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## THE OVERDETERMINED CAUCHY PROBLEM

by C. BOITI and M. NACINOVICH

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### Introduction.

In the classical Cauchy problem for a linear partial differential equation with initial data on a hypersurface, smooth initial data together with the equation allow to compute the Taylor series of a smooth solution at any given point of the hypersurface.

This leads to the notion of a formally non-characteristic hypersurface for a system of linear partial differential equations, that was considered in [AHLM], [AN3], [N2].

This remark suggests further generalizations of the Cauchy problem, where the assumption that the initial data are given on a formally non-characteristic initial manifold is dropped, and we allow formal solutions (in the sense of Whitney) of the given system on any closed subset as initial data.

The problem is then to find classical smooth solutions of the system, whose restrictions in the sense of Whitney are the given initial data.

This point of view was particularly fruitful while investigating initial value problem for overdetermined systems with constant coefficients and data on a hypersurface in [N2], [N3], and for systems of partial differential equations related to complex analysis in [N4].

In this paper we continue this investigation of the Cauchy problem for a pair of convex subsets of  $\mathbb{R}^N$ .

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*Key words:* Evolution – Overdetermined systems – Cauchy problem – Phragmén- Lindelöf principle.

*Math. classification:* 35N05 – 32A15.

The introduction of causality, evolution and hyperbolic pairs further generalizes the notions introduced in the previous works and gives a unifying point of view encompassing several different problems, ranging from questions of smoothness of the solutions, to the classical Cauchy problem, to the comparison of formal to actual solutions, to Hartog's type phenomena.

In the first part of this paper we discuss classes of entire functions on irreducible affine algebraic varieties.

By the use of Fourier-Laplace transform and of the fundamental principle of Ehrenpreis-Palamodov, many questions concerning evolution pairs can be translated into the problem of establishing a priori estimates of the Phragmén-Lindelöf type for such classes of entire functions.

Then we apply this method to discuss conditions for evolution in one space variable and several time variables, and Petrowski-type conditions for evolution from an affine submanifold of arbitrary codimension.

Finally we apply this theory for extending Hörmander's necessary and sufficient condition for evolution for partial differential equations with constant coefficients and Cauchy data on a hypersurface, to the case of general systems and Cauchy data on a closed submanifold of arbitrary codimension.

## 1. Algebraic preliminaries.

Let  $\mathfrak{P}$  be a regular unitary Noetherian commutative ring, of global finite homological dimension  $N$ . Let  $\mathfrak{M}$  be a  $\mathfrak{P}$ -module of finite type. We denote by  $\text{Supp}(\mathfrak{M})$  and  $\text{Ass}(\mathfrak{M})$  respectively its support and the set of its associated prime ideals:

(1.1)

$$\text{Supp}(\mathfrak{M}) = \{\mathfrak{p} \in \text{Spec}(\mathfrak{P}) \mid \mathfrak{M}_{\mathfrak{p}} \neq 0\},$$

$$\text{Ass}(\mathfrak{M}) = \{\mathfrak{p} \in \text{Spec}(\mathfrak{P}) \mid \mathfrak{M} \text{ contains a } \mathfrak{P}\text{-submodule isomorphic to } \mathfrak{P}/\mathfrak{p}\}.$$

Note that  $\text{Ass}(\mathfrak{M}) \subset \text{Supp}(\mathfrak{M})$  and the two sets contain the same minimal elements, as  $\text{Supp}(\mathfrak{M})$  is the set of all prime ideals of  $\mathfrak{P}$  that contain an associated prime ideal of  $\mathfrak{M}$ .

For the proof of the following two propositions we refer to [N1]:

**PROPOSITION 1.1.** — *Let  $\mathfrak{M}, \mathcal{F}$  be two  $\mathfrak{P}$ -modules, with  $\mathfrak{M}$  of finite type and  $p$  be a nonnegative integer. Then the following statements are equivalent:*

- (i)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{M}, \mathcal{F}) = 0 \quad \forall j \leq p;$
- (ii)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \leq p, \forall \mathfrak{p} \in \text{Ass}(\mathfrak{M});$
- (iii)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \leq p, \forall \mathfrak{p} \in \text{Supp}(\mathfrak{M});$
- (iv)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{N}, \mathcal{F}) = 0 \quad \forall j \leq p, \text{ and for every } \mathfrak{P}\text{-submodule } \mathfrak{N} \text{ of } \mathfrak{M}.$

PROPOSITION 1.2. — *Let  $\mathfrak{M}, \mathcal{F}$  be two  $\mathfrak{P}$ -modules, with  $\mathfrak{M}$  of finite type and  $p$  be a nonnegative integer. Assume moreover that*

$$(1.2) \quad \text{Ext}_{\mathfrak{P}}^j(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j > p \quad \text{and} \quad \forall \mathfrak{p} \in \text{Supp}(\mathfrak{M}).$$

*Then the following statements are equivalent:*

- (i)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{M}, \mathcal{F}) = 0 \quad \forall j \geq p;$
- (ii)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \geq p, \forall \mathfrak{p} \in \text{Ass}(\mathfrak{M});$
- (iii)  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{N}, \mathcal{F}) = 0 \quad \forall j \geq p, \text{ and for every } \mathfrak{P}\text{-submodule } \mathfrak{N} \text{ of } \mathfrak{M}.$

Analogous propositions hold for the Tor functor. We have indeed:

PROPOSITION 1.3. — *Let  $\mathfrak{M}, \mathcal{F}$  be two  $\mathfrak{P}$ -modules, with  $\mathfrak{M}$  of finite type and  $p$  be a nonnegative integer. Then the following statements are equivalent:*

- (i)  $\text{Tor}_j^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) = 0 \quad \forall j \leq p;$
- (ii)  $\text{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \leq p, \forall \mathfrak{p} \in \text{Ass}(\mathfrak{M});$
- (iii)  $\text{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \leq p, \forall \mathfrak{p} \in \text{Supp}(\mathfrak{M});$
- (iv)  $\text{Tor}_j^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) = 0 \quad \forall j \leq p, \text{ and for every } \mathfrak{P}\text{-submodule } \mathfrak{N} \text{ of } \mathfrak{M}.$

*Proof.* — To show that (iii)  $\Rightarrow$  (i), we consider a composition series for  $\mathfrak{M}$ :

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \dots \subset \mathfrak{M}_k \subset \mathfrak{M}_{k+1} = \mathfrak{M}.$$

For each  $0 \leq h \leq k$  the  $\mathfrak{P}$ -module  $\mathfrak{M}_{h+1}/\mathfrak{M}_h$  is isomorphic to a  $\mathfrak{P}$ -module  $\mathfrak{P}/\mathfrak{p}_h$  for a prime ideal  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ . From the long exact sequence associated to the quotient:

$$\begin{array}{ccccccc} & & & & \dots & \longrightarrow & \text{Tor}_{j+1}^{\mathfrak{P}}(\mathfrak{M}_{h+1}/\mathfrak{M}_h, \mathcal{F}) \\ \longrightarrow & \text{Tor}_j^{\mathfrak{P}}(\mathfrak{M}_{h+1}, \mathcal{F}) & \longrightarrow & \text{Tor}_j^{\mathfrak{P}}(\mathfrak{M}_h, \mathcal{F}) & \longrightarrow & \text{Tor}_j^{\mathfrak{P}}(\mathfrak{M}_{h+1}/\mathfrak{M}_h, \mathcal{F}) \\ \longrightarrow & & \dots & & & & \end{array}$$

we deduce from (iii) that

$$\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{M}_h, \mathcal{F}) \simeq \mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{M}_{h+1}, \mathcal{F}) \quad \text{for } 0 \leq h \leq k \quad \text{and } j < p,$$

while the homomorphisms

$$\mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}_h, \mathcal{F}) \longrightarrow \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}_{h+1}, \mathcal{F})$$

are surjective for  $0 \leq h \leq k$ . From these observations it follows that (i) holds true.

To show that (i)  $\Rightarrow$  (iii) we argue by contradiction. So we assume that (i) is valid, but  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) \neq 0$  for some  $\mathfrak{p} \in \mathrm{Supp}(\mathfrak{M})$  and some  $0 \leq j \leq p$ . Let  $q$  be the smallest positive integer for which is possible to find a  $\mathfrak{p} \in \mathrm{Supp}(\mathfrak{M})$  with  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) \neq 0$ . Having fixed  $q$ , using the assumption that  $\mathfrak{P}$  is Noetherian, there is a maximal  $\mathfrak{p}_0 \in \mathrm{Supp}(\mathfrak{M})$  such that  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}_0, \mathcal{F}) \neq 0$ . By the properties of the ideals in  $\mathrm{Supp}(\mathfrak{M})$  there is an exact sequence of  $\mathfrak{P}$ -modules of finite type of the form:

$$0 \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{M} \longrightarrow \Omega \longrightarrow 0$$

for a nonzero submodule  $\Omega$  of  $\mathfrak{P}/\mathfrak{p}_0$ . Then from the exact sequence:

$$\dots \longrightarrow \mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) \longrightarrow \mathrm{Tor}_q^{\mathfrak{P}}(\Omega, \mathcal{F}) \longrightarrow \mathrm{Tor}_{q-1}^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) \longrightarrow \dots$$

we deduce that  $\mathrm{Tor}_q^{\mathfrak{P}}(\Omega, \mathcal{F}) = 0$ .

Indeed  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) = 0$  by assumption (i) and  $\mathrm{Tor}_{q-1}^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) = 0$  by the first part of the proof, since  $\mathrm{Supp}(\mathfrak{N}) \subset \mathrm{Supp}(\mathfrak{M})$  and for every prime ideal  $\mathfrak{p}$  in  $\mathrm{Supp}(\mathfrak{M})$  we have  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0$  for every  $j \leq (q-1)$  by the choice of the integer  $q$ .

Let  $\Omega \simeq \mathfrak{I}/\mathfrak{p}_0$  for an ideal  $\mathfrak{I}$  of  $\mathfrak{P}$  containing  $\mathfrak{p}_0$ . From the exact sequence:

$$0 \longrightarrow \mathfrak{I}/\mathfrak{p}_0 \longrightarrow \mathfrak{P}/\mathfrak{p}_0 \longrightarrow \mathfrak{P}/\mathfrak{I} \longrightarrow 0$$

we obtain the long exact sequence:

$$\dots \longrightarrow \mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{I}/\mathfrak{p}_0, \mathcal{F}) \longrightarrow \mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}_0, \mathcal{F}) \longrightarrow \mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{I}, \mathcal{F}) \longrightarrow \dots$$

Then we have  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{I}/\mathfrak{p}_0, \mathcal{F}) = 0$  by the argument above; moreover  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{I}, \mathcal{F}) = 0$  by the implication (iii)  $\Rightarrow$  (i) because every prime ideal in  $\mathrm{Supp}(\mathfrak{P}/\mathfrak{I})$  belongs to  $\mathrm{Supp}(\mathfrak{M})$  and properly contains  $\mathfrak{p}_0$ . This contradicts  $\mathrm{Tor}_q^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}_0, \mathcal{F}) \neq 0$ . The proof of (i)  $\Rightarrow$  (iii) is complete.

From the equivalence (i)  $\Leftrightarrow$  (iii) the equivalence of these two statements with (ii) and (iii) easily follows.

PROPOSITION 1.4. — *Let  $\mathfrak{M}, \mathcal{F}$  be two  $\mathfrak{P}$ -modules, with  $\mathfrak{M}$  of finite type and  $p$  be a nonnegative integer. Assume moreover that*

$$(1.3) \quad \mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j > p \quad \text{and} \quad \forall \mathfrak{p} \in \mathrm{Supp}(\mathfrak{M}).$$

*Then the following statements are equivalent:*

- (i)  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) = 0 \quad \forall j \geq p$ ;
- (ii)  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0 \quad \forall j \geq p, \forall \mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$ ;
- (iii)  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) = 0 \quad \forall j \geq p$ , and for every  $\mathfrak{P}$ -submodule  $\mathfrak{N}$  of  $\mathfrak{M}$ .

*Proof.* — We first prove that (ii)  $\Rightarrow$  (i). We argue by descending induction on  $p$ . The statement is indeed trivial if  $p$  is larger than the homological dimension  $N$  of  $\mathfrak{P}$ . So we fix the integer  $p \leq N$  and assume that the statement is true for larger  $p$ .

Assume that  $\mathfrak{M}$  is  $\mathfrak{p}$ -coprimary. We argue by induction on the smallest integer  $k$  such that  $\mathfrak{p}^k \mathfrak{M} = 0$ . If  $k = 1$ , then  $\mathfrak{M}$  can be thought of as a torsion free  $\mathfrak{P}/\mathfrak{p}$ -module and hence there is an exact sequence of  $\mathfrak{P}$ -modules:

$$0 \longrightarrow \mathfrak{M} \longrightarrow (\mathfrak{P}/\mathfrak{p})^r \longrightarrow \Omega \longrightarrow 0.$$

From the exact sequence

$$\dots \longrightarrow \mathrm{Tor}_{p+1}(\Omega, \mathcal{F}) \longrightarrow \mathrm{Tor}_p(\mathfrak{M}, \mathcal{F}) \longrightarrow \left( \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}) \right)^r \longrightarrow \dots$$

we obtain that  $\mathrm{Tor}_p(\mathfrak{M}, \mathcal{F}) = 0$  because  $\mathrm{Tor}_p(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) = 0$  by assumption (ii) and  $\mathrm{Tor}_{p+1}(\Omega, \mathcal{F}) = 0$  by the inductive assumption, as  $\mathrm{Ass}(\Omega) \subset \mathrm{Supp}(\mathfrak{M})$ .

Let now  $k > 0$  and suppose that  $\mathrm{Tor}_j^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) = 0$  for all  $\mathfrak{p}$ -coprimary  $\mathfrak{P}$ -modules of finite type  $\mathfrak{N}$  for which  $\mathfrak{p}^{k-1} \mathfrak{N} = 0$ . Let  $\mathfrak{M}_0 = \{m \in \mathfrak{M} \mid \mathfrak{p} \cdot m = 0\}$ . This is a  $\mathfrak{p}$ -coprimary submodule of  $\mathfrak{M}$  for which  $\mathfrak{p} \mathfrak{M}_0 = 0$ , while  $\mathfrak{M}/\mathfrak{M}_0$  is also  $\mathfrak{p}$ -coprimary and  $\mathfrak{p}^{k-1}(\mathfrak{M}/\mathfrak{M}_0) = 0$ . The long exact sequence associated to the quotient yields:

$$\dots \longrightarrow \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}_0, \mathcal{F}) \longrightarrow \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) \longrightarrow \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}/\mathfrak{M}_0, \mathcal{F}) \longrightarrow \dots$$

and therefore  $\mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) = 0$  because  $\mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}_0, \mathcal{F}) = \mathrm{Tor}_p^{\mathfrak{P}}(\mathfrak{M}/\mathfrak{M}_0, \mathcal{F}) = 0$  by the inductive assumption.

To drop the assumption that  $\mathfrak{M}$  is  $\mathfrak{p}$ -coprimary, we note that if  $\phi$  is a part of  $\text{Ass}(\mathfrak{M})$  there is a  $\mathfrak{P}$ -submodule  $\mathfrak{N}$  of  $\mathfrak{M}$  such that

$$\text{Ass}(\mathfrak{N}) = \phi, \quad \text{Ass}(\mathfrak{M}/\mathfrak{N}) = \text{Ass}(\mathfrak{M}) \setminus \phi.$$

As we have the exact sequence

$$\text{Tor}_{\mathfrak{p}}^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) \longrightarrow \text{Tor}_{\mathfrak{p}}^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) \longrightarrow \text{Tor}_{\mathfrak{p}}^{\mathfrak{P}}(\mathfrak{M}/\mathfrak{N}, \mathcal{F})$$

the conclusion follows by induction on the number of prime ideals in  $\text{Ass}(\mathfrak{M})$ .

