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SL_2-EQUIVARIANT POLYNOMIAL AUTOMORPHISMS OF THE BINARY FORMS

by Alexandre KURTH

1. Introduction.

Let V be a finite dimensional vector space and G an algebraic group acting linearly on V where everything is defined over the field of complex numbers $\mathbb{C}$. We ask the following question:

Problem. — What is $\mathrm{Aut}_G(V)$, the group of bijective polynomial maps $V \to V$ that commute with the G-action?

The elements of $\mathrm{Aut}_G(V)$ are called G-automorphisms. Let $R_n := \mathbb{C}[x,y]_n$ denote the simple SL_2-module of the binary forms of degree $n \geq 0$. The sequence $\{R_n\}_{n\geq 0}$ is a complete system of representatives of all simple rational SL_2-modules. Instead of the complex numbers $\mathbb{C}$ we could take any arbitrary algebraically closed field of characteristic 0.

We would like to determine $\mathrm{Aut}_{\mathrm{SL}_2}(R_n)$. For $1 \leq n \leq 4$ we have a complete result whose elementary proof is based on the form of the generic stabilizer.

PROPOSITION (2.1). — *For $1 \leq n \leq 4$ every automorphism $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ is a scalar multiple of the identity* id_{R_n}.

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In general, this is only possible if we make some extra assumption: Let $f = \sum_{j=0}^{n} a_j x^{n-j} y^j \in R_n$ and suppose $a_0 \neq 0$. Let $z_1, \dots, z_n$ be the roots of f after setting $y = 1$. Then $\Delta(f) := a_0^{2n-2} \prod_{i<j} (z_i - z_j)^2$ is a polynomial SL_2-invariant function on R_n called the discriminant.

Now we can state the main result of this work:

THEOREM (6.1). — *Every automorphism $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ with $\varphi^*(\Delta) \in (\Delta)$ is a scalar multiple of the identity id_{R_n}.*

Notice that φ^* denotes the induced algebra automorphism on the algebra of functions $\mathbb{C}[R_n]$. This result generalizes the famous Lemma of Schur stating that all linear SL_2-automorphisms are multiples of the identity. As a consequence we obtain:

COROLLARY. — *If $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ has the property that the restriction of φ^* to the ring of invariant polynomials $\mathbb{C}[R_n]^{\mathrm{SL}_2}$ is a multiple of* id, *then so is φ.*

One aspect of the problem above is the relation to a rationality question of linearizing G-actions. First of all let us recall the linearization problem: When does an algebraic action of a reductive group on the affine space $\mathbb{A}^n$ become linear after a suitable (polynomial) change of coordinates? We refer to [7] for a survey of this question and related ones. Now let us give an account of a Galois criterion for rationality of the linearization problem.

Let k be a field of characteristic 0 and $K := \overline{k}$ its algebraic closure. Suppose we are given a G_k-action on the affine space $\mathbb{A}_k^n$ (defined over k), which is linearizable as affine G_K-space $\mathbb{A}_K^n$. The question arises whether the action is still linearizable over the smaller field k. It can be shown ([13], X; [12], III.1; [9]) that the set of isomorphism classes of G_k-actions on $\mathbb{A}_k^n$ that are G_K-isomorphic to a G_K-module V_K (notice that these isomorphisms are polynomial) is described by the non-abelian cohomology $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{Aut}_{G_K}(V_K))$ on the Galois group $\mathrm{Gal}(K/k)$ with values in $\mathrm{Aut}_{G_K}(V_K)$.

If for example $\mathrm{Aut}_{G_K}(V_K) = K^* \,\mathrm{id}_{V_K}$, then

$$\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{Aut}_{G_K}(V_K)) = 0$$

which shows that the G_k-action on $\mathbb{A}_k^n$ is also linearizable over the subfield k. If we assume that G_K is reductive and the quotient $\mathbb{A}_K^n /\!\!/ G_K$ is trivial, then the G_K-action on $\mathbb{A}_K^n$ as well as the G_k-action on $\mathbb{A}_k^n$ are linearizable ([9], 4.1.3). The linearization over K is a consequence of Luna's Slice Theorem [11].

