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INDEX AND DYNAMICS OF QUANTIZED CONTACT TRANSFORMATIONS

by Steven ZELDITCH(*)

1. INTRODUCTION

The problem of quantizing symplectic maps and of analyzing the dynamics of the quantum system is a very basic one in mathematical physics, and has been studied extensively, by both mathematicians and physicists, from many different points of view. The present article is concerned with one such quantization method, that of *Toeplitz quantization*, and with the semiclassical viewpoint towards the ergodicity and mixing properties of the quantized maps, as examples of *quantized Gelfand-Naimark-Segal systems* in the sense of [Z1]. We will describe a method of quantizing contact transformations of a contact manifold (X, α) with periodic contact flow as unitary operators on an associated Hardy space $H^2(X)$, and prove a number of results on the index and dynamics of the quantized contact transformations. The method, essentially a unitarized version of Boutet de Monvel's Toeplitz quantization [B] [BG], is closely related to the geometric quantization of symplectic maps on Kähler manifolds and produces new examples of quantized GNS systems. The quantum ergodicity theorems follow in part from the general results of [Z1], but also include some sharper ergodicity and mixing theorems analogous to those of [Z2], [Z3] in the case of wave groups.

To illustrate the method and ergodicity results we will also study in detail the Toeplitz quantization of symplectic torus automorphisms ('cat maps') (§5), undoubtedly the most popular of maps to undergo

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quantization– see [Ad'PW] [BNS] [HB] [BdB] [d'EGI] [KP] [K] [Ke] [We] for just a few among the many treatments. As the reader is surely aware, *quantization* is not a uniquely defined process and it is not a priori clear how the plethora of quantizations defined in these articles are related to each other or to the quantization presented here. In fact, although it is not quite obvious, all but that of [BNS] are equivalent to the Toeplitz quantization studied here. We will describe the relations between them more precisely at the end of the introduction.

What is more, it will be proved in §5 that the Toeplitz quantization of $SL(2,\mathbb{Z})$ reproduces what must be the quantization of most ancient vintage—namely, the Hermite-Jacobi action of $SL(2,\mathbb{Z})$ (or more precisely its theta-subgroup $SL_\theta(2,\mathbb{Z})$) on spaces Θ_N of theta functions of any degree N (cf. [Herm], [Kloo] or, for a modern treatment, [K] [KP]). We construct this action by lifting $g \in SL_\theta(2,\mathbb{Z})$ to a contact transformation χ_g of $N_{\mathbb{R}}/N_{\mathbb{Z}}$, the quotient of the Heisenberg group by its integral subgroup, and compressing the latter to the spaces Θ_N. This connection develops the long chain of links between theta functions and harmonic analysis on the Heisenberg group (e.g. [A] [M]), and is perhaps of independent interest. For one thing, it gives a framework for analysing asymptotic properties of theta functions in the semi-classical (large N) limit. It may also be used, together with the explicit formula of the Cauchy-Szegö kernel on the Heisenberg group, to give a Selberg-type trace formula for the trace of an element $g \in SL_\theta(2,\mathbb{Z})$ acting on the space of theta functions of degree N (§6).

It should also be mentioned that the quantization of $SL_\theta(2,\mathbb{Z})$ as unitary operators $U_{g,N}$ on Θ_N is just a concrete realization of the metaplectic representation of the finite metaplectic group $Mp(2,\mathbb{Z}/N)$. That is, the Toeplitz-quantization of an element $g \in SL_\theta(2,\mathbb{Z})$ is equivalent to reducing it mod N, and then applying the metaplectic representation μ_N of $SL(2,\mathbb{Z}/N)$. Hence the trace formula alluded to above is giving the characters of the finite metaplectic representations. We further mention that when $N = p^k$ is a power of a prime, $U_{g,N}$ may be described in terms of the metaplectic representation over the field of p-adic numbers, indeed as the quantization of g viewed as a symplectic map on the p-adic torus. We will not develop this point of view here, but we hope it may help clarify the number-theoretic aspects of the spectral theory of $U_{g,N}$ (cf. [Ke] [d'EGI]).

Although our main aim in this article is to discuss the quantum dynamics of Toeplitz-quantized maps, we would also like to mention an interesting index problem associated to them. Namely, the Toeplitz-quantization of a symplectic or contact map χ will be an operator U_χ which is unitary modulo finite rank operators. It therefore has an index, which depends only on χ and on the principal symbol of U_χ. The problem of calculating this index $\text{ind}(\chi)$ was raised in [Wei] in the closely related context of Fourier Integral operators but it does not seem to have been calculated before in any example. Hence it may be of interest to observe that the index $\text{ind}(\chi_g)$ of $g \in SL(2,\mathbb{Z})$ is always zero, as follows the unitarity of the Hermite-Jacobi 'transformation laws'. This vanishing of the index has a very simple alternative explanation, so it is not clear how generally to expect the index to vanish (see the Remarks at the end of §5).

This article will presume a degree of familiarity with the machinery of Toeplitz operators as presented in the book of Boutet de Monvel and Guillemin [BG]. This machinery involves some language and ideas from symplectic geometry, microlocal analysis, several complex variables, CR functions and from the representation theory of the Heisenberg and metaplectic groups. We hope that the explicit calculations of symbols, quantizations, traces and so on in the case of the symplectic torus automorphisms will provide elementary examples of how this machinery works, in a form accessible to those studying quantum maps from other points of view. In an obvious sense, which should be clearer by the end of §5-6, the cat maps are among the basic linear models for the general theory.

We will also assume some familiarity with quantum dynamics, especially from the semi-classical viewpoint. This is actually a rather broad assumption, since there are many different approaches to quantum dynamics. With the aim of clarifying the relation between our set-up, methods and results with those of other articles on the dynamics of quantum maps, we end this introduction with a rapid comparison to the works cited above.

0.1. Comparison to prior articles.

First, let us compare quantization methods. Besides the Toeplitz method of quantizing a symplectic map on a compact symplectic manifold, which requires the map to lift as a contact map of the 'prequantum circle bundle', the only general method is that of geometric quantization. Traditionally, this is a method only of quantizing symplectic manifolds and observables; but in the last few years it has been extended to include a variety of symplectic maps. In particular, motivated by the needs of topological field theory, there are many articles using the method of Kähler quantization to quantize elements of $SL(2,\mathbb{Z})$. By Kähler quantization we mean geometric quantization on a Kähler manifold in the presence of a complex polarization. This is the method used in [AdPW] [We], among many other places, and discussed in the book of Atiyah [At]. As in the Toeplitz construction, the symplectic torus automorphisms are quantized as translation operators on theta functions. However, such translations change the complex structure and so do not preserve a fixed space of holomorphic theta functions. In the language of geometric quantization, one has to define a BKS (Blattner-Kostant-Sternberg) pairing between the different complex polarizations to return to the original space. It is at this point that the Toeplitz and Kähler methods differ: In the Toeplitz method, one uses orthogonal projection back to the original space (times a normalizing factor) while in the Kähler method, one uses a parallel translation along the moduli space of complex structures on the torus. In the case of torus automorphisms, both methods produce the classical transformation laws for theta functions (as was pointed out by Weitsman in the Kähler case (loc.cit.)). Hence the Toeplitz and Kähler methods are equivalent in this case. They are surely equivalent in much greater generality, but to the author's knowledge this has never been studied systematically.

The other quantizations of the cat maps [HB] [Kea] [dEGI] [dBB] [BNS] are based (implicitly or explicitly) on the special representation theory of the Heisenberg and metaplectic groups. This is also true in the many physics articles on other quantum maps such as kicked rotors and tops. It is the author's impression that the methods of geometric and Kähler quantization are seldom used in the semi-classical physics literature, wherein quantization seems to be equated with canonical quantization (i.e. with representation theory of the Heisenberg group). It may therefore be useful to point out that the Toeplitz method gives equivalent quantizations to 'Weyl' or 'canonical quantization', not only for the cat maps but also for all other symplectic maps mentioned above.

Now let us turn to the comparison of dynamical notions such as ergodicity, mixing, K and so on.

These notions are often left undefined in the semi-classical literature, since the main problem there is to determine the impact of dynamical properties of the classical limit on the spectral data of the quantum system. However, one can also introduce intrinsic notions of quantum ergodicity, mixing, complete integrability (and so on) which capture the behaviour of quantizations of classically ergodic (etc.) systems. The definitions used in this paper are of this kind; they are based on [Z1], [Z2] (see also [Su]) and will be reviewed in §2.

In contrast, there are the definitions of ergodicity, mixing (etc.) in the theory of C^*- or W^*-dynamical systems. These are more analogous to the classical notions and are applied to open or infinite quantum systems. In this framework, a quantum dynamical system is defined by a C* or W* algebra $\mathcal{A}$ of observables, together with an action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ of a group G by automorphisms of $\mathcal{A}$. The system $(\mathcal{A}, G, \alpha)$ is generally covariantly represented on a Hilbert space $\mathcal{H}$, so that $\alpha_g(A) = U_g^* A U_g$ of $\mathcal{A}$, with U_g a unitary representation of G on $\mathcal{H}$. Dynamical notions are non-commutative analogues, often at the von-Neumann algebra (W^*-) level, of the usual notions for abelian systems. In particular, the spectra of mixing systems must be continuous. For some recent references in the mathematical physics literature, see [B] [JP] [Th].

As mentioned above, our interest is in the semi-classical aspects of quantum dynamics: The quantum systems studied in this paper will have discrete spectra and the ergodicity and mixing properties will be reflected (by definition) in the asymptotics of the eigenfunctions and the eigenvalues. To clarify the relation between this point of view and that of the C*-dynamical point of view, we will also state definitions in terms of the relevant C* algebras and their automorphism groups. It is hoped that this approach will also clarify the nature of the dynamical properties at issue in the semi-classical literature.

Let us contrast the two kinds of dynamical notions in the example of the cat maps, using the articles [B] [BNS] [NT1] [NT2] [Th] to represent the C* and W* approach. In these artricles, the cat maps are quantized as automorphisms of the rotation algebras $\mathcal{M}_\theta$ (the non-commutative torus), and have precisely one invariant state. The GNS representation with respect to this state determines a covariant representation of this system by translations by the classical cat map on functions on the torus. In their words, this gives a "radically different" quantization from the semi-classical one, in that the quantized cat maps have the same multiple Lesbesgue spectrum, hence the same mixing properties, as the classical maps.

The relation of this to the Toeplitz (or other semi-classical) quantizations is as follows: first, in the semi-classical quantizations, the Planck constant θ varies only over the rational values $\frac{1}{N}$, corresponding to the space Θ_N of theta functions of degree N. The finite Heisenberg group Heis $(\mathbb{Z}/N)$

acts irreducibly on this space and its group algebra $\mathbb{C}[\mathrm{Heis}(\mathbb{Z}/N)]$ defines the relevant C* algebra. This algebra is not the rotation algebra $\mathcal{M}_{\frac{1}{N}}$ but is rather the quotient $\mathcal{M}_{\frac{1}{N}}/\mathcal{Z}_N$ by its center $\mathcal{Z}_N$. The elements of $SL(2,Z)$ define automorphisms of $\mathcal{M}_{\frac{1}{N}}$ which (under a parity assumption) preserve the center. Hence they also define automorphisms of the quotient algebra. The quotient automorphisms are the ones studied in the semi-classical literature. Unlike the automorphisms of the full rotation algebras, the quotient ones have discrete spectra and many invariant states, and hence are not ergodic in the C* sense. However, they are quantum ergodic in the semi-classical sense whenever the classical cat map is ergodic. Finally, we note that the quantized cap map systems in the sense of [BNS] are also quantized GNS systems in the sense of [Z1], and are trivially quantum ergodic because the only invariant state is the unique tracial state. Hence they do not have distinct classical limits in the sense of this paper. The quantizations in [B] [BNS] [NT1], [NT2] appear essentially as classical dynamical systems, albeit involving non-commutative algebras. See §5 for a more complete discussion.

The organization of this article is as follows:

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1. STATEMENTS OF RESULTS

In this article, the terms *quantum ergodicity* and *quantum mixing* refer to the properties of *quantized abelian systems* defined in [Z1], [Z2]. They will be briefly reviewed in §2.

We will be concentrating on one kind of example of such quantized abelian systems. The setting will consist of a compact contact manifold (X, α) with an periodic contact flow ϕ^t, together with a contact transformation

$$\chi : X \to X \quad \chi^*(\alpha) = \alpha \quad \chi \cdot \phi^t = \phi^t \cdot \chi$$

commuting with ϕ^t. The S^1 action defined by ϕ^t will be assumed elliptic, so that its isotypic spaces are finite dimensional. The map χ will be quantized as a Toeplitz- Fourier Integral operator

$$U_\chi : H^2_\Sigma(X) \to H^2_\Sigma(X)$$

acting on a Hilbert space $H^2_\Sigma(X)$ of generalized CR functions on X called the Hardy space. The motivating example is where the symplectic quotient $(\mathcal{O}, \omega)$ is a Kähler manifold and where (X, α) is the principal $U(1)$-bundle (with connection) associated to the pre-quantum line bundle (with connection) $(L, \nabla) \to \mathcal{O}$ such that $\mathrm{curv}(\nabla) = \omega$. Relative to the given complex structure J, the quantum Hilbert spaces are the spaces $\mathcal{H}^N_J$ of holomorphic sections of $L^{\otimes N}$, which are canonically isomorphic to the spaces $H^2_\Sigma(N)$ of $U(1)$- equivariant CR functions on X, in the CR structure induced by J. For precise definitions and references, see §2-3.

We first give some general results on the spectrum and on the quantum ergodicity and mixing properties of quantized contact transformations. The quantization U_χ and the orthogonal projection Π_Σ on H^2_Σ will be constructed so that they commute with the operator W_t of translation by ϕ^t; under this $U(1)$-action, H^2_Σ breaks up into finite dimensional "weight" spaces $H^2_\Sigma(N)$ of dimensions d_N and U_χ breaks up into rank d_N unitary operators $U_{\chi,N}$. Hence the quantum system decomposes into finite dimensional systems. From the semi-classical point of view, the focus is on the eigenvalue problems:

$$\left\{ \begin{array}{ll} U_{\chi,N}\phi_{Nj} = e^{i\theta_{Nj}}\phi_{Nj} & (\phi_{Nj} \in \mathcal{H}_\Sigma(N)) \\ (\phi_{Ni}, \phi_{Mj} >= \delta_{MN}\delta_{ij}. & \end{array} \right\}.$$

We will prove the following statements about the eigenvalues and eigenfunctions in §4. The first is a rather basic and familiar kind of eigenvalue distribution theorem, which will be stated more precisely in §4.

THEOREM A. — *With the above notation and assumptions: The spectrum $\sigma(U_\chi)$ is a pure point spectrum. The following dichotomy holds:*

(i) *aperiodic case. If the set of periodic points of χ on the symplectic quotient $\mathcal{O}$ has measure zero (w.r.t. μ), then as $N \to \infty$, the eigenvalues $\{e^{i\theta_{Nj}}\}$ become uniformly distributed on S^1;*

(ii) *periodic case. If $\chi^p = \mathrm{id}$ for some $p > 0$ then there exists a χ-invariant Toeplitz structure Π_Σ so that $\sigma(U_\chi)$ is contained in the pth roots of unity.*

Here, μ is the symplectic volume measure of $(\mathcal{O}, \omega)$.

Next comes a series of general results on the quantum dynamics of Toeplitz systems. The rationale for viewing them as quantum ergodicity and mixing theorems will be reviewed in §2 (see also [Z1] [Z2] for extended discussions).

THEOREM B. — *With the same notation and assumptions: Suppose that (ϕ^t, χ) defines an ergodic action of $G = S^1 \times \mathbb{Z}$ on $(X, \alpha \wedge (d\alpha)^{n-1})$, and let $(\mathcal{O}, \omega)$ denote the symplectic quotient. Then the quantized action $(W_t, U_{\chi,a})$ of G has the following properties: for any $\sigma \in C^\infty(\mathcal{O})$*

$$(\mathcal{E}P) \qquad \lim_{N\to\infty} \frac{1}{d_N} \sum_{j=1}^{d_N} |(\sigma\phi_{Nj}, \phi_{Nj}) - \bar{\mu}(\sigma)|^2 = 0.$$

$$(\mathcal{E}P!) \qquad (\forall\epsilon)(\exists\delta) \limsup_{N\to\infty} \frac{1}{d_N} \sum_{\substack{i\neq j: \\ |e^{i\theta_{Ni}} - e^{i\theta_{Nj}}|<\delta}} |(\sigma\phi_{Ni}, \phi_{Nj})|^2 < \epsilon.$$

Here, $\bar{\mu}(\sigma) = \frac{1}{\mu(\mathcal{O})} \int_{\mathcal{O}} a d\mu$ is the average of σ on $(\mathcal{O}, d\mu)$

COROLLARY B. — *For each N there is a subset $J_N \subset \{1, \dots, d_N\}$ such that:*

(a) $\lim_{N\to\infty} \frac{\#J_N}{d_N} = 1$;

(b) *w-* $\lim_{N\to\infty, j\in J_N} |\phi_{Nj}|^2 = 1$ *on the quotient $\mathcal{O} := X/S^1$. Here, w-lim is the weak* limit on $C(\mathcal{O})$.*

THEOREM C. — *With the notations and assumptions of Theorem B: If the action is also weak-mixing, then in addition to $\mathcal{E}P, \mathcal{E}P!$, we have, for any $\sigma \in C^\infty(\mathcal{O})$,*

$$(\mathcal{M}P)(\forall\tau \in \mathbf{R}) \quad (\forall\epsilon)(\exists\delta) \limsup_{N\to\infty} \frac{1}{d_N} \sum_{\substack{i\neq j: \\ |e^{i\theta_{Ni}} - e^{i\theta_{Nj}} - e^{i\tau}|<\delta}} |(\sigma\phi_{Ni}, \phi_{Nj})|^2 < \epsilon.$$

The restriction $i \neq j$ is of course redundant unless $\tau = 0$, in which case the statement coincides with $\mathcal{E}P!$. For background on these mixing properties see §2 and [Z2], [Z3].

The third series of results concerns the special case of quantized symplectic torus automorphisms, or quantum 'cat maps' (as they are known in the physics literature). In this case, the phase space is the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$, equipped with the standard symplectic structure $\sum dx_i \wedge d\xi_i$. The cat maps are defined by elements $g \in Sp(2n, \mathbb{Z})$ (or more precisely, elements of the "theta-group" $Sp_\theta(2n, \mathbb{Z})$, see §5). As will be seen in §5 (and as is easy to prove) these symplectic maps are "contactible": i.e. can be lifted to the prequantum $U(1)$- bundle X as contact transformations χ_g. The resulting situation is very nice (and very well-studied) because of its relation to the representation theory of the Heisenberg group: This stems from the fact that X is the compact nil-manifold $\mathbb{H}_n^{\rm red}/\bar{\Gamma}$ where $\mathbb{H}_n^{\rm red} \sim \mathbb{R}^{2n} \times S^1$is the reduced Heisenberg group and where Γ is the integral lattice $\mathbb{Z}^{2n} \times \{1\}$.

