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ON THE HAAGERUP INEQUALITY AND GROUPS ACTING ON $\tilde{A}_n$ -BUILDINGS^(*)

by Alain VALETTE

1. Introduction.

Let $\mathbf{F}_n$ be the free group on n generators; for $\gamma \in \mathbf{F}_n$, denote by $|\gamma|$ the word length of γ with respect to a free generating subset; for f a function with finite support on $\mathbf{F}_n$, denote by $\lambda(f)$ the operator of left convolution by f on the Hilbert space $\ell^2(\mathbf{F}_n)$. In Lemma 1.5 of [Haa79], U. Haagerup proved the following remarkable inequality on the operator norm $\|\lambda(f)\|$:

$$\|\lambda(f)\| \leq 2 \sqrt{\sum_{\gamma \in \mathbf{F}_n} |f(\gamma)|^2 (1 + |\gamma|)^4}.$$

In other words, the convolution norm of f , which is in general quite hard to compute (see e.g. [AO76]), can be estimated by a weighted ℓ^2 -norm - or Sobolev norm - which is much easier to calculate.

Haagerup's inequality was studied in a systematic way by P. Jolissaint (see [Jol90], [Jol89]) in the setup of a group Γ endowed with a length function L . A *length function* is a function $L : \Gamma \rightarrow \mathbf{R}^+$ such that $L(1) = 0$, $L(\gamma^{-1}) = L(\gamma)$, $L(\gamma_1 \gamma_2) \leq L(\gamma_1) + L(\gamma_2)$ for every $\gamma_1, \gamma_2, \gamma \in \Gamma$, and for every $R > 0$ the set $\{\gamma \in \Gamma : L(\gamma) \leq R\}$ is finite (i.e. L is a proper function).

Apart from length functions given by word length with respect to a finite generating subset in a finitely generated Γ , examples of length

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functions are obtained by letting Γ act properly isometrically on a metric space (X, d) with base-point x_o , and setting

$$L(\gamma) = d(\gamma x_o, x_o)$$

for $\gamma \in \Gamma$ (actually this last example is general).

Denote by $\mathbf{C}\Gamma$ the group algebra of Γ , i.e. the space of complex-valued finitely supported functions on Γ , endowed with the convolution product; for $f \in \mathbf{C}\Gamma$ and $s > 0$, define the weighted ℓ^2 -norm of f as:

$$\|f\|_{s,L} = \sqrt{\sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + L(\gamma))^{2s}}.$$

DEFINITION 1. — *We say that Γ satisfies the Haagerup inequality, or has the (RD)-property with respect to L , if there exists constants $C, s > 0$ such that, for any $f \in \mathbf{C}\Gamma$, one has*

$$\|\lambda(f)\| \leq C \|f\|_{s,L}.$$

The papers [Jol90] and [dlH88] give the main known examples of (RD)-groups; among finitely generated groups with a word length, these are groups with polynomial growth and hyperbolic groups “à la Gromov”.

The main feature of (RD)-groups (which explains the acronym RD) appears in [Jol89]: for an (RD)-group Γ , the space of rapidly decreasing functions on Γ (i.e. functions ϕ on Γ such that $\|\phi\|_{s,L} < \infty$ for every $s > 0$) is a dense subalgebra of the reduced C^* -algebra $C_r^*(\Gamma)$, such that the inclusion induces isomorphisms in topological K-theory. (This fact played a crucial role in the Connes-Moscovici proof [CM90] of Novikov’s conjecture for hyperbolic groups.) Applications of Haagerup’s inequality to harmonic analysis were given in [Haa79] and [JV91]. More recently came other applications to spectra of Markov operators [dlHRV93].