Outline and organization. — For $n \geq 1$ let us consider the SL_2-equivariant morphism

$$\pi : (\mathbb{C}^2)^n \longrightarrow R_n, \qquad \pi\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix}\right) = \prod_{i=1}^{n} (b_i x - a_i y).$$

Let T_{n-1} denote the maximal torus in SL_n consisting of diagonal matrices; then we define the semidirect product $N := S_n \ltimes T_{n-1} \subset \mathrm{GL}_n(\mathbb{C})$ where S_n denotes the symmetric group on n letters. S_n acts on T_{n-1} by permuting the elements on the diagonal. The linear $\mathrm{SL}_2 \times N$-action on $(\mathbb{C}^2)^n$ is defined by:

$$(g, \sigma, \mathrm{diag}(t_1, \dots, t_n)) \cdot (v_1, \dots, v_n) = (t_1 g v_{\sigma(1)}, \dots, t_n g v_{\sigma(n)})$$

where $(g, \sigma, \mathrm{diag}(t_1, \dots, t_n)) \in \mathrm{SL}_2 \times (S_n \ltimes T_{n-1})$ and $(v_1, \dots, v_n) \in (\mathbb{C}^2)^n$. The morphism π is the quotient by the reductive group N which decomposes as follows:

$$(\mathbb{C}^2)^n \xrightarrow{Q_{T_{n-1}}} Y := (\mathbb{C}^2)^n /\!\!/ T_{n-1} \xrightarrow{Q_{S_n}} R_n$$

where Q_G denotes the algebraic quotient by the reductive group G. Since $T_{n-1} \subset N$ is a normal subgroup, S_n is acting on the T_{n-1}-quotient Y.

In §3 we prove that $(\mathbb{C}^2)^n$ does not allow any nonlinear $\mathrm{SL}_2 \times N$-automorphisms. In §4 (resp.§5) we show how to lift automorphisms over the quotient Q_{S_n} resp. $Q_{T_{n-1}}$; and finally in §6 we put these facts together to prove Theorem 6.1.

Remark 1.1. — It remains open whether $\mathrm{Aut}_{\mathrm{SL}_2}(R_n) = \mathbb{C}^* \,\mathrm{id}_{R_n}$. For $1 \leq n \leq 4$ this is true (see 2.1). Even in general it is a problem to find nonlinear equivariant automorphisms of simple modules. In a subsequent article I will describe nonlinear automorphisms of a simple module (with an open orbit) which is the first example of this kind.

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2. Equivariant automorphisms of R_n for $1 \leq n \leq 4$.

For $1 \leq n \leq 4$ Theorem 6.1 even works without the assumption involving the discriminant.

PROPOSITION 2.1. — *For $1 \leq n \leq 4$ every automorphism $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ is a scalar multiple of the identity id_{R_n}.*

Proof. — For $n = 4$ the generic isotropy group is the binary dihedral group $H = \langle \left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right), \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \rangle$. By equivariance every $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_4)$ restricts to an $\mathrm{Nor}_{\mathrm{SL}_2}(H)$-automorphism $\overline{\varphi}$ on $R_4^H \cong \mathbb{C}^2$. The finite group $\mathrm{Nor}_{\mathrm{SL}_2}(H)$ acts on R_4^H as a reflection group, so by equivariance $\overline{\varphi}$ stabilizes two lines. An automorphism of $\mathbb{C}^n$ stabilizing hyperplanes in general position (i.e., whose intersection is $\{0\}$) is linear. In fact, the induced automorphism on $\mathbb{C}[\mathbb{C}^n]$ maps each defining linear function of those hyperplanes to a multiple ([9], 7.2.1). It follows that for every $\lambda \in \mathbb{C}$ the relation $\lambda\,\mathrm{id} \circ \varphi - \varphi \circ \lambda\,\mathrm{id} = 0$ holds on $\mathrm{SL}_2 \cdot R_4^H$, even on R_4 by denseness, for H is the generic stabilizer. This means φ induces an automorphism on the projective space $\mathbb{P}R_4$ which is linear ([2], II. Example 7.1.1). Schur's Lemma yields $\mathrm{Aut}_{\mathrm{SL}_2}(R_4) = \mathbb{C}^* \,\mathrm{id}_{R_4}$.