The spectral theory of the classical cat map is that of the unitary translation operator T_{χ_g} by χ_g on $L^2(X)$. Its quantization U_g will be more or less its compression to the Hardy space $H^2_\Sigma(X)$ of CR functions associated to the standard CR structure on X. That is, essentially $U_g = \Pi_\Sigma T_{\chi_g} \Pi_\Sigma$ where $\Pi_\Sigma : L^2(X) \to H^2_\Sigma(X)$ is the Szego projector. (As will be explained in §2 and §5, this definition has only to be adjusted by a constant so that U_g is unitary.) The projector will often be denoted more simply by Π when the complex or CR structure is fixed. In a well-known way, this space of CR functions can be identified with the space of theta functions of all degrees for the lattice $\mathbb{Z}^n$, and thus the quantized cat maps will correspond to a sequence $U_{g,N}$ of unitary operators on the spaces of theta functions of degree N. As mentioned above, they are of a classic vintage and appear in the transformation laws of theta-functions. Equivalently, they arise in the metaplectic representation of the finite symplectic groups $Sp(2n, \mathbb{Z}/N)$. The CR structure plays the role of the complex polarization in Kähler quantization, with the standard CR structure corresponding to the choice of complex structure $J = iI$ on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

Postponing complete definitions until §2 and §5, we may state our results on theta functions as follows:

THEOREM D. — *Let $g \in SL_\theta(2,\mathbb{Z}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, with Nac, Nbd even $\}$. Then:*

(a) *There exists a constant multiplier $m(g)$ such that the Toeplitz operator $U_g := m(g)\Pi T_{\chi_g}\Pi$ is unitary. The space of elements $H^2_\Sigma(N)$ of weight N relative to the center $\mathcal{Z}$ of $\mathbb{H}_n^{\mathrm{red}}$ may be identified with the space $\Theta_N = \tilde{Th}_N^i$ of theta functions of degree N and the restriction $U_{g,N} := U_g|_{H^2_\Sigma(N)}$ defines the standard action (transformation law) of the element $g \in SL_\theta(2,\mathbb{Z})$ on $\tilde{Th}_N^i$.*

(b) *The multipliers $m(g)$ may be chosen so that the quantization maps $g \to U_{g,N}$ are projective representations of $SL_\theta(2,\mathbb{Z}/N)$, and indeed so that $U_{g,N}$ is the metaplectic representation of $Mp_\theta(2,\mathbb{Z}/N)$.*

(c) *The index of the symplectic map g and contact transformation χ_g in the sense of [Wei] equal zero.*

(d) *If no eigenvalue of g is a root of unity, then the spectral data $\{e^{i\theta_{Nj}}, \phi_{Nj}\}$ of $U_{g,N}$ satisfy the quantum mixing properties ($\mathcal{M}P!$) (cf. Definition 2a.6).*

(e) *One has the exact Egorov theorem: For $\sigma \in C^\infty(\mathbb{R}^{2n}/\mathbb{Z}^{2n})$, $U_g^*\Pi\sigma\Pi U_g = \Pi(\Pi_g\sigma \cdot \chi_g\Pi_g)\Pi$, where Π_g is the Toeplitz projector for the complex structure $g \cdot i$.*

The statements in (b)-(c) follow from that in (a). The main point is that the Toeplitz method produces the metaplectic representations. This is the periodic analogue of the result of Daubechies [D], which shows that the Toeplitz method produces the real metaplectic representation.

In §6 we will present an exact trace formula for the traces of the quantized symplectic torus automorphisms. As noted in the introduction, the trace formula is classical ([Kloo]), although the method of proof appears to be new.

THEOREM E. — *In the notation of Theorem D: The multiplier $m(g)$ can be chosen so that the trace of the quantized cat map $U_{g,N}$ is given by*

$$TrU_{g,N} = \frac{1}{\sqrt{\det(I-g)}} \sum_{[(m,n)]\in\mathbb{Z}^{2n}/(I-g)^{-1}\mathbb{Z}^{2n}} e^{i\pi N[\langle m,n\rangle - \sigma((m,n),(I-g)^{-1}(m,n))]}.$$

The square root is defined by the standard analytic continuation (cf. [F], §6).

This trace formula can be (and has been) used to analyse the fine structure of the spectra of quantized cat maps. The simplest case is that of the elliptic element $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Its quantization (on theta functions of degree N) is equivalent to the finite Fourier transform $F(N)$ on $L^2(\mathbb{Z}/N)$ (cf. [AT]. The trace formula then reads:

$$\frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} e^{2\pi i \frac{r^2}{N}} = \frac{1}{\sqrt{2}} e^{i\pi/4}(1+(-i)^N).$$

From this formula one can deduce that the eigenvalues of $F(N)$ are ± 1, $\pm i$ with essentially equal multiplicities (loc. cit.). It follows that the 'pair correlation function' for the quantization U_S of S is a sum of delta functions.

The above trace formula has also been studied previously in the physics literature ([Ke], loc.cit) to analyse the fine structure of the spectra of the $U_{g,N}$'s when g is a hyperbolic automorphism. In this case the eigenvalues become uniformly distributed on the circle as $N \to \infty$. On the scale of the mean level spacing, however, the spectra of the $U_{g,N}$'s behave very erratically as $N \to \infty$: For each N, there exists a minimal positive integer $\tau(N)$, known as the quantum period, with the property that $U_{g,N}^{\tau(N)} = e^{i\phi(N)}\mathrm{Id}$. The eigenvalues $e^{i\theta_{Nj}}$ are therefore among the translates by $e^{i\phi(N)}$ of the $\tau(N)$th roots of unity. The erratic aspect is that the period $\tau(N)$ depends on the factorization of N into primes and hence is very irregular as a function of N. Moreover the multiplicities m_{Nj} of the eigenvalues $e^{2\pi i \frac{j}{\tau(N)} + \phi(N)}$ seem to be evenly distributed as j varies over $\{0, 1, \ldots, \tau(N)-1\}$ [Ke]. It follows that they tend to infinity at the erratic rate of $N/\tau(N)$.

The eigenvalue pair correlation problem for quantized hyperbolic cat maps thus involves some intricate questions of number theory, while that for quantized elliptic maps is rather trivial. There are however some relatively interesting intermediate cases whose pair correlation functions can be analyzed by means of the trace formulae of the above type (and we hope to do so in a future article).

2. BACKGROUND

2a. Review of quantum ergodicity and mixing.

We begin by reviewing the notions of *quantized abelian system* and of *quantum ergodic system* from [Z1], and explain how they apply in the present context. We also review the mixing notions of [Z2], [Z3].

A quantum dynamical system is a C^*-dynamical system $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is a unital, separable C^*-algebra and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a representation of G by automorphisms of $\mathcal{A}$. We assume that $\mathcal{A}$ acts effectively on a Hilbert space $\mathcal{H}$ and that there exists a unitary representation U of G such that $\alpha_g(A) = U_g^* A U_g$. In other words, we assume the system is covariantly represented on $\mathcal{H}$.

Since $G = S^1 \times \mathbb{Z}$ in this paper, we will assume G is an abelian; moreover we will assume that the spectrum $\sigma(U)$ is discrete in the set $\mathrm{Irred}(G)(=\mathbb{Z} \times S^1$ here) of irreducible representations of G. (In fact, in the Toeplitz examples the spectrum will not only be discrete but will have the strong asymptotic properties described in Theorem A.) We denote by $\mathcal{H} = \bigoplus_{\sigma \in \sigma(U)} \mathcal{H}_\sigma$ the isotypic decomposition of $\mathcal{H}$, by Π_σ the orthogonal projection onto $\mathcal{H}_\sigma$, and by ω_σ the invariant state $\omega_\sigma(A) = \frac{1}{rk\Pi_\sigma} \mathrm{Tr}\, \Pi_\sigma A$.

We then say:

Definition (2a.1). — *$(\mathcal{A}, G, \alpha)$ is quantized abelian if the micro-canonical ensembles*

$$\omega_E := \frac{1}{N(E)} \sum_{E(\sigma) \leqslant E} rk(\Pi_\sigma)\omega_\sigma$$

have a unique weak- limit as $E \to \infty$, and if the C^*-dynamical system $(\pi_\omega(\mathcal{A}), G, \alpha_\omega)$ associated to ω by the GNS construction is abelian.

Here, rk is short for "rank", $N(E) = \sum_{|\sigma| \leqslant E} rk(\Pi_\sigma)$, and the sum runs over $\sigma \in \sigma(U)$ of energy $E(\sigma)$ less than E, with $E(\sigma)$ essentially the distance from σ to the trivial representation. We regard ω as the classical limit (state) of the system, and $(\mathcal{A}, G, \alpha)$ as the quantization of the associated

GNS system. The relevant notions simplify a good deal in the case of the Toeplitz systems of this paper, and will be further discussed in §2.b. For general discussion, including generalities on classical limits of Toeplitz systems, see [Z1].

We also say:

DEFINITION (2a.2). — *A quantized abelian system* $(\mathcal{A}, G, \alpha)$ *is quantum ergodic if there exists an operator K in the von-Neumann algebra closure of* $\mathcal{A}$ *such that*

$$< A >= \omega(A)I + K \qquad \textit{with} \quad \omega_E(K^*K) \to 0.$$

Here, $< A >$ is the time average of A,

$$< A >= w - \lim_{T\to\infty} < A >_T$$

where

$$< A >_T := \int_G \psi_T(g)\alpha_g(A)dg,$$

with ψ_T an "M-net" (approximate mean) for G. In the case $G = S^1 \times \mathbb{Z}$, $\psi_T(g)dg = \frac{1}{T}\chi_{[-T,T]}(t)dtd\theta$ where $d\theta$ (resp. dt) is Lebesgue measure on S^1 (resp. counting measure on $\mathbb{Z}$).

This notion of quantum ergodicity is equivalent to a condition on the eigenfunctions of the quantum system. To state it, we recall that in a generalized *quantized abelian* system the group G is assumed to have the form $T^n \times\ R^k \times \mathbb{Z}^m$ (with T^n the n-torus). Hence the irreducibles are 1-dimensional, of the form $\mathbb{C}\phi_\chi$ where ϕ_χ is an eigenfunction corresponding to a character χ of G. By our assumptions above, the set of such characters is discrete in the dual group $\hat{G}$ and we enumerate them in a sequence χ_j according to their distance $E(\chi_j)$ to the trivial representation. The corresponding eigenfunctions will be denoted ϕ_j. To each is associated an ergodic invariant state ρ_j of the quantum system, namely the vector state $\rho_j(A) = (A\phi_j, \phi_j)$. The criterion above of quantum ergodicity is equivalent to the following:

$$\exists \mathcal{S} \subset \mathrm{spec}(U) : D^*(\mathcal{S}) = 1 \qquad w - \lim_{j\to\infty, \chi_j \in \mathcal{S}} \rho_j = \omega.$$

Here $D^*(\mathcal{S})$ is the density of $\mathcal{S}$ (see [Z1]).

The general quantum ergodicity theorem are as follows. First:

THEOREM (2a.3) ([Z1], Theorems 1-2). — *Suppose $(\mathcal{A}, G, \alpha)$ is quantized abelian. Then: if ω is an ergodic state, the system is quantum ergodic.*

There is a more refined result due to Sunada [Su] and to the author [Z1].

THEOREM (2a.4) ([Su] [Z1], Theorem 3). — *With the same notation and assumptions as in Theorem A.1: ergodicity of ω is equivalent to quantum ergodicity of $(\mathcal{A}, G, \alpha)$ together with the following strong ergodicity property:*

$$(\mathcal{E}P!) \qquad \lim_{T\to\infty}\lim_{E\to\infty}\omega_E(\langle A\rangle_T^* A) = \lim_{E\to\infty}\lim_{T\to\infty}\omega_E(\langle A\rangle_T^* A) = |\omega(A)|^2.$$

Further, $\mathcal{E}P!$ is equivalent to:

$$(\mathcal{E}P!) \qquad (\forall\epsilon)(\exists\delta)\limsup_{E\to\infty}\frac{1}{N(E)}\sum_{\substack{j\neq k: E(\chi_j), E(\chi_k)\leqslant E\\ |E(\chi_j)-E(\chi_k)|<\delta}}|\rho_{jk}(A)|^2 < \epsilon$$

where $\rho_{ij}(A) = (A\phi_i, \phi_j) = TrA\cdot\phi_i\otimes\phi_j$.

There are analogous definitions and results in the case of weak-mixing systems.

Quantum weak mixing has to do with the mean Fourier transform

$$\hat{A}(\chi) := w - \lim_{T\to\infty}\hat{A}_T(\chi)$$

of observables $A \in \mathcal{A}$, where $\chi \in \hat{G}$ where

$$\hat{A}_T(\chi) = \int_G \psi_T(g)\alpha_g(A)\overline{\chi}(g)dg$$

is the partial mean Fourier transform, and where the limit is taken in the weak operator topology. (When the expression for A gets too complicated we often write $\mathcal{F}_T(A)(\chi)$ for this transform and similarly for the limit as $T \to \infty$). The following generalizes the definition of a quantum weak mixing system given in [Z2] for the case of the systems $(\Psi^o(M), \mathbb{R}, \alpha)$, with $\Psi^o(M)$ the C^*-algebra of bounded pseudodifferential operators over a compact manifold M and with $\alpha_t(A) = U_t^* A U_t$ the automorphisms of conjugation by the wave group $U_t := e^{it\sqrt{\Delta}}$ of a metric g on M:

DEFINITION (2a.5). — *A quantized abelian system is quantum weak mixing if, for $\chi \neq 1$,*

$$(\mathcal{M}P) \qquad \limsup_{E\to\infty}\omega_E(\hat{A}(\chi)\hat{A}(\chi)^*) = 0.$$

As in the case of ergodicity there is a sharper weak mixing condition which involves the partial mean Fourier transform and the eigenfunctionals $\rho_{ij}(A) := (A\phi_i, \phi_j)$ above. We note that the eigenfunctionals of the automorphism group correspond to eigenvalues in the "difference spectrum" $\{\chi_j\overline{\chi}_k\}$. For motivation and background on quantum weak mixing we refer to [Z2], [Z3]. In the following $\hat{G}^*$ denotes the set of non-trivial characters of G.

DEFINITION (2a.6). — *A quantized abelian system has the full weak mixing property if in addition to $\mathcal{M}P$ it satisfies, for all $\chi \in \hat{G}^*$:*

$$(\mathcal{M}P!) \qquad \lim_{T\to\infty} \lim_{E\to\infty} \frac{1}{N(E)} \omega_E(\hat{A}_T(\chi)^* \hat{A}_T(\chi)) = 0.$$

In the case of wave groups, we have:

THEOREM (2a.7) ([Z2], [Z3]). — *Let $(\mathcal{A}, G, \alpha) = (\Psi^o(M), \mathbb{R}, \alpha)$ with α the automorphism of conjugation by the wave group U_t of a Riemannian metric g on M. Then: the geodesic flow G^t is weak mixing on the unit cosphere bundle S^*M with respect to Liouville meassure μ if and only if the quantum system $(\mathcal{A}, G, \alpha)$ has the mixing properties $\mathcal{M}P$ and $\mathcal{M}P!$.*

COROLLARY (2a.8) ([Z2], [Z3]). — *With the same notations and assumptions as in Theorem (2a.7), we have, for all $\chi \in \hat{G}^*$:*

$$(\mathcal{M}P!^*) \qquad (\forall\epsilon)(\exists\delta) \limsup_{E\to\infty} \frac{1}{N(E)} \sum_{\substack{j\neq k: E(\chi_j), E(\chi_k)\leqslant E \\ |E(\chi_j\overline{\chi_k}\cdot\chi)|<\delta}} |\rho_{ij}(A)|^2 < \epsilon.$$

This theorem is generalized in Theorem C to Toeplitz systems.

2b. Periodic contact manifolds and Toeplitz algebras.

We now introduce the *quantized abelian* systems which play the principal role in this article: the ones generated by periodic contact flows and quantized contact transformations. The proof that the quantizations can be unitarized will be postponed to §3, where a good deal of further background on Toeplitz operators and their symbols will be reviewed.

The setting opens with a compact contact manifold (X, α). The characteristic distribution ker $d\alpha$ of α is one dimensional, and hence there exists a vector field Ξ on X such that $\alpha(\Xi) = 1$ and $\Xi \perp d\alpha = 0$. We will make the

ASSUMPTION 1 (2b.1). — *The characteristic flow ϕ^t of Ξ is periodic.*

This assumption is satisfied in the motivating examples from geometric quantization theory and complex analysis. Thus, suppose $L \to M$ is a holomorphic line bundle over a compact complex manifold, and let $\|\cdot\|$ be a hermitian metric on L. Then the disc bundle $\Omega = \{(x, v) : |v|_x < 1\}$ is a strictly pseudoconvex domain, whose boundary $\partial\Omega$ has a natural contact structure with periodic characteristic flow. Of particular interest here are the line bundles which arise as "pre-quantum line bundles" over Kähler manifolds in Kähler quantization. See [AdPW] [BG] [We] for examples and further discussion.

Now let

$$\Sigma := \{(x, r\alpha_x) : r > 0\} \subset T^*X \backslash 0\,. \tag{2b.2}$$

Then Σ is a symplectic cone, and according to Boutet de Monvel and Guillemin [BG] always has a Toeplitz structure Π_Σ, that is, an orthogonal projection with wave front along the graph of the identity on Σ, and with the microlocal properties of the Szegö projector onto boundary values of holomorphic functions on a strictly pseudoconvex domain. For the precise definition and local models we refer to [BG], Definition 2.10.

The algebra of concern is then the Toeplitz algebra $\mathcal{T}^\circ_\Sigma$ associated to Π_Σ. By definition, this is the algebra of operators $\Pi_\Sigma A \Pi_\Sigma$ on $L^2(X)$ with $A \in \Psi^o(X)$ (i.e. the algebra of zeroth order pseudodifferential operators over X). The range of Π_Σ will be denoted H^2_Σ, i.e.

$$\Pi_\Sigma : L^2(X) \to H^2_\Sigma(X).$$

It is clear that the Toeplitz algebra is effectively represented on this Hilbert space.

As mentioned above, the group G of concern will be $S^1 \times \mathbb{Z}$. The circle S^1 will operate on $L^2(X, d\nu)$ by: $W_t \cdot f(x) = f(\phi^{-t}x)$. Here $d\nu$ is the normalized volume form determined by α, i.e. $d\nu = c\alpha \wedge (d\alpha)^{n-1}$ for some $c > 0$, where $\dim X = 2n+1$. We may (and will) assume that Π_Σ is chosen so that $[\Pi_\Sigma, W_t] = 0$ ([BG], Appendix). Then S^1 will operate on H^2_Σ.