Jolissaint gave a purely algebraic obstruction to property (RD): if Γ contains a subgroup which is solvable with exponential growth, then there is no length function on Γ for which Haagerup’s inequality holds (combine 1.1.7, 2.1.1 and 3.1.8 in [Jol90]); this applies in particular to $SL_n(\mathbf{Z})$, with $n \geq 3$ ([Jol90], 3.1.9); more generally, this holds true for any non-uniform lattice in a simple real Lie group with real rank at least 2 (private communication of E. Leuzinger and C. Pittet). In contrast, a

uniform lattice in such a Lie group (or in a simple p-adic group with split rank at least 2) has no solvable subgroup with exponential growth (see [GW71]). Thus the question was raised in the problem section of [FFR95] whether such a uniform lattice has property (RD); that was the motivation for the present paper. While this article was under completion, we received a very interesting preprint by J. Ramagge, G. Robertson and T. Steger [RRS] providing a proof of property (RD) for $\tilde{A}_2$ -groups - these groups will be defined below.

In this paper, we first generalize Definition 1 as follows:

DEFINITION 2. — *Let E be a linear subspace of $\mathbf{C}\Gamma$; we say that E satisfies the Haagerup inequality if there exists constants $C, s > 0$ such that, for any $f \in E$, one has*

$$\|\lambda(f)\| \leq C \|f\|_{s,L}.$$

(Somewhat pedantically: $\mathbf{C}\Gamma$ satisfies the Haagerup inequality, according to Definition 2, if and only if Γ satisfies the Haagerup inequality, according to Definition 1.) The purpose of this generalization is twofold. First, even if Γ does not have property (RD), it may happen that some interesting subspaces of $\mathbf{C}\Gamma$ satisfy the Haagerup inequality (as an illustration, see [Jol96] for the case of a free product $\Gamma = G * \mathbf{Z}$, with G arbitrary). Second, it may be easier to prove Haagerup's inequality for a subspace, as we will show.

Our main results are as follows:

(1) A $*$ -subspace E of $\mathbf{C}\Gamma$ satisfies the Haagerup inequality if and only if there exists constants $C, s > 0$ such that, for any self-adjoint $f \in E$ and any $k \in \mathbf{N}$:

$$f^{(2k)}(1) \leq C^{2k} \|f\|_{s,L}^{2k},$$

where $f^{(j)}$ is the j -th convolution power of f in $\mathbf{C}\Gamma$.

(2) We get the following new characterization of property (RD): Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any symmetric, finitely supported probability measure μ on Γ and any $k \in \mathbf{N}$:

$$\mu^{(2k)}(1) \leq C^{2k} \|\mu\|_{s,L}^{2k}.$$

Noticing that $\mu^{(2k)}(1) = \sup\{\mu^{(2k)}(x) : x \in \Gamma\}$ measures the decay of the random walk on Γ associated with μ , one sees that this is close to results linking decay of random walks with growth properties of Γ , as they appear e.g. in Chapters VI and VII of [VSCC92].

(3) Denote by $\text{Rad}_L(\Gamma)$ the space of *radial functions*, i.e. the space of functions in $\text{C}\Gamma$ that depend only on L . If L is a word length function on a finitely generated group Γ , we are able to relate growth and amenability as follows. Suppose that $\text{Rad}_L(\Gamma)$ satisfies Haagerup's inequality; we prove that Γ is non-amenable if and only if Γ has superpolynomial growth. (This was known to Jolissaint [Jol90], under the stronger assumption that Γ has property (RD)).

(4) Assume that L is integer valued (e.g. L is a word length). It turns out that Haagerup's inequality for $\text{Rad}_L(\Gamma)$ has a purely combinatorial interpretation. Define a *strict N -loop with length $2k$* in Γ as a sequence $(v_0 = 1, v_1, \dots, v_{2k-1}, v_{2k} = 1)$ such that $L(v_{i-1}^{-1}v_i) = N$ for $i = 1, \dots, 2k$; the *sphere* S_N of radius N is the level set $S_N = L^{-1}(N)$. We show that $\text{Rad}_L(\Gamma)$ satisfies Haagerup's inequality if and only if there exists constants $C, s > 0$ such that for any $k, N \in \mathbf{N}$:

$$\text{card}\{\text{strict } N\text{-loops with length } 2k \text{ in } \Gamma\} \leq C^{2k}(1+N)^{2ks}(\text{card } S_N)^k.$$

