L \in \mathbb{R}$ such that, for each path $\gamma: [0,1] \to D$, $\gamma(1) = 0$, the limit $\lim_{t \to 1} f(\gamma(t))$ exists and it is equal to L.

This means that the function $\widetilde{f}: D \to \mathbb{R}$ defined by $\widetilde{f}(p) = f(p)$ for each $p \in D^*$ and $\widetilde{f}(0) = L$ is a continuous extension of f.

First step. — The Gauss-Kronecker curvature S_n of a hypersurface of $\mathbb{R}^{n+1}$, is the product of the principal curvatures of the hypersurface. Since we are assuming there is at least one strictly convex point, the hypersurface can be oriented so that all the principal curvatures are positive at this point. Since $S_n > 0$, they are positive at every point.

The signed distance d from a point $Q = F(p)$ to the tangent space $T_{P_0}S^*$ is given by $d = \langle F(p) - F(p_0), \boldsymbol{n}(P_0)\rangle$ where

$$\boldsymbol{n}(P_0) = \Big(\frac{-f_1(p_0)}{\sqrt{1+|\nabla f(p_0)|^2}}, \dots, \frac{-f_n(p_0)}{\sqrt{1+|\nabla f(p_0)|^2}}, \frac{1}{\sqrt{1+|\nabla f(p_0)|^2}}\Big)$$

is the unit normal vector to the surface at P_0 (see Figure 1).

Let ℓ_0^+ be the half $(n-1)$-plane passing through the origin and p_0 and let

$$L_0^+ = \{Q \in S^* \mid Q = F(p),\ p \in \ell_0^+ \cap D^*\}.$$

We prove that d has constant sign on $S^* \setminus L_0^+$.

By Taylor's formula, for points $p = (x_1, \dots, x_n) \notin \ell_0$, ($\ell_0$ is the $(n-1)$-plane that contains ℓ_0^+) we have

$$F(p) = F(p_0) + \sum_{i=1}^{n} F_i(p_0)\big(x_i - x_i(P_0)\big) + \tfrac{1}{2} \sum_{i,j=1}^{n} F_{ij}(\zeta)\big(x_i - x_i(P_0)\big)\big(x_j - x_j(P_0)\big)$$

where $\zeta = (\zeta_1, \dots, \zeta_n)$ is a point of the segment from p_0 to p. We remark that if $p \in \ell_0$, then ζ may be the origin and F is not defined there. It follows that:

$$d = \sum_{i=1}^{n} \big\langle F_i(p_0), \boldsymbol{n}(P_0)\big\rangle \big(x_i - x_i(P_0)\big) + \tfrac{1}{2} \sum_{i,j=1}^{n} \big\langle F_{ij}(\zeta), \mathbf{n}(P_0)\big\rangle \big(x_i - x_i(P_0)\big)\big(x_j - x_j(P_0)\big).$$

As $\boldsymbol{n}(P_0) \perp F_i$ for $i = 1, \dots, n$, we have

$$\begin{aligned} d &= \tfrac{1}{2} n_{n+1} \sum_{i.j=1}^{n} f_{ij}(\zeta)\big(x_i - x_i(P_0)\big)\big(x_j - x_j(P_0)\big) \\ &= \tfrac{1}{2} n_{n+1} \Big[\big(x_1 - x_1(P_0), \dots, x_n - x_n(P_0)\big)\big[H(f)(\zeta)\big] \\ &\qquad \times {}^t\big(x_1 - x_1(P_0), \dots, x_n - x_n(P_0)\big)\Big] \end{aligned}$$

where n_{n+1} is the last coordinate of the unit normal vector to S^* and $H(f)$ is the Hessian matrix of the function f.

Since all the points of S^* are strictly convex, we can assume that $H(f)$ is negative definite; so, as $n_{n+1} > 0$, d is negative.

It follows that all the points $Q \in S^* \setminus L_0^+$ are contained in one of the two halfspaces determined by $T_{P_0} S^*$. By continuity, all the points of L_0^+ are contained in the same halfspace.

Since $f \in C^2(D^*)$, no tangent space to S^* is vertical (except eventually on ∂S^*). Thus we can say that S^* lies under each tangent space, hence f is bounded above.

Second step. — By contradiction, there exists a sequence $\{p_m\} \subset D^*$ such that $p_m \to 0$ and $|\nabla f(p_m)| \to \infty$, as $m \to \infty$. Hence $n_{n+1}(p_m) \to 0$ and $T_{P_m}S^*$ tends to a vertical plane.

Let

$$s_m = \min\{x_{n+1}(Q) \mid Q = (q, x_{n+1}(Q)) \in T_{P_m}S^*,\ q \in \partial D\};$$

then, $T_{P_m}S^*$ tends to a vertical hyperplane if and only if $\lim_{m\to\infty} s_m = -\infty$.