The proofs for $n = 1, 2, 3$ are similar: For $n = 3$ the generic stabilizer is $H = \{ \left(\begin{smallmatrix} \zeta & \\ & \zeta^{-1} \end{smallmatrix}\right) \mid \zeta^3 = 1 \} \cong \mathbb{Z}_3$ and $R_3^H = \mathbb{C}x^3 \oplus \mathbb{C}y^3$. Since $\mathbb{C}x^3$ and $\mathbb{C}y^3$ are each fixed by a maximal unipotent subgroup, and thus stabilized by an equivariant automorphism, we can now make the same conclusion as for $n = 4$.

For $n = 1$ (resp. $n = 2$) the generic stabilizer H is $\{ \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \}$ (resp. a maximal torus). In both cases R_n^H is one-dimensional and so the restricted automorphism on R_n^H is linear. Again, we follow the same arguments as for $n = 4$. □

R_0 is the trivial SL_2-module; its automorphism group consists of linear elements, but translations are also allowed.

3. Equivariant automorphisms of $(\mathbb{C}^2)^n$.

By classical invariant theory we know that (cf. [15], Theorem 2.6.A):

$$\mathbb{C}[(\mathbb{C}^2)^n]^{\mathrm{SL}_2} = \mathbb{C}[\,[i, j] \mid 1 \leq i < j \leq n] \tag{1}$$

(2) $\mathbb{C}[(\mathbb{C}^2)^n \oplus (\mathbb{C}^2)^*]^{\mathrm{SL}_2} = \mathbb{C}[\,[i,j], \varepsilon_l \mid 1 \le i < j \le n,\ 1 \le l \le n]$

where $[i,j](v_1,\dots,v_n){:=}\det(v_i,v_j)$ and $\varepsilon_l(v_1,\dots,v_n,f){:=}f(v_l)$. As introduced above let $N{:=}S_n \ltimes T_{n-1}$. Every automorphism $\varphi\in \mathrm{Aut}_{\mathrm{SL}_2 \times N}((\mathbb{C}^2)^n)$ can be seen as an n-tuple $(\varphi_1,\dots,\varphi_n)$ of SL_2-covariants (of type $\mathbb{C}^2$) $\varphi_s : (\mathbb{C}^2)^n \longrightarrow \mathbb{C}^2$, $s = 1,\dots,n$. By determining the restitution of the multilinear invariants of (2) it follows that

$$\varphi_s(v_1,\dots,v_n) = \sum_{r=1}^{n} p_{rs}v_r, \qquad s = 1,\dots,n \tag{3}$$

where $p_{rs} \in \mathbb{C}[(\mathbb{C}^2)^n]^{\mathrm{SL}_2}$ (cf. [6], §6). In the following proposition we show that all p_{rs} are constant polynomials.

PROPOSITION 3.1. — *Every* $\mathrm{SL}_2 \times N$*-automorphism of* $(\mathbb{C}^2)^n$ *is a scalar multiple of the identity.*

Proof. — We keep the notations from above. Taking the SL_2-quotient of $(\mathbb{C}^2)^n$, an automorphism $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2 \times N}((\mathbb{C}^2)^n)$ induces an N-automorphism $\overline{\varphi} \in \mathrm{Aut}_N((\mathbb{C}^2)^n /\!\!/ \mathrm{SL}_2)$. For $k \le n$ the quotient $(\mathbb{C}^k)^n /\!\!/ \mathrm{SL}_k$ is the affine cone over the Grassmannian $G_{k,n}$. The ring of algebraic functions of $R := (\mathbb{C}^2)^n /\!\!/ \mathrm{SL}_2$ is described by equation (1). We claim that every $\phi \in \mathrm{Aut}_N(R)$ is a multiple of $\mathbb{C}^* \,\mathrm{id}_R$.

The relations among the generating invariants $[i,j]$ involving $[1,2]$ are given by the Plücker relations:

(4) $[1,2][i,j] - [1,i][2,j] + [1,j][2,i] = 0$ for all $3 \le i < j \le n$.