To show that (i)  $\Rightarrow$  (ii), we note that the implication (ii)  $\Rightarrow$  (i), together with assumption (1.3) gives  $\text{Tor}_j(\Omega, \mathcal{F}) = 0$  for all  $\mathfrak{P}$ -modules  $\Omega$  of finite type satisfying  $\text{Ass}(\Omega) \subset \text{Supp}(\mathfrak{M})$  and all  $j > p$ . If  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ , then  $\mathfrak{M}$  contains a submodule  $\mathfrak{N}$  isomorphic to  $\mathfrak{P}/\mathfrak{p}$ . If  $\Omega$  is the quotient  $\mathfrak{M}/\mathfrak{N}$ , we have  $\text{Supp}(\Omega) \subset \text{Supp}(\mathfrak{M})$  and then the exact sequence

$$\text{Tor}_{p+1}^{\mathfrak{P}}(\Omega, \mathcal{F}) \longrightarrow \text{Tor}_p^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) \longrightarrow \text{Tor}_p^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F})$$

gives that  $\text{Tor}_p^{\mathfrak{P}}(\mathfrak{P}/\mathfrak{p}, \mathcal{F}) \simeq \text{Tor}_p^{\mathfrak{P}}(\mathfrak{N}, \mathcal{F}) = 0$ .

The equivalence with condition (iii) is clear, because  $\text{Ass}(\mathfrak{N}) \subset \text{Ass}(\mathfrak{M})$  for every  $\mathfrak{P}$ -submodule of  $\mathfrak{M}$ .

*Remark 1.1.* — In the applications of Propositions 1.1, 1.2, we will consider often the situation where  $\mathcal{F}$  is the kernel of an epimorphism of unitary injective  $\mathfrak{P}$  modules: when we have a short exact sequence

$$(1.4) \quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

we obtain for every unitary  $\mathfrak{P}$ -module  $\mathfrak{M}$  of finite type the long exact sequence

$$(1.5) \quad \dots \longrightarrow \text{Ext}_{\mathfrak{P}}^{j-1}(\mathfrak{M}, \mathcal{F}_2) \longrightarrow \text{Ext}_{\mathfrak{P}}^j(\mathfrak{M}, \mathcal{F}) \longrightarrow \text{Ext}_{\mathfrak{P}}^j(\mathfrak{M}, \mathcal{F}_1) \longrightarrow \dots$$

which yields  $\text{Ext}_{\mathfrak{P}}^j(\mathfrak{M}, \mathcal{F}) = 0$  for every  $j > 1$ . In particular the question of the surjectivity of the map  $\text{Ext}_{\mathfrak{P}}^0(\mathfrak{M}, \mathcal{F}_1) \longrightarrow \text{Ext}_{\mathfrak{P}}^0(\mathfrak{M}, \mathcal{F}_2)$  reduces to the surjectivity of the maps  $\text{Ext}_{\mathfrak{P}}^0(\mathfrak{P}/\mathfrak{p}, \mathcal{F}_1) \longrightarrow \text{Ext}_{\mathfrak{P}}^0(\mathfrak{P}/\mathfrak{p}, \mathcal{F}_2)$  for all associated prime ideals  $\mathfrak{p}$  of  $\mathfrak{M}$ . This reduction is especially convenient when using the Fourier-Laplace transform to investigate the properties of systems of p.d.e.'s with constant coefficients.

In the same way, Propositions 1.3, 1.4 can be applied in particular to  $\mathfrak{P}$ -modules  $\mathcal{F}$  that are quotients of flat  $\mathfrak{P}$ -modules. The short exact sequence of unitary  $\mathfrak{P}$ -modules

$$(1.6) \quad 0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \longrightarrow 0$$

yields for every unitary  $\mathfrak{P}$ -module  $\mathfrak{M}$  a long exact sequence

$$(1.7) \quad \dots \longrightarrow \operatorname{Tor}_j^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}_2) \longrightarrow \operatorname{Tor}_j^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}) \longrightarrow \operatorname{Tor}_{j-1}^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}_1) \longrightarrow \dots$$

When  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are flat, the question of the injectivity of the map  $\operatorname{Tor}_0^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}_1) \longrightarrow \operatorname{Tor}_0^{\mathfrak{P}}(\mathfrak{M}, \mathcal{F}_2)$  for a given unitary  $\mathfrak{P}$ -module  $\mathfrak{M}$  of finite type reduces to the analogous question where  $\mathfrak{M}$  is substituted by  $\mathfrak{P}/\mathfrak{p}$  for the associated prime ideals  $\mathfrak{p}$  of  $\mathfrak{M}$ .

## 2. The Cauchy problem for a pair of convex subsets of $\mathbb{R}^N$ .

Let  $\mathfrak{M}$  be a unitary module of finite type over the ring  $\mathcal{P} = \mathbb{C}[\zeta_1, \dots, \zeta_N]$  of polynomials in  $N$  indeterminates with coefficients in  $\mathbb{C}$ . Then  $\mathfrak{M}$  has a Hilbert resolution:

$$(2.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P}^{a_d} & \xrightarrow{{}^t A_{d-1}(\zeta)} & \mathcal{P}^{a_{d-1}} & \longrightarrow & \mathcal{P}^{a_{d-2}} \longrightarrow \dots \\ & & \dots & \xrightarrow{{}^t A_1(\zeta)} & \mathcal{P}^{a_1} & \xrightarrow{{}^t A_0(\zeta)} & \mathcal{P}^{a_0} \longrightarrow \mathfrak{M} \longrightarrow 0 \end{array}$$

of length  $d \leq N$ . We define  $D_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x^j}$ , where  $x^1, \dots, x^N$  are Euclidean coordinates in  $\mathbb{R}^N$ . Let  $\mathcal{F}$  be a  $\mathbb{C}$ -linear space of (generalized) functions or distributions defined on a subset of  $\mathbb{R}^N$  and such that  $D_j \mathcal{F} \subset \mathcal{F}$  for every  $j = 1, \dots, N$ . It becomes a unitary  $\mathcal{P}$ -module by the action

$$(2.2) \quad p(\zeta) \cdot f = p(D)f \quad \forall p \in \mathcal{P}, \quad \forall f \in \mathcal{F}.$$

Then the groups  $\operatorname{Tor}_j^{\mathcal{P}}(\mathfrak{M}, \mathcal{F})$  and  $\operatorname{Ext}_{\mathcal{P}}^j(\mathfrak{M}, \mathcal{F})$  are isomorphic to the cohomology groups of the complexes

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}^{a_d} & \xrightarrow{{}^t A_{d-1}(D)} & \mathcal{F}^{a_{d-1}} & \xrightarrow{{}^t A_{d-3}(D)} & \mathcal{F}^{a_{d-2}} \longrightarrow \dots \\ & & \dots & \xrightarrow{{}^t A_1(D)} & \mathcal{F}^{a_1} & \xrightarrow{{}^t A_0(D)} & \mathcal{F}^{a_0} \longrightarrow 0 \end{array}$$

and

$$(2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}^{a_0} & \xrightarrow{A_0(D)} & \mathcal{F}^{a_1} & \xrightarrow{A_1(D)} & \dots \\ & & \dots & \xrightarrow{A_{d-2}(D)} & \mathcal{F}^{a_{d-1}} & \xrightarrow{A_{d-1}(D)} & \mathcal{F}^d \longrightarrow 0 \end{array}$$

respectively. The  $\mathcal{P}$ -module  $\mathcal{F}$  is *injective* if  $\text{Ext}_{\mathcal{P}}^j(\mathfrak{M}, \mathcal{F}) = 0$  for every  $j > 0$  and every unitary  $\mathcal{P}$ -module  $\mathfrak{M}$  of finite type; it is *flat* if  $\text{Tor}_j^{\mathcal{P}}(\mathfrak{M}, \mathcal{F}) = 0$  for every  $j > 0$  and every unitary  $\mathcal{P}$ -module of finite type  $\mathfrak{M}$ .

If  $\Omega$  is an open subset of  $\mathbb{R}^N$  we denote by  $\mathcal{E}(\Omega)$  the space of complex valued  $\mathcal{C}^\infty$  functions on  $\Omega$ , endowed with the usual Fréchet-Schwartz topology. It is a  $\mathcal{P}$ -module by the action (2.2) and is injective when  $\Omega$  is convex.

Let  $F$  be a locally closed subset of  $\mathbb{R}^N$ . Then there is some open subset  $\Omega$  of  $\mathbb{R}^N$  such that  $F \subset \Omega$  and  $\overline{F} \cap \Omega = F$ . Denote by  $\mathcal{I}^\infty(F, \Omega)$  the subspace of functions in  $\mathcal{E}(\Omega)$  which vanish with all derivatives on  $F$ . It is a closed subspace and a  $\mathcal{P}$ -submodule of  $\mathcal{E}(\Omega)$ . The exact sequence

$$(2.5) \quad 0 \longrightarrow \mathcal{I}^\infty(F, \Omega) \longrightarrow \mathcal{E}(\Omega) \longrightarrow \mathcal{W}_F \longrightarrow 0$$

can be taken as a definition of the space of Whitney functions on  $F$ , showing both its topological structure of a space of Fréchet-Schwartz and of a  $\mathcal{P}$ -module by the action induced by (2.2). Whitney's extension theorem shows that these structures are independent of the choice of the open neighborhood  $\Omega$  of  $F$  in  $\mathbb{R}^N$  (subject to the condition that  $F = \overline{F} \cap \Omega$ ). Note that for a convex  $F$  the  $\mathcal{P}$ -module  $\mathcal{W}_F$  is injective (cf. [N1]).

When  $F$  is *regular* in the sense of Schwartz the strong dual of  $\mathcal{W}_F$  can be identified to the space  $\mathcal{E}'_F$  of distributions in  $\mathbb{R}^N$  having compact support contained in  $F$ . This is a  $\mathcal{P}$ -module by (2.2) and is flat when  $F$  is convex.

If  $f \in \mathcal{W}_F$ , all its partial derivatives  $D^\alpha f$  (for  $\alpha \in \mathbb{N}^N$ ) are well defined at points of  $F$ . If  $F \subset \mathbb{R}^N$  is locally closed and  $S$  is a closed subset of  $F$ , we define  $\mathcal{I}^\infty(S, F)$  as the space of Whitney functions  $f$  on  $F$  such that  $D^\alpha f(x) = 0$  for every  $x \in S$  and  $\alpha \in \mathbb{N}^N$ . Then we have the exact sequence:

$$(2.6) \quad 0 \longrightarrow \mathcal{I}^\infty(S, F) \longrightarrow \mathcal{W}_F \longrightarrow \mathcal{W}_S \longrightarrow 0.$$

In the classical formulation of the Cauchy problem, one seeks for a solution of a partial differential equation on a manifold with boundary, requiring that the solution and some of its normal derivatives take assigned values on the boundary. Usually the requirements on the given partial differential operator are such that, for a solution smooth up to the boundary, the differential equation and the initial data allow to compute all its partial derivatives at points of the boundary. In our formulation, we will drop this assumption, using as initial data Whitney functions and look then to the possibility of extending *formal* solutions on the initial manifold to solutions in a larger manifold that contains the initial one in its boundary.

Let  $K_1 \subset K_2$  be convex subsets of  $\mathbb{R}^N$ , with  $K_2$  locally closed in  $\mathbb{R}^N$  and  $K_1$  closed in  $K_2$ . We think of  $K_1$  as the set where the initial data are given and of  $K_2$  as the set where we want to find the solution of a (generalized) Cauchy problem. Thus, given a unitary  $\mathcal{P}$ -module  $\mathfrak{M}$  of finite type, we are concerned with the  $\mathcal{P}$ -homomorphism:

$$(2.7) \quad \text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{W}_{K_2}) \longrightarrow \text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{W}_{K_1}).$$

Using the Hilbert resolution (2.1) the homomorphism (2.7) translates into the continuous restriction map

$$(2.8) \quad \{u \in (\mathcal{W}_F)^{a_0} \mid A_0(D)u = 0\} \ni u \longrightarrow u|_S \in \{u \in (\mathcal{W}_S)^{a_0} \mid A_0(D)u = 0\}.$$

We introduce the following notions:

The pair  $(K_1, K_2)$  is a *causality pair* for the unitary  $\mathcal{P}$ -module  $\mathfrak{M}$  if (2.7) is injective;

The pair  $(K_1, K_2)$  is an *evolution pair* for the unitary  $\mathcal{P}$ -module  $\mathfrak{M}$  if (2.7) is surjective;

The pair  $(K_1, K_2)$  is a *hyperbolic pair* for the unitary  $\mathcal{P}$ -module  $\mathfrak{M}$  if (2.7) is an isomorphism.

From Propositions 1.1, 1.2 we obtain:

**PROPOSITION 2.1.** — *Let  $\mathfrak{M}$  be a unitary  $\mathcal{P}$ -module of finite type and let  $K_1 \subset K_2$  be convex subsets of  $\mathbb{R}^N$  with  $K_2$  locally closed and  $K_1$  closed in  $K_2$ . Then the following statements are equivalent:*

- (1)  $(K_1, K_2)$  is a causality pair for  $\mathfrak{M}$ ;
- (2)  $\text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{I}^\infty(K_1, K_2)) = 0$ ;
- (3)  $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = 0$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ ;

- (4)  $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = 0$  for all  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ ;
- (5)  $(K_1, K_2)$  is a causality pair for  $\mathcal{P}/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ ;
- (6)  $(K_1, K_2)$  is a causality pair for  $\mathcal{P}/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ .

PROPOSITION 2.2. — *Let  $\mathfrak{M}$  be a unitary  $\mathcal{P}$ -module of finite type and let  $K_1 \subset K_2$  be convex subsets of  $\mathbb{R}^N$  with  $K_2$  locally closed and  $K_1$  closed in  $K_2$ . Then the following statements are equivalent:*

- (1)  $(K_1, K_2)$  is an evolution pair for  $\mathfrak{M}$ ;
- (2)  $\text{Ext}_{\mathcal{P}}^1(\mathfrak{M}, \mathcal{I}^\infty(K_1, K_2)) = 0$ ;
- (3)  $\text{Ext}_{\mathcal{P}}^1(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = 0$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ ;
- (4)  $(K_1, K_2)$  is an evolution pair for  $\mathcal{P}/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .

PROPOSITION 2.3. — *Let  $\mathfrak{M}$  be a unitary  $\mathcal{P}$ -module of finite type and let  $K_1 \subset K_2$  be convex subsets of  $\mathbb{R}^N$  with  $K_2$  locally closed and  $K_1$  closed in  $K_2$ . Then the following statements are equivalent:*

- (1)  $(K_1, K_2)$  is a hyperbolic pair for  $\mathfrak{M}$ ;
- (2)  $\text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{I}^\infty(K_1, K_2)) = \text{Ext}_{\mathcal{P}}^1(\mathfrak{M}, \mathcal{I}^\infty(K_1, K_2)) = 0$ ;
- (3)  $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = \text{Ext}_{\mathcal{P}}^1(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = 0$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ ;
- (4)  $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = \text{Ext}_{\mathcal{P}}^1(\mathcal{P}/\mathfrak{p}, \mathcal{I}^\infty(K_1, K_2)) = 0$  for all  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ ;
- (5)  $(K_1, K_2)$  is a hyperbolic pair for  $\mathcal{P}/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ ;
- (6)  $(K_1, K_2)$  is a hyperbolic pair for  $\mathcal{P}/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ .

The dual map of (2.7) is the map

$$(2.9) \quad \text{Tor}_0^{\mathcal{P}}(\mathfrak{M}, \mathcal{E}'_S) \longrightarrow \text{Tor}_0^{\mathcal{P}}(\mathfrak{M}, \mathcal{E}'_F).$$

When  $S$  and  $F$  are convex, the subspaces  ${}^tA_0(D)((\mathcal{E}'_S)^{a_1})$  and  ${}^tA_0(D)((\mathcal{E}'_F)^{a_1})$  are closed respectively in  $(\mathcal{E}'_S)^{a_0}$  and  $(\mathcal{E}'_F)^{a_0}$ . The map (2.9) can be described in terms of the Hilbert resolution (2.1) by the continuous linear map:

$$(2.10) \quad \frac{(\mathcal{E}'_S)^{a_0}}{{}^tA_0(D)((\mathcal{E}'_S)^{a_1})} \longrightarrow \frac{(\mathcal{E}'_F)^{a_1}}{{}^tA_0(D)((\mathcal{E}'_F)^{a_1})}$$

induced by the inclusion  $S \hookrightarrow F$ . Given an ideal  $\mathfrak{p}$  in  $\mathcal{P}$  and a locally closed subset  $S$  of  $\mathbb{R}^N$ , we denote by

$$(2.11) \quad \mathfrak{F}(\mathfrak{p}, S) = \{u \in \mathcal{W}_S \mid p(D)u = 0 \ \forall p \in \mathfrak{p}\}$$

the space of Whitney functions on  $S$  which are annihilated by every partial differential operator associated to a polynomial in  $\mathfrak{p}$  and by  $\mathfrak{F}^o(\mathfrak{p}, S)$  its annihilator in  $\mathcal{E}'_S$ :

$$(2.12) \quad \mathfrak{F}^o(\mathfrak{p}, S) = \{T \in \mathcal{E}'_S \mid \langle u, T \rangle = 0 \ \forall u \in \mathfrak{F}(\mathfrak{p}, S)\}.$$

Since, when  $S$  and  $F$  are convex, (2.8) is a continuous linear map between Fréchet-Schwartz spaces having a dense image, we obtain:

PROPOSITION 2.4. — *Let  $\mathfrak{M}$  be a unitary  $\mathcal{P}$ -module of finite type and let  $K_1 \subset K_2$  be convex subsets of  $\mathbb{R}^N$  with  $K_2$  locally closed and  $K_1$  closed in  $K_2$ . Consider the maps*

$$(2.13) \quad \frac{\mathcal{E}'_{K_1}}{\mathfrak{F}^o(\mathfrak{p}, K_1)} \longrightarrow \frac{\mathcal{E}'_{K_2}}{\mathfrak{F}^o(\mathfrak{p}, K_2)}$$

where  $\mathfrak{p}$  is a prime ideal in  $\mathcal{P}$ . Then we have:

(1) *A necessary and sufficient condition in order that  $(K_1, K_2)$  be a causality pair for  $\mathfrak{M}$  is that (2.13) has a dense image for every  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$ ; it suffices that (2.13) has a dense image for every  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .*

(2) *A necessary and sufficient condition in order that  $(K_1, K_2)$  be an evolution pair is that (2.13) has a closed image for every  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .*

(3) *A necessary and sufficient condition in order that  $(K_1, K_2)$  be a hyperbolic pair is that (2.13) is an isomorphism for every  $\mathfrak{p} \in \text{Supp}(\mathfrak{M})$  and is sufficient that (2.13) is an isomorphism for every  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .*

The advantage of the formulation given in Proposition 2.4 stems from the good characterization of the spaces involved in (2.13) which we obtain by the use of Fourier-Laplace transform and Ehrenpreis fundamental principle.

We define the Fourier-Laplace transform of a distribution with compact support  $T \in \mathcal{E}'(\mathbb{R}^N)$  by

$$(2.14) \quad \hat{T}(\zeta) = \langle \exp(-\sqrt{-1}\langle \cdot, \zeta \rangle), T \rangle \quad \forall \zeta \in \mathbb{C}^N.$$

This is an entire function of exponential type.

Introducing the support function  $H_K$  of the convex subset  $K$  of  $\mathbb{R}^N$ :

$$(2.15) \quad H_K(\zeta) = \sup_{x \in K} \operatorname{Im} \langle x, \zeta \rangle \quad \text{for } \zeta \in \mathbb{C}^N$$

the Fourier-Laplace transform of  $T$  is characterized by the theorem of Paley-Wiener-Schwartz by the estimate

$$(2.16) \quad |\hat{T}(\zeta)| \leq C(1 + |\zeta|)^m \exp(H_K(\zeta)) \quad \forall \zeta \in \mathbb{C}^N$$

where  $K$  is the convex hull of the support of  $T$  and  $C, m$  are suitable nonnegative constants.

Let  $V = V(\mathfrak{p})$  denote the affine algebraic variety of  $\mathbb{C}^N$  of common zeros of the prime ideal  $\mathfrak{p}$  of  $\mathcal{P}$ . We denote by  $\mathcal{O}(V)$  the space of holomorphic functions on  $V$ , i.e. the space of restrictions to  $V$  of entire functions in  $\mathbb{C}^N$ . It is a Fréchet space for the topology of uniform convergence on compact subsets of  $V$ . Let  $K$  be a locally closed convex subset of  $\mathbb{R}^N$  and denote by  $\mathcal{O}_{[K]}(V)$  the linear subspace of  $\mathcal{O}(V)$  of functions  $F$  satisfying an inequality of the form

$$(2.17) \quad \begin{cases} |F(\zeta)| \leq C_F(1 + |\zeta|)^{m_F} \exp H_{\sigma_F}(\zeta) \\ \text{for some nonnegative constants } C_F, m_F \\ \text{and a convex compact subset } \sigma_F \subset K. \end{cases}$$

From the Ehrenpreis fundamental principle we deduce the following:

**PROPOSITION 2.5.** — *Let  $V$  be the irreducible affine algebraic variety of common zeros in  $\mathbb{C}^N$  of a prime ideal  $\mathfrak{p}$  of  $\mathcal{P}$  and let  $K$  be a locally closed convex subset of  $\mathbb{R}^N$ . The Fourier-Laplace transform induces a  $\mathbb{C}$ -linear isomorphism*

$$(2.18) \quad \frac{\mathcal{E}'_K}{\mathfrak{F}^o(\mathfrak{p}, K)} \xrightarrow{\simeq} \mathcal{O}_{[K]}(V).$$

We shall further investigate the topology and the properties of spaces of holomorphic functions on an affine algebraic variety of  $\mathbb{C}^N$  in the following section. Although in this paper we will restrain to the applications to the Cauchy problem for Whitney functions, we keep a more general point of view, suitable for developing the study of the Cauchy problem in different classes of functions and distributions.

### 3. Spaces of entire functions.

Let  $V \subset \mathbb{C}^N$  be a reduced affine algebraic variety. We denote by  $\mathcal{O}(V)$  the space of holomorphic functions on  $V$ , i.e. the space of complex valued continuous functions on  $V$  which are restrictions of entire functions in  $\mathbb{C}^N$ . By the Weierstrass theorem (cf [GR]),  $\mathcal{O}(V)$  is a Fréchet space for the topology of uniform convergence on compact subsets of  $V$ . We note that  $\mathcal{O}(V)$  is a Montel space.

Given an upper semicontinuous function  $\phi : V \rightarrow \mathbb{R}$ , we denote by  $\mathcal{O}_\phi(V)$  the space

$$(3.1) \quad \mathcal{O}_\phi(V) = \{f \in \mathcal{O}(V) \mid \sup_{\zeta \in V} |f(\zeta)e^{-\phi(\zeta)}| < \infty\}.$$

This is a Banach space for the norm

$$\mathcal{O}_\phi(V) \ni f \rightarrow \|f\|_\phi = \sup_{\zeta \in V} |f(\zeta)e^{-\phi(\zeta)}| \in \mathbb{R}$$

and the natural inclusion  $\mathcal{O}_\phi(V) \hookrightarrow \mathcal{O}(V)$  is compact. Moreover, we have:

LEMMA 3.1. — *Let  $\phi, \psi : V \rightarrow \mathbb{R}$  be upper semicontinuous functions on  $V$  such that*

$$\lim_{\substack{\zeta \in V \\ |\zeta| \rightarrow \infty}} e^{\phi(\zeta) - \psi(\zeta)} = 0.$$

*Then the inclusion map*

$$\mathcal{O}_\phi(V) \hookrightarrow \mathcal{O}_\psi(V)$$

*is compact, as bounded subsets of  $\mathcal{O}_\phi(V)$  are relatively compact in  $\mathcal{O}_\psi(V)$ .*

Let  $\mathcal{P}_N = \mathcal{P}$  denote the ring  $\mathbb{C}[\zeta_1, \dots, \zeta_N]$  of complex valued polynomials in  $\mathbb{C}^N$ . We are interested in spaces of holomorphic functions which are  $\mathcal{P}$ -modules for the multiplication by polynomials. Hence we introduce sequences  $\psi = \{\psi_n\}_{n \in \mathbb{N}}$  of upper semicontinuous functions  $\psi_n : V \rightarrow \mathbb{R}$  having the property:

$$(3.2) \quad \log(1 + |\zeta|) + \psi_n(\zeta) \leq \psi_{n+1}(\zeta) \quad \forall n \in \mathbb{N}, \forall \zeta \in V$$

and consider the direct limits:

$$(3.3) \quad \mathcal{O}_\psi(V) = \varinjlim_n \mathcal{O}_{\psi_n}(V).$$

By Lemma 3.1 the space  $\mathcal{O}_\psi(V)$  is a compact inductive limit of the sequence  $\mathcal{O}_{\psi_n}(V)$ .

Therefore  $\mathcal{O}_\psi(V)$  is the strong dual of a Fréchet-Schwartz space and each bounded subset  $B$  of  $\mathcal{O}_\psi(V)$  is already contained and bounded in one of the Banach spaces  $\mathcal{O}_{\psi_n}(V)$  for some nonnegative integer  $n$  (cf. [FW], [Ko]).

We associate to the sequence  $\psi$  the set of weights

$$(3.4) \quad \mathcal{L}_\psi = \{\lambda(\zeta) = \sum_{n=0}^{\infty} \epsilon_n e^{\psi_n(\zeta)} \mid \epsilon_n > 0 \forall n, \lambda(\zeta) < \infty \forall \zeta \in V\}.$$

The topology of  $\mathcal{O}_\psi(V)$  is described (cf. [BMS], Theorem 1.6) by the following:

LEMMA 3.2. — For  $\lambda$  varying in  $\mathcal{L}_\psi$ , the sets

$$(3.5) \quad U_\psi^\lambda = \{f \in \mathcal{O}_\psi(V) \mid |f(\zeta)| < \lambda(\zeta) \forall \zeta \in V\}$$

form a fundamental system of open circled convex neighborhoods of 0 in  $\mathcal{O}_\psi(V)$ . A subset  $G$  of  $\mathcal{O}_\psi(V)$  is open if and only if  $G \cap \mathcal{O}_{\psi_n}(V)$  is open in  $\mathcal{O}_{\psi_n}(V)$  for every  $n \in \mathbb{N}$ . Moreover, the sequence of compact sets

$$K_n = \{f \in \mathcal{O}_{\psi_n}(V) \mid \|f\|_{\psi_n} \leq n\} \quad \text{for } n = 1, 2, \dots$$

is a fundamental covering of  $\mathcal{O}_\psi(V)$ .

We collect the topological properties of the spaces  $\mathcal{O}_\psi(V)$  which follow from the discussion above in the following:

THEOREM 3.1. — Let  $\psi = \{\psi_n\}$  be an increasing sequence of upper semicontinuous functions defined on the reduced affine algebraic variety  $V \subset \mathbb{C}^N$  and satisfying (3.2). Then the space  $\mathcal{O}_\psi(V)$  is

(1) an  $(\mathcal{LF})$  space, being the compact inductive limit of an increasing sequence of Banach spaces;

(2) is barreled, being a locally convex topological vector space in which absorbing closed convex circled subsets containing 0 are neighborhoods of 0;

(3) is a  $(\mathcal{DF})$  space, since it admits a fundamental sequence of compact subsets and every intersection of a countable family of convex circled neighborhoods of 0 that absorbs strongly bounded subsets is a neighborhood of 0;

(4) is an  $(\mathcal{S})$  space, since every convex circled neighborhood  $U$  of 0 contains an open neighborhood of 0 which is relatively compact in the completion of  $\mathcal{O}_\psi(V)$  with respect to the Minkowski norm defined by  $U$ ;

(5) is an  $(\mathcal{M})$  space, i.e. bounded subsets are relatively compact in  $\mathcal{O}_\psi(V)$ ;

(6) is bornological, i.e. all seminorms that are bounded on bounded subsets are continuous;

(7) is reflexive;

(8) is complete.

We refer to [AT], [Gr], [Sc] for the standard implications which connect the results obtained in the preceding lemmas to the statement of this theorem.

#### 4. The abstract Phragmén–Lindelöf principle.

Let  $V$  be an irreducible affine algebraic variety in  $\mathbb{C}^N$  and consider, for two sequences  $\psi = \{\psi_n\}$ ,  $\phi = \{\phi_n\}$  of upper semicontinuous functions on  $V$  satisfying (3.2), the spaces  $\mathcal{O}_\psi(V)$  and  $\mathcal{O}_\phi(V)$ . In the applications, these spaces will be identified via the Fourier-Laplace transform to dual spaces of spaces of solutions of a homogeneous system of l.p.d.o's with constant coefficients. The statements about hyperbolicity, causality and evolution will translate then into the following notions. Let us consider the inclusion map

$$(4.1) \quad \mathcal{O}_\phi(V) \cap \mathcal{O}_\psi(V) \hookrightarrow \mathcal{O}_\psi(V).$$

We say that the pair  $(\phi, \psi)$  is

- (i) *hyperbolic* if (4.1) is an isomorphism;
- (ii) *of causality* if (4.1) has a dense image;
- (iii) *of evolution* if the map (4.1) has a closed image.

While discussing these notions, we note that  $\mathcal{O}_\phi(V) \cap \mathcal{O}_\psi(V)$  is equal to the space  $\mathcal{O}_{\phi \wedge \psi}(V)$  where  $\phi \wedge \psi = \{\phi_n \wedge \psi_n\}$ . Since  $\phi \wedge \psi$  also satisfies conditions (3.2) we shall for simplicity assume that  $\phi_n \leq \psi_n$  on  $V$  for every  $n$  and consider then the inclusion:

$$(4.1') \quad \mathcal{O}_\phi(V) \hookrightarrow \mathcal{O}_\psi(V).$$

We also note that, because of the topological properties of the spaces  $\mathcal{O}_\phi(V)$  and  $\mathcal{O}_\psi(V)$  the condition that (4.1') has a closed image is equivalent to the fact that it is a topological homomorphism. In particular:

THEOREM 4.1. — *The pair  $(\phi, \psi)$  is of evolution if and only if:*

$$\forall \lambda \in \mathcal{L}_\phi \exists \lambda' \in \mathcal{L}_\psi \text{ such that } U_{\psi}^{\lambda'} \cap \mathcal{O}_\phi(V) \subset U_{\phi}^{\lambda}.$$

LEMMA 4.1. — *Let  $\alpha < \beta$  and let  $K$  be a compact subset of  $V$  with a nonempty interior. Then for every  $\epsilon > 0$  there is a constant  $c_\epsilon > 0$  such that*

$$(4.2) \quad \|f\|_{\phi_\beta} \leq \epsilon \|f\|_{\phi_\alpha} + c_\epsilon \sup_{\zeta \in K} |f(\zeta)| \quad \forall f \in \mathcal{O}_{\phi_\alpha}(V).$$

*Proof.* — Assume by contradiction that the statement is not true for some  $\epsilon > 0$ . Then we can find a sequence  $\{f_\nu\} \subset \mathcal{O}_{\phi_\alpha}(V)$  such that

$$\|f_\nu\|_{\phi_\beta} > \epsilon \|f_\nu\|_{\phi_\alpha} + \nu \sup_{\zeta \in K} |f_\nu(\zeta)|.$$

We can assume that  $\|f_\nu\|_{\phi_\beta} = 1$ . Then we have

$$\|f_\nu\|_{\phi_\alpha} < \epsilon^{-1}, \quad \sup_{\zeta \in K} |f_\nu(\zeta)| < \nu^{-1}.$$

Since the inclusion  $\mathcal{O}_{\phi_\alpha}(V) \hookrightarrow \mathcal{O}_{\phi_\beta}(V)$  is compact, by passing to a subsequence we can assume that  $f_\nu \rightarrow f \in \mathcal{O}_{\phi_\beta}(V)$ . We have  $\|f\|_{\phi_\beta} = 1$  and therefore  $f \neq 0$ . But  $f$  vanishes at all points of  $K$  and therefore is 0 on  $V$  by the unique continuation principle. This gives a contradiction, proving our contention.