This last result allows us to make the link with J. Swiatkowski's paper [Swi], to which the present paper is an appendix. Indeed, let Γ be an $\tilde{A}_n$ -group, i.e. a group acting simply transitively on the vertices of a thick euclidean building Δ of type $\tilde{A}_n$; $\tilde{A}_2$ -groups have been studied for some years now, first from a combinatorial point of view [CMSZ93], then from the point of view of harmonic analysis [CMS93]; for $n \geq 3$ the existence of $\tilde{A}_n$ -groups has been established by D. Cartwright and T. Steger [CS]. Let $v_0 \in \Delta$ be a base-vertex; consider the length function $L(\gamma) = d_\Delta(\gamma v_0, v_0)$, where d_Δ is the combinatorial distance on the 1-skeleton of Δ . Swiatkowski's loop inequality (Theorem 0.6.(a) in [Swi]) is nothing but our combinatorial criterion, equivalent to the Haagerup inequality for $\text{Rad}_L(\Gamma)$. Moreover, the fact that Γ has exponential growth (proved in Proposition 1.9 of [Swi]) gives a direct, combinatorial proof of the non-amenability of Γ .

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2. Property (RD) for a subspace of $\mathbf{C}\Gamma$.

We shall consider two involutions on $\mathbf{C}\Gamma$:

$$f \rightarrow f^* \text{ where } f^*(\gamma) = \overline{f(\gamma^{-1})};$$

$$f \rightarrow \check{f} \text{ where } \check{f}(\gamma) = f(\gamma^{-1}).$$

We say that f is *self-adjoint* if $f = f^*$, and *symmetric* if $f = \check{f}$. A linear subspace E of $\mathbf{C}\Gamma$ is a **-subspace* if $E^* = E$.

PROPOSITION 1. — *For a *-subspace E of $\mathbf{C}\Gamma$, the following conditions are equivalent:*

- (i) *E satisfies the Haagerup inequality;*
- (ii) *there exists constants $C_1, s > 0$ such that for any self-adjoint $f \in E$, one has*

$$\|\lambda(f)\| \leq C_1 \|f\|_{s,L};$$

- (iii) *there exists constants $C_2, s > 0$ such that for any $k \in \mathbf{N}$ and any self-adjoint $f \in E$, one has*

$$f^{(2k)}(1) \leq C_2^{2k} \|f\|_{s,L}^{2k}.$$

Proof. — (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i) This follows easily from the fact that the involution $f \mapsto f^*$ on $\mathbf{C}\Gamma$ is an isometry both for the norm $\|\lambda(f)\|$ and $\|f\|_{s,L}$.

(ii) $\Rightarrow$ (iii) Notice that $g^* \star g(1) = \|g\|_2^2$ for any $g \in \mathbf{C}\Gamma$. Then, for a self-adjoint $f \in E$:

$$f^{(2k)}(1) = \|f^{(k)}\|_2^2 \leq \|\lambda(f^{(k)})\|^2 = \|\lambda(f)\|^{2k} \leq C_1^{2k} \|f\|_{s,L}^{2k}.$$

(iii) $\Rightarrow$ (ii) It follows from the spectral theorem (see e.g. [Kes59], lemma 2.2) that, for any self-adjoint $g \in \mathbf{C}\Gamma$:

$$\lim_{k \rightarrow \infty} \left(g^{(2k)}(1) \right)^{\frac{1}{2k}} = \|\lambda(g)\|.$$

This concludes the proof of Proposition 1.

PROPOSITION 2. — *Let E be a $*$ -subspace of $\mathbf{C}\Gamma$ which is stable under the map $f \rightarrow |f|$. The subspace E has property (RD) if and only if there exists constants $C, s > 0$ such that for any symmetric non-negative $f \in E$ and any $k \in \mathbf{N}$:*

$$f^{(2k)}(1) \leq C^{2k} \|f\|_{s,L}^{2k}.$$

Proof. — The direct implication follows from Proposition 1. For the converse, notice that for g a self-adjoint element in E , we have $|g^{(2k)}| \leq |g|^{(2k)}$ pointwise, and $|g|$ is non-negative and symmetric in E . Then

$$g^{(2k)}(1) = |g^{(2k)}(1)| \leq |g|^{(2k)}(1) \leq C^{2k} \| |g| \|_{s,L}^{2k} = C^{2k} \|g\|_{s,L}^{2k},$$

so that the result follows from (iii) $\Rightarrow$ (i) in Proposition 1.