Consider the subgroup $H := \{\mathrm{diag}(t,t^{-1},1,\dots,1) \in T_{n-1} \mid t \in \mathbb{C}^*\} \subset N$ acting on R. By using the Plücker relations we have:

$$\begin{aligned}\mathbb{C}[R]^H &= \mathbb{C}[[1,2],[1,i][2,j],[i,j] \mid 3 \le i < j \le n] \\ &= \mathbb{C}[[1,2],[1,i][2,j] + [1,j][2,i],[i,j] \mid 3 \le i < j \le n].\end{aligned}$$

All elements of the latter system of generators are invariant under the transposition $\tau = (12)$ except $[1,2]$ which is mapped to $-[1,2]$. By the H-equivariance we have $\phi^*(\mathbb{C}[R]^H) \subset \mathbb{C}[R]^H$ for every $\phi \in \mathrm{Aut}_N(R)$; moreover, $\phi^*([1,2]) = [1,2]I$ using the τ-equivariance where $I \in \mathbb{C}[R]^H$. Since $[1,2]$ is irreducible and by the S_n-equivariance the claim $\phi \in \mathbb{C}^* \,\mathrm{id}_R$ follows; in particular $\overline{\varphi} \in \mathbb{C}^* \,\mathrm{id}_R$.

Let us define the $n\times n$-matrix $P := (p_{ij})$ with $p_{ij} \in \mathbb{C}[(\mathbb{C}^2)^n]^{\mathrm{SL}_2}$ from equation (3). Set $m := \binom{n}{2}$. We have just shown that the $m\times m$-matrix $\wedge^2 P$ consisting of all 2×2-minors of P is a scalar multiple of the identity matrix

E_m. Since the kernel of the canonical homomorphism $\mathrm{GL}(V) \to \mathrm{GL}(\wedge^2 V)$ is $\{\pm\,\mathrm{id}\}$ ($\dim V > 2$), it follows that $P \in \mathbb{C}^* E_n$, i.e., φ is a scalar multiple of $\mathrm{id}_{(\mathbb{C}^2)^n}$.

In case $n = 2$ let $n_t := (\mathrm{diag}(t, t^{-1}), \mathrm{id}, \mathrm{diag}(t, t^{-1})) \in \mathrm{SL}_2 \times N$ for $t \in \mathbb{C}^*$. By considering $\lim_{t \to 0} \varphi(n_t(v_1, v_2))$, we get that φ stabilizes a hyperplane. Thus $\varphi \in \mathbb{C}^*\,\mathrm{id}$ because $(\mathbb{C}^2)^2$ is a simple $\mathrm{SL}_2 \times N$-module. □

4. Lifting automorphisms to finite coverings.

Let us consider the factorization from the introduction

$$\pi : (\mathbb{C}^2)^n \xrightarrow{Q_{T_{n-1}}} Y = (\mathbb{C}^2)^n /\!\!/ T_{n-1} \xrightarrow{Q_{S_n}} R_n.$$

π is the quotient morphism by the group $N = S_n \ltimes T_{n-1} \subset \mathrm{GL}_n$. Let $D := \{f \in R_n \mid \Delta(f) = 0\}$ the zero locus of the discriminant $\Delta \in \mathbb{C}[R_n]$. The aim of this section is to prove the following result:

Proposition 4.1. — *Every $\mathbb{C}^*$-action on R_n which stabilizes D can be lifted to a unique $\mathbb{C}^*$-action on Y commuting with the S_n-action and satisfying*

$$Q_{S_n}(ty) = t^{n!} Q_{S_n}(y) \text{ for all } (t, y) \in \mathbb{C}^* \times Y.$$

If the given action on R_n commutes with SL_2, then so does the lifted one.

The proof requires some preparation. Let π_1 denote the homotopy functor which associates to an analytic variety X (endowed with the $\mathbb{C}$-topology) the fundamental group $\pi_1(X)$. For a continuous map $f : X \to Z$ between analytic varieties X, Z the induced group homomorphism on the fundamental groups is denoted by $f_\sharp : \pi_1(X) \to \pi_1(Z)$.