From this lemma (cf. also [Bae]) we deduce:

THEOREM 4.2 (PHRAGMÉN-LINDELÖF FOR EVOLUTION). — *A necessary and sufficient condition in order that the pair  $(\phi, \psi)$  be of evolution is that one of the following equivalent conditions be satisfied:*

$$(PhL I) \quad \forall \alpha \in \mathbb{N} \exists \beta \in \mathbb{N}, c_\alpha > 0 \text{ such that } f \in \mathcal{O}_\phi(V), \|f\|_{\psi_\alpha} \leq 1 \Rightarrow \|f\|_{\phi_\beta} \leq c_\alpha.$$

$$(PhL II) \quad \forall \alpha \in \mathbb{N} \exists \beta \in \mathbb{N} \text{ such that } \mathcal{O}_\phi(V) \cap \mathcal{O}_{\psi_\alpha}(V) \subset \mathcal{O}_{\phi_\beta}(V).$$

*Proof.* — Assume that the pair  $(\phi, \psi)$  be of evolution. Then, for every fixed  $\alpha \in \mathbb{N}$ , the set  $\{f \in \mathcal{O}_\phi(V) \mid \|f\|_{\psi_\alpha} \leq 1\}$  is compact in  $\mathcal{O}_\phi(V)$  and

therefore is contained and bounded in  $\mathcal{O}_{\phi_\beta}(V)$  for some  $\beta \in \mathbb{N}$ . Therefore we obtain (PhL I) and (PhL II). It is obvious that (PhL I)  $\Rightarrow$  (PhL II) and also that (PhL I) is a sufficient condition in order that  $(\phi, \psi)$  be an evolution pair. To complete the proof, we only need to prove that (PhL II) implies (PhL I).

Assume then that (PhL II) is valid. Let  $\alpha \in \mathbb{N}$  be fixed and let  $\beta \in \mathbb{N}$  be such that  $\mathcal{O}_\phi(V) \cap \mathcal{O}_{\psi_\alpha}(V) \subset \mathcal{O}_{\phi_\beta}(V)$ . Then we have

$$\mathcal{O}_{\phi_\gamma}(V) \cap \mathcal{O}_{\psi_\alpha}(V) = \mathcal{O}_{\phi_\beta}(V) \cap \mathcal{O}_{\psi_\alpha}(V) \quad \forall \gamma \geq \beta.$$

We consider on these Banach spaces the intersection norm. By the Banach open mapping theorem we obtain, with some constant  $c(\gamma) > 1$ , the estimate:

$$\|f\|_{\phi_\beta} + \|f\|_{\psi_\alpha} \leq c(\gamma) (\|f\|_{\phi_\gamma} + \|f\|_{\psi_\alpha}) \quad \forall f \in \mathcal{O}_{\phi_\gamma}(V) \cap \mathcal{O}_{\psi_\alpha}(V).$$

Now we apply Lemma 4.1 choosing  $\gamma = \beta + 1$  and  $\epsilon > 0$  such that  $\epsilon \cdot c(\gamma) < 1$ . With  $K \subset V$  compact and with a nonempty interior we obtain:

$$(1 - c(\gamma)\epsilon)\|f\|_{\phi_\beta} \leq c(\gamma)\|f\|_{\psi_\alpha} + c_\epsilon \cdot c(\gamma) \sup_{\zeta \in K} |f(\zeta)| \quad \forall f \in \mathcal{O}_\phi(V) \cap \mathcal{O}_{\psi_\alpha}(V)$$

and this shows that (PhL II)  $\Rightarrow$  (PhL I), because  $e^{-\psi_\alpha}$  is bounded from below in  $K$ .

As a corollary, we obtain:

**THEOREM 4.3 (PHRAGMÉN-LINDELÖF PRINCIPLE FOR HYPERBOLICITY).** — *A necessary and sufficient condition in order that  $(\phi, \psi)$  be a hyperbolic pair is that one of the following equivalent statements be valid:*

(PhL I')  $\forall \alpha \in \mathbb{N} \exists \beta \in \mathbb{N}, c_\alpha > 0$  such that  $\|f\|_{\phi_\beta} \leq c_\alpha \|f\|_{\psi_\alpha} \quad \forall f \in \mathcal{O}(V)$ ;

(PhL II')  $\forall \alpha \in \mathbb{N} \exists \beta \in \mathbb{N}$  such that  $\mathcal{O}_{\psi_\alpha}(V) \subset \mathcal{O}_{\phi_\beta}(V)$ .

## 5. Further remarks on hyperbolicity.

In this section we shall consider equivalent formulations of the hyperbolicity conditions for special pairs  $(\phi, \psi)$ .

$\alpha)$  We consider first the case where the sequence  $\{\psi_\alpha\}$  is defined in the following way:

$$(5.1) \quad \psi : \mathbb{C}^N \longrightarrow \mathbb{R} \text{ is a real valued convex function}$$

and the functions  $\psi_\alpha : V \longrightarrow \mathbb{R}$  are restrictions to  $V$  of the continuous plurisubharmonic functions  $\zeta \longrightarrow \psi(\zeta) + \alpha \log(e + |\zeta|)$ .

We introduce the dual convex function

$$\psi^*(z) = \sup_{\zeta \in \mathbb{C}^N} -\text{Im}\langle z, \zeta \rangle - \psi(\zeta).$$

This is an upper semicontinuous convex function in  $\mathbb{C}^N$ , with values in  $\mathbb{R} \cup \{+\infty\}$ . We define  $D(\psi^*) = \{z \in \mathbb{C}^N \mid \psi^*(z) < \infty\}$ . The set  $D(\psi^*)$  is nonempty and we have

$$\psi(\zeta) = \sup_{z \in D(\psi^*)} -\text{Im}\langle z, \zeta \rangle - \psi^*(z) \quad \forall \zeta \in \mathbb{C}^N.$$

**THEOREM 5.1.** — *A necessary and sufficient condition in order that the pair  $(\phi, \psi)$ , with  $\psi = \{(\psi + \alpha \log(e + \cdot))|_V\}$  for a convex function  $\psi : \mathbb{C}^N \longrightarrow \mathbb{R}$  be hyperbolic is that there exist  $\alpha \in \mathbb{N}$  and a constant  $c \geq 0$  such that*

$$(5.2) \quad \psi(\zeta) \leq c + \phi_\alpha(\zeta) \quad \forall \zeta \in V.$$

*Proof.* — We note that (5.2) is obviously a sufficient condition for hyperbolicity. To prove that the condition is also necessary, we use condition (PhL I') of the previous section: we can find  $\alpha \in \mathbb{N}$  and  $\kappa > 0$  such that

$$(*) \quad |f(\zeta)| \leq \kappa e^{\phi_\alpha(\zeta)} \quad \forall \zeta \in V$$

for every function  $f \in \mathcal{O}(V)$  satisfying

$$(**) \quad |f(\zeta)| \leq e^{\psi(\zeta)} \quad \forall \zeta \in V.$$

For every  $z \in D(\psi^*)$  we consider the entire function on  $\mathbb{C}^N$ :

$$\mathbb{C}^N \ni \zeta \longrightarrow e^{-\psi^*(z)} e^{i\langle z, \zeta \rangle} \in \mathbb{C}.$$

Its restriction to  $V$  satisfies (\*\*) and therefore we have by (\*):

$$-\text{Im}\langle z, \zeta \rangle - \psi^*(z) \leq \log \kappa + \phi_\alpha(\zeta) \quad \forall \zeta \in V.$$

Taking the supremum of the left hand side for  $z \in D(\psi^*)$  for fixed  $\zeta \in V$ , we obtain (5.2).

$\beta$ ) Next we consider the case where  $\psi : \mathbb{C}^N \rightarrow \mathbb{R}$  is a continuous plurisubharmonic function satisfying for some constants  $a \geq 0, c > 0, r > 0$ :

$$(5.3) \quad |\psi(\zeta) - \psi(\eta)| \leq c(1 + |\zeta|)^a |\zeta - \eta| \quad \forall \zeta, \eta \in \mathbb{C}^N \text{ with } |\zeta - \eta| < r$$

and  $\psi_\alpha$  is the restriction to  $V$  of the continuous plurisubharmonic function on  $\mathbb{C}^N$ :

$$\mathbb{C}^N \ni \zeta \rightarrow \psi(\zeta) + \alpha \log(e + |\zeta|) \in \mathbb{R}.$$

We have the following:

LEMMA 5.1. — *Let  $\psi$  be a continuous plurisubharmonic function on  $\mathbb{C}^N$  satisfying (5.3). Then, if  $g \in \mathcal{O}(\mathbb{C}^N)$  satisfies*

$$\|g\|_{L^2(\mathbb{C}^N, \psi)}^2 = \int_{\mathbb{C}^N} |g(\zeta)|^2 e^{-2\psi(\zeta)} d\lambda(\zeta) < \infty,$$

*it also satisfies the pointwise estimate:*

$$(5.4) \quad |g(\zeta)| \leq \sqrt{\frac{N!}{\pi^N}} e^N \left( \max \left( \frac{c}{N}, \frac{1}{r} \right) \right)^N \|g\|_{L^2(\mathbb{C}^N, \psi)} (1 + |\zeta|)^{aN} e^{\psi(\zeta)} \quad \forall \zeta \in \mathbb{C}^N.$$

*Proof.* — For every fixed  $\theta \in \mathbb{C}^N$  and  $g \in \mathcal{O}(\mathbb{C}^N)$  we obtain by the mean value theorem

$$g(\theta) e^{-\psi(\theta)} = \frac{N!}{\pi^N} \rho^{-2N} \int_{\zeta \in B(\theta, \rho)} g(\zeta) e^{-\psi(\zeta)} d\lambda(\zeta)$$

for every  $\rho > 0$ . When  $0 < \rho < r$ , using (5.3) we obtain by the Cauchy inequality

$$\begin{aligned} |g(\theta) e^{-\psi(\theta)}| &\leq \frac{N!}{\pi^N} \rho^{-2N} e^{c\rho(1+|\theta|)^a} \int_{B(\theta, \rho)} |g(\zeta)| e^{-\psi(\zeta)} d\lambda(\zeta) \\ &\leq \sqrt{\frac{N!}{\pi^N}} e^{c\rho(1+|\theta|)^a} \rho^{-N} \|g\|_{L^2(\mathbb{C}^N, \psi)}. \end{aligned}$$

We can assume that  $c \geq Nr$ . Then we can take in the inequality  $\rho = \frac{N}{c(1+|\theta|)^a}$ , obtaining (5.4).

Using the  $L^2$  existence theory for  $\bar{\partial}$ , we obtain, using the lemma above and a standard argument (cf. [Hö1]):

LEMMA 5.2. — *Let  $\psi$  be a continuous plurisubharmonic function satisfying (5.3). Then there are positive constants  $c, m > 0$  such that for every  $\theta \in \mathbb{C}^N$  we can find an entire function  $F_\theta \in \mathcal{O}(\mathbb{C}^N)$  with:*

$$(5.5) \quad F_\theta(\theta) = e^{\psi(\theta)}, \quad |F_\theta(\zeta)| \leq c(1 + |\zeta|)^m e^{\psi(\zeta)} \quad \forall \zeta \in \mathbb{C}^N.$$

Using this lemma we obtain the criterion:

THEOREM 5.2. — *Let  $\psi : \mathbb{C}^N \rightarrow \mathbb{R}$  be a continuous plurisubharmonic function satisfying (5.3) and let  $\psi_\alpha$  be the restriction to  $V$  of the function  $\psi + \alpha \log(e + |\cdot|)$ . Then a necessary and sufficient condition in order that the pair  $(\phi, \psi)$  be hyperbolic is that there exist  $\alpha \in \mathbb{N}$  and  $c > 0$  such that*

$$\psi(\zeta) \leq c + \phi_\alpha(\zeta) \quad \forall \zeta \in V.$$

*Proof.* — The condition is obviously sufficient. To obtain the necessity, it suffices to apply the condition (PhL I) of the previous section to the restriction to  $V$  of the entire functions  $F_\theta$ , for  $\theta \in V$ , given in the previous lemma.

## 6. The Phragmén-Lindelöf principle for plurisubharmonic functions.

To investigate the conditions for a pair  $(\phi, \psi)$  to be of evolution, it is convenient to translate the results of § 4, formulated in terms of holomorphic functions, into statements involving instead weakly plurisubharmonic functions on the irreducible affine algebraic variety  $V$ . In this section we show that this is in fact possible under some additional conditions on the sequences  $\phi$  and  $\psi$ . This result is analogous to the corresponding one in [MTV] and is suggested by the treatment of analytic convexity in [Hö2] and [AN2].

As before, we assume that  $V \subset \mathbb{C}^N$  be an irreducible affine algebraic variety, of positive dimension.

We recall that a function  $u : V \rightarrow [-\infty, +\infty[$  is said to be *weakly plurisubharmonic* if it is plurisubharmonic on the complement of the singular set  $S(V)$  of  $V$  and moreover

$$u(\zeta) = \limsup_{z \rightarrow \zeta} u(z) \quad \forall \zeta \in V.$$

We will denote by  $P(V)$  the space of weakly plurisubharmonic functions on  $V$ .

We say that a sequence  $\psi = \{\psi_n\}$  of real valued functions on  $V$  is *admissible* if the following conditions are satisfied:

(i) for every integer  $n \geq 0$  the function  $\psi_n$  is the restriction to  $V$  of a plurisubharmonic function  $\tilde{\psi}_n$  in  $\mathbb{C}^N$ ;

(ii) for every nonnegative integer  $n$  and for every constant  $c > 0$  there are an integer  $n' \geq 0$  and a constant  $c' \geq 0$  such that

$$(6.1) \quad \tilde{\psi}_n(\zeta) + c \log(e + |\zeta|) \leq \tilde{\psi}_{n'}(\zeta) + c' \quad \forall \zeta \in \mathbb{C}^N;$$

(iii) for every integer  $n \geq 0$  there are a real number  $0 < \theta < 1$ , an integer  $n' \geq 0$  and a constant  $c' > 0$  such that

$$(6.2) \quad \tilde{\psi}_n(\zeta) \leq \theta \tilde{\psi}_{n'}(\zeta) + c' \quad \forall \zeta \in \mathbb{C}^N;$$

(iv) for every nonnegative integer  $n$  there are constants  $b_n, c_n \geq 0$  such that

$$(6.3) \quad |\tilde{\psi}(\zeta) - \tilde{\psi}_n(\zeta')| \leq c_n(1 + |\zeta|)^{b_n} |\zeta - \zeta'| \quad \text{if } \zeta, \zeta' \in \mathbb{C}^N \text{ and } |\zeta - \zeta'| \leq 1.$$

We note that condition (6.3) implies that  $\tilde{\psi}_n$  is bounded by a constant times  $(1 + |\zeta|)^{b_n+1}$  on  $\mathbb{C}^N$ . In the following, while considering admissible sequences  $\psi$ , we will think the functions  $\{\psi_n\}$  as given and write for simplicity  $\psi_n$  instead of  $\tilde{\psi}_n$ .

We have the following:

**THEOREM 6.1.** — *Let  $\phi$  and  $\psi$  be two admissible sequences of plurisubharmonic functions defined on an irreducible affine algebraic variety  $V \subset \mathbb{C}^N$ . Then the Phragmén-Lindelöf conditions (PhL I), (PhL II) of Theorem 4.2 are equivalent to the following:*

(PhL III) *for every integer  $n \geq 0$  there are an integer  $n' \geq 0$  and a constant  $c_n > 0$  such that, for  $u \in P(V)$ :*

$$\left\{ \begin{array}{l} \text{(i)} \quad u(\zeta) \leq \psi_n(\zeta) \quad \forall \zeta \in V \\ \text{(ii)} \quad \exists \nu \in \mathbb{N}, \exists c > 0 \text{ such that } \Rightarrow u(\zeta) \leq \phi_{n'}(\zeta) + c_n \quad \forall \zeta \in V. \\ \quad \quad u(\zeta) \leq \psi_\nu(\zeta) + c \quad \forall \zeta \in V \end{array} \right.$$

The proof of Theorem 6.1 follows the general pattern used in [Hö2], [AN2] for discussing analytic convexity and can be obtained by repeating with slight variants the arguments in [MTV] and [F]. It will therefore be omitted, referring the reader to [MTV], [F], [Hö3], [Hö4] for the general results on plurisubharmonic functions that are needed to fill in the details.