Its generator $\frac{1}{i}D_{\Xi}$ compresses to the Toeplitz operator $\Pi_{\Sigma}\frac{1}{i}D_{\Xi}\Pi_{\Sigma}$. The symbol of this Toeplitz operator is the function $\Phi : \Sigma \to \mathbb{R}, \Phi(x,\xi) = \langle \xi, \Xi \rangle$.

The group $\mathbb{Z}$ will act by powers of a quantized contact transformation. By definition this will be a unitary operator $U_{\chi} : H^2_{\Sigma}(X) \to H^2_{\Sigma}(X)$ of the form:

(2b.3) $$U_{\chi} = \Pi_{\Sigma} T_{\chi} A \Pi_{\Sigma}$$

where $T_{\chi} f(x) = f(\chi^{-1}(x))$ and where $A \in \Psi^{\circ}_{\Pi_{\Sigma}}$. We will make the

ASSUMPTION 2 (2b.4). — $\Phi > 0$ *and* $[T_{\chi}, W_t] = 0$.

These assumptions are also satisfied in the examples from Kähler quantization theory. They imply that χ descends to a symplectic automorphism $\chi_{\mathcal{O}}$ of the quotient $\mathcal{O} = X/S^1$ of X by the action of ϕ^t. They also imply that $\Pi_{\Sigma}\frac{1}{i}D_{\Xi}\Pi_{\Sigma}$ is an elliptic Toeplitz operator, and hence that the isotypic subspaces $H^2_{\Sigma}(N)$ are finite dimensional. Vice-versa they hold if (X, α) is the prequantum S^1- bundle of an integral symplectic manifold [BG], Lemma 14.9.

To show that (2b.3) is non-vacuous we will prove the:

UNITARIZATION LEMMA 1 (2b.5). — *Let χ be a contact transformation of a contact manifold (X, α) satisfying the assumptions 1-2. Then there exists a symbol $\sigma_A \in C^{\infty}(\mathcal{O})$ determined in a canonical way from χ, Π_{Σ} and a canonically constructed operator $A \in \Psi^{\circ}_{\Pi_{\Sigma}}$ with principal symbol σ_A such that $[A, \frac{1}{i}D_{\Xi}] = 0$ and such that U_{χ}, in (2b.3) is unitary on H^2_{Σ} (at least on the complement of a finite dimensional subspace).*

Granted the Unitarization Lemma, the C^*-dynamical system of concern will be $(\mathcal{T}^o_{\Sigma}, S^1 \times \mathbb{Z}, \alpha)$ where $\alpha_{k,t}$ is conjugation by $U^k_{\chi} W_t$. By the composition theorem of [BG], such conjugations are automorphisms of the Toeplitz algebra.

The principal symbol of $\Pi_{\Sigma} A \Pi_{\Sigma}$ may be identified with $\sigma_A|_{\Sigma}$; a more complete description of the symbol will follow in the next section. The symbol algebra of $\mathcal{T}^o_{\Sigma}$ (zeroth order Toeplitz operators) may then be identified with smooth homogenous functions of degree 0 on Σ; hence with functions on X. In the C^*-closure one gets all the continuous functions.

Many of the notions involved in the definition of quantized abelian systems in § 2.a simplify a good deal for these Toeplitz systems. First, the irreducibles correspond to characters $\chi_{(N,\tau)} := e^{2\pi iNt} \otimes e^{ik\tau}$ of $S^1 \times \mathbb{Z}$ and have the form $\mathbb{C}\phi_{(N,i)}$ where $U_\chi^k W_t.\phi_{(N,j)} = e^{2\pi iNt}e^{ik\theta_{(N,j)}}\phi_{(N,j)}$. The energy $E(\chi_{(N,\tau)})$ is defined to be N. Hence the microcanonical ensembles ω_E have the form

$$\omega_E = \frac{1}{N(E)}\sum_{N=1}^{E} d_N\omega_N$$

where $E \in \mathcal{N}$ and where ω_N is the degree n ensemble defined by

$$\omega_N := \frac{1}{d_N}\sum_{i=1}^{d_N}\rho_{(N,i)} \qquad \text{with} \qquad \rho_{(N,i)}(A) := (A\phi_{(N,i)}, \phi_{(N,i)}).$$

The ensembles ω_E are equivalent to the ensembles

$$\tilde{\omega}_E := \frac{1}{E}\sum_{N=1}^{E}\omega_N$$

in the sense that $\omega_E(A) = \tilde{\omega}_E(a) + o(1)$ as $E \to \infty$. This follows easily from the fact that $d_N \sim N^{\dim X - 1}$ has polynomial growth (for similar assertions see [Z1], [Z4].) In fact, the order N ensembles ω_N have sufficiently well-behaved asymptotics that we will not need to further average over all N. This accounts for the stronger kinds of results available in the Toeplitz case.

PROPOSITION (2b.6). — *$(\mathcal{T}_\Sigma^o, S^1 \times \mathbb{Z}, \alpha)$ is a quantized abelian system, with classical limit system $(C(X), S^1 \times Z, \alpha_\omega)$, where $\alpha_{\omega(t,k))}$ is conjugation by $T_\chi^k \cdot W_t$.*

Proof. — With the assumptions (1)-(2) above, as well as the temporary assumption of the Unitarization Lemma, the isotypic decomposition of H_Σ^2 is just its decomposition into joint eigenspaces for (U_χ, W_t), and the weight spaces $H_\Sigma^2(N)$ are just the eigenspaces of W_t corresponding to the characters $e^{2\pi iNt}$. Let Π_N denote the associated orthogonal projection, and let $d_N = \dim H_\Sigma^2(N) = rk(\Pi_N)$. Then we have (with $E \in \mathbf{N}), A \in \mathcal{T}_\Sigma^o$,

$$\omega_N(A) = \frac{1}{d_N}\mathrm{Tr}\Pi_N A$$

where of course

$$\omega_E(A) = \frac{1}{N(E)}\sum_{0 \leqslant N \leqslant E}\mathrm{Tr}A\Pi_N$$

and of course $N(E) = \sum\limits_{N \leqslant E} d_N$. The asymptotics of $\omega_N(A)$ follow in a standard way from the singularity asymptotics at $t = 0$ of the dual sum

$$\sum_{N \geqslant 0} (\mathrm{Tr} A \Pi_N) e^{2\pi i N t} = \sum_{N \geqslant 0} d_n \omega_N(A) e^{2\pi i N t} = \mathrm{Tr} \Pi_\Sigma A W_t. \tag{2b.7}$$

The composition theorem for Fourier Integral operators and Hermite Fourier Integral operators [BG], §7, shows that the trace is a Lagrangian distribution with singularity only at $t = 0$ and with principal symbol at the singularity given by

$$\omega(A) = \int_X \sigma_A d\nu.$$

Here, $d\nu = \alpha \wedge (d\alpha)^{n-1}$, and as above σ_A is identified with a scalar function on X. In fact the trace (2b.7) is a Hardy distribution on S^1 so one can conclude, simply by comparing Fourier series expansions, that

$$\omega_N(A) \sim \omega(A) + O(N^{-1}) \tag{2b.8}$$

for smooth Toeplitz operators A (see [BG], §13, for details of this argument). It obviously follows that $\omega_E(A), \tilde{\omega}_E(A) \to \omega(A)$. Since (2b.8) is much stronger we will henceforth use it as the key property of the Toeplitz system.

To complete the proof, we only need to identify the classical limit system precisely. From the composition theorem we have

$$\sigma(\alpha_{t,k}(A)) = \sigma_A \cdot (\phi^t \cdot \chi^k), \tag{2b.9}$$

and hence need only to identify the GNS representation with the symbol map. However, it is clear that for smooth elements $A \in \mathcal{T}_\Sigma^o$ (i.e. not in the norm-closure), $\omega(A^*A) = 0$ if and only if $\sigma_A = 0$, hence the ideal $\mathcal{N} = \{A : \omega(A^*A) = 0\}$ is the norm closure of $\mathcal{T}_\Sigma^{-1}$, namely the ideal $\mathcal{K}$ of compact operators in the algebra. However one has the exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_\Sigma^o \to C(X) \to 0$$

where the last map is the symbol map [Do]. Hence $\mathcal{T}_\Sigma^o / \mathcal{N}$, closed under the inner product induced by ω is precisely $L^2(X, d\nu)$, and the induced automorphisms are those of (2b.9). □

3. SYMPLECTIC SPINORS AND PROOF OF THE UNITARIZATION LEMMA

The main point of this lemma is to determine the principal symbol σ_A of the pseudodifferential operator A in (2b.3) which unitarizes the Toeplitz translation by T_χ. This symbol will reappear in the course of the proof of Theorem D. The rest follows by use of the functional calculus. The length of the proof is due mainly to the review it contains of symplectic spinors and symbols of Toeplitz operators.

Proof. — By [BG], Theorem 7.5, any operator of the form (2b.3) is a Fourier Integral operator of Hermite type with wave front along the graph of $\chi|_\Sigma$. Here, χ is understood to extend to Σ as a homogeneous map of order 1. Therefore, the main point is to construct A so that U_χ is unitary, and so that it commutes with the other operators. Consider first the unitarity. At the principal symbol level, this requires

$$\sigma(\Pi_\Sigma A^* T_\chi^{-1} \Pi_\Sigma T_\chi A \Pi) = \sigma(\Pi_\Sigma). \tag{3.1}$$

To solve (3.1), we will have to go further into the symbol algebra of $\mathcal{T}_\Sigma^{\cdot}$: We first recall that the principal symbol of a Hermite operator is a "symplectic spinor" on Σ. In other words, a homogeneous section of the bundle

$$\mathrm{Spin}(\Sigma^\#) \simeq \Lambda^{\frac{1}{2}}(\Sigma^\#) \otimes \mathcal{S}(\Sigma^{\#\perp}/\Sigma^\#)$$

where ([BG], p. 41, [G2])

(3.2)(i) $\Sigma^\# = \{x, \xi, x, -\xi) : (x, \xi) \in \Sigma\}$

(ii) $\Lambda^{\frac{1}{2}}$ is the $\dfrac{1}{2}$ form bundle

(iii) $\Sigma^{\#\perp}/\Sigma^\#$ is the symplectic normal bundle of $\Sigma^\#$

(iv) $\mathcal{S}(\Sigma^{\#\perp}/\Sigma^\#)$ is the bundle of Schwartz vectors along $\Sigma^{\#\perp}/\Sigma^\#$.

In the case at hand, $\Lambda^{\frac{1}{2}}(\Sigma^\#)$ has a natural trivialization coming from the symplectic volume $\frac{1}{2}$-form on X. Hence we can ignore it. Also, $(\Sigma^{\#\perp}/\Sigma^\#)_{(p,-p)}$ is the sum $(\Sigma_p)^\perp \oplus (\Sigma_p^\perp)$ where $(\Sigma_p)^\perp$ is the symplectic orthogonal complement of $\Sigma_p := T_p\Sigma$ in $T_p(T^*X)$. For each choice of symplectic basis of $(T_p\Sigma)^\perp$, one has identifications

$$\begin{aligned} &(T_p\Sigma)^\perp \simeq \mathbb{R}^\ell \otimes (\mathbb{R}^\ell)^* \\ &\mathcal{S}(\Sigma^{\#\perp}/\Sigma^\#)_{(p,-p)} \simeq \mathcal{S}(\mathbb{R}^\ell \oplus \mathbb{R}^\ell). \end{aligned} \tag{3.3}$$

Here, $\mathbb{R}^\ell \oplus \mathbb{R}^\ell$ is the Lagrangian subspace of $(T_p\Sigma)^\perp \oplus (T_p\Sigma)^\perp$ indicated in (3.3) and $\mathcal{S}$ is the usual space of Schwartz functions. A symplectic spinor σ can be identified in this way with the kernel $\kappa_\sigma(p,\circ,\circ)$ of a smoothing operator $T(\sigma,p)$ on the Hilbert space $L^2(\mathbb{R}^\ell)$.

Consider in particular the Szegö–Toeplitz projector Π_Σ. According to [BG], Theorem 11.2, its symbol $\sigma(\Pi_\Sigma)$ may be described as follows: First, Π_Σ determines a homogeneous positive definite Lagrangian sub-bundle Λ of $\Sigma^\perp \otimes \mathbb{C}$ (the complexified normal bundle of Σ in $T^*(X)$). For each $x \in \Sigma$, Λ_x then determines a unique (up to multiples) vector $e_{\Lambda_x} \in \mathcal{S}(\Sigma_x^\perp)$, called the vacuum vector corresponding to e_{Λ_x}. Then $T(\sigma(\Pi_\Sigma),x) = e_{\Lambda_x} \otimes e^*_{\Lambda_x}$, i.e. $\sigma(\Pi_\Sigma)$ is the rank one projection onto $\mathbb{C}e_{\Lambda_x}$.

To bring this somewhat down to earth, we note that Π_Σ is annihilated by an involutive system of $d = \dfrac{1}{2}\dim\Sigma - 1$ equations

$$\begin{aligned} &D_j\Pi_\Sigma \sim 0 \quad (\text{modulo } \Psi^{-\infty}) \\ &[D_j, D_k] \sim \Sigma A^m_{jk} D_m \quad (A^m_{jk} \in \Psi^\circ) \end{aligned} \tag{3.4}$$

similar to the tangential Cauchy–Riemann equations for the Szegö projector of a strictly pseudo convex domain. Above, $\Psi^{-\infty}$ is the algebra of smoothing operators on X. The characteristic variety of this system is Σ, and the matrix $\dfrac{1}{i}\{\sigma(D_j), \sigma(D_k)\}$ is Hermitian positive (or negative) definite along Σ. Let H_{σ_j} be the Hamilton vector field of $\sigma_j := \sigma(D_j)$ and set

$$\Lambda_x = \operatorname{span}_{\mathbb{C}}\{H_{\sigma_j} : j = 1, \ldots, d\}. \tag{3.5}$$

One can check that $\Lambda_x \subset \Sigma_x^\perp \otimes \mathbb{C}$, that $\dim_{\mathbb{C}} \Lambda_x = \dfrac{1}{2}\dim_{\mathbb{C}} \Sigma^\perp \otimes \mathbb{C}$ and that Λ_x is involutive. Hence, Λ_x is a Lagrangian subspace of $\Sigma_x^\perp \otimes \mathbb{C}$.

Now let $Mp(\Sigma^\perp)$ be the metaplectic frame bundle of $\Sigma^\perp$: i.e. the double cover of the symplectic frame bundle of $\Sigma^\perp$ corresponding to the cover $Mp(2n,\mathbb{R}) \to Sp((2n,\mathbb{R})$. Then

$$\mathcal{S}(\Sigma^\perp) = Mp(\Sigma^\perp) \times_\mu \mathcal{S}(\mathbb{R}^\ell)$$

where μ is the metaplectic representation. From this one can transfer the Schrödinger representation ρ of the Heisenberg group on $\mathcal{S}(\mathbb{R}^\ell)$ to $\mathcal{S}(\Sigma^\perp)_x$ for each x, and each metaplectic frame of $\Sigma_x^\perp$. If $d\rho_x$ represents the derived representation at x, then one sees that the symbol equations corresponding to (3.4) are

$$d\rho_x(H_{\sigma_j})\sigma(\Pi_\Sigma) = 0 \quad (x \in \Sigma, \Xi_j \in \Lambda_x). \tag{3.6}$$

The vacuum state e_{Λ_x} is the unique solution of the similar system of equations on $\mathcal{S}(\Sigma^\perp)_x$. Since $\sigma(\Pi_\Sigma)$ is a projector it must be $e_{\Lambda_x} \otimes e^*_{\Lambda_x}$.

Next, return to (3.1). By the composition theorem [BG], 7.5,

$$\sigma(\Pi_\Sigma A^* T_\chi^{-1} \Pi_\Sigma T_\chi A \Pi) = |\sigma_A|^2 \cdot \sigma_{\Pi_\Sigma^\circ} \sigma(T_\chi^{-1} \Pi_\Sigma T_\chi) \circ \sigma_{\Pi_\Sigma}. \tag{3.7}$$

Now $T_\chi^{-1}\Pi_\Sigma T_\chi$ is also a Toeplitz structure on Σ, since χ is a symplectic diffeomorphism of Σ. Hence $\sigma(T_\chi^{-1}\Pi_\Sigma T_\chi)$ will be a rank one projection $e_{\Lambda_\chi} \otimes e_{\Lambda_\chi}$ for some Lagrangian sub-bundle Λ_χ of $\Sigma^\perp \otimes \mathbb{C}$. In fact

$$\Lambda_\chi = d\tilde{\chi}(\Lambda) \tag{3.8}$$

where $\tilde{\chi} : T^*(X) \to T^*(X)$ is the natural lift $(d\chi^t)^{-1}$ of χ to $T^*(X)$. Note that $\tilde{\chi}|_\Sigma = \chi|_\Sigma$, and since $\tilde{\chi}$ is symplectic,

$$d\tilde{\chi} : T\Sigma \otimes \mathbb{C} \to T\Sigma \otimes \mathbb{C},$$

$$d\tilde{\chi} : \Sigma^\perp \otimes \mathbb{C} \to \Sigma^\perp \otimes \mathbb{C}.$$

Also, $d\tilde{\chi}(\Lambda)$ is lagrangian sub-bundle of $\Sigma^\perp \otimes \mathbb{C}$. By the symbolic calculus, $\sigma(T_\chi^{-1}\Pi_\Sigma T_\chi)$ will have to solve (3.6) with Ξ_j replaced by $d\tilde{\chi}(\Xi_j)$; hence (3.8).

Carrying out the composition of projections in (3.7), we conclude that

$$\sigma(\Pi_\Sigma A^* T_\chi^{-1} \Pi_\Sigma T_\chi A \Pi_\Sigma) = |\sigma_A|^2 \; |\langle e_{\Lambda_\chi}, e_\Lambda \rangle|^2 e_\Lambda \otimes e^*_\Lambda. \tag{3.9}$$

To satisfy (3.1) it is sufficient to set:

$$\sigma_A(x) = \langle e_{\Lambda_\chi}, e_\Lambda \rangle^{-1}. \tag{3.10}$$

Of course, we must show that $\langle e_{\Lambda_x}, e_\Lambda \rangle(x)$ never vanishes. In the model case $\mathbb{R}^\ell$, e_{Λ_χ} and e_Λ correspond to a pair of Gaussians γ_{Z_1} and γ_{Z_2}, where $\gamma_Z = e^{i\langle ZX, X\rangle}$ for a complex symmetric matrix $Z = X + iY$ with $Y \gg 0$. It is obvious that $\langle \gamma_{Z_1}, \gamma_{Z_2} \rangle$ never vanishes, since the Fourier transform of a Gaussian is never zero.