Lemma 4.2. — *Let H be a finite group of order m acting freely on an irreducible, smooth, affine variety X. Let $Q : X \to X/H$ denote its algebraic quotient. Every algebraic $\mathbb{C}^*$-action on X/H can be uniquely lifted to a $\mathbb{C}^*$-action on X satisfying $Q(t \cdot x) = t^m \cdot Q(x)$ for all $(t, x) \in \mathbb{C}^* \times X$. Moreover, the lifted action commutes with the H-action.*

Proof. — Let $\varrho : \mathbb{C}^* \times X/H \to X/H$ denote the given action. Define $p : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^m$ and consider the diagram

$$\begin{array}{ccc} \mathbb{C}^* \times X & & X \\ \big\downarrow{\scriptstyle p \times Q} & & \big\downarrow{\scriptstyle Q} \\ \mathbb{C}^* \times X/H & \xrightarrow{\varrho} & X/H. \end{array}$$

The quotient Q is an unramified covering (with respect to the $\mathbb{C}$-topology), so by a standard argument in covering/homotopy theory (cf. [14], 2.4. Theorem 5) there is a topological lift of ϱ if and only if

$$\varrho_\sharp \circ (p \times Q)_\sharp(\pi_1(\mathbb{C}^* \times X)) \subset Q_\sharp(\pi_1(X)).$$

Let $[\gamma] = [\gamma_1, \gamma_2] \in \pi_1(\mathbb{C}^* \times X) \cong \pi_1(\mathbb{C}^*) \times \pi_1(X)$ denote the class of a closed path γ in $\mathbb{C}^* \times X$. We claim that $\varrho_\sharp \circ (p \times Q)_\sharp[\gamma] \in Q_\sharp(\pi_1(X))$. In fact, let e denote the neutral element of the corresponding fundamental group, then

$$(p \times Q)_\sharp[\gamma] = ([\gamma_1]^m, Q_\sharp[\gamma_2]) = ([\gamma_1], e)^m \cdot (e, Q_\sharp[\gamma_2]). \tag{5}$$

Moreover, since the order of H is m, it follows from the short exact sequence

$$\begin{array}{ccccccccc} & & & & \pi_1(\mathbb{C}^* \times X/H) & & & & \\ & & & & \Big\downarrow{\scriptstyle \varrho_\sharp} & & & & \\ 1 & \longrightarrow & \pi_1(X) & \xrightarrow{Q_\sharp} & \pi_1(X/H) & \xrightarrow{\alpha} & H & \longrightarrow & 1 \end{array}$$

that $\varrho_\sharp([\gamma_1], e)^m \in \ker\alpha = Q_\sharp(\pi_1(X))$. Thus applying $\varrho_\sharp$ to (5) gives the claim.

There is a unique lift $\widetilde{\varrho}$ if we require $\widetilde{\varrho}(1, x_0) = x_0$. Hence $\widetilde{\varrho}(1, x) = x$ for all $x \in X$ which is the equivalent condition to $\widetilde{\varrho}$ being a (topological) action. Moreover, $\widetilde{\varrho}$ commutes with H. In fact, the conjugate action $(t, x) \mapsto h\,\widetilde{\varrho}(t, h^{-1}x)$ is also a lift of ϱ and therefore equal to $\widetilde{\varrho}$.

To show that $\widetilde{\varrho}$ is an algebraic morphism, assume we are given a variety Z and a morphism $\varphi : Z \to X/H$ which admits a topological lift $\widetilde{\varphi} : Z \to X$. The graph $\Gamma_{\widetilde{\varphi}}$ is algebraic since it is a connected component of the fiber product $Z \times_{X/H} X = \{(z, x) \in Z \times X \mid \varphi(z) = q(x)\}$. X is affine and smooth so it follows that the projection $p_Z : \Gamma_{\widetilde{\varphi}} \to Z$ is an isomorphism ([5], II.3.4 Lemma); thus $\widetilde{\varphi} = p_X \circ p_Z^{-1}$ is algebraic. □

LEMMA 4.3. — *Let $Q : X \to Y$ be a finite surjective morphism where X is an irreducible variety. Let $\phi : Z' \to X$ be a morphism where $Z' \subset Z$ is an open subset of a normal variety Z. If $Q \circ \phi$ has an extension to Z then so does ϕ.*

Proof. — ϕ induces a homomorphism on the field of functions $\phi^* : \mathbb{C}(X) \to \mathbb{C}(Z) = \mathbb{C}(Z')$. $\mathcal{O}(X)$ is a finite module over $Q^*(\mathcal{O}(Y))$ ([5], AI.4.3). By applying ϕ^* to an integral equation $p^m + \sum_{j=1}^{m} Q^*(a_j)p^{m-j} = 0$

for $p \in \mathcal{O}(X)$, we easily obtain $\phi^*(p) \in \mathcal{O}(Z)$ since Z is normal. This induces an extension $Z \to X$. □

We are now ready to prove the main result of this section.