## 7. The Cauchy problem with data on an affine subspace.

Let  $\Sigma$  denote an  $n$  dimensional affine subspace of  $\mathbb{R}^N$ . By an affine change of coordinates we can as well assume that  $\Sigma$  is the coordinate  $n$ -plane:

$$\Sigma = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N \mid x^{n+1} = \dots = x^N = 0\}.$$

A typical closed wedge  $\Gamma$  with edge  $\Sigma$  can then be written, after another linear change of coordinates, in the form

$$\Gamma = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N \mid x^{n+1} \geq 0, \dots, x^N \geq 0\}.$$

We are interested in characterizing the  $\mathcal{P}$ -module of finite type  $\mathfrak{M}$  for which  $(\Sigma, \Gamma)$  is a hyperbolic or an evolution pair. We already observed in §4 that we can restrain to consider the  $\mathcal{P}$ -modules of the form  $\mathcal{P}/\mathfrak{p}$  with  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .

Let us denote by  $\mathcal{P}_n$  the ring  $\mathbb{C}[\zeta_1, \dots, \zeta_n]$  of polynomials in the first  $n$  indeterminates. This is a subring of  $\mathcal{P}$  and therefore every  $\mathcal{P}$ -module  $\mathfrak{M}$  can be considered also as a  $\mathcal{P}_n$ -module by change of the base ring. Let (4.1) be a Hilbert resolution of  $\mathfrak{M}$  and identify  $\text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{E}(\mathbb{R}^N))$  to the space of smooth solutions of the homogeneous system  $A_0(D)u = 0$ , with  $u \in \left(\mathcal{E}(\mathbb{R}^N)\right)^{a_0}$ . In [AN3] and [N2] it was observed that a necessary and sufficient condition in order that the Taylor series at points of  $\Sigma$  of the elements of  $\text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{E}(\mathbb{R}^N))$  be determined by the restriction to  $\Sigma$  of a finite number of their transversal derivatives is that  $\mathfrak{M}$ , considered as a  $\mathcal{P}_n$ -module, be of finite type. When this is the case, we say that  $\Sigma$  is *formally noncharacteristic* for  $\mathfrak{M}$ .

Denote by  $\pi_n : \mathbb{C}^N \rightarrow \mathbb{C}^n$  the natural projection onto the first  $n$  coordinates. Then we obtain:

LEMMA 7.1. — *A necessary and sufficient condition in order that  $\Sigma$  be formally noncharacteristic for  $\mathfrak{M}$  is that for every  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$  the map*

$$(7.1) \quad V(\mathfrak{p}) \ni \zeta \rightarrow \pi_n(\zeta) \in \mathbb{C}^n$$

*be finite.*

For the proof of this lemma we refer to [N2].

Let us denote by  $(\mathfrak{M})_n$  the  $\mathcal{P}_n$ -module obtained from  $\mathfrak{M}$  by change of the base ring. When  $\Sigma$  is formally noncharacteristic, we obtain a Hilbert resolution for  $(\mathfrak{M})_n$  of the form

$$(7.2) \quad 0 \longrightarrow \mathcal{P}_n^{r_{d'}} \xrightarrow{^t R_{d'-1}(\xi)} \dots \xrightarrow{^t R_1(\xi)} \mathcal{P}_n^{r_1} \xrightarrow{^t R_0(\xi)} \mathcal{P}_n^{r_0} \longrightarrow (\mathfrak{M})_n \longrightarrow 0$$

where we set  $\xi = (\zeta_1, \dots, \zeta_n)$  and  $d' \leq n$ . The Cauchy data for  $\mathfrak{M}$  on  $\Sigma$  can then be identified to the solutions  $v \in (\mathcal{E}(\mathbb{R}^n))^{r_0}$  of  $R_0(D_1, \dots, D_n)v = 0$  and, using the homotopy formulas relating (4.1) to (7.2) we obtain a one-to-one correspondence between  $\text{Ext}_{\mathcal{P}_n}^0((\mathfrak{M})_n, \mathcal{E}(\mathbb{R}^n))$  and  $\text{Ext}_{\mathcal{P}}^0(\mathfrak{M}, \mathcal{W}_\Sigma)$ .

We say that  $\mathfrak{M}$  has *free Cauchy data* on  $\Sigma$  if  $(\mathfrak{M})_n$  is a free  $\mathcal{P}_n$ -module. We obtain:

LEMMA 7.2. — *A necessary and sufficient condition in order that  $\mathfrak{M}$  have free Cauchy data on  $\Sigma$  is that*

$$V(\mathfrak{p}) \ni \zeta \longrightarrow \pi_n(\zeta) \in \mathbb{C}^n$$

*be surjective for every  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$ .*

## 8. The Cauchy problem with data on a formally noncharacteristic free affine subspace of $\mathbb{R}^N$ : a necessary Hörmander's type condition.

In [Hö1] Hörmander investigates the necessary and sufficient conditions for the pair consisting of a closed half space of  $\mathbb{R}^N$  and its boundary to be of evolution for a  $\mathcal{P}$ -module of the form  $\mathcal{P}/\mathcal{I}$ , where  $\mathcal{I} = (p)$  is a principal ideal generated by a polynomial  $p \in \mathcal{P}$ .

There is no loss of generality in considering the case where the boundary  $\Sigma$  of the closed half space  $\Omega$  is a hyperplane containing the origin. Then the pair  $(\Sigma, \Omega)$  is of evolution for  $\mathcal{P}/(p)$  if and only if the scalar partial differential operator  $p(D)$  admits a fundamental solution  $E \in \mathcal{D}'(\mathbb{R}^N)$  with support contained in  $\Omega$ . Using this fact, the characterization is obtained in terms of properties of the complex affine variety of the zeros of the polynomial  $p$ .

In our formulation there is a difference in sign with respect to the one in [Hö1], due to the fact that we are taking as the main object for investigation the irreducible affine algebraic variety  $V$ , rather than the

system of differential operators attached to it: starting with a scalar partial differential operator  $p(D)$  the variety  $V$  is given by the equation  $p(-\zeta) = 0$ .

In this section we extend this result to the case of overdetermined systems, showing how conditions generalizing the one in [Hö1] are related to the Phragmén-Lindelöf principle.

They are indeed equivalent for the pair consisting of an affine hyperplane and the half space it bounds.

In the case of pairs  $(S, \Omega)$ , where  $S$  is an affine subspace of  $\mathbb{R}^N$  of arbitrary codimension and  $\Omega$  a closed wedge with edge equal to  $S$ , we prove the necessity of the generalized Hörmander condition, whereas for the sufficiency we need a stronger condition, which is no more necessary, but which coincides with the previous one in the codimension 1 case for pairs consisting of compact convex sets  $K_1 \subset S$ ,  $K_2 \subset \Omega$ .

However, in the case in which the associated variety is an algebraic curve, we prove that our generalized Hörmander condition is necessary and sufficient for semi-global evolution, i.e. for pairs consisting of compact convex sets  $K_1 \subset S$ ,  $K_2 \subset \Omega$ .

Note that global evolution implies semi-global evolution (for data on a free noncharacteristic affine subspace of  $\mathbb{R}^N$ ). The two concepts agree for a scalar operator and initial data on a hypersurface, but are distinct in general.

Let the pair  $(S, \Omega)$  consist of an affine subspace  $S$  of  $\mathbb{R}^N$  of arbitrary codimension and a closed wedge  $\Omega$  with edge equal to  $S$ . By an affine change of coordinates we can always reduce to the case where

$$(8.1) \quad S = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N : x^{n+1} = \dots = x^N = 0\}$$

for some  $1 \leq n \leq N-1$  and

$$(8.2) \quad \Omega = \{x = (x^1, \dots, x^N) \in \mathbb{R}^N : x^{n+1} \geq 0, \dots, x^N \geq 0\}.$$

We assume that  $\mathfrak{M}$  is a given  $\mathcal{P}$ -module of finite type and that  $S$  is formally noncharacteristic and free for  $\mathfrak{M}$ . In particular, for every irreducible affine algebraic variety  $V = V(\mathfrak{p})$  with  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$  the map

$$\pi_n : V \ni (\zeta_1, \dots, \zeta_N) \rightarrow (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$$

is finite and surjective. Let us set, for  $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$ :

$$\zeta' = (\zeta_1, \dots, \zeta_n), \quad \zeta'' = (\zeta_{n+1}, \dots, \zeta_N).$$

Since  $\pi_n$  is finite and surjective, there are positive constants  $k, b$  such that

$$(8.3) \quad |\zeta''| \leq k(1 + |\zeta'|)^b \quad \forall \zeta = (\zeta', \zeta'') \in V.$$

Moreover, there is a proper affine algebraic variety  $Z \subset \mathbb{C}^n$  such that  $V \setminus \pi_n^{-1}(Z)$  is smooth and

$$V \setminus \pi_n^{-1}(Z) \ni \zeta \rightarrow \zeta' \in \mathbb{C}^n \setminus Z$$

is an  $m$ -sheeted cover of  $\mathbb{C}^n \setminus Z$ .

We consider the following condition on  $V$ :

$$(H) \quad \left\{ \begin{array}{l} \exists R > 0, c_o \in \mathbb{R} \text{ such that} \\ \text{for every } \theta \in \mathbb{R}^n \text{ with } B_n(\theta, R) = \{|\zeta' - \theta| \leq R\} \subset \mathbb{C}^n \setminus Z \\ \text{and every connected component } \omega \text{ of } \pi_n^{-1}(B_n(\theta, R)) \\ \text{there is } \zeta \in \omega \text{ such that} \\ \text{Im} \zeta_j \leq c_o \quad \text{for } j = n+1, \dots, N. \end{array} \right.$$

It coincides with the one given by Hörmander in [Hö1] in case  $V$  is an affine algebraic hypersurface.

Then we obtain the following criterion:

**THEOREM 8.1.** — *Let  $\mathfrak{M}$  be a  $\mathcal{P}$ -module of finite type for which  $S$  is formally noncharacteristic and free. Then a necessary condition in order that the pair  $(S, \Omega)$  be of evolution for  $\mathfrak{M}$  is that for every  $\mathfrak{p} \in \text{Ass}(M)$  condition (H) be satisfied on  $V = V(\mathfrak{p})$ .*

*Proof.* — Let us consider the plurisubharmonic function in  $\mathbb{C}^N$ :

$$\kappa(\zeta) = \sum_{j=n+1}^N \max\{\text{Im} \zeta_j, 0\}.$$

By Theorems 4.2 and 6.1 a necessary and sufficient condition for the pair  $(S, \Omega)$  to be of evolution for  $\mathfrak{M}$  is that, for every  $V = V(\mathfrak{p})$  with  $\mathfrak{p} \in \text{Ass}(\mathfrak{M})$  we have:

$$(8.4) \quad \begin{array}{l} \forall A, B, \alpha \geq 0 \exists A', \beta \geq 0 \text{ with the property:} \\ \text{if } u \in P(V) \text{ satisfies, with constants } A_u \text{ and } \alpha_u \text{ depending on } u: \\ \left\{ \begin{array}{ll} u(\zeta) \leq A|\text{Im} \zeta'| + B\kappa(\zeta) + \alpha \log(1 + |\zeta|) & \text{on } V \\ u(\zeta) \leq A_u|\text{Im} \zeta'| + \alpha_u \log(1 + |\zeta|) & \text{on } V \end{array} \right. \\ \text{then it also satisfies:} \\ u(\zeta) \leq A'|\text{Im} \zeta'| + \beta \log(e + |\zeta|) & \text{on } V. \end{array}$$

To show that (8.4)  $\Rightarrow$  (H), we argue by contradiction. Let  $\mathcal{U}$  denote the set of pairs  $(\theta, s)$  with  $\theta \in \mathbb{R}^n$  and  $s \in \mathbb{R}$  with  $s > 0$  such that  $B_n(\theta, s) \subset \mathbb{C}^n \setminus Z$  and there is a connected component  $\omega$  of  $\pi_n^{-1}(B_n(\theta, s))$  in  $V$  such that  $\kappa(\zeta) > s$  for every  $\zeta \in \omega$ . This set is semialgebraic and therefore the function

$$f(t) = \sup\{s \in \mathbb{R} : (\theta, s) \in \mathcal{U}, |\theta| = t\}$$

is semialgebraic. Statement (H) is equivalent to the fact that  $f(t)$  is bounded for  $t \rightarrow +\infty$ . Assuming the contrary, there are positive constants  $c, q$  such that

$$f(t) = t^q(2c + o(1)) \quad \text{for } t > t_0 \geq 0.$$

Thus there is  $t_1 \geq t_0$  such that for  $t > t_1$  there is  $\theta_t \in \mathbb{R}^n$  with  $|\theta_t| = t$ ,  $B_n(\theta_t, ct^q) \subset \mathbb{C}^n \setminus Z$  and a connected component  $\omega_t$  of  $\pi_n^{-1}(B_n(\theta_t, ct^q))$  on which  $\kappa(\zeta) > ct^q$ .

Let  $A$  be a positive real number and fix a real valued nonnegative function  $\chi \in \mathcal{D}(\mathbb{R}^n)$ , with support contained in the ball  $\{x' = (x^1, \dots, x^n) \in \mathbb{R}^n : |x'| < A/2\}$ , and  $\hat{\chi}(0) = 1$ . By Paley-Wiener theorem for every integer  $\ell$  there is a constant  $c_\ell$  such that

$$(8.5) \quad |\hat{\chi}(\zeta')| \leq c_\ell (1 + |\zeta'|)^{-\ell} \exp\left(\frac{A}{2} |\operatorname{Im} \zeta'|\right) \quad \forall \zeta' \in \mathbb{C}^n.$$

We consider for every positive integer  $\ell$  the plurisubharmonic function in  $\mathbb{C}^N$  defined by

$$u_{t,\ell}(\zeta) = \log(|\hat{\chi}(\zeta' - \theta_t)|) + \ell \log(1 + |\zeta|).$$

Clearly we have, as (8.5) holds with  $c_0 = 1$  when  $\ell = 0$ :

$$u_{t,\ell}(\zeta) \leq \frac{A}{2} |\operatorname{Im} \zeta'| + \ell \log(1 + |\zeta|) \quad \text{on } \mathbb{C}^N.$$

Using (8.3) we obtain that  $\log(1 + |\zeta|) \leq \gamma \log(e + |\zeta'|)$  on  $V$ , for a constant  $\gamma > 0$ . Hence for each fixed  $B > 0$ , we can find  $t_2 \geq t_1$  such that

$$u_{t,\ell}(\zeta) < A |\operatorname{Im} \zeta'| + B \kappa(\zeta) \quad \forall \zeta \in \omega_t$$

provided  $t > t_2$ . Indeed  $\log(e + |\zeta'|) \leq \log(e + t + ct^q)$  on  $\omega_t$ , while  $\kappa(\zeta) > ct^q$  on  $\omega_t$ .