Now let S^1 act via W_t. Since Π_Σ commutes with the action, one may assume the operators D_j in (3.4) commute with the action (otherwise, one can average them). Hence, the Lagrangian sub-bundle Λ is S^1-invariant, and since χ commutes with the S^1 action, Λ_χ is also S^1-invariant. It follows from uniqueness of the vacuum vectors (up to multiples) that e_Λ and e_{Λ_χ} are eigenvectors of the S^1 action. Since they correspond under $\tilde{\chi}$, they must transform by the same character. It follows that $\langle e_\Lambda, e_{\Lambda_\chi} \rangle$ is S^1 invariant. Hence, we have:

(3.11) *σ_A is S^1-invariant.*

Now extend σ_A (in any smooth way) as a homogeneous function of degree 0 on $T^*X\backslash 0$, and let A_1 be any operator in Ψ°_{Π} with symbol σ_A. By operator averaging against W_t, we may assume $\left[A, \frac{1}{i}D_\Xi\right] = 0$. At this point, A_1 satisfies:

$$\begin{gathered}[A_1, \Pi_\Sigma] = 0 \\ \left[A_1, \frac{1}{i}D_\Xi\right] = 0 \\ U_1 := \Pi_\Sigma T_\chi A_1 \Pi_\Sigma \quad \text{is unitary modulo} \quad \mathcal{T}_\Sigma^{-1}.\end{gathered} \tag{3.12}$$

We now employ a simple argument of Weinstein [Wei] to correct A_1 to define an operator $U_{(\chi,a)}$ which is unitary. In the following we will pretend that the index $\operatorname{ind}(U_1) = 0$. If it is not, one has to work on the orthogonal complement of a finite dimensional subspace. This index is an invariant of the contact transformation χ and hence is called the *index of* χ. We will discuss it further in §5.

By (3.12), $U_1^*U_1$ and $U_1U_1^*$ are elliptic Toeplitz operators, with principal symbols $\sigma(\Pi_\Sigma)$. Hence their kernels are finite dimensional. Let K be an S^1-invariant isometric operator from $\ker U_1 \to \ker U_1^*$; let P denote the orthogonal projection onto $\ker U_1$. Then KP is a finite rank operator and

$$B_1 = U_1 + KP$$

is an injective Fourier Integral operator of Hermite type. It follows that $B_1^*B_1$ is a positive Toeplitz operator with symbol $\sigma(\Pi_\Sigma)$. Just as for pseudo-differential operators, there is a functional calculus for $\mathcal{T}_\Sigma^\circ$. We may express

$$B_1^*B_1 = \Pi_\Sigma C \Pi_\Sigma \qquad C \in \Psi_\Sigma^\circ$$

and then define

$$G = (B_1^*B_1)^{\frac{-1}{2}} = \Pi_\Sigma C^{\frac{-1}{2}} \Pi_\Sigma \in \mathcal{T}_\Sigma^\circ \, .$$

Then set $U_\chi = B_1 G$. It is unitary and satisfies all the conditions of the lemma. □

4. QUANTUM ERGODICITY AND MIXING: PROOFS OF THEOREMS A, B, C

We begin with the proof of the spectral dichotomy:

Proof of Theorem A.

Proof of (i). — The precise statement of (i) is that the eigenvalues $\{e^{2\pi i\theta_{Nj}} : j = 1, \ldots, d_N\}$ with $d_N = \dim H^2_\Sigma(N)$ become uniformly distributed on S^1 as $n \to \infty$ in the sense that

$$w - \lim_{N\to\infty} \frac{1}{d_N} \Sigma_{j=1}^{d_N} \delta(e^{2\pi i\theta_{Nj}}) = d\theta \tag{4.1}$$

or equivalently for, $f \in C(S^1)$,

$$\lim_{N\to\infty} \frac{1}{d_N} \sum_{j=1}^{d_N} f(e^{2\pi i\theta_{Nj}}) = \int_0^1 f(e^{2\pi i\theta})d\theta. \tag{4.2}$$

Of course, it suffices to let $f(z) = z^k (k \in \mathbb{Z})$, and to prove that the left side tends to 0 if $k \neq 0$. But if $f = z^k$, the left side is

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Tr}(U^k_{\chi,N})\Pi_N \tag{4.3}$$

where as above $\Pi_N : H^2_\Sigma(X) \to H^2_\Sigma(N)$ is the orthogonal projection. The limit can be obtained from the singularity at $\theta = 0$ of the trace

$$\mathrm{Tr} W_\theta U^k_\chi = \sum_{N=0}^{\infty} e^{2\pi i N\theta} \mathrm{Tr}(U^k_{\chi,N})\Pi_N.$$

By the composition calculus of Fourier Integral and Hermite operators [BG], the singularities of the trace occur at values of θ for which $e^{2\pi i\theta} \cdot \chi^k$ has non-empty fixed point set. It is clear that θ must equal zero, and that the fixed point sets consists of the fibers over the fixed points of χ^k on $\mathcal{O}$. This is a finite subset if $k \neq 0$, and hence the singularity is of the type $(t + i0)^{-1}$ (compare [BG], Theorem 12.9). It follows that $\mathrm{Tr}(U_{\chi,N})^k$ is bounded as $N \to \infty$ if $k \neq 0$ (compare [BG], Proposition 13.10). On the other hand, $d_N = \dim H^2_\Sigma(N) \sim N^{(\dim X+1)/2-1}$ [loc. cit.], so the limit (4.3) is zero unless $k = 0$. □

Proof of (ii). — If χ is periodic, then the whole group G generated by the contact flow and by χ is compact, and as mentioned above the

Toeplitz structure Π_Σ may be constructed to be invariant under it. Hence the unitary quantization of χ is simply

$$U_\chi := \Pi_\Sigma T_\chi \Pi_\Sigma$$

and $U_\chi^k = \Pi_\Sigma T_{\chi^k}\Pi_\Sigma$. It follows that $U_\chi^p = \Pi_\Sigma$ (the identity operator on H^2_Σ) and hence its eigenvalues are pth roots of unity. □

We now turn to the quantum ergodicity and mixing theorems. The ergodicity theorems follow almost immediately from the results of [Z1].

Proof of Theorem B.

Proof. — By Proposition (2b.6), $(\mathcal{T}^o_\Sigma, S^1\times\mathbb{Z}, (W_t, U_\chi))$ is a quantized abelian system and by assumption the classical limit system is ergodic. Except for one gap, the statement then follows from Theorems 1-3 of [Z1].

The gap is that we are using the more localized ensembles ω_n rather than the microcanonical ensembles ω_E. However, the only properties of ω_E used in [Z1] are that they form a sequence of invariant states satisfying $\omega_E \to \omega$. Since this was also proved for the degree N ensembles ω_N in Proposition (2b.6) (see (2b.8), the proof of Theorem B is complete. □

Remark. — The ergodicity assumption is equivalent to the ergodicity of χ on $(\mathcal{O}, \mu)$

The following theorem states that if χ is weak mixing on $(\mathcal{O}, \mu)$, then the quantum system has the full weak mixing property of Definition (2a.6) in the even stronger form involving the degree n ensembles. There is a notational overlap in that we are writing χ both for characters and for the contact map; both are conventional and we do not believe this should cause any confusion.

Proof of Theorem C.

Proof. — First, the weak mixing property of χ on $(\mathcal{O}, \mu)$ is equivalent to the statement that

$$\lim_{M\to\infty} ||\mathcal{F}_M(\tau)f||_{L^2} = 0 \qquad (\forall f \perp 1) \tag{4.4}$$

where

$$\mathcal{F}_M(\tau) : L^2(\mathcal{O}, d\mu) \to L^2(\mathcal{O}, d\mu)$$

is the partial mean Fourier transform

$$\mathcal{F}_M(\tau)f = \frac{1}{2M}\sum_{-M}^{M} e^{-im\tau}T_\chi^m f.$$

Indeed, we have

$$\lim_{M\to\infty} ||\mathcal{F}_M(\tau)f - P_\tau f||_{L^2} = 0$$

for all $f \in L^2$, where P_τ is the orthogonal projection onto the eigenspace of T_χ of eigenvalue P_τ. On the other hand if χ is weak mixing, then $P_\tau f = 0$ for all $f \perp 1$ since the unitary operator

$$T_\chi : L^2(\mathcal{O}, d\mu) \to L^2(\mathcal{O}, d\mu)$$

$$T_\chi f(o) = f(\chi^{-1}(o))$$

has no L^2-eigenfunctions other than constants. Henceforth we only consider non-trivial characters ($\tau \neq 0$) since the case $\tau = 0$ is covered in the ergodicity theorem above.

One connection to the quantum theory is thru the partial mean Fourier transforms

$$\hat{A}_M(\chi) = \mathcal{F}_M(\chi)A := \int_G \psi_M(g)\alpha_g(A)\overline{\chi}(g)dg$$

of observables $A \in \mathcal{T}_\Sigma^o$. To simplify, we recall that without loss of generality, an element of $\mathcal{T}_\Sigma^o$ may be assumed to be of the form $\Pi_\Sigma A \Pi_\Sigma$ with $A \in \Psi^o(X)$, with $[A, W_t] = 0, [A, \Pi_\Sigma] \sim 0$ [BG]. As above, we also write characters χ in the form $e^{2\pi i N t} \otimes e^{ik\tau}$ with $e^{2\pi i N t} \in S^1$. We further note that the quantum mixing condition stated in the theorem concerns only the diagonal blocks $\Pi_N A \Pi_N$ whose partial mean Fourier transforms have the form

$$\mathcal{F}_M(\chi)\Pi_N A \Pi_N = \frac{1}{2M}\sum_{m=-M}^{M} e^{-im\tau}\int_{S^1} e^{-2\pi i N t}W_t^* U_\chi^{-m}\Pi_N A \Pi_N U_\chi^m W_t dt.$$

Since $[U_\chi, \Pi_N] = [W_t, \Pi_N] = 0, W_t\Pi_N = e^{2\pi i N t}\Pi_N$, the conjugates $\alpha_g(\Pi_N A \Pi_N)$ are constant in t and hence $\mathcal{F}_M(\chi)\Pi_N A \Pi_N = 0$ unless the character χ has the form $1 \otimes e^{ik\tau}$. In the latter case, the partial mean Fourier transforms of the blocks simplifies to

$$\mathcal{F}_M(\tau)\Pi_N A \Pi_N = \frac{1}{2M}\sum_{m=-M}^{M} e^{-im\tau}U_\chi^{-m}\Pi_N A \Pi_N U_\chi^m \tag{4.5}$$

which begin to look very much like their classical counterparts. The resemblence is of course made even closer by use of the Egorov theorem for Toeplitz operators [BG], which implies that $U_\chi^{-m}\Pi A\Pi U_\chi^m \in \mathcal{T}_\Sigma^o$ with principal symbol equal to $T_\chi^m \sigma_{\Pi A\Pi}$.

We now make the key observation:

$$\lim_{N\to\infty}\frac{1}{d_N}||\mathcal{F}_M(\tau)\Pi_N A\Pi_N||_{HS}^2 = ||\mathcal{F}_M(\tau)\sigma_A||_{L^2}^2 \tag{4.6}$$

where σ_A denotes the function on $\mathcal{O}$ induced by the principal symbol of $\Pi A\Pi$. This follows from the Egorov theorem combined with a special case of the Szego limit theorem for Toeplitz operators:

$$\lim_{N\to\infty}\frac{1}{d_N}||\Pi_N A\Pi_N||^2 = \frac{1}{\mu(\mathcal{O})}\int_{\mathcal{O}}|\sigma_A|^2 d\mu \tag{4.7}$$

which holds if $\sigma_A|_\Sigma$ is invariant under the contact flow (see [BG], Theorems 6 and 13.11.)

It follows from (4.4) and (4.6) that if χ is weak mixing and $\tau \neq 0$, then

$$\lim_{M\to\infty}\lim_{N\to\infty}\frac{1}{d_N}||\mathcal{F}_M(\tau)\Pi_N A\Pi_N||_{HS}^2 = 0 \tag{4.8}$$

for all smooth A in the Toeplitz algebra. Let us now express (4.8) in terms of the eigenfunctions $\phi_{(N,i)}$ of U_g and in terms of the eigenfunctionals $\rho_{(N,ij)}(A) := (A\phi_{(N,i)}, \phi_{(N,j)})$ of the automorphisms α_g. We have

$$\rho_{(N,ij)}(\mathcal{F}_M(\tau)\Pi_N A\Pi_N) = \frac{1}{2M}\sum_{m=-M}^{M} e^{im(\theta_{Ni}-\theta_{Nj}-\tau)}\rho_{(N,ij)}(A)$$

$$= \frac{1}{2M}D_M(\theta_{Ni}-\theta_{Nj}-\tau)\rho_{(N,ij)}(A)$$

where D_M is the Dirichlet kernel $D_M(x) = \dfrac{\sin\left(M+\frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}$. Hence (4.8) is equivalent to:

$$\lim_{M\to\infty}\lim_{N\to\infty}\frac{1}{d_N}\sum_{i,j=1}^{d_N}|\frac{1}{2M}D_M(\theta_{Ni}-\theta_{Nj}-\tau)|^2|\rho_{(N,ij)}(A)|^2 = 0. \tag{4.9}$$

Given $\epsilon > 0$ we choose M sufficient large so that (4.9) is $\leqslant \epsilon$. If we then choose $\delta > 0$ so that $\frac{1}{2M}D_M(x) \geqslant \frac{1}{2}$ for $x \leqslant \delta$, the statement of the theorem follows for A in place of $\sigma \in C^\infty(\mathcal{O})$.

This is actually the general case: the diagonal part $\bigoplus_{N=0}^{\infty} \Pi_N A \Pi_N$ of A is its average relative to W_t and hence its symbol is S^1-invariant and may be identified with a function σ on $\mathcal{O}$. Since the lower order terms in the symbol make no contribution in the limit $n \to \infty$, the statement is only non-trivial for the Toeplitz multiplier $\Pi\sigma\Pi$. □

COROLLARY. — *The Toeplitz system is quantum weak mixing in the sense that*

$$\lim_{N\to\infty} \omega_N(\hat{A}^*(\chi)\hat{A}(\chi)) = 0$$

for $\chi \neq 1$.

Proof. — The Szegö limit theorem cited above shows that

$$\frac{1}{d_N} ||\hat{A}_M^*(\chi)\hat{A}_M(\chi)||_{HS}^2 = \frac{1}{d_N} \omega_N(\hat{A}_M^*(\chi)\hat{A}_M(\chi)) + o(1). \tag{4.10}$$

Hence

$$\frac{1}{d_N} \omega_N(\hat{A}_M^*(\chi)\hat{A}_M(\chi)) \to 0$$

for $\tau \in \mathbb{R} - 0$ and the corollary follows from the fact that

$$\omega_N(\hat{A}_M^*(\chi)\hat{A}_M(\chi)) \geqslant \omega_N(\hat{A}^*(\chi)\hat{A}(\chi). \tag{4.11}$$

(For the proof of (4.11) see [Z2], Proposition (1.3iv).) □

Remark. — Although we will not prove it here, the quantum mixing property $\mathcal{M}!$ is actually equivalent to the weak mixing of χ on $(\mathcal{O}, \mu)$. The proof is essentially as in Theorem 1 of [Z2], given the modifications above to the 'if' half of that theorem. We also refer to [Z2] for other variants of the weak mixing conditions. All of these conditions generalize to the Toeplitz setting and even to the case of essentially general quantized abelian systems. For the sake of brevity we have only stated the condition which is most concrete in terms of the eigenfunctions of the system.

5. QUANTIZED SYMPLECTIC TORUS AUTOMORPHISMS: PROOF OF THEOREM D

In this section, we illustrate the general theory in §2-4 with the special case of quantized symplectic torus automorphisms $g \in Sp(2n, \mathbb{Z})$. As will be seen, if g lies in the theta-subgroup $Sp_\theta(2n, \mathbb{Z})$, then it lifts to a contact transformation χ_g of the circle bundle $N_{\mathbb{Z}}/N_{\mathbb{R}} \sim \mathbb{H}_n^{\text{red}}/\bar{\Gamma}$ over $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with respect to the natural contact structure α. Here, $N_{\mathbb{Z}}/N_{\mathbb{R}} \sim \mathbb{H}_n^{\text{red}}/\bar{\Gamma}$ is the quotient of the Heisenberg group $N_{\mathbb{R}}$ (or reduced Heisenberg group $\mathbb{H}_n^{\text{red}}$) by its integer lattice $N_{\mathbb{Z}}$ (or reduced lattice $\bar{\Gamma}$). The quantization will then be a unitary Toeplitz operator of the form $\Pi\chi_g\Pi$, operating on the Hardy space $H^2_\Sigma(N_{\mathbb{Z}}/N_{\mathbb{R}})$ of CR functions on the quotient.

As mentioned in the introduction, the action of the Toepltitz-quantized torus automorphisms on these CR functions will be identified with the classical action of the theta group $Sp_\theta(2n, \mathbb{Z})$ on the space of theta functions (of variable degree). The statements in Theorem D will follow directly from this link. To establish it, we will need to draw on the harmonic analysis of theta functions from [A] [AT], the transformation theory of theta functions from [KP], and the analysis of CR functions on $N_{\mathbb{Z}}/N_{\mathbb{R}}$ from [FS] [S]. The notational differences between these references explain, and we hope justify, the notational redundancies in this section.

5.1. Symplectic torus automorphisms.

The starting point is the affine symplectic manifold $(T^*\mathbb{R}^n, \sigma)$, where $\sigma = \Sigma_{j=1}^n dx_j \Lambda d\xi_j$, and with a co-compact lattice $\Gamma \subset T^*\mathbb{R}^n$ which we will take to be $\mathbb{Z}^{2n}$. The quotient $(T^*\mathbb{R}^n/\Gamma, \sigma)$ is then a symplectic torus. If $g \in Sp(T^*\mathbb{R}^n, \sigma) = Sp(2n, \mathbb{R})$ is a linear symplectic map satisfying $g(\mathbb{Z}^{2n}) = \mathbb{Z}^{2n}$, then g descends to symplectic automorphism of the torus (still denoted g).

It is convenient to express g in block form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{R}^n_x \oplus \mathbb{R}^n_\xi \to \mathbb{R}^n_x \oplus \mathbb{R}^n_\xi \tag{5.1.1}$$

relative the the splitting $T^*\mathbb{R}^n = \mathbb{R}^n_x \oplus \mathbb{R}^n_\xi \simeq \mathbb{R}^{2n}$. Then $g \in Sp(2n, \mathbb{R})$, i.e. g is a symplectic linear map of $\mathbb{R}^{2n}$, if and only if

(5.1.2) (i) $g^* \in Sp(n, \mathbb{R})$

(ii) $A^*C = C^*A, B^*D = D^*B, A^*D - C^*B = I$

(iii) $AB^* = BA^*, CD^* = DC^*, AD^* - BC^* = I$.