We single out as a corollary what Proposition 2 says for $E = \mathbf{C}\Gamma$.

COROLLARY 1. — *Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any symmetric, finitely supported probability measure μ on Γ and any $k \in \mathbf{N}$:*

$$\mu^{(2k)}(1) \leq C^{2k} \|\mu\|_{s,L}^{2k}.$$

(By homogeneity, the condition in the corollary is clearly equivalent to the one in Proposition 2.) On purpose, we expressed the corollary by appealing to probability measures μ on Γ ; indeed, $\mu^{(2k)}(1)$ is just the probability of return to 1, in $2k$ steps, of the random walk on Γ with probability transitions $p(x, y) = \mu(y^{-1}x)$. There are numerous results on the decay of $\mu^{(2k)}(1)$ as $k \rightarrow \infty$; see especially Chapters VI and VII of [VSCC92] for the relation between decay of random walks and growth of the group.

For the rest of the paper, we assume that the length function L is integer-valued (this will be the case if L comes from a proper isometric action of Γ on a graph). We denote by χ_N the characteristic function of the sphere S_N .

PROPOSITION 3. — *Let E be a $*$ -subspace of $\mathbf{C}\Gamma$.*

(a) *If there exists constants $C, s > 0$ such that, for any $N, k \in \mathbf{N}$ and any self-adjoint $f \in E$:*

$$(*) \quad (f\chi_N)^{(2k)}(1) \leq C^{2k} \|f\chi_N\|_{s,L}^{2k}$$

(where $f\chi_N$ denotes the pointwise product), then E satisfies the Haagerup inequality.

(b) Assume moreover that E is stable under $f \rightarrow |f|$. Then E satisfies the Haagerup inequality provided (*) holds for any non-negative symmetric $f \in E$.

Proof. — (a) The following computation is inspired by the proof of Lemma 1.5 in [Haa79]. First, as in the proof of Proposition 1 above, we have for any self-adjoint $f \in E$ and any $N \in \mathbb{N}$:

$$\|\lambda(f\chi_N)\| \leq C\|f\chi_N\|_{s,L}.$$

But $f = \sum_{N=0}^{\infty} f\chi_N$, hence

$$\begin{aligned} \|\lambda(f)\| &\leq \sum_{N=0}^{\infty} \|\lambda(f\chi_N)\| \leq C \sum_{N=0}^{\infty} \|f\chi_N\|_{s,L} = C \sum_{N=0}^{\infty} \|f\chi_N\|_{s,L} (1+N)(1+N)^{-1} \\ &\leq C \left(\sum_{N=0}^{\infty} \|f\chi_N\|_{s,L}^2 (1+N)^2 \right)^{\frac{1}{2}} \left(\sum_{N=0}^{\infty} (1+N)^{-2} \right)^{\frac{1}{2}} \quad (\text{by Cauchy-Schwarz}) \\ &= C \sqrt{\frac{\pi^2}{6}} \left(\sum_{N=0}^{\infty} \|f\chi_N\|_{s+1,L}^2 \right)^{\frac{1}{2}} = C \sqrt{\frac{\pi^2}{6}} \|f\|_{s+1,L}. \end{aligned}$$

One concludes as in the proof of Proposition 1, (ii) $\Rightarrow$ (i).

(b) This follows immediately from (a) and the proof of Proposition 2.

Taking $E = \mathbf{C}\Gamma$, one immediately sees that Corollary 1 may be improved:

COROLLARY 2. — *The group Γ has property (RD) if and only if there exists constants $C, s > 0$ such that, for any $k, N \in \mathbb{N}$ and any symmetric probability measure μ supported in S_N :*

$$\mu^{(2k)}(1) \leq C^{2k} \|\mu\|_{s,L}^{2k}.$$

3. Radial functions.

We restrict attention to the subspace $E = \text{Rad}_L(\Gamma)$ of radial functions in $\mathbf{CF}$; note that this is exactly the linear span of the χ_N 's. It turns out that property (RD) for $\text{Rad}_L(\Gamma)$ has a purely combinatorial meaning.