Proof of Proposition 4.1. — Let us define $R'_n := R_n \setminus D$ and $Y' := Q_{S_n}^{-1}(R'_n) \subset Y$. The induced morphism $Q'_{S_n} : Y' \to R'_n$ is an unramified Galois covering with Galois group S_n. By Lemma 4.2 the $\mathbb{C}^*$-action on R_n uniquely lifts to a $\mathbb{C}^*$-action on Y' satisfying the required relation and commuting with S_n. The extension to an action $\mathbb{C}^* \times Y \to Y$ is a consequence of Lemma 4.3; for $Y = (\mathbb{C}^2)^n /\!\!/ T_{n-1}$ is normal ([5], II.3.3, Satz 1) and Q_{S_n} is a finite surjective morphism (cf. [5], II.3.6, Satz 1). The last statement follows from the uniqueness of the lift. □

Remark 4.4. — (1) It is possible to lift a single SL_2-automorphism of R_n stabilizing the discriminant divisor D to an $\mathrm{SL}_2 \times S_n$-automorphism of Y (see [9], 8.3). The proof is based on the structure of $\pi_1(R'_n/\mathbb{C}^*)$ which is the n-th braid group over the sphere $\mathbb{P}^1\mathbb{C}$ containing the subgroup of pure braids as characteristic subgroup [1].

The idea of lifting an SL_2-automorphism of R'_n by using the structure of the braid group was inspired by the work [10] of Lin. This was the starting point of our considerations.

(2) With similar arguments one can prove the following generalization of Proposition 2.1. We will not need this result for our further arguments, though.

Let X be an irreducible normal variety and $Q : X \to Y$ a finite surjective morphism. Assume that a connected algebraic group G acts on Y and stabilizes the ramification locus of Q. Then there is a finite covering $p : \widetilde{G} \to G$ and a $\widetilde{G}$-action on X lifting the G-action on Y.

5. Lifting automorphisms to principal torus bundles.

The main tool for lifting $\mathbb{C}^*$-actions from $Y = (\mathbb{C}^2)^n /\!\!/ T_{n-1}$ to $(\mathbb{C}^2)^n$ is based on lifting them to principal torus bundles.

DEFINITION 5.1. — *Let G be an algebraic group. A principal G-bundle over a variety X is a morphism $\pi_P : P \to X$ where P is a (right)*

G-variety such that the fibers are the G-orbits and where π_P is locally trivial in the étale topology.

Let F be a variety whose automorphism group $G := \mathrm{Aut}(F)$ is algebraic. There is a known equivalence of categories (cf. [8], IV.1.3)

$$\{\text{principal } G\text{-bundles over } X\} \xleftrightarrow{\sim} \{\text{fiber bundles over } X \text{ with fiber } F\}$$

which is given by the functor $P \mapsto P *^G F := (P \times F)/G$. The G-action on the product $P \times F$ is defined by $g(p, z) = (pg^{-1}, gz)$.

Now we can formulate the main result of this section in a slightly more general setting than needed:

PROPOSITION 5.2. — *Let T be a torus and $P \to X$ be a principal T-bundle over the normal variety X. Assume that a connected algebraic group G with $\mathrm{Pic}(G) = 0$ acts on X. Then the G-action on X can be lifted to a G-action on P commuting with T.*

Recall that the set of isomorphism classes of line bundles on X is denoted by $\mathrm{Pic}(X)$. It has a group structure given by the tensor product. We want to mention the application we are mostly interested in:

Remark 5.3. — The proposition is applicable to $G \cong \mathbb{C}^*$ ($\mathrm{Pic}(\mathbb{C}^*) = 0$, [2], II.6) as well as for a unipotent group G, for as a variety G is isomorphic to $\mathbb{C}^k$ for a suitable $k \in \mathbb{N}$, therefore $\mathrm{Pic}(G) = 0$.

For the proof of the proposition we need the notion of a G-linearization of a line bundle $L \to X$ which is a lifting of the G-action from X to L being linear on the fibers ([3], 2.1, Lemma). L is also called a G-line bundle.