Also $g(\mathbb{Z}^{2n}) = \mathbb{Z}^{2n}$ is equivalent to $a \in Sp(2n, \mathbb{Z})$. ([F], Chapter 4)

5.2. Kähler and Toeplitz quantization of complex torii.

The quantization of g should be a unitary operator U_g on a Hilbert space $\mathcal{H}$ which quantizes $(T^*\mathbb{R}^n/\Gamma, \sigma)$. The method of geometric (Kähler) quantization constructs $\mathcal{H}$ as the space of holomorphic sections of a holomorphic line bundle $L \to T^*\mathbb{R}^n/\Gamma$, with respect to a complex structure Z on $T^*\mathbb{R}^n/\Gamma$. We will temporarily assume Z to be the affine complex structure J coming from the identification $\mathbb{R}^n_x \oplus \mathbb{R}^n_\xi \to \mathbb{C}^n((x,\xi) \mapsto x + i\xi)$. Later we will consider more general Z.

The line bundle L, and its powers $L^{\otimes N}$, are associated to the so-called prequantum circle bundle $p : X \to T^*\mathbb{R}^n/\Gamma$ by the characters χ_N of S^1. The definition of prequantum circle bundle also includes a connection α. As is well-known, in this example X is the compact nilmanifold $\mathbb{H}^{\text{red}}_n/\bar{\Gamma}$, where $\mathbb{H}^{\text{red}}_n$ is the reduced Heisenberg group $\mathbb{R}^{2n} \times S^1$ and where $\bar{\Gamma}$ is a maximal isotropic lattice. We pause to recall the precise definitions, since there are many (equivalent) definitions of these groups and lattices.

We will take the group law of $\mathbb{H}^{\text{red}}_n$ in the form

$$(x, \xi, e^{\mathrm{it}}) \cdot (x', \xi', e^{\mathrm{it}'}) = \left(x + x', \xi + \xi', e^{i(t+t'+\frac{1}{2}\sigma((x,\xi),(x',\xi')))}\right) \tag{5.2.1}$$

with $\sigma((x,\xi),(x',\xi'))) = \langle \xi, x' \rangle - \langle \xi', x \rangle$. The center Z of $\mathbb{H}^{\text{red}}_n$ is the circle factor S^1. Evidently, Z acts by left translations on X and its orbits are the fibers of p. The connection one form is given by

$$\alpha = dt + \frac{1}{2}\Sigma_{j=1}^n (x_j d\xi_j - \xi_j dx_j). \tag{5.2.2}$$

With the group law in the form (5.2.1), the integer lattice $\bar{\Gamma}$ is not $\mathbb{Z}^{2n} \times \{1\}$ (which is not a subgroup) but is rather its image under the splitting homomorphism

$$s : \mathbb{Z}^{2n} \to \mathbb{H}^{\text{red}}_n \qquad s(m,n) := (m, n, e^{i\pi\langle m,n \rangle}). \tag{5.2.3}$$

See subsection (5.8) for the terminology and further discussion.

Under the action of Z, $L^2(X)$ has the isotypic decomposition

$$L^2(X) = \bigoplus_{N=-\infty}^{\infty} H_N$$

where H_N is the set of vectors satisfying $W_t f = e^{2\pi i N t} f$; here W_t is the unitary representation of z by translations on L^2. In the standard way, we identify H_N with the sections of $L^{\otimes N}$. Thus, $\oplus_{N=0}^{\infty} H_N$ incorporates the sections of all the bundles $L^{\otimes N}$ at once.

The holomorphic sections of $L^{\otimes N}$ then correspond to the subspace $H^2_\Sigma(N)$ of CR functions in H_N. Let us recall the definition [FS]: First, one defines the left invariant vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \xi_j \frac{\partial}{\partial t} && (j = 1, \dots, n) \\ \Xi_j &= \frac{\partial}{\partial \xi_j} - x_j \frac{\partial}{\partial t} && (j = 1, \dots, n) \\ T &= \frac{\partial}{\partial t} \end{aligned} \tag{5.2.4a}$$

on $\mathbb{H}_n^{\mathrm{red}}$. They satisfy the commutation relations $[\Xi_j, X_k] = 2\delta_{jk} T$, all other brackets zero. Then set

$$\begin{aligned} Z_j &= \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t} = X_j - i\Xi_j && (j = 1, \dots, n) \\ \bar{Z}_j &= \frac{\bar{\partial}}{\partial z_j} - i z_j \frac{\partial}{\partial t} = X_j + i\Xi_j && (j = 1, \dots, n) \end{aligned} \tag{5.2.4b}$$

(with $Z_j = X_j + i\xi_j$). The commutation relations are $[Z_j, \bar{Z}_k] = -2i\delta_{jk} T$, all other brackets zero. One notes that $\alpha(Z_j) = \alpha(\bar{Z}_j) = 0 \quad (\forall j)$, so the sub-bundle $T_{1,0}$ of $T(\mathbb{H}_n^{\mathrm{red}}) \otimes \mathbb{C}$ defines a CR structure on $\mathbb{H}_n^{\mathrm{red}}$. The Levi form is given by $\langle Z_j, Z_k \rangle_L = \frac{1}{2} i \langle \alpha, [Z_j, \bar{Z}_k] \rangle = \delta_{jk}$, so $\mathbb{H}_n^{\mathrm{red}}$ is strongly pseudo convex. All of these structures descend to the quotient by $\bar{\Gamma}$ and define a CR structure on X. The CR functions are the solutions of the Cauchy–Riemann equations

$$\bar{Z}_j f = 0 \qquad (j = 1, \dots, n). \tag{5.2.5}$$

We will denote by $H^2_\Sigma(X)$ the CR functions which lie in $L^2(X)$. Under the action of Z we have the isotypic decomposition

$$H^2_\Sigma(X) = \bigoplus_{N=0}^{\infty} H^2_\Sigma(N) \tag{5.2.6}$$

where $H^2_\Sigma(N) := H^2_\Sigma \cap H_N$ is the space of CR vectors transforming by the N-th character χ_N. Under the identification of sections of $L^{\otimes N}$ with equivariant functions on X in H_N, the holomorphic sections correspond to $H^2_\Sigma(N)$([AT] [A] [M]). As is well-known, and will be reviewed below, the holomorphic sections $\Gamma_{\mathrm{hol}}(L^{\otimes N})$ are the theta functions of degree N.

5.3. Toeplitz quantization of symplectic torus automorphisms.

Thus far, we have followed the procedure of geometric quantization theory and have quantized $(T^*\mathbb{R}^n/\Gamma, \sigma)$ as the sequence of Hilbert spaces $H^2_\Sigma(N) \simeq \Gamma_{\mathrm{hol}}(L^{\otimes N})$. The next step is to quantize the symplectic map g. For this, geometric quantization offers no well-defined procedure in general, and indeed it is not possible to quantize general symplectic maps (even very simple ones) in a systematic way. In the case of certain $g \in Sp(2n, \mathbb{Z})$ we can use the Toeplitz method. These are the elements in the theta-subgroup $Sp_\theta(2n, \mathbb{Z}) := \{g \in Sp(2n, \mathbb{Z}) : AC \equiv 0(\text{mod } 2), BD \equiv 0(\text{mod } 2)\}$.

Proposition (5.3.1). — *Let $g \in Sp_\theta(2n, \mathbb{Z})$, and let $\chi_g : N_\mathbb{R} \to N_\mathbb{R}$ be defined by*

$$\chi_g(x, \xi, t) = (g(x, \xi), t).$$

Then χ_g descends to a contact diffeomorphism of (X, α).

Proof. — First, χ_g is well-defined on the quotient $\mathbb{H}^{\mathrm{red}}_n/\bar{\Gamma}$ of the Heisenberg group since the elements of $Sp_\theta(2n, \mathbb{Z})$ are the automorphisms of $\mathbb{H}^{\mathrm{red}}_n$ preserving $\bar{\Gamma}$. The last statement follows from the fact that $F(g(m, n)) \equiv F(m, n)$ (mod 2) if $g \in Sp_\theta(2n, \mathbb{Z})$ and if $F(m, n) := \langle m, n\rangle$.

It remains to show that $\chi_g^*\alpha = \alpha$. Let us write $\alpha = dt + \frac{1}{2}(\langle x, d\xi\rangle - \langle \xi, dx\rangle)$ where $x = (x, \ldots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n)$ and $\langle a, b\rangle = \Sigma a_i b_i$. Then $\chi_g^*\alpha = dt + \frac{1}{2}(\langle x^1, d\xi^1\rangle - \langle \xi^1, dx^1\rangle)$ where $x^1 = Ax + B\xi$, $\xi^1 = Cx + d\xi$ (in the notation of 3.1-2). We note that

$$\begin{aligned}
\langle x^1, d\xi^1\rangle - \langle \xi^1, dx^1\rangle &= \langle (A^*C - C^*A)x, dx\rangle \\
&\quad + \langle (D^*B - B^*D)\xi, d\xi\rangle \\
&\quad + \langle (D^*A - B^*C)x, d\xi\rangle \\
&\quad + \langle (C^*B - A^*D)\xi, dx\rangle \\
&= \langle x, d\xi\rangle - \langle \xi, dx\rangle
\end{aligned} \tag{5.3.2}$$

by the identities in (5.1.2). Hence $\chi_g^*\alpha = \alpha$. □

Remark. — Unfortunately, translations $T_{(x_o,\xi_o)}$ on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ do *not* lift to contact transformations of this contact structure. They do of course lift to translations of X by the elements $(x_o, \xi_o, 1) \in \mathbb{H}_n^{\text{red}}$, but these do not preserve α. Indeed, α is right-invariant but not bi-invariant under $\mathbb{H}_n^{\text{red}}$, and the invariance was used up in going to the quotient by $\bar{\Gamma}$. The only elements of $\mathbb{H}_n^{\text{red}}$ which lift to contact transformations are those which normalize $\bar{\Gamma}$, namely $N_{\mathbb{Z}}$ itself.

As above, we let $\Sigma = \{(x, r\alpha_x) : x \in X, r > 0\}$ denote the symplectic cone through (X, α) in $T^*X\backslash 0$. We also let $\Pi : L^2(X) \to H^2(X)$ denote the orthogonal projection (i.e. the Szegö projector) onto the space of L^2 CR functions. From the analysis of Π due to Boutet de Monvel and Sjöstrand [BS], one knows that Π is a Toeplitz structure on Σ. It is obvious that the contact manifold (X, α) has periodic characteristic flow (generated by $\Xi = T$), and that both Π and χ_g commute with T. Hence, by the Unitarization Lemma, we can quantize χ_g as a unitary operator on $H^2(X)$ of the form

$$U_g = \Pi T_{\chi_g} A \Pi \tag{5.3.3}$$

for some pseudodifferential operator over X commuting with T. More precisely, it will be unitary if the *index* of χ_a vanishes, a condition that we will discuss further below. Since U_g commutes with T, it is diagonal with respect to the decomposition (5.4) and hence is equivalent to sequence of finite rank unitary operators

$$U_{N,g} : H^2_\Sigma(N) \to H^2_\Sigma(N), \tag{5.3.3N}$$

the finite dimensional quantizations of g.

Since the Unitarization Lemma constructs U_g in a canonical fashion from the contact transformation χ_g, we should be able to determine it completely in a concrete example. The first step is to determine the principal symbol, or more precisely the function given in (3.10).

To calculate it, we introduce the coordinates $(x, \xi, t, p_x, p_\xi, p_t)$ on $T^*(X)$ with (x, ξ, t) the base coordinates used above and with (p_x, p_ξ, p_t) the sympletically dual fiber coordinates. Thus, the symplectic structure on $T^*(X)$ is given by

$$\Omega := \sum dx_j \wedge dp_{x_j} + d\xi_j \wedge dp_{\xi_j} + dt \wedge dp_t.$$

The cone Σ is then parametrized by $i : \mathbb{R}^+ \times X \to T^*X$, $(r, x, \xi, t) \to (x, \xi, t, 2r\alpha_x)$ and since this is a diffeomorphism we can use the parameters as coordinates on Σ. The equation of Σ is then given by

$$p_x = r\xi \qquad p_\xi = -x \qquad p_t = -r.$$

Hence,

$$i^*(\Omega) = \sum dx_j \wedge d\xi_j + \alpha \wedge dr.$$

We recall that Σ is the characteristic variety of the involutive system (5.2.5) and that the symbol σ_Π of the Szegö projector involves the positive Lagrangian sub-bundle (3.5) of $T\Sigma^\perp \otimes \mathbb{C}$. We now describe these objects concretely:

PROPOSITION (5.3.4). — *At a point $p = i(x_o, \xi_o, t_o, r_o) \in \Sigma$, we have:*

(a)

$$T_p\Sigma^\perp = sp\left\{X_j + r_o\frac{\partial}{\partial p_{x_j}}, \Xi_j - r_o\frac{\partial}{\partial p_{\xi_j}}\right\}$$

(b)

$$\Lambda_p = sp_{\mathbb{C}}\left\{\overline{Z}_j + r_o\left(\frac{\partial}{\partial p_{x_j}} + i\frac{\partial}{\partial p_{\xi_j}}\right)\right\}.$$

Proof. — (a) Using the above parametrization, we find that

$$\begin{aligned}
i_*\frac{\partial}{\partial x_j} &= \frac{\partial}{\partial x_j} - r_o\frac{\partial}{\partial p_{\xi_j}}\\
i_*\frac{\partial}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j} + r_o\frac{\partial}{\partial p_{x_j}}\\
i_*\frac{\partial}{\partial t} &= \frac{\partial}{\partial t}\\
i_*\frac{\partial}{\partial r} &= \xi_{oj}\frac{\partial}{\partial p_{x_j}} - x_{oj}\frac{\partial}{\partial p_{\xi_j}} - \frac{\partial}{\partial p_t}
\end{aligned}$$

from which it is simple to determine the vectors X such that $\Omega(X, T_p\Sigma) = 0$.

(b) The operators D_j of §3 are the operators $\overline{Z}_j$ of (5.4b) whose symbols are given by

$$\sigma_{D_j} = ip_{x_j} - p_{\xi_j} + (x_j + i\xi_j)p_t.$$

Their Hamilton vector fields

$$H_{\sigma_j} = \frac{1}{i}\left(X_j + i\Xi_j + ir_o\left(\frac{\partial}{\partial p_{x_j}} + i\frac{\partial}{\partial p_{\xi_j}}\right)\right)$$

are easily seen to agree (up to complex scalars) with the vector fields asserted to span Λ_p. □

We now wish to determine the vacuum states corresponding to Λ and $\chi(\Lambda)$. Recall that, given a symplectic frame of $T_p\Sigma^\perp$, we get a representation $d\rho_p$ of the Heisenberg algebra on the space $\mathcal{S}((\Sigma^\perp)_p$ (see §3) and that the vacuum state e_{Λ_p} is the unique state annihilated by the elements of Λ_p. To determine it, we choose the symplectic frame

$$\mathcal{B}_p := \left\{\frac{1}{\sqrt{r_o}}\mathrm{Re}\ H_{\sigma_j}, \frac{1}{\sqrt{r_o}}\ \mathrm{Im}\ H_{\sigma_j}, j = 1, \dots, n\right\}$$

and write a vector $V \in (T_p\Sigma)^\perp$ as $V = \sum \alpha_j \frac{1}{\sqrt{r_o}}\mathrm{Re}\ H_{\sigma_j} + \beta_j \frac{1}{\sqrt{r_o}}\ \mathrm{Im}\, H_{\sigma_j}$. We observe that $\left\{\frac{1}{\sqrt{r_o}}\mathrm{Re}H_{\sigma_j}, \frac{1}{\sqrt{r_o}}\ \mathrm{Im}\, H_{\sigma_j}, T\right\}$ form a Heisenberg algebra and that under the Schrödinger representation $d\rho_p$ they go over to $\left\{\frac{\partial}{\partial\alpha_j}, \alpha_j, 1\right\}$.

PROPOSITION (5.3.5). — *With the above notation: The vacuum state e_{Λ_p} equals the Gaussian $e^{-\frac{1}{2}|\alpha|^2}$.*

Proof. — The annihilation operators in the representation $d\rho_p$ are given by the usual expressions $\frac{\partial}{\partial\alpha_j} + \alpha_j$ and hence the vacuum state is the usual one in the Schrödinger representation. □

Now consider the image of Λ under the contact transformation χ_g, or more precisely its lift as the symplectic transformation

(5.3.6)

$$\tilde\chi_g(x, \xi, t, p_x, p_\xi, p_t) = (Ax + B\xi, Cx + D\xi, t, Dp_x - Cp_\xi, -Bp_x + Ap_\xi, p_t)$$

of T^*X. Of course, it is linear in the given coordinates. We would like to compare $d\tilde\chi_{g,p}(\Lambda_p)$ and $\Lambda_{\tilde\chi_g(p)}$.

PROPOSITION (5.3.7). — *Under $d\tilde\chi_{g,p}$ we have, in an obvious matrix notation:*

(a)

$$\begin{aligned} X &\to AX + C\Xi \\ \Xi_j &\to BX + D\Xi \end{aligned}$$

(b)
$$\frac{\partial}{\partial p_x} \to B\frac{\partial}{\partial p_x} + D\frac{\partial}{\partial p_\xi}$$
$$\frac{\partial}{\partial p_\xi} \to A\frac{\partial}{\partial p_x} + C\frac{\partial}{\partial p_\xi}$$

(c)
$$\mathrm{Re}\, iH_\sigma \to A(\mathrm{Re}\, iH_\sigma) + C(\mathrm{Im}\, iH_\sigma)$$
$$\mathrm{Im}\, iH_\sigma \to B(\mathrm{Re}\, iH_\sigma) + D(\mathrm{Im}\, iH_\sigma)$$

(d) $d\tilde{\chi}_{g,p}\mathcal{B}_p = g^*\mathcal{B}_{\chi(p)}$

(e) $e_{\Lambda_{\chi g}} = \mu(g^*)e_\Lambda$ *where μ is the metaplectic representation.*

Proof. — The formulae in (a)-(b) are easy calculations left to the reader. The ones in (c)-(d) are immediate consequences. The statement in (e) follows from the change in the Schrodinger representation under a change of metaplectic basis [BG]. □

The desired principal symbol is determined by the following proposition.