PROPOSITION 4. — $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality if and only if there exists constants $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\text{card}\{\text{strict } N\text{-loops with length } 2k \text{ in } \Gamma\} \leq C^{2k}(1+N)^{2ks}(\text{card } S_N)^k.$$

Proof. — It follows from Propositions 2 and 3(b) that $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality if and only if there exists $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\chi_N^{(2k)}(1) \leq C^{2k} \|\chi_N\|_{s,L}^{2k}.$$

Now

$$\|\chi_N\|_{s,L} = \sqrt{\sum_{\gamma: L(\gamma)=N} (1+L(\gamma))^{2s}} = (1+N)^s (\text{card } S_N)^{\frac{1}{2}}$$

and

$$\chi_N^{(2k)}(1) = \sum_{(s_1, s_2, \dots, s_{2k}): s_1 s_2 \dots s_{2k} = 1} \chi_N(s_1) \chi_N(s_2) \dots \chi_N(s_{2k}).$$

With $v_0 = 1 = v_{2k}$ and $v_{i-1}^{-1} v_i = s_i$ for $i = 1, \dots, 2k$, this yields:

$$\begin{aligned} \chi_N^{(2k)}(1) &= \sum_{(v_0, v_1, \dots, v_{2k}): v_0 = v_{2k} = 1} \chi_N(v_0^{-1} v_1) \chi_N(v_1^{-1} v_2) \dots \chi_N(v_{2k-1}^{-1} v_{2k}) \\ &= \text{card}\{\text{strict } N\text{-loops of length } 2k\} \end{aligned}$$

since $(v_0, v_1, \dots, v_{2k})$ contributes a non-zero term to the summation if and only if $L(v_0^{-1} v_1) = L(v_1^{-1} v_2) = \dots = L(v_{2k-1}^{-1} v_{2k}) = N$. This concludes the proof.

An N -loop with length $2k$ in Γ is a sequence $(v_0 = 1, v_1, \dots, v_{2k-1}, v_{2k} = 1)$ such that $L(v_{i-1}^{-1} v_i) \leq n$ for $i = 1, \dots, 2k$. Consider also the ball with radius N in Γ :

$$B_N = \{\gamma \in \Gamma : L(\gamma) \leq N\}.$$

LEMMA 1. — Assume that $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality. Then there exists constants $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\text{card}\{N - \text{loops with length } 2k \text{ in } \Gamma\} \leq C^{2k}(1+N)^{2ks}(\text{card } B_N)^k.$$

Proof. — Denote by η_N the characteristic function of B_N . Since $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality, we find by Proposition 1 constants $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\eta_N^{(2k)}(1) \leq C^{2k} \|\eta_N\|_{s,L}^{2k}.$$

But

$$\|\eta_N\|_{s,L}^{2k} = \left(\sum_{\gamma \in B_N} (1 + L(\gamma))^{2s} \right)^k \leq (1+N)^{2ks} (\text{card } B_N)^k.$$

On the other hand, the same calculation as in the proof of Proposition 4 yields:

$$\eta_N^{(2k)}(1) = \text{card}\{N - \text{loops with length } 2k \text{ in } \Gamma\}.$$

This concludes the proof of Lemma 1.

Suppose now that Γ is a finitely generated group, and that L is a word length function with respect to some finite, symmetric, generating subset. Lemma 1 exhibits a link between the Haagerup inequality and growth properties of Γ , i.e. the behaviour of the growth function $N \rightarrow \text{card } B_N$. It turns out that amenability also plays a subtle role, as the following two propositions illustrate.

PROPOSITION 5. — Suppose that Γ is not amenable. The following statements are equivalent:

- (i) $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality;
- (ii) There exists constants $C, s > 0$ such that, for any $k, N \in \mathbf{N}$:

$$\text{card}\{N - \text{loops with length } 2k \text{ in } \Gamma\} \leq C^{2k}(1+N)^{2ks}(\text{card } B_N)^k.$$

Proof. — (i) $\Rightarrow$ (ii) This is just Lemma 1 (which does not depend on amenability).