Set $M = \min_{p\in\partial D} f(p)$, then there exists $m(M)$ such that for each $m > m(M)$

$$s_m < M = \min_{p\in\partial D} f(p) \le f(q)$$

where $Q \in T_{P_m}S^*$ and $q \in \partial D$; this last inequality is a contradiction, because the surface S^* is under each tangent space by the first step.

Third step. — By the remark at the end of the first step f has an upper bound, so we have only to prove that f has a lower bound. If not, there exists a sequence $\{p_m\} \in D^*$ such that $p_m \to 0$ and $f(p_m) \to -\infty$, as $m \to \infty$.

Passing to a subsequence, we can assume that:

(i) $f(p_m) \to -\infty$ decreasing and $f(p_{m+1}) < -c + f(p_m)$ where c is a positive constant;

(ii) $|p_m - p_{m+1}| < 3(2^{-(m+1)})$ (we choose p_m in the disk of radius 2^{-m} centered at 0);

(iii) the origin does not belong to the segments $t(p_{m+1} - p_m) + p_m$, $t \in [0,1]$ for each $m \in \mathbb{N}$; it is enough to choose all the p_m in a set as in Figure 2.

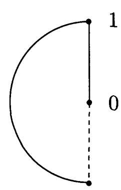

Figure 2

Let $g_m : [0,1] \to \mathbb{R}$ be the C^1 function defined as follows:

$$g_m(t) = f\big(t(p_{m+1} - p_m) + p_m\big).$$

This function is well defined because $t(p_{m+1} - p_m) + p_m \neq 0$ for each $t \in [0,1]$.

Then there exists a $t_0 \in (0,1)$ such that

$$g'_m(t_0) = g_m(1) - g_m(0) = f(p_{m+1}) - f(p_m).$$

Furthermore

$$\begin{aligned} \big|g'_m(t_0)\big| &= \big|\big\langle \nabla f(t(p_{m+1} - p_m) + p_m), (p_{m+1} - p_m)\big\rangle\big| \\ &\leq \big|\nabla f(t(p_{m+1} - p_m) + p_m)\big| \cdot |p_{m+1} - p_m| \\ &\leq \big|\nabla f(t(p_{m+1} - p_m) + p_m)\big| \frac{3}{2^m} \leq \frac{3C}{2^m} \end{aligned}$$

where the last two inequalities depend on (ii) and on step 2, respectively.

Hence, for each $m \in \mathbb{N}$

$$0 < c < \big|f(p_{m+1}) - f(p_m)\big| \leq \frac{3C}{2^m}$$

where the second inequality follows from (i); a contradiction.

Fourth step. — By contradiction, there exists a path γ satisfying the hypothesis such that $\lim_{t\to 1} f(\gamma(t))$ does not exist; then there exist two sequences $\{s_m\}$, $\{t_m\}$, $s_m, t_m \to 1$ as $m \to \infty$, such that

$$\lim_{m\to\infty} f\big(\gamma(t_m)\big) = L, \quad \lim_{m\to\infty} f\big(\gamma(s_m)\big) = \ell$$

and $L = \ell + c$ where c is a positive constant.

Let $p_m = \gamma(t_m)$ and $q_m = \gamma(s_m)$.

The equation of the tangent space to S^* at a point $Q_m = (q_m, f(q_m))$ is

$$\big\langle \boldsymbol{n}(Q_m), (x_1 - x_1(Q_m), \dots, x_n - x_n(Q_m), x_{n+1} - f(q_m))\big\rangle = 0$$

that is

$$\sum_{i=1}^{n} f_i(q_m)\big(x_i - x_i(Q_m)\big) - x_{n+1} + f(q_m) = 0.$$

The intersection of this hyperplane with the x_{n+1} axis is the point $(0, x^m_{n+1})$ given by

$$x^m_{n+1} - f(q_m) = -\sum_{i=1}^{n} f_i(q_m) x_i(Q_m).$$

As $|\nabla f|$ is bounded on D^*, there exists a positive constant C such that

$$\left|x^m_{n+1} - f(q_m)\right| \leq C\left|\sum_{i=1}^{n} x_i(Q_m)\right|,$$

hence $\lim_{m\to\infty}\left|x^m_{n+1} - f(q_m)\right| = 0$.

Let $\epsilon > 0$. By the fact that the tangent space to S^* is not "too vertical" (step 2), there exists a neighbourhood of the origin in D, say U_ϵ, such that for every $p \in U_\epsilon$, the point $P \in T_{Q_m}S^*$ such that $P = (p, x_{n+1}(P))$ satisfies

(1) $$\left|x_{n+1}(P) - x^m_{n+1}\right| < \epsilon.$$

Now, let $m(\epsilon)$ such that for each $m > m(\epsilon)$, we have $p_m \in U_\epsilon$ and

(2) $$\left|f(q_m) - x^m_{n+1}\right| < \epsilon, \quad \left|f(q_m) - \ell\right| < \epsilon, \quad \left|f(p_m) - L\right| < \epsilon.$$

For any $m > m(\epsilon)$, let $P \in T_{Q_m}S^*$ such that $(x_1(P), \dots, x_n(P)) = p_m = \gamma(t_m)$. Then

$$x_{n+1}(P) < x^m_{n+1} + \epsilon < f(q_m) + 2\epsilon < \ell + 3\epsilon = L - c + 3\epsilon < f(p_m) + 4\epsilon - c$$

where inequalities are given by (1), and (2) respectively.