PROPOSITION (5.3.8). — *Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then the inner product $\langle e_{\Lambda_{\chi g}}, e_\Lambda \rangle$ in the Schrodinger representation equals:*

$$\langle e_{\Lambda_{\chi g}}, e_\Lambda \rangle = 2^{\frac{n}{2}} (\det (A + D + iB - iC))^{-\frac{1}{2}}.$$

Proof. — Let $Z = X + iY$ be a complex symmetric matrix with $Y >> 0$, and let $\gamma_Z(x) := e^{\frac{i}{2}<Zx,x>}$ be the associated Gaussian. The action of an element $g \in Mp(2n, \mathbb{R})$ is the given by:

$$\mu(g^{*-1})\gamma_Z = m(g, Z)\gamma_{\alpha(g)Z}$$

where

$$m(g, Z) = \det{}^{-\frac{1}{2}}(CZ + D), \qquad \alpha(g)Z = (AZ + B)(CZ + D)^{-1}$$

(see [F], Ch.4.5). We may assume $e_\Lambda = \dfrac{\gamma_i}{||\gamma_i||}$ and since

$$\mu(g^*)\gamma_i = m(g^{-1}, i)\gamma_{g^{-1}i}$$

we have

$$\langle e_{\Lambda_{\chi g}}, e_\Lambda \rangle = m(g^{-1}, i)\langle \gamma_{g^{-1}i}, \gamma_i \rangle.$$

The inner product of two Gaussians is given by

$$\langle \gamma_\tau, \gamma_{\tau'} \rangle = \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle (\tau - \overline{\tau'})\xi, \xi \rangle} d\xi = \frac{1}{\sqrt{\det\left[-i(\tau - \overline{\tau'})\right]}} \tag{5.3.9}$$

with the usual analytic continuation of the square root [F]. Putting $\tau = g^{-1}iI$ and $\tau' = iI$ and simplifying we get the stated formula. □

For future reference we will rephrase the previous proposition in the following form:

COROLLARY (5.3.10). — *The Toeplitz operator*

$$U_g := m(g)\Pi\chi_g\Pi \qquad m(g) = 2^{\frac{-n}{2}}(det(A + D + iB - iC))^{\frac{1}{2}}$$

is unitary modulo compact operators.

We will now see that U_g is actually unitary if $g \in Sp_\theta(2, \mathbb{Z})$ or if g lies in the image of the natural embedding of $Sp_\theta(2, \mathbb{Z})$ in $Sp_\theta(2n, \mathbb{Z})$. The same statements are true for the other elements $Sp_\theta(2n, \mathbb{Z})$, but we will restrict to these elements so that we can easily quote from [KP].

5.4. Theta functions.

We begin with a rapid review of the transformation theory of theta functions under elements $g \in Sp_\theta(2, \mathbb{Z})$. As above, in dimensions larger than two, $Sp_\theta(2, \mathbb{Z})$ is understood to be embedded in $Sp_\theta(2n, \mathbb{Z})$ as the block matrices $\begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}$ with I_n the $n \times n$ identity matrix. For this case, we closely follow the exposition of Kac-Peterson [KP]. For more classical treatments of transformation laws, and in more general cases, see [Bai] [Kloo].

NOTATION. — *$\mathcal{H}_+ := \{\tau = x + iy | x, y \in \mathbb{R}, y > 0\}$ will denote the Poincaré upper half-plane and the standard action of $SL(2, \mathbb{R})$ on $\mathcal{H}_+$ will be written*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

$U_{\mathbb{R}} \equiv \mathbb{R}^n$ will denote a real vector space of dimension n, equipped with a positive definite symmetric bilinear form $<,>$, and $U = U_{\mathbb{R}} \otimes \mathbb{C}$. The Heisenberg group will be taken in the unreduced form $N_{\mathbb{R}} = U_{\mathbb{R}} \times U_{\mathbb{R}} \times \mathbb{R}$

with multiplication $(x,\xi,t)(x',\xi',t') = (x+x',\xi+\xi',t+t'+\frac{1}{2}[\langle x',\xi\rangle - \langle x,\xi'\rangle])$.

To quantize $SL(2,\mathbb{Z})$ as a group action, one has to lift to the metaplectic group

$$Mp(2,\mathbb{R}) := \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right) : j(\tau)^2 = c\tau + d, j : \mathcal{H}_+ \to \mathbb{C} \text{ holomorphic}\right\}.$$

Set

$$Y := \mathcal{H}_+ \times U \times \mathbb{C}$$

and let $Mp(2,\mathbb{R})$ act on Y by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right)(\tau,z,t) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, t + \frac{c}{2}\frac{< z,z >}{c\tau+d}\right). \tag{5.4.1}$$

Also let the Heisenberg group act by
(5.4.2)

$$(x,\xi,t_o)\cdot(\tau,z,t) := (\tau, z-x+\tau\xi, t- < \xi, z > -\frac{1}{2}\tau < \xi,\xi > + \frac{1}{2} < x,\xi > +t_o).$$

Let $G_{\mathbb{R}}$ be the semi-direct product of $Mp(2,\mathbb{R})$ with $N_{\mathbb{R}}$, with $gng^{-1} = g\cdot n$. It acts on functions on Y by

$$f\Big|_{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right)}(\tau,z,t) = j(\tau)^{-n} f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau,z,t)\right) \tag{5.4.3}$$

$$f|_n(\tau,z,t) = f(n(\tau,z,t)).$$

Now let L denote a lattice of full rank in $U_{\mathbb{R}}$ such that $\langle\gamma,\gamma'\rangle \in \mathbb{Z}$ for all $\gamma,\gamma' \in L$, let L^* be the dual lattice $\{\gamma : \langle\alpha,\gamma\rangle \in \mathbb{Z}(\forall\alpha \in L)\}$. For the sake of simplicity we will assume $L = L^*$ and in fact that $L = \mathbb{Z}^n$. Then define the integral subgroup

$$N_{\mathbb{Z}} = \left\{(x,\xi,t) \in N_{\mathbb{R}} : x,\xi \in L, t + \frac{1}{2}\langle x,\xi\rangle \in \mathbb{Z}\right\}. \tag{5.4.4}$$

The normalizer $G_{\mathbb{Z}}$ of $N_{\mathbb{Z}}$ in $N_{\mathbb{R}}$ is given by
(5.4.5)

$$G_{\mathbb{Z}} = \begin{cases} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right)(\alpha,\beta,t) \in G_{\mathbb{R}} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}); \\ bd\langle\gamma,\gamma\rangle \equiv 2\langle\alpha,\gamma\rangle \bmod 2\mathbb{Z}, ac\langle\gamma,\gamma\rangle \equiv 2\langle\beta,\gamma\rangle \bmod 2\mathbb{Z}, \forall\gamma \in \mathbb{Z}^n\}. \end{cases}$$

In particular, $Sp_\theta(2,\mathbb{Z}) \subset G_{\mathbb{Z}}$.

DEFINITION (5.4.4). — *The space of theta functions of degree N is the space $\tilde{Th}_N$ of holomorphic functions f on Y satisfying:*

$$f|_n = f \quad (\forall n \in N_{\mathbb{Z}}), \qquad f|_{(o,o,t)} = e^{-2\pi i N t} f.$$

The entire ring of theta functions is the space

$$\tilde{Th} := \bigoplus_{N \in N} \tilde{Th}_N.$$

We observe that $\tilde{Th}_1$ puts together the holomorphic (pre-quantum) line bundles

$$\mathcal{L}_\tau \to U/(L + \tau L) \tag{5.4.5}$$

over $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ -torus as the one-parameter family of complex structures parametrized by τ varies. Indeed, following [KP], p. 181 we observe that the natural projection

$$\pi : Y = \mathcal{H}_+ \times U \times \mathbb{C} \to \mathcal{H}_+ \times U$$

defines a holomorphic line bundle. The group $\overline{N}_{\mathbb{Z}} := N_{\mathbb{Z}}/\mathbb{Z}$ acts freely by bundle maps, so the quotient line bundle

$$\overline{\pi} : Y/\overline{N}_{\mathbb{Z}} \to (\mathcal{H}_+ \times U)/\overline{N}_{\mathbb{Z}} \tag{5.4.6}$$

defines a holomorphic line bundle which for each fixed τ restricts to (5.4.6τ). Similarly for the powers $\mathcal{L}^{\otimes N}$. Hence $\tilde{Th}$ simeltaneously puts together theta-functions of all degrees and complex structures in the one-parameter family above. If we fix τ we get the space Th_N^τ of holomorphic sections of $\mathcal{L}_\tau^{\otimes N}$, that is, the space of holomorphic theta functions of degree N relative to τ.

5.5. Classical theta functions of degree N and characteristic μ à la [KP].

We now introduce a specific basis of the theta functions of any degree and with respect to any complex structure τ. These are not yet the theta functions which will play the key role in Theorem D, but are a preliminary version of them. We follow the notation and terminology of [KP] except that we put $L = \mathbb{Z}^n = L^*$.

For $\mu \in \mathbb{Z}^n/N\mathbb{Z}^n$, define the *classical theta function of degree n* and *characteristic* μ with respect to the complex structure τ by

$$\Theta_{\mu,N}(\tau,z,t) := e^{-2\pi i N t} \sum_{\gamma \in \mathbb{Z}^n + \frac{\mu}{N}} e^{2\pi i N\{\frac{1}{2}\tau<\gamma,\gamma> - <\gamma,z>\}}. \tag{5.5.1}$$

When the degree N=1 and $\mu = 0$ this is the *Riemann theta function* for the lattice $\mathbb{Z}^n$,

$$\Theta(\tau,z,t) := e^{-2\pi i t} \sum_{\gamma \in \mathbb{Z}^n} e^{2\pi i\{\frac{1}{2}\tau<\gamma,\gamma> - <\gamma,z>\}} \tag{5.5.2}$$

while the general theta function of degree 1 and characteristic $\mu \in \mathbb{R}^n/\mathbb{Z}^n$ is given by

$$\Theta_{\mu}(\tau,z,t) := \Theta|_{(0,-\mu,0)} = e^{-2\pi i t} \sum_{\gamma \in \mathbb{Z}^n + \mu} e^{2\pi i\{\frac{1}{2}\tau<\gamma,\gamma> - <\gamma,z>\}}. \tag{5.5.2μ}$$

One has the following:

PROPOSITION (5.5.3) (see [KP], Lemma 3.12). — *Fix* τ. *Then:*

$$\{\Theta_{\mu,N}|_{Y_\tau}\}_{\mu \in \mathbb{Z}^n/N\mathbb{Z}^n} \text{ is a } \mathbb{C}\text{-basis of } Th_N^\tau.$$

5.6. Transformation laws.

The transformation laws for classical theta functions are given by the following:

TRANSFORMATION LAW (5.6.1) ([KP], Proposition 3.17). — *Let* $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right) \in Mp(2,\mathbb{R})$ *be an element satisfying:*

$$Nbd \equiv 0 \bmod 2\mathbb{Z} \qquad Nac \equiv 0 \bmod 2\mathbb{Z}.$$

Then there exists $\nu(N,g) \in \mathbb{C}$ *such that, for* $\mu \in \mathbb{Z}$,

$$\Theta^L_{\mu,N}|_g = \nu(N,g) \sum_{\substack{\alpha \in \mathbb{Z}^n \\ c\alpha \bmod N\mathbb{Z}^n}} e^{i\pi[N^{-1}cd\alpha^2 + 2N^{-1}bc\alpha\mu + N^{-1}ab\mu^2]}\Theta^L_{a\mu+c\alpha,N}.$$

The matrix of g with respect to the above $\mathbb{C}$*-basis is unitary.*

The multiplier $\nu(N,g)$ is described in detail in [KP], loc.cit and involves the Jacobi symbol.

For the generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL(2,\mathbb{Z})$, and for $n = 1$, the transformation law reads:
(5.6.2S)

$$\Theta_{\mu,N}\Big(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{z^2}{2\tau}\Big) = (-i\tau)^{\frac{1}{2}} (N)^{-\frac{1}{2}} \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} e^{-\frac{2\pi i \mu\alpha}{N}} \Theta_{\alpha,N}(\tau, z, t)$$

(5.6.2T) $$\Theta_{\mu,N}(\tau+1, z, t) = e^{2\pi i \frac{|\mu|^2}{N}} \Theta_{\mu,N}(\tau, z, t).$$

We note that (5.6.2S) is the formula for the finite Fourier transform on $\mathbb{Z}/N$ [AT], p. 853.

5.7. The space $\Theta^\tau(N)$ of theta functions $\vartheta^\tau_{\mu,N}$.

We now specify the theta functions which will play the key role in linking the classical transformation theory to the action of the quantized contact transformation U_{χ_g}. They are essentially the (variable degree) versions of the 'most natural and basic' theta functions of [M] and coincide with the span $\Theta(N)$ of the theta-functions denoted ϕ_{Nj} in [AT].

The reader should note that the expression $\Theta_{\mu,N}|_n(\tau, z, t)$ depends on many variables. In different articles, different sets of variables are viewed as the significant ones. Here we wish to regard theta-functions as functions on $N_{\mathbb{Z}}/N_{\mathbb{R}}$ so we emphasize the $n \in N_{\mathbb{R}}$ variable. In other contexts, (τ, z) are viewed as the significant variables (cf. [Bai] [M] [Kloo]).

DEFINITION (5.7.1). — *For $\mu \in \mathbb{Z}^n/N\mathbb{Z}^n$, put:*

$$\begin{aligned}\vartheta^\tau_{\mu,N}(x,\xi,t) &:= e^{-2\pi i N t}\Theta_{\mu,N}|_{(x,\xi,0)}(\tau,0,0) \\ &= e^{-2\pi i N t} \sum_{\gamma\in\mathbb{Z}^n} e^{2\pi i N[\frac{\tau}{2}\langle \xi + \frac{\mu}{N}+\gamma, \xi+\frac{\mu}{N}+\gamma\rangle + \langle \frac{\mu}{N}\xi+\gamma, x\rangle]}.\end{aligned}$$

The significance of these theta-functions is due to the following:

PROPOSITION (5.7.2). — *The theta functions $\vartheta^\tau_{\mu,N}$ satisfy:*

(i) $\vartheta^{\tau}_{\mu,N} \in H_N(N_{\mathbb{Z}}\backslash N_{\mathbb{R}})$;

(ii) *As* μ *runs over* $\mathbb{Z}^n/N\mathbb{Z}^n$, $\vartheta^{\tau}_{\mu,N}(x,\xi,t)$ *forms a basis of the CR functions of degree (= weight) N on* $N_{\mathbb{Z}}\backslash N_{\mathbb{R}}$.

Proof. — First, for $\mu \in \mathbb{Z}^n/N\mathbb{Z}^n, \Theta_{\mu,N}$ is $N_{\mathbb{Z}}$-invariant as a function on Y, that is, $\Theta_{\mu,N}|_n = \Theta_{\mu,N}$ for $n \in N_{\mathbb{Z}}$ ([KP], 3.2). It follows that

$$\vartheta^{\tau}_{\mu,N}(n(x,\xi,0)) = \Theta_{\mu,N}|_{n(x,\xi,0)}(\tau,0,0) = \Theta_{\mu,N}|_{(x,\xi,0)}(\tau,0,0)$$

since $|_n$ is a right action.

The CR property is a direct consequence of the fact that the theta functions are holomorphic on Y. To give a complete proof of this, one would have to introduce the CR structure $\overline{Z}_j$ on $N_{\mathbb{Z}}\backslash N_{\mathbb{R}}$ corresponding to a complex structure Z on the torus $N_{\mathbb{Z}}\backslash N_{\mathbb{R}}/\mathcal{Z}$ (with $\mathcal{Z}$ the center), and verify that differentiation of $\Theta_{\mu,N}|_{(x,-\xi,0)}(\tau,0,0)$ by $\overline{Z}_j$ in the (x,ξ,t)-variables is equivalent to differentation of $\Theta_{\mu,N}(\tau,z,t)$ in $\overline{\partial_z}$. For the details of this calculation we refer the reader to [M], p. 22 or [A].

Granted the CR property, the statement that the $\vartheta^{\tau}_{\mu,N}$'s form a basis for the CR funtions of weight N relative to the CR structure τ follows from Proposition (5.5.3). □

The proposition has the following representation-theoretic interpretation: $H_N(N_{\mathbb{Z}}\backslash N_{\mathbb{R}})$ is reducible as a unitary representation of $N_{\mathbb{R}}$ for $N > 1$, and the space $H^2_{\Sigma}(N)$ of CR functions in H_N consists of the lowest weight vectors. For the multiplicity theory, see [A] [AT].

We now record the modified transformation laws for the theta functions $\vartheta^{\tau}_{\mu,N}$ under elements $g \in Sp_{\theta}(2n,\mathbb{Z})$. It will be these transformation laws which will be used to prove Theorem D.

PROPOSITION (5.7.3) (Transformation laws for $\vartheta^{\tau}_{\mu,N}$). — *As above, let* $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, j\right) \in Mp(2,\mathbb{R})$ *be an element satisfying*

$$Nbd \equiv 0 \bmod 2\mathbb{Z} \qquad Nac \equiv 0 \bmod 2\mathbb{Z}.$$

Then there exists $\nu(N,g) \in \mathbb{C}$ *such that, for* $\mu \in \mathbb{Z}^n/N\mathbb{Z}^n$,

$$\vartheta^{\tau}_{\mu,N}(g\cdot(x,\xi,t)) = \nu(N,g)j(g^{-1}\tau)^n \sum_{\substack{\alpha\in\mathbf{Z}^n \\ c\alpha \bmod N\mathbf{Z}^n}} e^{i\pi[N^{-1}(cd)\alpha^2+2N^{-1}bc\alpha\mu+N^{-1}(ab)\mu^2]}\vartheta^{\tau'}_{a\mu-c\alpha,N}((x,\xi,t)$$

with $\tau' = g^{-1}\tau = \dfrac{d\tau - b}{-c\tau + a}$. *The matrix of* g *with respect to the above* $\mathbb{C}$*-basis is unitary.*

Proof. — We may (and will) set $t = 0$ on both sides. Then,

$$\begin{aligned}\vartheta^{\tau}_{\mu,N}(ax + b\xi, cx + d\xi, 0) :&= \Theta_{\mu,N}|_{(ax+b\xi,cx+d\xi,0)}(\tau, 0, 0)\\ &= \Theta_{\mu,N}|_{g\cdot(x,\xi,0)}(\tau, 0, 0).\end{aligned}$$

Here, $g \cdot (x, \xi, 0)$ is the action of $(g, j) \in Mp(2, \mathbb{R})$ on $(x, \xi, 0)$ as an automorphism of $N_{\mathbb{R}}$.

Now recall that in the semi-direct product $Mp(2, \mathbb{R})N_{\mathbb{R}}$, we have $(g, j)n(g, j)^{-1} = (g \cdot n)$. Hence

$$\begin{aligned}\Theta_{\mu,N}|_{g\cdot(x,\xi,0)}(\tau, 0, 0) &= \Theta_{\mu,N}|_{(g,j)(x,\xi,0)(g,j)^{-1}} \cdot (\tau, 0, 0))\\ &= j(g^{-1}\tau)^n[\Theta_{\mu,N}|_{(g,j)}]|_{(x,\xi,0)}(g^{-1}\tau, 0, 0).\end{aligned} \tag{5.7.4}$$

Applying the transformation laws (5.6.1), the last expression becomes

$$= \nu(N, g)j(g^{-1}\tau)^n \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} e^{i\pi[N^{-1}(cd)\alpha^2+2N^{-1}bc\alpha\mu+N^{-1}(ab)\mu^2]} \Theta_{a\mu-c\alpha,N}|_{(x,\xi,0)}(g^{-1}\tau, 0, 0)$$

$$= \nu(N, g)j(g^{-1}\tau)^n \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} e^{i\pi[N^{-1}(cd)\alpha^2+2N^{-1}bc\alpha\mu+N^{-1}(ab)\mu^2]} \vartheta^{\tau'}_{a\mu-c\alpha,N}(x, \xi, 0).$$

The unitarity of the matrix of coefficients follows from the usual transformation rule. □

5.8. Θ^{τ}_N as a Heis $(\mathbb{Z}^n/N)$-module.