(ii) $\Rightarrow$ (i) We assume that (ii) holds. Since Γ is non-amenable, by Folner's property there exists $\epsilon > 0$ such that $\text{card } S_N \geq \epsilon \cdot \text{card } B_N$ for any $N \in \mathbf{N}$. Then, for $k, N \in \mathbf{N}$:

$$\begin{aligned} \text{card}\{\text{strict } N - \text{loops of length } 2k\} &\leq \text{card}\{N - \text{loops of length } 2k\} \\ &\leq C^{2k}(1+N)^{2ks}(\text{card } B_N)^k \\ &\leq \left(\frac{C}{\sqrt{\epsilon}}\right)^{2k} (1+N)^{2ks}(\text{card } S_N)^k. \end{aligned}$$

It follows from Proposition 4 that $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality.

The following proposition extends Jolissaint's result that an amenable group with property (RD) (with respect to a word length function) necessarily has polynomial growth; see Corollary 3.1.8 in [Jol90]. Following [VSCC92], we say that a finitely generated group is *superpolynomial* if its growth function grows faster than any polynomial.

PROPOSITION 6. — *Assume that, for some word length function L , the space $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality. The following are then equivalent:*

- (i) Γ is not amenable;
- (ii) Γ has exponential growth;
- (iii) Γ is superpolynomial.

Proof. — (i) $\Rightarrow$ (ii) It is a general fact that any non-amenable group has exponential growth.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Assume that Γ is superpolynomial. Let $C, s > 0$ be such that $\|\lambda(f)\| \leq C\|f\|_{s,L}$ for any $f \in \text{Rad}_L(\Gamma)$. Take $f = \frac{\eta_N}{\text{card } B_N}$, the uniform probability measure on B_N . Then:

$$\left\| \lambda\left(\frac{\eta_N}{\text{card } B_N}\right) \right\| \leq \frac{C}{\text{card } B_N} \sqrt{\sum_{\gamma \in B_N} (1 + L(\gamma))^{2s}} \leq \frac{C(1+N)^s}{(\text{card } B_N)^{\frac{1}{2}}}.$$

Since Γ is superpolynomial, we have $\left\| \lambda\left(\frac{\eta_N}{\text{card } B_N}\right) \right\| < 1$ for N big enough. By Kesten's well-known characterization of amenability [Kes59], the group Γ has to be non-amenable.

It is often useful to have criteria for non-amenability that do not depend on the presence inside the group of a free group on two generators. Proposition 6 provides such a criterion. It will be used in the next section to deduce that $\tilde{A}_n$ -groups are non-amenable. It would be interesting to use this criterion to prove non-amenability for other finitely generated groups.

4. From Jolissaint to Tits: groups acting on buildings.

Here we make the connection with the companion paper by J. Swiatkowski [Swi]. It is noticed in [CMSZ93] that an irreducible euclidean building with a vertex-transitive group of automorphisms is necessarily of type $\tilde{A}_n$. So let Δ be a locally finite, thick euclidean building of type $\tilde{A}_n$. Following Definition 0.1.2 in [Swi], we say that Δ is *uniformly thick* if there exists $q \in \mathbb{N}$ such that any codimension 1 face in Δ is contained in $q + 1$ chambers. We thank the referee for suggesting the next lemma, that improves a previous version.

LEMMA 2. — *For $n \geq 2$, a thick building of type $\tilde{A}_n$ is uniformly thick.*

Proof. — For $n \geq 3$, this follows from Tits' result [Tit86] that a thick building of type $\tilde{A}_n$ is "classical", i.e. comes from a (not necessarily commutative) field K endowed with a discrete valuation v (see §2 of Chapter 9 in [Ron89] for a construction of the building $\tilde{A}_n(K, v)$). For $n = 2$, the lemma follows from the fact that the link of a vertex in an $\tilde{A}_2$ -building is a generalized 3-gon (see §2 in Chapter 3 in [Ron89]), and all vertices in a thick generalized 3-gon have the same valency (Proposition (3.3) in [Ron89]).