Let $\varrho : G \times X \to X$ be a G-action. Let $p : L \to X$ be a line bundle (we sometimes write L if no confusion occurs) and consider the diagram:

$$\begin{array}{ccc} G \times L & & L \\ \downarrow{\scriptstyle \mathrm{id} \times p} & & \downarrow{\scriptstyle p} \\ G \times X & \xrightarrow{\varrho} & X. \end{array}$$

It is an elementary fact that L admits a G-linearization if and only if $\varrho^* L \cong G \times L = p_X^* L$ ([3], 2.3 Lemma) where $p_X : G \times X \to X$ is the projection on X.

The following lemma gives sufficient conditions for a line bundle being G-linearizable.

LEMMA 5.4. — *Let X be a normal G-variety where G is a connected algebraic group with* $\mathrm{Pic}(G) = 0$. *Then every line bundle $p : L \to X$ is G-linearizable. Moreover, the G-action on* $\mathrm{Pic}(X)$ *(induced by the G-action on X) is trivial.*

Proof. — The first statement follows from the exact sequence (see [4], 2.2, Lemma)

$$0 \to \mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*) \to \mathrm{Pic}_G(X) \to \mathrm{Pic}(X) \to \mathrm{Pic}(G) = 0$$

where $\mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*)$ is the group of algebraic cocycles and $\mathrm{Pic}_G(X)$ the set of isomorphism classes of G-line bundles on X. Using the notation introduced above we now have $g^*L{:=}\varrho^*L|_{\{g\}\times X} \cong p_X^*L|_{\{g\}\times X} \cong L$ by restricting to $\{g\} \times X$ for any $g \in G$. Therefore the class of L in $\mathrm{Pic}(X)$ is fixed by G. □

Remark 5.5. — If G is an arbitrary connected algebraic group, one can show that there is an $n \in \mathbb{N}$ such that $L^{\otimes n}$ is G-linearizable line bundle on X ([3], 2.4).

Assuming G is not connected, the statement is still valid if $g_j^*L \cong L$ for a decomposition $G = \bigcup_{j=1}^{s} g_j G^0$ (G^0 denotes the connected component of $\mathrm{id} \in G$).

Now we finish the preparations and prove Proposition 5.2.

Proof of Proposition 5.2. — Let us fix a decomposition $T = (\mathbb{C}^*)^d$ where $d = \dim T$. The functor $P \mapsto P{*^T}\mathbb{C}^d =: \bigoplus_{j=1}^{d} L_j$ defines an equivalence between principal T-bundles and direct sums of line bundles where $\mathbb{C}^d$ is the standard representation of T. Under this equivalence lifting the G-action from X to P commuting with T corresponds to G-linearizations of the line bundles L_j. Their existence was constructed in Lemma 5.4. □

The equivariance of the lifting with another action is the subject of the following lemma:

LEMMA 5.6. — *Let T be a torus and $\pi_P : P \to X$ a principal T-bundle. Let us be given algebraic groups G, H acting on P and both commuting with T. Assume that G is connected and every invertible function on X is constant. If the induced actions of G and H on X commute then so do their actions on P.*

Proof. — For $h \in H$ consider the conjugate action

$$\psi_h : G \times P \longrightarrow P, \quad (g,p) \mapsto hgh^{-1}(p).$$

By assumption all ψ_h induce the same action on X, i.e., it exists a morphism $\Omega : H \times G \times X \to T$ such that $\psi_h(g,p) = \Omega(h,g,\pi_P(p))g(p)$. Ω does not depend on X since all invertible functions on X are constant. Clearly, for every $g \in G$ the map $h \mapsto \Omega(h,g)$ is a group homomorphism $H \to T$. Since the set of group homomorphisms $H \to T$ is discrete and G is connected it follows $\Omega(h,g) = \Omega(h,1) = 1$ for all $h \in H$ which implies the claim. □

6. Proof of the main theorem.

We now come to the proof of Theorem 6.1. We keep the notations from the previous sections. First of all we recall the statement:

THEOREM 6.1. — *Every automorphism* $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ *with* $\varphi^*(\Delta) \in (\Delta)$ *is a scalar multiple of the identity* id_{R_n}.

Proof. — For $n = 1,2,3,4$ we have already shown that $\mathrm{Aut}_{\mathrm{SL}_2}(R_n) = \mathbb{C}^* \,\mathrm{id}_{R_n}$ (see 2.1).