As mentioned above, Θ^{τ}_N is an irreducible representation for the finite Heisenberg group Heis$(\mathbb{Z}^n/N)$. We pause to define this group and its action on Θ^{τ}_N. This will clarify the distinguished role of the classical theta functions as a basis for Θ^{τ}_N and hence will make explicit the isomorphism to $L^2(\mathbb{Z}/N)$, which is the setting for the quantized cat maps in [HB] [dEG.I] [Kea]. It will also clarify the relation between the dynamics of cat maps as studied in the semi-classical literature and those studied in [B] [BNS].

In the following, $\mathbb{C}_1^*$ denotes the unit circle in $\mathbb{C}$, $\mathbb{C}_1^*(N)$ denotes the group of Nth roots of unity, and $\pm C_1^*(N)$ denotes the group of elements $\pm e^{2\pi i \frac{j}{N}}$.

DEFINITION (5.8.1). — *The finite Heisenberg group* Heis$(\mathbb{Z}^n/N)$ *is the subset of elements of*

$$\mathbb{Z}^n/N\mathbb{Z}^n \times \mathbb{Z}^n/N\mathbb{Z}^n \times (\pm\mathbb{C}_1^*(N))$$

generated by $\mathbb{Z}^n/N \times \mathbb{Z}^n/N$ *and* $\mathbb{C}_1^*(N)$ *under the group law*

$$(m, n, e^{i\phi}) \cdot (m', n', e^{i\phi'}) = (m + m', n + n', e^{\frac{2\pi i}{N}\sigma((m,n),(m',n'))} e^{i(\phi+\phi')}).$$

In the terminology of [M], Heis$(\mathbb{Z}^n/N)$ is a generalized Heisenberg group in the following sense: A general Heisenberg group $G = \text{Heis}\,(K, \psi)$ is a central extension by $\mathbb{C}_1^*$ of a locally compact abelian group K:

$$1 \to \mathbb{C}_1^* \to G \to K \to 0 \tag{5.8.2}$$

satisfying the following conditions:

(i) As a set $G = K \times \mathbb{C}_1^*$;

(ii) The group law is given by

$$(x, \lambda) \cdot (\mu, y) = (\lambda\mu\psi(x, y), x + y)$$

where $\psi : K \times K \to \mathbb{C}_1^*$ is a 2-cocycle:

$$\psi(x, y)\psi(x + y, z) = \psi(x, y + z)\psi(y, z);$$

(iii) Define a map $e : K \times K \to \mathbb{C}_1^*$ by

$$e(x, y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$$

where $\tilde{x}, \tilde{y}$ are any lifts of x, y to G ($e(x, y)$ is independent of the choice). Also define $\phi : K \to \hat{K}$ by $\phi(x)(y) = e(x, y)$. Then ϕ is an isomorphism. Here, $\hat{K}$ is the group of characters of K.

In the case of Heis$(\mathbb{Z}^n/N)$, $K = \mathbb{Z}^n/N\mathbb{Z}^n \times \mathbb{Z}^n/N\mathbb{Z}^n$ and ψ is given by $\psi(v, w) := e^{\frac{2\pi i}{N}\sigma(v,w)}$ where σ is the restriction of the symplectic form to $\mathbb{Z}^{2n}$. Also, we consider the finite subgroup generated by K and by $\mathbb{C}_1^*(N)$.

The analogues of Lagrangian subspaces in the case $K = T^*\mathbb{R}^n$ are the maximally isotropic subgroups. Here, a subgroup $H \subset K$ is called *isotropic* if $e_{H\times H} \equiv 1$ and is *maximally isotropic* if it is maximal with this property. Examples of maximal isotropic subgroups of Heis$(\mathbb{Z}^n/N)$ are given by $\mathbb{Z}^n/N\mathbb{Z}^n$ and by the character group $\widehat{\mathbb{Z}^n/N\mathbb{Z}^n}$.

Given any isotropic subgroup, there is a (splitting) homomorphism

$$s : H \to G \qquad s(h) = (h, F(h)) \tag{5.8.3}$$

such that $\pi \cdot s = \mathrm{id}_H$. Here, $\pi : G \to K$ is the map in (5.8.2). The map $F(h)$ satisfies:

$$\frac{F(a+b)}{F(a)\cdot F(b)} = \psi(a,b) \qquad (a,b \in H).$$

Given a maximal isotropic subgroup $H \subset K$ and a splitting homomorphism $s : H \to G$, one defines the Hilbert space:

(5.8.4)

$$\mathcal{H} = \{f : K \to \mathbf{C} : f \in L^2(K/H), f(x+h) = F(h)^{-1}\psi(h,x)^{-1}f(x) \ \ \forall h \in H\}$$

and the representation ρ of G on $\mathcal{H}$

$$\rho(k,\lambda)f(x) := \lambda\psi(x,k)f(x+k).$$

Then: ρ is an irreducible representation, and is the unique irreducible with the given central character.

The choice of $\widehat{\mathbb{Z}^n/N\mathbb{Z}^n}$ gives a close analogue of the Schrödinger representation in the real case. The associated Hilbert space may be identified with $L^2(\mathbb{Z}^n/N\mathbb{Z}^n)$ and the representation is given by

(5.8.5)

$$U_{(0,0\lambda)}f(b) = \lambda f(b) \qquad U_{(a,0,0)}f(b) = e^{\frac{2\pi i}{N}\langle a,b\rangle}f(b) \qquad U_{(0,\chi_a,0)} = f(b+a).$$

Now let us return to Θ_N^τ. We first observe that $\Theta_{\mu,N}$ is constructed from Θ by:

$$\begin{aligned}\Theta_{\mu,N}(\tau,z,t) &= \Theta^N_{\frac{\mu}{M}}(\tau,z,Nt)\\ \Theta_\mu(\tau,z,t) &= \Theta|_{(o,-\mu,o)}(\tau,z,t)\end{aligned} \tag{5.8.6}$$

where Θ^N is the same as Θ except that the complex quadratic form $< \cdot,\cdot >$ is replaced by $N < \cdot,\cdot >$.

We further observe with [KP] (3.10), that

$$(5.8.7)\qquad \begin{aligned}\Theta_{\mu,N}|_{(\frac{\mu'}{N},0,0)} &= e^{\frac{2\pi i}{N}<\mu,\mu'>}\Theta_{\mu,N}\\ \Theta_{\mu,N}|_{(0,\frac{\mu'}{N},0)} &= \Theta^L_{\mu-\mu',N}\end{aligned}$$

where $\mu, \mu' \in$ vary over $\mathbb{Z}^n/N\mathbb{Z}^n$. Since $\Theta_{\mu,N}$ depends only on μ mod $N\mathbb{Z}^n$, we see that (5.8.7) defines the same representation of Heis $(\mathbb{Z}^n/N)$ as in (5.8.5).

The same situation holds for the theta functions $\vartheta^\tau_{\mu,N}$, but we rephrase things slightly. First, with [AT] let us set

$$(5.8.7)\qquad \vartheta^{\tau,N}(x,\xi,t) := e^{-2\pi it}\sum_{\gamma\in\mathbb{Z}^n} e^{2\pi i[N\frac{\tau}{2}\langle\xi+\gamma,\xi+\gamma\rangle+\langle\gamma,x\rangle]}.$$

One can verify that $\vartheta^{\tau,1} = \vartheta^\tau_{0,1}$. Then define the Heisenberg dilations

$$(5.8.8)\qquad D_m : N_{\mathbb{R}} \to N_{\mathbb{R}} \qquad D_m(x,\xi,t) = (mx,\xi,mt)$$

which are automorphisms of $N_{\mathbb{R}}$. Associated to them are the dilation operators

$$D_N : \Theta_m \to \Theta_{Nm} \qquad D_N f = f\cdot D_N.$$

Then we have

$$(5.8.9)\qquad D_N\vartheta^{\tau,N} = \vartheta^\tau_{0,N} = e^{-2\pi iNt}\sum_{\gamma\in\mathbb{Z}^n} e^{2\pi iN[\frac{\tau}{2}\langle\xi+\gamma,\xi+\gamma\rangle+\langle\gamma,x\rangle]}.$$

(5.8.9μ)

$$\vartheta^\tau_{\mu,N} = \vartheta^\tau_{0,N}|_{(0,-\frac{\mu}{N},0)} = e^{-2\pi iNt}\sum_{\gamma\in\mathbb{Z}^n} e^{2\pi iN[\frac{\tau}{2}\langle-\xi+\frac{\mu}{N}+\gamma,-\xi+\frac{\mu}{N}+\gamma\rangle+\langle\frac{\mu}{N}-\xi+\gamma,x\rangle]}$$

as stated in (5.7.1).

Consider in particular the case of dimension $n = 1$. Then relative to the basic theta functions $\vartheta^\tau_{\mu,N}$ the elements $V = \left(0,\frac{1}{N},1\right)$ and $U = \left(\frac{1}{N},0,1\right)$ of Heis $(\mathbb{Z}/N)$ are representated by the matrices

$$V : e_1 \to e_2, \dots, e_n \to e_1 \qquad U := \operatorname{diag}(1, e^{2\pi i\frac{1}{N}}, \dots, e^{2\pi i(N-1)\frac{1}{N}})$$

where $\{e_i\}$ denotes the standard basis of $\mathbb{C}^N$. These elements satisfy $UV = e^{\frac{2\pi i}{N}}VU$ hence generate the rational rotation algebra $\mathcal{M}_{\frac{1}{N}}$ with

Planck constant $h = \frac{1}{N}$. Hence Θ_N determines a finite dimensional representation π of this algebra, with image the group algebra $\mathbb{C}[\text{Heis}\,(\mathbb{Z}/N)]$ of the finite Heisenberg group. Moreover, the transformation laws define $Sp_\theta(2, \mathbb{Z}/N)$ as a covariant group of automorphisms of $\mathbb{C}[\text{Heis}\,(\mathbb{Z}/N)]$. From the dynamical point of view, these automorphisms are very different from the automorphisms defined by $Sp_\theta(2, \mathbb{Z}/N)$ on $\mathcal{M}_{\frac{1}{N}}$ (as in [B] [BNS]): Indeed, the representation π kills the center of $\mathcal{M}_{\frac{1}{N}}$, and since it is finite dimensional representation the automorphisms have discrete spectra.

5.9. Finite degree Cauchy-Szegö projectors and change of complex structure.

As mentioned several times above, we would like to view the transformation laws as defining a unitary operator on the space Θ_N^i of theta functions with a fixed complex structure. However, as things stand, the transformation laws (5.7.3) change the complex structure τ into $\tau' = \frac{a\tau - b}{-c + d\tau}$. The purpose of this section is to use the degree N Cauchy-Szegö projector to change the complex structure back to τ.

It is right at this point that the Toeplitz method differs most markedly from the Kähler quantization method of [AdPW] [We]. In the Kähler scheme, the unitary (BKS) operator carrying $\tilde{Th}_N^{\tau'}$ back to $\tilde{Th}_N^{\tau}$ is parallel translation with respect to a natural flat connection on the vector bundle $\tilde{\Theta}_N$ over the moduli space of complex structures, whose fiber over τ is the space $\tilde{Th}_N^{\tau}$. As discussed in these articles, the connection is defined by the heat equation for theta functions. Since the *classical theta functions* are solutions of this equation, they are already a parallel family with respect to the connection–hence the unitary BKS operator in the Kähler setting is simply to 'forget' the change in complex structure $\tau \to \tau'$. Thus, the unitary matrix defined relative to the classical theta functions is precisely the quantization of g in the Kähler sense. It is also the quantization of [BH] [dBB] [dEGI] [Ke], as the interested reader may confirm by comparing their formulae for the quantized cat maps with the expressions in the transformation formulae.

Our purpose now is to show that the Toeplitz method leads to the same result.

LEMMA (5.9.1). — *Let Π^τ_N be the orthogonal projection onto $H^2_{\Sigma_\tau}(N) \equiv \Theta^\tau_N$ and let $\Pi^{\tau,\tau'}_N := \Pi^\tau_N \Pi^{\tau'}_N : \Theta^{\tau'}_N \to \Theta^\tau_N$. Then: $\Pi^{\tau,\tau'*}_N \Pi^{\tau,\tau'}_N = (4\pi)^n \Big| \frac{(\mathrm{Im}\tau \mathrm{Im}\,\tau')^{\frac{n}{4}}}{(-2\pi i(\tau - \overline{\tau'}))^{\frac{n}{2}}}\Big|^2 \Pi^\tau_N$.*

Proof. — Let $f \in \Theta^\tau_N$, and $g \in \Theta^{\tau'}_N$, for any pair of complex structure τ, τ'. As elements of $L^2(N_{\mathbb{Z}}/N_{\mathbb{R}})$ their inner product is given by

$$(f|g) := \int_{N_{\mathbb{Z}}\backslash N_{\mathbb{R}}} (f|_n)\overline{(g|_n)} dn$$

with $dn = dx d\xi dt$ the $N_{\mathbb{R}}$-invariant measure on $N_{\mathbb{Z}}\backslash N_{\mathbb{R}}$. Our main task is to calculate the inner products

$$(\vartheta^\tau_{\mu,N} | \vartheta^{\tau'}_{\mu',N})$$

in $H_N(N_{\mathbb{Z}}\backslash N_{\mathbb{R}})$. The lemma is equivalent to the following:

CLAIM (5.9.2).

$$(\vartheta^\tau_{\mu,N} \mid \vartheta^{\tau'}_{\mu',N}) = \delta_{\mu,\mu'} \mathrm{vol}\,(\mathbb{R}^n/\mathbb{Z}^n)(-2\pi i N(\tau - \overline{\tau'}))^{-\frac{1}{2}n}.$$

Proof of Claim. — Using the expressions in (5.7.1)-(5.8.9μ), we can rewrite the inner product in the form

(5.9.3)

$$(\vartheta^\tau_{\mu,N} \mid \vartheta^{\tau'}_{\mu',N}) = \sum_{\gamma,\gamma'\in\mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} e^{2\pi i N[\frac{1}{2}\tau\langle\gamma+\frac{\mu}{N}+\xi,\gamma+\frac{\mu}{N}+\xi\ \rangle + \langle\gamma+\frac{\mu}{n}+\xi,x\rangle]}$$

$$e^{-2\pi i N[\frac{1}{2}\overline{\tau'}\langle\gamma'+\frac{\mu'}{N}+\xi,\gamma'+\frac{\mu'}{N}+\xi\rangle + \langle\gamma'+\frac{\mu'}{N}+\xi,x\rangle]} \quad dx d\xi.$$

The dx integral equals

$$\int_{\mathbb{R}^n/\mathbb{Z}^n} e^{2\pi i\langle x, N(\gamma-\gamma')+(\mu-\mu')\rangle} dx = \delta_{N\gamma+\mu,N\gamma'+\mu'}.$$

Since

$$\delta_{N\gamma+\mu,N\gamma'+\mu'} = \delta_{\mathbb{Z}^n+\frac{\mu}{N},\mathbb{Z}^n+\frac{\mu'}{N}} \delta_{\gamma,\gamma'}$$

the expression in (5.9.3) simplifies to

$$\sum_\gamma \int_{\mathbb{R}^n/\mathbb{Z}^n} e^{2\pi i N\{\frac{1}{2}(\tau-\overline{\tau'})\}\langle\gamma+\frac{\mu}{N}+\xi,\gamma+\frac{\mu}{N}+\xi\rangle} d\xi = \int_{\mathbb{R}^n} e^{2\pi i N\{\frac{1}{2}(\tau-\overline{\tau'})\}\langle\xi,\xi\rangle} d\xi.$$

The last expression is an inner product of Gaussians, so by (5.3.9) it equals

$$\langle\gamma_\tau,\gamma_{\tau'}\rangle = (-2\pi i N(\tau-\overline{\tau'}))^{-\frac{1}{2}n} \tag{5.9.4}$$

proving the claim.

It follows first that for each τ the basis $\{\vartheta^\tau_{\mu,N},\mu\in\mathbb{Z}^n/N\mathbb{Z}^n\}$ is orthonormal up to the factor $(4\pi N\mathrm{Im}\,\tau)^{-\frac{1}{2}n}$. If we normalize the basis to $\tilde{\vartheta}^\tau_{\mu,N} := (4\pi N\mathrm{Im}\,\tau)^{\frac{1}{4}n}\vartheta^\tau_{\mu,n}$ then the projection Π^τ_N onto the space $H^2_{\Sigma_\tau}(N)$ of degree N ϑ^τ's may be written in the form

$$\Pi^\tau_N = \sum_{\mu\in\mathbb{Z}^n/N\mathbb{Z}^n} \tilde{\vartheta}^\tau_{\mu,N}\otimes\tilde{\vartheta}^{\tau *}_{\mu,N}. \tag{5.9.5}$$

We then have

$$\Pi^{\tau,\tau' *}_N\Pi^{\tau,\tau'}_N = \Pi^\tau_N\Pi^{\tau'}_N\Pi^\tau_N$$

$$= \sum_{\mu\in\mathbb{Z}^n/N\mathbb{Z}^n} |(\tilde{\vartheta}^\tau_{\mu,N}|\tilde{\vartheta}^{\tau'}_{\mu,N})|^2\tilde{\vartheta}^\tau_{\mu,N}\otimes\tilde{\vartheta}^{\tau *}_{\mu,N}$$

$$= (4\pi)^n\left|\frac{(\mathrm{Im}\tau\mathrm{Im}\,\tau')^{\frac{n}{4}}}{(-2\pi i(\tau-\overline{\tau'}))^{\frac{n}{2}}}\right|^2\sum_\mu \tilde{\vartheta}^\tau_{\mu,N}\otimes\tilde{\vartheta}^{*\,\tau}_{\mu,N}$$

proving the lemma. □

Corollary. — *Let*

$$\mathcal{U}^{\tau,\tau'}_N : H^2_{\Sigma_{\tau'}}(N)\to H^2_{\Sigma_\tau}(N)$$

be the unitary opeator

$$\mathcal{U}^{\tau,\tau'}_N := \sum_{\mu\in\mathbb{Z}/N\mathbb{Z}} \tilde{\vartheta}^\tau_{\mu,N}\otimes\tilde{\vartheta}^{\tau' *}_{\mu,N}.$$

Then

$$\Pi^{\tau,\tau'}_N = (4\pi)^{\frac{1}{2}n}\left[\frac{(\mathrm{Im}\tau\mathrm{Im}\,\tau')^{\frac{n}{4}}}{(-2\pi i(\tau-\overline{\tau'}))^{\frac{n}{2}}}\right]\mathcal{U}^{\tau,\tau'}_N.$$

We can now complete the

5.10. Proof of Theorem D.

(a) Since $\Pi T_{\chi_g}\Pi$ is block diagonal relative to the decomposition (5.2.6) it suffices to show that each block $\Pi^i_N T_{\chi_g}\Pi^i_N$ is unitary up to a constant independent of N.