On the other hand, using the Paley-Wiener estimate, we obtain for some  $t_3 \geq t_2$  that

$$u_{t,\ell}(\zeta) < A|\operatorname{Im}\zeta'| \quad \text{near the boundary of } \omega_t \text{ if } t > t_3.$$

Indeed, with a fixed  $s > \gamma\ell \left(1 + \frac{1}{q}\right)$  we obtain on the boundary of  $\omega_t$ :

$$\log(|\hat{\chi}(\zeta' - \theta_t)|) \leq \log c_s - s \log(1 + ct^q) + \frac{A}{2} |\operatorname{Im}\zeta'|,$$

whereas

$$\ell \log(1 + |\zeta|) \leq \ell \gamma \log(e + |\zeta'|) \leq \ell \gamma \log(e + t + ct^q).$$

We define a function  $v_{t,\ell} \in P(V)$  by setting

$$v_{t,\ell}(\zeta) = \begin{cases} \max\{u_{t,\ell}(\zeta), A|\operatorname{Im}\zeta'|\} & \text{for } \zeta \in \omega_t \\ A|\operatorname{Im}\zeta'| & \text{for } \zeta \in V \setminus \omega_t. \end{cases}$$

When  $t > t_3$  we have

$$\begin{cases} v_{t,\ell}(\zeta) \leq A|\operatorname{Im}\zeta'| + B\kappa(\zeta) & \text{on } V \\ v_{t,\ell}(\zeta) \leq A|\operatorname{Im}\zeta'| + \ell \log(1 + |\zeta|) & \text{on } V \end{cases}$$

but (8.4) cannot hold true because for every  $\ell$  and  $t > t_3$  we also have at points  $\tilde{\theta}_t \in \omega_t$  with  $\pi_n(\tilde{\theta}_t) = \theta_t$ :

$$v_{t,\ell}(\tilde{\theta}_t) \geq \ell \log(1 + |\theta_t|).$$

*Remark 8.1.* — By repeating the same arguments used in the proof of the previous theorem, we also show that a necessary condition in order that the pair  $(K_1, K_2)$  defined by

$$(8.6) \quad K_2 = \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n : |x| \leq A, 0 \leq t_j \leq B \text{ for } j = 1, \dots, k\}$$

$$(8.7) \quad K_1 = \{(t, x) \in K_2 : t = 0\},$$

for constants  $A, B > 0$ , be of evolution for  $\mathfrak{M}$  is that for every  $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{M})$  condition (H) be satisfied on  $V = V(\mathfrak{p})$ .

Indeed, it suffices to put in (8.4)  $A_u = A' = A$ .

We can thus say that condition (H) is necessary for global evolution (from a formally non-characteristic affine subspace of  $\mathbb{R}^N$  to  $\mathbb{R}^N$ ), but also for semi-global evolution, i.e. for pairs  $(K_1, K_2)$  of compact subsets as above.

### 9. The case of algebraic curves.

In this section we investigate the Cauchy problem in  $K_2$  with initial data in  $K_1$ , when  $K_1$  and  $K_2$  are defined by (8.7) and (8.6) with  $n = 1$ .

The general discussion allows to reduce to the investigation about the validity of a Phragmén-Lindelöf principle on algebraic varieties  $V = V(\mathfrak{p}) \subset \mathbb{C}^{k+1}$  where  $\mathfrak{p}$  is a prime ideal in  $\mathbb{C}[\tau_1, \dots, \tau_k, \zeta]$ .

In this particular case we obtain more precise results. Namely, we prove that condition (H) is necessary and sufficient for the pair  $(K_1, K_2)$  to be of evolution for a system of linear partial differential operators with constant coefficients.

We assume that  $V = V(\mathfrak{p})$  is an irreducible affine algebraic curve in  $\mathbb{C}^{k+1}$  and that the projection into the last coordinate  $\pi : V \rightarrow \mathbb{C} = \mathbb{C}_\zeta$  is finite and surjective. The closure  $\bar{V}$  of  $V$  in  $\mathbb{CP}^{k+1}$  is an irreducible projective curve and  $\pi$  extends to a finite surjective map  $\bar{\pi} : \bar{V} \rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ . We note that  $\bar{V} \setminus V = \bar{\pi}^{-1}(\infty)$ .

The normalization  $\tilde{V} \xrightarrow{\sigma} \bar{V}$  is an irreducible smooth projective curve and the birational isomorphism  $\sigma$  is regular.

Let  $\sigma^{-1} \circ \bar{\pi}^{-1}(\infty) = \{P_1, \dots, P_s\}$ .

Then we can fix pairwise disjoint connected open neighbourhoods  $\tilde{V}_1, \dots, \tilde{V}_s$  of  $P_1, \dots, P_s$  respectively in  $\tilde{V}$ , in such a way that, setting  $V_j = \sigma(\tilde{V}_j \setminus \{P_j\}) \subset V$  for  $1 \leq j \leq s$  we obtain:

- (i)  $V_j \cap V_h = \emptyset$  for  $1 \leq j < h \leq s$ ;
- (ii)  $\sigma : \tilde{V}_j \setminus \{P_j\} \rightarrow V_j$  is biholomorphic;
- (iii)  $\pi : V_j \rightarrow \pi(V_j) \subset \mathbb{C}_\zeta$  is an  $m_j$ -fold covering for some integer  $m_j \geq 1$ ;
- (iv)  $\pi(V_j) \cup \{\infty\}$  is an open neighbourhood of  $\infty$  in  $\mathbb{CP}^1$ .
- (v) for each  $j$  (with  $1 \leq j \leq s$ ),  $\pi^{-1}(\mathbb{R}) \cap V_j$  consists of  $2m_j$  connected components.

We can also assume that for a fixed  $r > 0$  and every  $j = 1, \dots, s$  we have:

$$\pi(V_j) = \{\zeta \in \mathbb{C} : |\zeta| > r\}.$$

For each  $j = 1, \dots, s$  we obtain a Puiseux parametric description of  $V_j$  of the form:

$$\begin{cases} \zeta = z^{m_j} \\ \tau_h = \sum_{\alpha \leq \nu(h,j)} \tau_{h,j,\alpha} z^\alpha, \end{cases} \quad \text{for } h = 1, \dots, k.$$

Denote by  $E_R$ , for  $R > 0$ , the set

$$E_R = \{(\tau, \zeta) = (\tau_1, \dots, \tau_k, \zeta) \in V : |\operatorname{Im} \zeta| \leq R\}.$$

For each fixed  $R > 0$  and  $R' > 0$  the set

$$\{(\tau, \zeta) \in E_R : |\zeta| \geq R'\}$$

is semialgebraic and for  $R' > r$  sufficiently large for each  $j = 1, \dots, s$

$$V_j \cap E_R \cap \{(\tau, \zeta) \in V : |\zeta| \geq R'\}$$

consists of  $2m_j$  connected components.

**PROPOSITION 9.1.** — *Let  $\mathfrak{p}$  be a prime ideal in  $\mathbb{C}[\tau_1, \dots, \tau_k, \zeta]$  such that the natural projection  $\pi : V(\mathfrak{p}) \rightarrow \mathbb{C}_\zeta$  is finite and surjective.*

*Then the pair  $(K_1, K_2)$ , defined by (8.7) and (8.6), is of evolution for  $\mathcal{P}/\mathfrak{p}$  in the Whitney class if and only if the following condition is satisfied for every  $R' > 0$ :*

(\*) *there exist  $c_0 \in \mathbb{R}$  and  $R > 0$  such that on every connected component of  $\{(\tau, \zeta) \in V : |\operatorname{Im} \zeta| \leq R, |\zeta| \geq R'\}$  there is a sequence  $\{(\tau_\nu, \zeta_\nu)\}$  such that  $|\zeta_\nu| \rightarrow +\infty$  and  $|(\operatorname{Im} \tau_\nu)^+| \leq c_0$ .*

*Proof. — Sufficiency.* We can assume  $R > 1$ . For  $\nu$  sufficiently large, condition (\*) implies that for every  $j = 1, \dots, s$  on each of the  $2m_j$  connected components of

$$\{(\tau, \zeta) \in V_j : |\zeta| \geq R', |\operatorname{Im} \zeta| \leq R\},$$

we can find a sequence  $\{(\tau_\nu, \zeta_\nu)\}$  with  $|\zeta_\nu| \rightarrow +\infty$  and such that  $|(\operatorname{Im} \tau_\nu)^+| \leq c_0$ .

Let us fix  $j = 1, \dots, s$  and let us omit the index  $j$  for simplicity.

We can then find  $2m$  sequences  $\{z_\nu^{(h)}\}_{h \in \mathbb{N}} \subset \mathbb{C} \setminus B(0, r)$  such that, for every  $h = 0, \dots, 2m - 1$ ,  $|z_\nu^{(h)}|^m \geq R$ ,  $|\operatorname{Im}(z_\nu^{(h)} e^{-i \frac{h\pi}{m}})| \leq R$ ,  $|z_\nu^{(h)}| \nearrow +\infty$  and  $|(\operatorname{Im} \tau)^+(z_\nu^{(h)})| \leq c_0$ .

Let us consider the semi-algebraic set

$$E = \{(s, z^{(o)}, \dots, z^{(2m-1)}) \in [r, +\infty) \times \mathbb{C}^{2m} : \\ |z^{(h)}| \geq s, |\operatorname{Im}(z^{(h)} e^{-i \frac{h\pi}{m}})| \leq R, |(\operatorname{Im}\tau)^+(z^{(h)})| \leq c_o \quad \forall h = 0, \dots, 2m-1\}.$$

By assumption the projection  $\pi : E \rightarrow [r, +\infty)$  is onto for large positive  $s$ .

Therefore by Theorem A.2.8 of [Hö1] we can find  $2m$  Puiseux series  $z^{(o)}(s), \dots, z^{(2m-1)}(s)$  converging for large positive  $s$  and such that  $(s, z^{(o)}(s), \dots, z^{(2m-1)}(s)) \in E$ , i.e.

$$|\operatorname{Im}(z^{(h)}(s) e^{-i \frac{h\pi}{m}})| \leq R \quad \text{and} \quad |(\operatorname{Im}\tau)^+(z^{(h)}(s))| \leq c_o \quad \forall h = 0, \dots, 2m-1$$

for  $s$  large enough.

Extending these curves up to  $0 \in \mathbb{C}$  we can then find  $2m$  real analytic curves  $w_h(t)$ , such that, for some constant  $c'_o \geq c_o$ ,

$$|\operatorname{Im}(w_h(t) e^{-i \frac{h\pi}{m}})| \leq R \quad \text{and} \quad |(\operatorname{Im}\tau)^+(w_h(t))| \leq c'_o, \quad \forall h = 0, \dots, 2m-1.$$

In order to prove that the following Phragmén-Lindelöf principle is valid: given  $A, \bar{B} > 0$ ,

(9.1)

$\forall M \in \mathbb{N} \exists M' \in \mathbb{N}$  such that  $\forall u \in P(V)$  with

$$\begin{cases} u(\tau, \zeta) \leq A|\operatorname{Im}\zeta| + \bar{B}|(\operatorname{Im}\tau)^+| + M \log(1 + |\zeta| + |\tau|) & \forall (\tau, \zeta) \in V \\ u(\tau, \zeta) \leq A|\operatorname{Im}\zeta| + M_u \log(1 + |\zeta| + |\tau|) + C_u & \forall (\tau, \zeta) \in V \end{cases}$$

we also have:

$$u(\tau, \zeta) \leq A|\operatorname{Im}\zeta| + M' \log(1 + |\zeta| + |\tau|) + C' \quad \forall (\tau, \zeta) \in V,$$

it suffices to prove the following Phragmén-Lindelöf principle: given  $A, B' > 0$ ,

$\forall M \in \mathbb{N} \exists M' \in \mathbb{N}$  such that  $\forall u \in P(\mathbb{C})$  with

$$\begin{cases} u(z) \leq A|\operatorname{Im}z^m| + B' + M \log(1 + |z^m|) & \text{for } z = w_h(t), h = 0, \dots, 2m-1 \\ u(z) \leq A|\operatorname{Im}z^m| + M_u \log(1 + |z^m|) + C_u & \text{for } z \in \mathbb{C} \end{cases}$$

we also have:

$$u(z) \leq A|\operatorname{Im}z^m| + M' \log(1 + |z^m|) + C' \quad \forall z \in \mathbb{C}.$$

Indeed:

1) Let us first assume  $M_u = 0$  and prove that

$\forall M \in \mathbb{N} \exists M' \in \mathbb{N}$  such that  $\forall u \in P(\mathbb{C})$  with

$$\begin{cases} u(z) \leq A|\operatorname{Im} z^m| + B' + M \log(1 + |z^m|) & \text{for } z = w_h(t), \quad h = 0, \dots, 2m-1 \\ u(z) \leq A|z^m| + C_u & \text{for } z \in \mathbb{C} \end{cases}$$

we also have:

$$u(z) \leq A|\operatorname{Im} z^m| + M' \log(1 + |z^m|) + C' \quad \forall z \in \mathbb{C}.$$

The  $2m$  real analytic curves  $w_h(t)$  can be chosen to divide the  $\mathbb{C}_\zeta$  plane into  $2m$  components. We obtain the Phragmén-Lindelöf estimate by applying the maximum principle to each of these components.

As the argument is the same on each of them, we give the proof of the estimate for the sector  $\bar{S}$  bounded by the curves  $w_o(t)$  and  $w_1(t)$ .

Let us first consider the sector  $S$  bounded by the curve  $w_o(t)$  and the half ray  $\{\rho e^{i\frac{\pi}{2m}} : \rho \geq 0\}$ .

We assume, as we can, that

$$-\frac{\pi}{2m} < \arg w_o(t) < \frac{\pi}{2m}$$

for every  $t$ .

By construction we can find  $k > 0$  and  $\varphi$  such that for  $\theta = \arg z$  with  $z \in S$  we have

$$-\frac{\pi}{2} < (m+k)(\theta - \varphi) < \frac{\pi}{2}$$

and hence  $\cos(m+k)(\theta - \varphi) \geq \lambda > 0$ .

Let us set, for  $\varepsilon > 0$ ,

$$\begin{aligned} v_\varepsilon(z) &= u(z) - \varepsilon \operatorname{Re}((ze^{-i\varphi})^{m+k}) - 2M \operatorname{Re} \log(i + z^m) \\ &\quad - A|\operatorname{Im} z^m| - 2AR - B' - M \log 4 - C_u. \end{aligned}$$

For  $z = w_o(t) = \rho_o(t)e^{i\theta_o(t)}$ ,  $\rho_o(t) \geq 0$ , we have:

$$\begin{aligned} v_\varepsilon(z) &= u(w_o(t)) - \varepsilon \rho_o^{m+k}(t) \cos(m+k)(\theta_o(t) - \varphi) - 2M \log|i + w_o^m(t)| \\ &\quad - A|\operatorname{Im} w_o^m(t)| - 2AR - B' - M \log 4 - C_u \\ &\leq A|\operatorname{Im} w_o^m(t)| + B' + M \log(1 + \rho_o^m(t)) \\ &\quad - M \log[\rho_o^{2m}(t) \cos^2 m\theta_o(t) + (1 + \rho_o^m(t) \sin m\theta_o(t))^2] \\ &\quad - A|\operatorname{Im} w_o^m(t)| - 2AR - B' - M \log 4 \\ &\leq A\{|\operatorname{Im} w_o^m(t)| - \operatorname{Im} w_o^m(t) - 2R\} + M \log(1 + \rho_o^m(t)) \\ &\quad - M \log(1 + \rho_o^{2m}(t) + 2\rho_o^m(t) \sin m\theta_o(t)) - M \log 4 \\ &\leq 0. \end{aligned}$$

For  $\theta = \frac{\pi}{2m}$ :

$$\begin{aligned} v_\varepsilon(z) &= u(z) - \varepsilon \rho^{m+k} \cos(m+k) \left( \frac{\pi}{2m} - \varphi \right) - 2M \log |i + i\rho^m| \\ &\quad - A\rho^m - 2AR - B' - M \log 4 - C_u \\ &\leq A\rho^m + C_u - 2M \log(1 + \rho^m) \\ &\quad - A\rho^m - 2AR - B' - M \log 4 - C_u \\ &\leq 0. \end{aligned}$$

Moreover, for  $z \in S$ , we have

$$A|z|^m = A\rho^m \leq \varepsilon \rho^{m+k} \cos(m+k)(\theta - \varphi)$$

$$\text{for } |z| = \rho \geq R_\varepsilon = \left( \frac{A}{\varepsilon \lambda} \right)^{1/k} \geq \left[ \frac{A}{\varepsilon \cos(m+k)(\theta - \varphi)} \right]^{1/k}.$$

Therefore for  $|z| = \rho \geq R_\varepsilon$ ,  $z \in S$ , we have

$$\begin{aligned} v_\varepsilon(z) &= u(z) - \varepsilon \rho^{m+k} \cos(m+k)(\theta - \varphi) - 2M \log |i + z^m| \\ &\quad - A\text{Im}z^m - 2AR - B' - M \log 4 - C_u \\ &\leq A|z^m| + C_u - \varepsilon \rho^{m+k} \cos(m+k)(\theta - \varphi) - A(\text{Im}z^m + R) - AR - C_u \\ &\leq -A(\text{Im}w_o^m(t) + R) \\ &\leq 0. \end{aligned}$$

It follows, by the maximum principle, that

$$v_\varepsilon(z) \leq 0 \quad \text{in } S \quad \forall \varepsilon > 0.$$

For  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} u(z) &\leq 2M \log |1 + z^m| + A\text{Im}z^m + 2AR + B' + M \log 4 + C_u \\ &\leq 2M \log(1 + |z^m|) + A|\text{Im}z^m| + C' + C_u \quad \forall z \in S. \end{aligned}$$

Arguing in the same way in the other sectors we have that

$$(9.2) \quad u(z) \leq 2M \log(1 + |z^m|) + A|\text{Im}z^m| + C' + C_u \quad \forall z \in \mathbb{C}.$$

Let us now get rid of the constant  $C_u$ .