Let $\varphi \in \mathrm{Aut}_{\mathrm{SL}_2}(R_n)$ stabilizing the discriminant divisor D and consider the $\mathbb{C}^*$-action on R_n obtained from the scalar multiplication by conjugating with φ:

$$\varrho : \mathbb{C}^* \times R_n \longrightarrow R_n, \quad (s,f) \mapsto sf := \varphi(s\varphi^{-1}(f)).$$

We want to lift ϱ over the N-quotient $\pi : (\mathbb{C}^2)^n \to R_n$ where N is the semidirect product $S_n \ltimes T_{n-1} \subset \mathrm{GL}_n(\mathbb{C})$. Recall that π has a factorization

$$\pi : (\mathbb{C}^2)^n \xrightarrow{Q_{T_{n-1}}} Y = (\mathbb{C}^2)^n /\!\!/ T_{n-1} \xrightarrow{Q_{S_n}} R_n$$

where $Q_{T_{n-1}}$, Q_{S_n} is the quotient by the torus T_{n-1}, by the symmetric group S_n, respectively (see section 1). By Proposition 4.1 ϱ lifts to a (unique) $\mathbb{C}^*$-action ψ on Y commuting with $\mathrm{SL}_2 \times S_n$ and such that $Q_{S_n}(sy) = s^{n!}Q_{S_n}(y)$ for all $(s,y) \in \mathbb{C}^* \times Y$. ψ stabilizes $Q_{T_{n-1}}(0)$ which is the only SL_2-fixed point. We define $V_i := \{(v_1,\dots,v_n) \in (\mathbb{C}^2)^n \mid v_i = 0\}$ for $i = 1,\dots,n$. Then

$$\mathcal{N} := Q_{T_{n-1}}^{-1}(Q_{T_{n-1}}(0)) = \bigcup_{i=1}^{n} V_i$$

is the nilcone of the T_{n-1}-module $(\mathbb{C}^2)^n$. Clearly, $(T_{n-1})_v = \{\mathrm{id}\}$ for every $v \in P := (\mathbb{C}^2)^n \setminus \mathcal{N}$. The T_{n-1}-quotient

$$Q_{T_{n-1}}|_P : P \longrightarrow Y \setminus \{Q_{T_{n-1}}(0)\}$$

is a principal T_{n-1}-bundle. In fact, the quotient by the diagonal $(\mathbb{C}^*)^n$-action on $P = (\mathbb{C}^2 \setminus \{0\})^n$ is the principal $(\mathbb{C}^*)^n$-bundle $P \to \mathbb{P}^1$. Then the associated bundle $P \to P/T_{n-1}$ for the subgroup $T_{n-1} \subset (\mathbb{C}^*)^n$ is a principal T_{n-1}-bundle.

By Proposition 5.2 there is an action $\Psi : \mathbb{C}^* \times P \to P$ which lifts ψ and commutes with T_{n-1}. Even more, Ψ commutes with $\mathrm{SL}_2 \times S_n$ using Lemma 5.6.

Since $\operatorname{codim} \mathcal{N} = 2$, it holds $\mathcal{O}((\mathbb{C}^2)^n) = \mathcal{O}(V')$ ([5], AI.6.1, Lemma 1), and therefore Ψ can be extended to a $\mathbb{C}^*$-action on $(\mathbb{C}^2)^n$ commuting with $\mathrm{SL}_2 \times N$. By Proposition 3.1 this action is given by scalar multiplication. It follows that the same holds for the $\mathbb{C}^*$-actions on Y and R_n implying $(\varphi \circ \mathbb{C}^* \mathrm{id}_{R_n} \circ \varphi^{-1})^{n!} = \mathbb{C}^* \mathrm{id}_{R_n}$. We obtain that $\varphi \circ \mathbb{C}^* \mathrm{id}_{R_n} \circ \varphi^{-1} = \mathbb{C}^* \mathrm{id}_{R_n}$. By taking differentials we see that $\varphi \circ \lambda \mathrm{id}_{R_n} = \lambda \mathrm{id}_{R_n} \circ \varphi$ for every $\lambda \in \mathbb{C}^*$. Therefore φ induces an automorphism on $\mathbb{P}R_n$ which has to be linear (cf. [2], II, Example 7.1.1). □

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