For simplicity of notation let us rewrite the unitary coefficients under the sum in (5.7.3) as $u_{\mu,\alpha}(g,N)$. Let us also observe that the norm of $||\theta^\tau_{\mu,N}||$ varies with τ. Hence the transformation laws (5.7.3) take the following form in terms of the $\tilde{\vartheta}^\tau_{\mu,N}$'s:

(5.10.1)

$$\Pi^i T_{\chi_g}\Pi^i\tilde{\vartheta}^i_{\mu,N} = j_g(g^{-1}\cdot i)^n \frac{||\vartheta^{g^{-1}\cdot i}_{\mu,N}||}{||\vartheta^i_{\mu,N}||}\nu(g,N)\Pi^i \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} u_{\mu,\alpha}(g,N)\tilde{\vartheta}^{g^{-1}\cdot i}_{a\mu-c\alpha,N}.$$

Using Corollary (5.9.6) and simplifying, (5.10.1) becomes

$$\langle\gamma_{g^{-1}\cdot i},\gamma_i\rangle j_g(g^{-1}\cdot i)^n\nu(g,N)\mathcal{U}_N^{i,g^{-1}i} \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} u_{\mu,\alpha}(g,N)\tilde{\vartheta}^{g^{-1}\cdot i}_{a\mu-c\alpha,N}$$

$$= \langle\gamma_{g^{-1}\cdot i},\gamma_i\rangle j_g(g^{-1}\cdot i)^n\nu(g,N) \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} u_{\mu,\alpha}(g,N)\tilde{\vartheta}^i_{a\mu-c\alpha,N}. \tag{5.10.2}$$

Noting that $j_g(g^{-1}\cdot i) = \mu(g^{-1},i)$ and comparing with Proposition (5.3.8) we see that

$$\Pi^i T_{\chi_g}\Pi^i = \langle\mu(g^*)e_\Lambda, e_\Lambda\rangle U_{g,N} \tag{5.10.3}$$

where

$$U_{g,N}\tilde{\vartheta}^i_{\mu,N} := \nu(g,N) \sum_{\substack{\alpha\in\mathbf{Z}^n\\ c\alpha \bmod N\mathbf{Z}^n}} u_{\mu,\alpha}(g,N)\tilde{\vartheta}^i_{a\mu-c\alpha,N}. \tag{5.10.4}$$

Hence by Corollary (5.3.10) we have

$$U_{g,N} = m(g)\Pi^i T_{\chi_g}\Pi^i$$

with $m(g) = \langle\mu(g^*)e_\lambda, e_\Lambda\rangle^{-1} = 2^{-\frac{n}{2}}(\det(A+D+iB-iC))^{\frac{1}{2}}$. □

Comparing (5.10.2) and Corollary (5.3.10) we see that the principal symbol is indeed the complete symbol.

(b) It is a classical fact that the transformation laws define the metaplectic representation of $SL_\theta(2,\mathbb{Z}/N)$. We have defined the multiplier $m(g)$ precisely to obtain this representation.

Remark. — In the case of the real metaplectic representation, Daubechies [D] finds that $W_J(S) = \eta_{J,S} P_J U_S|_{\mathcal{H}_J}$, where: $W_J(S)$ denotes the metaplectic representation, realized on the Bargmann space $\mathcal{H}_J$ of J-holomorphic functions; U_S denotes left translation by S^{-1}, P_J; P_J denotes the orthogonal onto $\mathcal{H}_J$; and $\eta_{J,S} := (\Omega_J, W_J(S)\Omega_J)^{*-1}$ [D], p. 1388. It is evident that in our notation $g = S^{-1}$ and that $m(g) = \eta_{J,S}$, corroborating that $m(g)$ is the correct multiplier to get the metaplectic representation.

(c) The index of χ_g is by definition the index of any Toepltiz Fourier Integral operator $\Pi A T_{\chi_g} \Pi$ quantizing χ_g with unitary principal symbol. We have seen that $m(g)\Pi T_\chi \Pi$ has a unitary principal symbol, and by (a) it is actually a unitary operator. Hence its index is zero. □

(d) The ergodicity and mixing statements follow from Theorem B together with the fact that symplectic torus automorphisms are mixing if no eigenvalue is a root of unity [W].

(e) We have:

$$U_g^* \Pi \sigma \Pi U_g = \Pi T_{\chi_g}^* \Pi \sigma \Pi T_{\chi_g} \Pi$$

as the remaining constant factors cancel. The formula in (e) follows since $T_{\chi_g}^* \Pi T_{\chi_g}$ is precisely the Toeplitz structure corresponding to the complex structure $g \cdot i$. It also follows that the matrix elements of a Toeplitz operator relative to the eigenfunctions $\vartheta_{k,N}^i$ of $U_{g,N}$ satisfy:

$$\langle \Pi \sigma \Pi \vartheta_{k,N}^i | \vartheta_{k,N}^i \rangle = \langle U_{g,N} \Pi \sigma \Pi \vartheta_{k,N}^i | U_{g,N} \vartheta_{k,N}^i \rangle = \langle \Pi \sigma \cdot \chi_g \Pi \vartheta_{k,N}^{gi} | \theta_{k,N}^{gi} \rangle$$

where $\vartheta_{k,N}^{gi} = \mathcal{U}_N^{i,gi} \theta_{k,N}^i$. □

Remarks.

1. On the index problem: Weinstein's index problem actually concerns Fourier Integral operators quantizing homogeneous canonical transformations on T^*M [Wei]. Of course, such a transformation is the same as a contact transformation on S^*M. Moreover, it is known that any FIO can be expressed in the form $\Pi A T_\chi \Pi$ where Π is a Toeplitz structure on the symplectic cone generated by the canonical contact form on S^*M in $T^*(S^*M)$ and where A is a pseudodifferential operator on S^*M. Thus Π is a Szegö projector to a space $H^2(S^*M)$ of CR functions on S^*M. The Boutet de Monvel index theorem for pseudodifferential Toeplitz operators and the logarithm law for the index reduce Weinstein's index problem to that of calculating indices of operators of the form $\Pi T_\chi \Pi$. It is possible that the index of such an operator always vanishes; we have just seen a non-trivial example of this (i.e. an example not homotopic to the identity thru contact transformations).

The fact that the index vanishes for the symplectic torus automorphisms above is due to the fact that their quantizations commute with an elliptic circle action. Hence they are direct sums of finite rank operators and the index, being the sum of indices of finite rank operators, has to vanish. It would be interesting to see if the seemingly more difficult index problem for Zoll surfaces (the original problem in [Wei]) cannot be solved by a similar argument. The main difference is that the contact map arising there intertwines two different elliptic circle actions.

2. On the quantum ergodicity: The quantum ergodicity theorem for cat maps of $\mathbb{R}^2/\mathbb{Z}^2$ has previously been proved in [d'EGI] and [BdB] by a different method. These papers also allow for non-trivial characters of the fundamental group.

6. TRACE FORMULAE FOR QUANTIZED TORUS AUTOMORPHISMS

The purpose of this section is to prove an exact trace formula for the trace $\mathrm{Tr} U_{g,N}$ of a quantized cat map in the space of theta functions Θ_N of degree N. The standard complex structure $\tau = iI$ is fixed throughout. In the following we assume that g is non-degenerate in the sense that $\ker(I-g)$ is trivial. The square root $\sqrt{\det(I-g)}$ is defined by the usual analytic continuation [F].

THEOREM E (6.1). — *With the notations and assumptions of Theorem D, and with the assumption that g is non-degenerate, we have*

$$\mathrm{Tr} U_{g,N} = \frac{1}{\sqrt{\det(I-g)}} \sum_{[(m,n)]\in\mathbb{Z}^{2n}/(I-g)^{-1}\mathbb{Z}^{2n}} e^{i\pi N[\langle m,n\rangle - \sigma((m,n),(I-g)^{-1}(m,n))]}.$$

Proof. — Our starting point is the explicit form of the Szego kernel S from $L^2(N_{\mathbb{R}})$ on $N_{\mathbb{R}}$ (cf. [S]). It is a convolution kernel $S(x,y) = K(x^{-1}y)$ with

$$K(x) = c_n \partial_t (t + i|\zeta|^2)^{-n} \tag{6.2}$$

where c_n is a constant whose value we will not need to know, and where $x = (\zeta, t)$. The Szegö kernel $S(x, y)$ is singular along the diagonal, but it can be regularized in a well-known way (see [S]) and we can safely pretend that it is regular. In fact, we will not need the full Szegö kernel, but only the part of degree N, and this is regular.

The kernel of $\Pi T_{\chi_g} \Pi$ on $N_{\mathbb{R}}$ is then given by $S(x, g(y)) = c_n K(x^{-1} g(y))$. Since g is an automorphism, the kernel on the quotient is

$$c_n \sum_{\tau \in N_{\mathbb{Z}}} K(x^{-1} \tau g(y)).$$

Actually, it will prove convenient to put the quotient kernel in a slightly different form by passing to the quotient in two stages. First, we sum over the central lattice $N_{\mathbb{Z}} \cap Z_{N_{\mathbb{R}}}$ to get the kernel of the Szegö projector on the reduced Heisenberg group $\mathbb{H}^{\mathrm{red}}$:

$$S_{\mathrm{red}}(x, y) := \sum_{k \in \mathbb{Z}} S(x, (0, 0, k) y).$$

Since the part of degree N on $\mathbb{H}^{\mathrm{red}}$ is given by

$$S_N(x, y) = \int_o^1 S_{\mathrm{red}}(x, y(0, 0, \theta)) e^{-2\pi i N \theta} d\theta, \tag{6.3}$$

we may express it in the form

$$S_N(x, y) = \int_{\mathbb{R}} S(x, y(0, 0, \theta)) e^{-2\pi i N \theta} d\theta.$$

The degree N part of $\Pi T_g \Pi$ on $\mathbb{H}^{\mathrm{red}}$ is therefore given by

$$S_N(x, g(y)) = \int_{\mathbb{R}} S(x, g(y)(0, 0, \theta)) e^{-2\pi i N \theta} d\theta. \tag{6.4}$$

To pass to the full quotient we must further divide by the covering group $\bar{\Gamma}$ of $\mathbb{H}_n^{\mathrm{red}}$ over $N_{\mathbb{R}}/N_{\mathbb{Z}}$. It is not quite $\mathbb{Z}^{2n}$ since the latter is not a subgroup of the Heisenberg group. Rather $\mathbb{Z}^{2n}$ is a maximal isotropic subgroup of $K = \mathbb{R}^{2n}$ and we must embed it in $\mathbb{H}_n^{\mathrm{red}}$ by the splitting homomorphism

$$s : \mathbb{Z}^{2n} \to \mathbb{H}_n^{\mathrm{red}} \qquad s(m, n) = (m, n, e^{i \frac{1}{2} F(m,n)})$$

with $F(x, y) = \langle x, y \rangle$. (cf. §5.8).

Since g is an automorphism of the reduced Heisenberg group, the kernel of the degree N part of $\Pi T_{\chi_g}\Pi$ on the full quotient can then be expressed in the form

$$\Pi_N T_{\chi_g}\Pi_N = c_n \sum_{\gamma\in\bar{\Gamma}} \int_{\mathbb{R}} K(x^{-1}\gamma g(y)(0,0,\theta))e^{-2\pi i N\theta}d\theta. \tag{6.5N}$$

Now denote a fundamental domain for $N_{\mathbb{Z}}$ in $N_{\mathbb{R}}$ by $\mathcal{D}$. Then we have

$$\mathrm{Tr}\Pi_N T_{\chi_g}\Pi_N = c_n \sum_{\gamma\in\bar{\Gamma}} \int_{\mathbb{R}}\int_{\mathcal{D}} K(x^{-1}\gamma g(x)(0,0,\theta))e^{-2\pi i N\theta}d\theta dx. \tag{6.6N}$$

To simplify (6.6N), we define an equivalence relation on $\bar{\Gamma}$:

$$\gamma\sim\gamma' \equiv \exists M\in\Gamma: \tag{6.7}$$

$$\gamma' = M^{-1}\gamma g(M).$$

Here $g(M)$ denotes the value of $g\in Sp(2n,\mathbb{Z})$ on M in $\mathbb{H}_n^{red}$. We denote the set of equivalence classes $[\gamma]$ by $[\bar{\Gamma}]$.

It follows from (6.6N), and (6.7) that the trace may be re-written in the form

$$\mathrm{Tr}\,\Pi_N T_{\chi_g}\Pi_N = c_n \sum_{[\gamma]}\sum_{M\in\bar{\Gamma}} \int_{\mathcal{D}}\int_{\mathbb{R}} K(x^{-1}M^{-1}[\gamma]g(M)g(x)(0,0,\theta)) \\ e^{-2\pi i N\theta}d\theta dx. \tag{6.8N}$$

We now use that g is an automorphism to rewrite $g(M)g(x)$ as $g(Mx)$. Changing variables to $x' = Mx$ and noting that $\bigcup M\mathcal{D} = \mathbb{R}^{2n}\times S^1$, we have

(6.9N)

$$\mathrm{Tr}\,\Pi_N T_{\chi_g}\Pi_N = c_n \sum_{[\gamma]\in[\bar{\Gamma}]} \int_{\mathbb{R}^{2n}\times S^1}\int_{\mathbb{R}} K(x^{-1}[\gamma]g(x)(0,0,\theta))e^{-2\pi i N\theta}d\theta dx.$$

We now observe that the central part of x cancels out, so that we may replace the reduced Heisenberg group by $\mathbb{R}^{2n}$. We henceforth denote points in this space by $\zeta = (x,\xi)$.

Since $s:\mathbb{Z}^{2n}\to\bar{\Gamma}$ is an isomorphism, the equivalence classes $[\gamma]$ in $\bar{\Gamma}$ are in 1-1 correspondence with the cosets $[m,n]$ in $\mathbb{Z}^{2n}/(g-I)\mathbb{Z}^{2n}$. We denote the latter set of equivalence classes by $[\mathbb{Z}^{2n}]$ and rewrite (6.9N) in the form

$$(6.10N)\quad c_n \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} K((-\zeta,0)(m,n,\frac{1}{2}F(m,n))(g\zeta,0)(0,0,\theta)) e^{-2\pi i N\theta} d\theta dx d\xi.$$

We now multiply out the argument in K. Since it is somewhat more convenient, we express the result for the reduced form of the Heisenberg group:

$$(6.11)\quad (-\zeta,1)((m,n),e^{i\pi F(m,n)})(g\zeta,1) = ((g-I)\zeta+(m,n), e^{i\pi\sigma((m,n)-\zeta,g\zeta)}e^{-i\pi\sigma(\zeta,(m,n))}e^{i\pi F(m,n)}).$$

Then write $(m,n)=(g-I)v$ and change variables $\zeta\to\zeta+v$. Then (6.10N) becomes

$$(6.12N)\quad c_n \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} e^{i\pi NF(m,n)} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} K((g-I)\zeta,0)(0,0,\theta)) e^{i\pi N\sigma(g(\zeta-v),\zeta-v)} e^{i\pi N\sigma((g-I)v,(g+I)[\zeta-v])} e^{-2\pi i N\theta} d\theta d\zeta.$$

Next we recall (cf. [S]) that the Fourier transform $\hat{K}$ as a function on $\mathbb{R}^{2n+1}$ is given by

$$\hat{K}(u,v,\tau) = 2^n e^{-\pi\frac{|(u,v)|^2}{2\tau}} \qquad (\tau>0).$$

Hence the partial Fourier transform in the θ-variable equals

$$\hat{K}_\theta(\zeta,N) = 2^n c'_n N^n e^{-\pi N|\zeta|^2}$$

for another constant c'_n. Therefore, (6.12 N) has the form

$$(6.13N)\quad 2^n c''_n N^n \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} e^{i\pi NF(m,n)} \int_{\mathbb{R}^{2n}} e^{-\pi N|(g-I)\zeta|^2} e^{i\pi N\sigma(g(\zeta-v),\zeta-v)} e^{i\pi N\sigma((g-I)v,(g+I)[\zeta-v])} d\zeta.$$

This is a Gaussian integral, and hence can be explicitly evaluated. To do so, we first simplify the exponent.

First, the quadratic terms in ζ in the exponent are:

$$-\pi N[|(g-I)\zeta|^2 - i\sigma(g\zeta,\zeta)]$$

while the linear terms are:

$$i\pi N[\sigma(g\zeta, -v) + \sigma(-gv, \zeta) + \sigma((g-I)v, (g+I)\zeta).$$

The terms independent of ζ come to

$$i\pi N[\sigma(gv, v) + \sigma((g-I)v, (g+I)(-v)) + F(m,n)].$$

which simplify to

$$i\pi N[F(m,n) - \sigma((m,n), v)]$$

since g is symplectic. The terms linear in ζ cancel out.

Hence,

$$\text{(6.14}N\text{)} \quad \mathrm{Tr}\Pi_N T_{\chi_g}\Pi_N = 2^n c_n'' N^n I_{g,N} \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} e^{i\pi N[F(m,n)-\sigma((m,n),v)]}$$

with

$$I_{g,N} = \int_{\mathbb{R}^{2n}} e^{-\pi N|(g-I)\zeta|^2} e^{i\pi N\sigma(g\zeta,\zeta)} d\zeta.$$

This integral has been evaluated in [D], p. 1386, and equals

$$\text{(6.15)} \qquad N^{-n}c_n'''[\det(I - g - iJ(I+g))]^{-\frac{1}{2}}[\det(I-g)]^{-\frac{1}{2}}$$

for some normalizing factor c_n'''. It follows that

$$\text{(6.16}N\text{)} \quad \mathrm{Tr}\Pi_N T_{\chi_g}\Pi_N = 2^n C_n[\det(I - g - iJ(I+g))]^{-\frac{1}{2}}[\det(I-g)]^{-\frac{1}{2}} \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} e^{i\pi N[F(m,n)-\sigma((m,n),v)]}$$

for some constant C_n. Using the remark after Theorem D(b) and using the formula

$$\text{(6.17)} \qquad 2^n[\{\det(I - g - iJ(I+g))]^{-\frac{1}{2}} = m(g)^{-1}$$

from [D], p. 1388, we see that

$$\text{(6.18}N\text{)} \quad \mathrm{Tr}\, U_{g,N} = C_n[\det(I-g)]^{-\frac{1}{2}} \sum_{[(m,n)]\in[\mathbb{Z}^{2n}]} e^{i\pi N[F(m,n)-\sigma((m,n),v)]}$$

for some constant C_n. We can determine this constant by computing one non-degenerate example; the example we choose is the finite Fourier transform $F(N)$, whose trace is given after the statement of Theorem E in §1. Comparing with (6.18N) we find that $C_n = 1$. □

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