Of course this lemma does not hold for $n = 1$, since an $\tilde{A}_1$ -building is just a tree. We shall use the fact that the lemma is (trivially!) true if this tree admits a vertex-transitive group of automorphisms.

Let Γ be an $\tilde{A}_n$ -group, i.e. a group acting simply transitively on the vertices of a thick $\tilde{A}_n$ -building Δ (examples of such groups appear in [CMSZ93], [CS]). Fix a base-vertex $v_o \in \Delta$; let S be the set of elements $\gamma \in \Gamma$ such that γv_o is a neighbour of v_o in the 1-skeleton $\Delta^{(1)}$ of Δ . Then S is a finite, symmetric, generating subset for Γ , and the Cayley

graph of Γ with respect to S identifies with $\Delta^{(1)}$. Consider the length function $L(\gamma) = d_\Delta(\gamma v_o, v_o)$, where d_Δ is the combinatorial distance in $\Delta^{(1)}$; alternatively, L is the word length function with respect to S . From Swiatkowski's loop inequality (Theorem 0.6.(a) in [Swi]) together with our Proposition 4, we immediately get:

THEOREM 1. — *Let Γ be an $\tilde{A}_n$ -group, with L as above. Then the space $\text{Rad}_L(\Gamma)$ satisfies the Haagerup inequality.*

COROLLARY 3. — *An $\tilde{A}_n$ -group is non-amenable.*

Proof. — From Claims 1 and 2 in the proof of Proposition 1.9 in [Swi], it follows that an $\tilde{A}_n$ -group has exponential growth. Then combine Proposition 6 with Theorem 1.

Of course this corollary is known, and we indicate two other possible proofs.

First, for $n \geq 2$, one may prove the stronger statement that an $\tilde{A}_n$ -group Γ has Kazhdan's property (T). For $n = 2$, this is done in [CMS93] when Γ acts in a type-rotating way and the building is locally Desarguesian (these assumptions were dropped in [Pan] and [Zuk96]); for $n \geq 3$, first use Tits' result [Tit86] (see also p.137 in [Ron89]) that a euclidean building with dimension at least 3 is "classical", i.e. comes from some simple algebraic group G with F -rank at least 2, defined over some non-archimedean local field F . So Γ is essentially a lattice in G , and one may prove as in [dlHV89] that G and Γ have property (T).

Alternatively, one may construct free subgroups inside Γ . For $n = 1$, this is a simple exercise. For $n = 2$, this is a recent result of W. Ballmann and M. Brin (Theorem E in [BB]). For $n \geq 3$, using the fact that Γ is essentially a lattice in G , one may appeal to the celebrated Tits alternative [Tit72].

As a final remark, we mention that Swiatkowski proves in Proposition 1.9 of [Swi] that, for any uniformly thick building of type $\tilde{A}_n$, one has

$$\text{card } B_N(v_o) \leq C(1 + N)^{\dim \Delta} \text{card } S_N(v_o)$$

for any $N \in \mathbb{N}$. If Γ acts simply transitively on the vertices of Δ , then Γ is non-amenable (by Corollary 3), so that the above inequality may be improved to the strong isoperimetric inequality

$$\text{card } B_N(v_o) \leq C' \text{card } S_N(v_o)$$

for any $N \in \mathbf{N}$. Actually the latter inequality holds for any thick building Δ that admits a discrete group Γ acting properly co-compactly. Indeed, such a Γ is non-amenable (Theorem F in [BB]), so that $\text{card } B_N \leq K \text{ card } S_N$ by Følner's property. But the assumptions are such that Γ is quasi-isometric to Δ ; and it is known that satisfying a strong isoperimetric inequality is a quasi-isometry invariant among graphs (see Proposition 4.1 in [Pit] for a recent proof of this fact). If Δ comes from a simple algebraic group G defined over some non-archimedean local field of characteristic zero, then such co-compact lattices do exist [BH78].

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