If $\epsilon < \frac{1}{4}c$ the point $(p_m, f(p_m)) \in S^*$ lies above the hyperplane $T_{Q_m}S^*$; a contradiction by step 1.

Analogously one proves that two paths with different limits do not exist. □

4. Isolated singularities of hypersurfaces with $S_2 > 0$.

As remarked in [R] (see also [ACC]), $S_2 > 0$ yields an elliptic equation on any hypersurface in a space form. We briefly recall the proof of this fact.

Let $A = \{a_{ij}\}$ be the second fundamental form of a positive scalar curvature hypersurface M and let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be its eigenvalues list. Let $S_2 = \sum_{i<j=1}^{n} \lambda_i \lambda_j$ and $S_1 = \sum_{j=1}^{n} \lambda_j$. Then for each $j = 1, \ldots, n$,

$$S_1^2 = \sum_{j=1}^{n} \lambda_j^2 + 2S_2 > \lambda_j^2.$$

Hence $S_1 - \lambda_j = (\partial S_2/\partial \lambda_j) > 0$ and the matrix $\{\partial S_2/\partial \lambda_j\}$ is positive definite on M. So M satisfies the maximum principle (*cf.* Theorem 1.3.4 of [N]).

Also the previous equation clearly implies that the sign of the mean curvature function of a positive scalar curvature hypersurface is constant.

An important tool in the proof of Theorem 2.1 is the classification of rotational hypersurfaces with constant scalar curvature. Leite classifies such hypersurfaces in space forms of dimension bigger than three (*cf.* [L]). We recall the part of her result that we need.

THEOREM 4.1. — *Let $n \geq 3$. For any constant $S > 0$, there exists a one parameter family of complete embedded rotational hypersurfaces of $\mathbb{R}^{n+1}$ (resp. $\mathbb{H}^{n+1}$) with constant scalar curvature S, (resp. computed in the metric of $\mathbb{H}^{n+1}$) all periodic and cylindrically bounded, which converges to a stack of geodesic spheres i.e. a sequence of spheres, tangent two by two.*

Classically such hypersurfaces are called *Delaunay hypersurfaces.*

In the following, where there is no ambiguity, we will refer to hyperplanes as planes and to hypersurfaces as surfaces *etc.*

Proof of Theorem 2.2.

The proof in $\mathbb{R}^3$ is a particular case of Theorem 2.1. Let us restrict to $n \geq 3$.

The technique of the proof in hyperbolic space clearly yields the result in Euclidean space.

Let Ω be a domain in the geodesic plane P of $\mathbb{H}^{n+1}$ and let γ_p be the geodesic through p orthogonal to P. Consider the family of rotational surfaces around γ_p with constant scalar curvature S.

Each Delaunay surface divides hyperbolic space into two connected components and its mean curvature vector points towards the component $\mathfrak{U}$

containing the geodesic γ_p. Furthermore, we can find a geodesic disk D_R in P centered at p, of radius R contained in Ω such that there exists a portion Del_ε of a Delaunay surface with the following properties:

(A) Del_ε has scalar curvature S;

(B) Del_ε is a Killing graph over the annulus $A_\varepsilon = D_R \setminus D_\varepsilon$;

(C) Del_ε is tangent to the Killing cylinder over ∂D_ε.

Consider the family of hyperbolic translation $\{H_t\}_{t\in\mathbb{R}}$ along γ_p. The function f is bounded on A_ε, hence we can assume that, by increasing t, $H_t(\mathrm{Del}_\varepsilon)$ is disjoint from the graph of f. Now decrease t in order to find a first point of contact between the two surfaces. The mean curvature vector of any $H_t(\mathrm{Del}_\varepsilon)$ points towards $\mathfrak{U}$, hence so does the mean curvature vector of the graph of f at the point where they first touch. Then, by the maximum principle, the first point of contact cannot be interior. If it is on the internal Killing cylinder (the Killing cylinder over ∂D_ε) then by (C), the graph of f must be tangent to the cylinder there and this is a contradiction by the fact that f is a C^2 Killing graph over the punctured disk.

So, the first contact point is on the Killing cylinder over ∂D_R. This gives a bound on f. □

Remark. — The same technique cannot work in the space form of positive constant curvature, *i.e.* the sphere. In fact by the classification theorem (*cf.* [L]) the family of rotational surfaces with positive constant scalar curvature in the sphere is at most countable.

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