Let us set, for  $z \in \bar{S}$  and  $\varepsilon > 0$ ,

$$\begin{aligned} w_\varepsilon(z) &= u(z) - \varepsilon \text{Re} \left[ (ze^{-i\varphi})^{mk} \right] - 4M \text{Re} \log(i + z^m) \\ &\quad - A\text{Im}z^m - 2AR - B' - 2M \log 4 - C' \end{aligned}$$

with  $k > 0$  and  $\varphi$  such that for  $\theta = \arg z$  with  $z \in \bar{S}$  we have

$$-\frac{\pi}{2} < mk(\theta - \varphi) < \frac{\pi}{2}$$

and hence  $\cos mk(\theta - \varphi) \geq \lambda' > 0$ . For  $z = w_o(t)$  or  $z = w_1(t)$  we have

$$\begin{aligned} w_\varepsilon(z) &= u(w_j(t)) - \varepsilon \rho_j^{mk}(t) \cos mk(\theta_j(t) - \varphi) - 4M \log |i + w_j^m(t)| \\ &\quad - A \operatorname{Im} w_j^m(t) - 2AR - B' - 2M \log 4 - C' \\ &\leq A |\operatorname{Im} w_j(t)| + B' + M \log(1 + \rho_j^m(t)) \\ &\quad - M \log[\rho_j^{2m}(t) \cos^2 m\theta_j(t) + (1 + \rho_j^m(t) \sin m\theta_j(t))^2] \\ &\quad - A \operatorname{Im} w_j^m(t) - 2AR - B' - M \log 4 \\ &\leq 0. \end{aligned}$$

Moreover for  $z \in \bar{S}$  we have

$$C_u \leq \varepsilon \rho^{mk} \cos mk(\theta - \varphi)$$

$$\text{for } |z| = \rho \geq R_u = \left( \frac{C_u}{\varepsilon \lambda'} \right)^{\frac{1}{mk}} \geq \left( \frac{C_u}{\varepsilon \cos mk(\theta - \varphi)} \right)^{\frac{1}{mk}}.$$

Therefore for  $z \in \bar{S}$  with  $|z| \geq R_u$ , by (9.2) we have

$$\begin{aligned} w_\varepsilon(z) &= u(z) - \varepsilon \rho^{mk} \cos mk(\theta - \varphi) - 4M \log |i + z^m| \\ &\quad - A \operatorname{Im} z^m - 2AR - B' - 2M \log 4 - C' \\ &\leq A |\operatorname{Im} z^m| + 2M \log(1 + |z^m|) + C' + C_u \\ &\quad - \varepsilon \rho^{mk} \cos mk(\theta - \varphi) - 4M \log |i + z^m| \\ &\quad - A \operatorname{Im} z^m - 2AR - B' - 2M \log 4 - C' \\ &\leq A(|\operatorname{Im} z^m| - \operatorname{Im} z^m - 2R) \\ &\leq 2A(\max_{j=0,1} |\operatorname{Im} w_j^m(t)| - R) \\ &\leq 0. \end{aligned}$$

By the maximum principle we have

$$w_\varepsilon(z) \leq 0 \quad \forall z \in \bar{S} \quad \forall \varepsilon > 0.$$

For  $\varepsilon \rightarrow 0$  we obtain, for  $z \in \bar{S}$ :

$$\begin{aligned} u(z) &\leq 4M \log(1 + |z^m|) + A \operatorname{Im} z^m + 2AR + B' + 2M \log 4 + C' \\ &\leq M' \log(1 + |z^m|) + A |\operatorname{Im} z^m| + C''. \end{aligned}$$

Arguing in the same way in the other sectors, we finally have

$$u(z) \leq M' \log(1 + |z^m|) + A |\operatorname{Im} z^m| + C'' \quad \forall z \in \mathbb{C}.$$

2) Let us consider now the general case  $M_u \in \mathbb{R}$ .

We can easily see that for each  $\varepsilon > 0$  there is a constant  $B_{u,\varepsilon} > 0$  such that

$$M_u \log(1 + |z^m|) \leq \varepsilon |z^m| + B_{u,\varepsilon} \quad \forall z \in \mathbb{C}$$

and hence

$$u(z) \leq (A + \varepsilon) |z^m| + (C_u + B_{u,\varepsilon}) \quad \forall z \in \mathbb{C}.$$

By the first step we have

$$u(z) \leq (A + \varepsilon) |\operatorname{Im} z^m| + M' \log(1 + |z^m|) + C' \quad \forall z \in \mathbb{C}$$

for every  $\varepsilon > 0$ .

For  $\varepsilon \rightarrow 0$  we have the thesis.

*Necessity.* Condition (H) is necessary in view of the remark following Theorem 8.1 and it clearly implies condition (\*).

*Remark 9.1.* — Theorem 8.1 and Proposition 9.1 show that, in the special case where  $V(\mathfrak{p})$  is an algebraic curve, condition (H) is necessary and sufficient in order that the pair  $(K_1, K_2)$  given by (8.7) and (8.6) be of evolution for  $\mathcal{P}/\mathfrak{p}$  in the Whitney class.

## 10. The Cauchy problem with data on a formally noncharacteristic free affine subspace of $\mathbb{R}^N$ : a sufficient Hörmander's type condition.

Now we consider a stronger condition (H') which is sufficient for evolution for the pair  $(K_1, K_2)$  considered in the previous sections, but which in general is not necessary. In the case  $n = N - 1$  this stronger condition (H') coincides with Hörmander's condition (H), and hence we obtain another proof of Hörmander's characterization of evolution in the case of scalar operators.

Let us then state condition (H') by:

$$(H') \quad \begin{cases} \exists R, r > 0, c_1 \in \mathbb{R} \text{ such that} \\ \text{for every } \theta \in \mathbb{R}^n \text{ with } \mathfrak{B}_n(\theta, R) \subset \mathbb{C}^n \setminus Z \\ \text{and every connected component } \omega \text{ of } \pi_n^{-1}(B_n(\theta, R)) \\ \text{there is } B(\zeta'_\theta, r) \subset B_n(\theta, R) \text{ such that} \\ \operatorname{Im} \zeta_j \leq c_1 \quad \forall \zeta \in \pi_n^{-1}(B(\zeta'_\theta, r)), j = n+1, \dots, N. \end{cases}$$

By translation we can assume that condition (H) holds with  $c_0 = 0$ .

Then the following lemma shows that Hörmander's condition (H) is equivalent to (H') when  $n = N - 1$ .

LEMMA 10.1. — *Let  $F_m(B(\theta, R))$  be the set of analytic functions in  $B(\theta, R) \subset \mathbb{C}^n$  algebraic of degree  $\leq m$ , i.e. the set of all analytic functions  $f$  in  $B(\theta, R)$  such that for some polynomial  $R \neq 0$  of degree  $\leq m$  in  $\mathbb{C}^{n+1}$  the equation  $R(\zeta, f(\zeta)) = 0$  is valid in  $B(\theta, R)$ , and let  $0 < R_0 < R_1 < R$  and  $\delta > 0$ .*

*Then there is  $r > 0$  such that if  $f \in F_m(B(\theta, R))$  and if*

$$\exists \xi_0 \in \overline{B(\theta, R_0)} \text{ s.t. } \operatorname{Im} f(\xi_0) \leq 0$$

*it follows that*

$$\operatorname{Im} f(\xi) \leq 0 \quad \forall \xi \in B$$

*where  $B$  is some ball in  $B(\theta, R_1)$  of radius  $r$ .*

This is Lemma 12.8.8 of [Hö1] after substituting  $-f$  to  $f$ .

THEOREM 10.1. — *Condition (H') on  $V(\mathfrak{p})$  implies that the pair  $(K_1, K_2)$  is of evolution for  $\mathcal{P}/\mathfrak{p}$ .*

*Proof.* — By translation we can assume  $c_1 = 0$ .

We have to prove that condition (8.4) holds with  $Au = A$ .

Let  $u$  be a plurisubharmonic function which satisfies the first two inequalities of (8.4) with  $Au = A$ , then by the Hadamard three circle theorem (cf. [Hö1]) we can find  $0 < R_1 < R$  and  $0 < \delta < 1$  such that

$$u(\tau(\xi), \xi) \leq (1 - \delta) \sup_{B(\theta, R)} u + \delta \sup_B u \quad \forall \xi \in B(\theta, R_1).$$

Therefore, for  $\zeta \in B(\theta, R)$  we have

$$\begin{aligned} \sup_{B(\theta, R)} u &\leq \sup_{B(\theta, R)} \{m_u \log(1 + |\zeta|) + A|\operatorname{Im} \xi| + c_u\} \\ &\leq m_u [\log(1 + |\zeta|) + R + R_1] + AR + c_u, \end{aligned}$$

since  $\log(1 + |\theta| + R) \leq \log(1 + |\theta| - R_1) + R + R_1 \leq \log(1 + |\zeta|) + R + R_1$ .

Moreover,

$$\begin{aligned} \sup_{B(\zeta_\theta, r)} &\leq \sup_{B(\zeta_\theta, r)} \{m \log(1 + |\zeta|) + A|\operatorname{Im}\xi| + c\} \\ &\leq m[\log(1 + |\zeta|) + R + R_1 + r] + AR + c \end{aligned}$$

since

$$\begin{aligned} \log(1 + |\zeta_\theta| + r) &\leq \log(1 + |\theta| + R + r) \\ &\leq \log(1 + |\theta| - R_1) + R + R_1 + r \\ &\leq \log(1 + |\zeta|) + R + R_1 + r \end{aligned}$$

for  $\zeta \in B(\theta, R_1)$ . Therefore for  $\zeta \in V \cap \mathbb{R}^N$  we have

$$\begin{aligned} u(\zeta) &\leq (1 - \delta)\{m_u \log(1 + |\zeta|) + m_u(R + R_1) + AR + c_u\} \\ &\quad + \delta\{m \log(1 + |\zeta|) + m(R + R_1 + r) + AR + c\} \\ &= [(1 - \delta)m_u + \delta m] \log(1 + |\zeta|) \\ &\quad + (1 - \delta)[m_u(R + R_1) + AR + c_u] + \delta[m(R + R_1 + r) + AR + c]. \end{aligned}$$

This inequality, together with the second inequality of (8.4), implies, by the classical Phragmén-Lindelöf theorem, that

$$\begin{aligned} u(\zeta) &\leq n[(1 - \delta)m_u + \delta m] \log(1 + |\zeta|) + A|\operatorname{Im}\xi| \\ &\quad + (1 - \delta)[m_u(R + R_1) + AR + c_u] + \delta[m(R + R_1 + r) + AR + c]. \end{aligned}$$

After  $\ell$  steps we obtain:

$$u(\zeta) \leq c_\ell \log(1 + |\zeta|) + A|\operatorname{Im}\xi| + \lambda_\ell + \ell\eta$$

where  $c_\ell$  and  $\lambda_\ell$  are defined by recurrence by

$$(10.1) \quad \begin{cases} c_1 = m_u \\ c_\ell = n[(1 - \delta)c_{\ell-1} + \delta m] \end{cases}$$

$$(10.2) \quad \begin{cases} \lambda_1 = c_u \\ \lambda_\ell = (1 - \delta)(c_{\ell-1}(R + R_1) + AR + \lambda_{\ell-1}) \end{cases}$$

and  $\eta = m(R + R_1 + r) + AR + c$ .

If we apply the above considerations to

$$v(\zeta) = u(\zeta) - c - m(R + R_1 + r) - AR$$

instead of  $u$ , we obtain

$$v(\zeta) \leq c_\ell \log(1 + |\zeta|) + A|\operatorname{Im}\xi| + \lambda_\ell.$$

Let

$$L = \limsup_{\ell \rightarrow \infty} c_\ell, \quad \Lambda = \limsup_{\ell \rightarrow \infty} \lambda_\ell.$$

By (10.1) we compute:

$$L = n[(1 - \delta)L + \delta m]$$

$$L = \frac{\delta m}{1 - n(1 - \delta)} < +\infty$$

(it is always possible to chose  $\delta$  such that  $(1 - \delta) \neq \frac{1}{n}$  if  $n > 1$ ).

By (10.2)

$$\Lambda = (1 - \delta)[L(R + R_1) + AR + \Lambda]$$

$$= L(R + R_1) + AR + \Lambda - \delta[L(R + R_1) + AR] - \delta\Lambda$$

$$\Lambda = \frac{1 - \delta}{\delta}[L(R + R_1) + AR].$$

Letting  $\ell$  tend to  $+\infty$ , we finally obtain

$$v(\zeta) \leq L \log(1 + |\zeta|) + A|\operatorname{Im}\xi| + \Lambda$$

and hence

$$u(\zeta) \leq L \log(1 + |\zeta|) + A|\operatorname{Im}\xi| + \Lambda + c + m(R + R_1 + r) + AR.$$

This theorem proves that condition (H') is sufficient for evolution for the pair  $(K_1, K_2)$  defined by (8.7) and (8.6).

However, condition (H') is not necessary for evolution neither for the pair  $(S, \Omega)$  nor for the pair  $(K_1, K_2)$ .

*Example 1.* — Let us consider the following system:

$$(10.3) \quad \begin{cases} \frac{\partial}{\partial t_1} - i \frac{\partial^2}{\partial x^2} \\ \frac{\partial}{\partial t_2} + i \frac{\partial^2}{\partial x^2}. \end{cases}$$

The associated algebraic variety is

$$V = \{(\tau_1, \tau_2, \zeta) \in \mathbb{C}^3 : \tau_1 = \zeta^2, \tau_2 = -\zeta^2\},$$

and

$$(\operatorname{Im}\tau_1)^+ + (\operatorname{Im}\tau_2)^+ = 2|\operatorname{Re}\zeta| \cdot |\operatorname{Im}\zeta| \text{ on } V.$$

In this case condition (H) is valid, but not condition (H').

However the system (10.3) is of evolution since adding the two equations of the system we obtain

$$\frac{\partial}{\partial t_1} = -\frac{\partial}{\partial t_2}.$$

A solution of such an equation is of the form  $u = u(t_1 - t_2)$ .

Therefore, if  $u(t_1, x)$  is a solution of the Schrödinger operator

$$\frac{\partial}{\partial t_1} - i \frac{\partial^2}{\partial x^2}$$

(note that this solution exists since we have already proved necessity and sufficiency of Hörmander's condition (H) in the case  $n = N - 1$ ), we have that

$$v(t_1, t_2, x) = u(t_1 - t_2, x)$$

is a solution of the system (10.3).

This example proves that condition (H') is sufficient but not necessary for the pair  $(K_1, K_2)$  to be of evolution.

We will show by the following example that semi-global evolution does not imply global evolution, i.e. the Phragmén-Lindelöf principle

$\forall \alpha \geq 0 \exists \beta \geq 0$  with the property:

if  $u \in P(V)$  satisfies, with a constant  $\alpha_n$  depending on  $n$ :

$$(10.4) \quad \begin{cases} u(\zeta) \leq A|\operatorname{Im}\zeta'| + Bk(\zeta) + \alpha \log(e + |\zeta|) & \text{on } V \\ u(\zeta) \leq A|\operatorname{Im}\zeta'| + \alpha_u \log(e + |\zeta|) & \text{on } V \\ \text{then it also satisfies:} & \\ u(\zeta) \leq A|\operatorname{Im}\zeta'| + \beta \log(e + |\zeta|) & \text{on } V \end{cases}$$

does not imply the Phragmén-Lindelöf principle (8.4).

*Example 2.* — Let us consider the system

$$\begin{cases} \frac{\partial}{\partial t_1} - i \frac{\partial^2}{\partial x^2} \\ \frac{\partial}{\partial t_2} + i \frac{\partial^2}{\partial x^2} \\ \frac{\partial}{\partial t_3} - \frac{\partial^2}{\partial x^2} \end{cases}.$$

The associated algebraic variety is given by

$$V = \{(\tau_1, \tau_2, \tau_3, \zeta) \in \mathbb{C}^4 : \tau_1 = \zeta^2, \tau_2 = -\zeta^2, \tau_3 = -i\zeta^2\}.$$

For  $\zeta = \xi \in \mathbb{R}$  we have

$$\operatorname{Im}\tau_1 = \operatorname{Im}\tau_2 = 0$$

$$\operatorname{Im}\tau_3 = -\xi^2 \leq 0$$

and hence condition (\*) of Proposition 9.1 is satisfied.

Therefore the pair  $(K_1, K_2)$  defined by (8.7) and (8.6) is of evolution for the given system, i.e. the Phragmén-Lindelöf principle (10.4) is satisfied.

However, we assert that there is a sequence  $\{u_n\}_{n \in \mathbb{N}}$  of plurisubharmonic functions in  $\mathbb{C}^4$  which satisfy

$$\begin{cases} u_n(\tau, \zeta) \leq A|\operatorname{Im}\zeta| + B \sum_{j=1}^3 (\operatorname{Im}\tau_j)^+ & \forall (\tau, \zeta) \in V \\ u_n(\tau, \zeta) \leq A_n |\operatorname{Im}\zeta| & \forall (\tau, \zeta) \in V \end{cases}$$

but which do not satisfy

$$u_n(\tau, \zeta) \leq A' |\operatorname{Im}\zeta| + \beta \log(e + |\tau| + |\zeta|) \quad \forall (\tau, \zeta) \in V$$

for any  $\beta \in \mathbb{N}$ ,  $A' \geq 0$ .

Indeed, for  $(\tau, \zeta) \in V$  we have:

$$\begin{aligned} \operatorname{Im}\tau_1 &= 2\operatorname{Re}\zeta \cdot \operatorname{Im}\zeta \\ \operatorname{Im}\tau_2 &= -2\operatorname{Re}\zeta \cdot \operatorname{Im}\zeta \\ \operatorname{Im}\tau_3 &= (\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2 \end{aligned}$$

and hence

$$\sum_{j=1}^3 (\operatorname{Im}\tau_j)^+ = 2|\operatorname{Re}\zeta| \cdot |\operatorname{Im}\zeta| + ((\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2)^+ \geq \frac{3}{4}(\operatorname{Im}\zeta)^2,$$

since for  $|\operatorname{Re}\zeta| \leq \frac{1}{2}|\operatorname{Im}\zeta|$

$$\begin{aligned} ((\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2)^+ &= (\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2 \\ &\geq (\operatorname{Im}\zeta)^2 - \frac{1}{4}(\operatorname{Im}\zeta)^2 \\ &= \frac{3}{4}(\operatorname{Im}\zeta)^2 \end{aligned}$$

and for  $|\operatorname{Re}\zeta| \geq \frac{1}{2}|\operatorname{Im}\zeta|$

$$2|\operatorname{Re}\zeta| \cdot |\operatorname{Im}\zeta| \geq |\operatorname{Im}\zeta|^2 \geq \frac{3}{4}(\operatorname{Im}\zeta)^2.$$

It is then sufficient to find a sequence  $\{u_n\}_{n \in \mathbb{N}}$  of plurisubharmonic functions which satisfy

$$(10.5) \quad \begin{cases} u_n(\tau, \zeta) \leq A|\operatorname{Im}\zeta| + B|\operatorname{Im}\zeta|^2 & \text{on } V \\ u_n(\tau, \zeta) \leq A_n|\operatorname{Im}\zeta| & \text{on } V \end{cases}$$

but which do not satisfy

$$(10.6) \quad u_n(\tau, \zeta) \leq \beta \log(e + |\tau| + |\zeta|) + A'|\operatorname{Im}\zeta| \quad \text{on } V$$

for any  $\beta \in \mathbb{N}$ ,  $A' \geq 0$ .

Let us construct a sequence  $\{\varphi_n\}_{n \in \mathbb{N}}$  of convex increasing functions by

$$\varphi_n(x) = \begin{cases} 0 & \text{if } -\infty < x \leq 0 \\ \frac{x}{n^2} & \text{if } 0 \leq x \leq n \\ x + \frac{1}{n} - n & \text{if } n \leq x < +\infty. \end{cases}$$

Then we have a sequence  $\{u_n\}_{n \in \mathbb{N}}$  of plurisubharmonic functions defined by

$$u_n(\tau, \zeta) = u_n(\zeta) = n\varphi_n(\operatorname{Im}\zeta) = \begin{cases} 0 & \text{if } \operatorname{Im}\zeta \leq 0 \\ \frac{1}{n}\operatorname{Im}\zeta & \text{if } 0 \leq \operatorname{Im}\zeta \leq n \\ n\operatorname{Im}\zeta + 1 - n^2 & \text{if } \operatorname{Im}\zeta \geq n. \end{cases}$$

These functions  $u_n(\zeta)$  cannot satisfy (10.6) for  $\operatorname{Im}\zeta$  large enough, however they obviously satisfy the second inequality of (10.5), and they also satisfy the first inequality of (10.5) since

$$\text{for } \operatorname{Im}\zeta \leq 0 \quad u_n(\zeta) = 0 \leq (\operatorname{Im}\zeta)^2 + 1,$$

$$\text{for } 0 \leq \operatorname{Im}\zeta \leq n \quad u_n(\zeta) = \frac{1}{n}\operatorname{Im}\zeta \leq \begin{cases} (\operatorname{Im}\zeta)^2 & \text{if } \frac{1}{n} \leq \operatorname{Im}\zeta \leq n \\ \frac{1}{n^2} \leq 1 & \text{if } 0 \leq \operatorname{Im}\zeta \leq \frac{1}{n} \end{cases} \leq (\operatorname{Im}\zeta)^2 + 1,$$

$$\text{and for } \operatorname{Im}\zeta \geq n \quad u_n(\zeta) = n\operatorname{Im}\zeta + 1 - n^2 \leq (\operatorname{Im}\zeta)^2 + 1.$$

Therefore

$$u_n(\zeta) \leq (\operatorname{Im}\zeta)^2 + 1 \quad \forall \zeta \in \mathbb{C},$$

and hence the Phragmén-Lindelöf principle (8.4) is violated.

We summarize the discussion in the last three sections, by saying that whereas in the cases  $n = N - 1$  or  $n = 1$  condition (H) is necessary and

sufficient for semi-global evolution, in the general case we only proved it is necessary, whereas for sufficiency we require the stronger condition (H') (which coincides with (H) when  $n = N - 1$ ) which is not necessary.

Let us translate the results obtained so far in terms of Cauchy problems for systems of differential operators.

Let us first recall that the hypothesis that  $\pi_n : V \rightarrow \mathbb{C}_\zeta^n$  be finite, proper and onto, for

$$V = V(\mathfrak{p}) = \{\zeta = (\tau, \xi) \in \mathbb{C}^k \times \mathbb{C}^n \simeq \mathbb{C}^N : p(\zeta) = 0 \ \forall p \in \mathfrak{p}\},$$

means that the module  $\mathfrak{M} = \mathcal{P}_N/\mathfrak{p}$ , considered as a  $\mathcal{P}_n$ -module  $(\mathfrak{M})_n$  via the natural inclusion  $\mathcal{P}_n \hookrightarrow \mathcal{P}_N$ , is finitely generated and free, i.e.  $(\mathfrak{M})_n \simeq \mathcal{P}_n^\nu$  for some  $\nu \in \mathbb{N} \setminus \{0\}$ .

This means that:

(i)  $\mathbb{R}^n$  is formally non-characteristic for the complex

$$(10.7) \quad \mathcal{W}_{K_2} \xrightarrow{{}^t A(D)} \mathcal{W}_{K_2}^r \xrightarrow{{}^t B(D)} \mathcal{W}_{K_2}^s \longrightarrow \dots$$

where  ${}^t A(D) = ({}^t p_1(D), \dots, {}^t p_r(D))$  for generators  $p_1(\zeta), \dots, p_r(\zeta)$  of the ideal  $\mathfrak{p}$ , and  $D = (D_t, D_x)$ ;

(ii) A complex of Cauchy data

$$\mathcal{W}_{K_1}^{a_0} \xrightarrow{{}^t \tilde{A}(D_x)} \mathcal{W}_{K_1}^{a_1} \xrightarrow{{}^t \tilde{B}(D_x)} \mathcal{W}_{K_1}^{a_2} \longrightarrow \dots$$

for the complex (10.7) on  $\mathbb{R}^n$  reduces to the trivial complex

$$\mathcal{W}_{K_1} \longrightarrow 0,$$

i.e. we have a set of free Cauchy data for  ${}^t A(D)$ .

If  ${}^t \tau_o : \mathcal{P}_n \rightarrow \mathcal{P}_N$  is a  $\mathcal{P}_n$ -homomorphism such that the diagram

$$\begin{array}{ccc} \mathfrak{M} & \longleftarrow & \mathcal{P}_N \\ \parallel & & \uparrow {}^t \tau_o \\ (\mathfrak{M})_n & \longleftarrow & \mathcal{P}_n \end{array}$$

commutes, then the Taylor series in  $t$  along  $t = 0$  of a solution  $f \in W_{K_2}$  of  ${}^t A(D)f = 0$  is uniquely determined by the Cauchy data

$$\tau_o(f) = \tau_o(D_t, D_x)f(t)|_{t=0}.$$

We refer the reader to [AN1] for more details.

We can finally formulate the following theorems:

THEOREM 10.2. — *Let*

$$(10.8) \quad S = \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n \mid t = 0\}$$

$$(10.9) \quad \Omega = \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n \mid t_j \geq 0 \text{ for } j = 1, \dots, k\}.$$

Then condition (H) is necessary in order that the following generalized Cauchy problem have a solution:

$$\begin{cases} \text{given } f \in \mathcal{W}_\Omega^r \text{ and } \varphi \in \mathcal{W}_S \\ \text{find } u \in \mathcal{W}_\Omega \text{ such that } {}^t A(D)u = f \text{ and } u|_S = \varphi. \end{cases}$$

If  $k = 1$  and  $r = 1$  condition (H) is also sufficient.

THEOREM 10.3. — *Let*

$$(10.10) \quad K_1 = \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n \mid |x| \leq A, t = 0\}$$

$$(10.11) \quad K_2 = \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n \mid |x| \leq A, 0 \leq t_j \leq B \text{ for } j = 1, \dots, k\}.$$

Then condition (H) is necessary in order that the following generalized Cauchy problem have a solution:

$$\begin{cases} \text{given } f \in \mathcal{W}_{K_2}^r \text{ and } \varphi \in \mathcal{W}_{K_1} \\ \text{find } u \in \mathcal{W}_{K_2} \text{ such that } {}^t A(D)u = f \text{ and } u|_{K_1} = \varphi. \end{cases}$$

If  $k = 1$  or  $n = 1$  condition (H) is also sufficient.

THEOREM 10.4. — *Let  $K_1$  and  $K_2$  as in (10.10) and (10.11). Then condition (H') is sufficient (but not necessary) in order that the following generalized Cauchy problem have a solution:*

$$\begin{cases} \text{given } f \in \mathcal{W}_{K_2}^r \text{ and } \varphi \in \mathcal{W}_{K_1} \\ \text{find } u \in \mathcal{W}_{K_2} \text{ such that } {}^t A(D)u = f \text{ and } u|_{K_1} = \varphi. \end{cases}$$

### 11. The Petrowski condition for evolution.

We want to show that the following *Petrowski condition*:

$$(11.1) \quad \exists c_j \in \mathbb{R}^+ : (\operatorname{Im} \tau_j)^+ \leq c_j \quad \forall (\tau, \zeta) \in V, \zeta = \xi \in \mathbb{R}^n$$

for  $j = 1, \dots, k$ , is a sufficient condition for the pair  $(K_1, K_2)$  to be of evolution in the Whitney class, for  $K_1$  and  $K_2$  defined as in (8.7) and (8.6).

We will prove that (11.1) implies the following Phragmén-Lindelöf principle:

$$(11.2) \quad \begin{aligned} & \forall m \exists M, c > 0 \text{ s.t. } \forall F \in \mathcal{O}(\mathbb{C}^n) \text{ with} \\ & \begin{cases} |F(\tau, \zeta)| \leq (1 + |\tau| + |\zeta|)^m \exp \left( \sum_{j=1}^k B(\operatorname{Im} \tau_j)^+ + A|\operatorname{Im} \zeta| \right) \\ |F(\tau, \zeta)| \leq c_F (1 + |\tau| + |\zeta|)^{m_F} \exp(A|\operatorname{Im} \zeta|) \end{cases} \\ & \text{we also have} \\ & |F(\tau, \zeta)| \leq c(1 + |\tau| + |\zeta|)^M \exp(A|\operatorname{Im} \zeta|) \quad \forall (\tau, \zeta) \in V. \end{aligned}$$

Indeed, by (8.3) and (11.1), from (11.2) we obtain

$$\begin{cases} |F(\tau, \xi)| \leq c'(1 + |\xi|)^{m'} & \text{on } V, \zeta = \xi \in \mathbb{R}^n \\ |F(\tau, \zeta)| \leq c'_F(1 + |\zeta|)^{m'_F} \exp(A|\operatorname{Im} \zeta|) & \text{on } V. \end{cases}$$

If we set

$$u(\zeta) = \sup_{(\tau, \zeta) \in V} \log |F(\tau, \zeta)|,$$

then  $u$  is plurisubharmonic in  $\mathbb{C}^n$  and

$$\begin{cases} u(\zeta) \leq c_u + m_u \log(1 + |\zeta|) + A|\operatorname{Im} \zeta| & \forall \zeta \in \mathbb{C}^n \\ u(\xi) \leq c'' + m' \log(1 + |\xi|) & \forall \xi \in \mathbb{R}^n. \end{cases}$$

We want to show that

$$u(\zeta) \leq c + M \log(1 + |\zeta|) + A|\operatorname{Im} \zeta| \quad \forall \zeta \in \mathbb{C}^n.$$

For this purpose we prove the following Phragmén-Lindelöf principle for plurisubharmonic functions:

**THEOREM 11.1.** — *For every positive constants  $M, B$  we can find some positive constants  $M', B'$  such that every plurisubharmonic function  $u$  in  $\mathbb{C}^n$  which satisfies*

$$\begin{cases} u(\zeta) \leq A|\zeta| + m_u \log(1 + |\zeta|) + c_u & \forall \zeta \in \mathbb{C}^n, \text{ for some } m_u, c_u \\ u(\xi) \leq M \log(1 + |\xi|) + B & \forall \xi \in \mathbb{R}^n \end{cases}$$

then is also satisfies

$$u(\zeta) \leq A|\operatorname{Im}\zeta| + M' \log(1 + |\zeta|) + B' \quad \forall \zeta \in \mathbb{C}^n.$$

*Proof.* — Let us first remark that the case  $n = 1$  is fairly well-known and moreover has already been treated in the proof of the sufficiency of condition (\*) of Proposition 9.1. The general case follows by a standard reduction argument from the case  $n = 1$ .

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