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LINEAR HOLONOMY GROUPS OF ALGEBRAIC SOLUTIONS OF POLYNOMIAL DIFFERENTIAL EQUATIONS

by Paulo SAD

INTRODUCTION

Let $\mathcal{F}$ be an analytic foliation of $\mathbf{C}P(2)$ of degree $k \in \mathbf{N}$. Given an affine coordinate system of $\mathbf{C}P(2)$ such that its line at infinity is not invariant by $\mathcal{F}$, the foliation is defined by a polynomial differential equation

$$\omega = -Qdx + Pdy = -(Q_k + yg)dx + (P_k + xg)dy = 0 \tag{1.1}$$

where $g \neq 0$ is a homogeneous polynomial of degree $k \in \mathbf{N}$ and P_k, Q_k are polynomials of degree at most $k \in \mathbf{N}$. If the line at infinity is $\mathcal{F}$-invariant, we still may use (1.1) to define $\mathcal{F}$ with $g \equiv 0$, but at least one of the polynomials P_k, Q_k must have degree $k \in \mathbf{N}$. Setting

$$H(u,v) = (u^{-1}, vu^{-1}) = (x,y) \tag{1.2}$$

we get

$$H^*\omega = u^{-(k+2)}[(\tilde{Q}_k(u,v) - v\tilde{P}_k(u,v))du + (u\tilde{P}_k(u,v) + g(1,v))dv] \tag{1.3}$$

where $\tilde{P}_k, \tilde{Q}_k$ are polynomials of degree $k \in \mathbf{N}$ at most. We see from (1.3) that the line at infinity $L_\infty := \overline{\{u=0\}}$ is the polar curve of ω, $(\omega)_\infty = (k+2)L_\infty$.

We may also define $\mathcal{F}$ by the meromorphic vector field

$$Z = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} = (P_k + xg)\frac{\partial}{\partial x} + (Q_k + yg)\frac{\partial}{\partial y} \tag{1.4}$$

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(1.5)
$$H^*Z = -u^{-(k-1)}\left[(u\tilde{P}_k(u,v) + g(1,v))\frac{\partial}{\partial u} - (\tilde{Q}_k(u,v) - v\tilde{P}_k(u,v))\frac{\partial}{\partial v}\right].$$
Therefore $(Z)_\infty = (k-1)L_\infty$. We define inclination functions

(1.6)
$$\psi(x,y) = \frac{Q(x,y)}{P(x,y)} = \frac{Q_k + yg}{P_k + xg}$$
$$\xi(u,v) = -\frac{\tilde{Q}_k(u,v) - v\tilde{P}_k(u,v)}{u\tilde{P}_k(u,v) + g(1,v)}.$$
It follows easily that if $\psi_\infty(u,v) := \psi(u^{-1}, u^{-1}v)$ we have

(1.7)
$$u\xi(u,v) + \psi_\infty(u,v) = v$$
wherever the functions ξ, ψ_∞ are finite.

Let $C \subset \mathbf{C}P(2)$ be a smooth algebraic curve, invariant for $\mathcal{F}$; we may always assume that the two affine coordinates systems $(x,y), (u,v) = (x^{-1}, x^{-1}y)$ cover C, and that C is transversal to L_∞. If $f(x,y) = 0$ is the polynomial equation which defines C, then

(1.8)
$$\psi\Big|_C = -\frac{f_x}{f_y}\Big|_C$$
holds except for a finite number of points in $C \setminus L_\infty$.

Given a leaf F of $\mathcal{F}$ and $p = (x_0, y_0) \in F$ such that $P(x_0, y_0) \neq 0$ (that is, F is not vertical at $p \in F$), the holonomy group $\mathcal{H}(F,p)$ is the set of germs of holomorphic diffeomorphisms defined as follows: to each continuous closed curve $\gamma = (\gamma_1, \gamma_2) \in \pi_1(F,p)$, $\gamma\colon [0,1] \to F$, $\gamma(0) = \gamma(1) = (x_0, y_0)$, we associate a continuous family of liftings $\gamma^{(y)}\colon [0,1] \to F_y$ of γ, where F_y is the leaf through (x_0, y) and $\gamma^{(y)}(0) = (x_0, y)$; putting $\gamma^{(y)}(1) := (x_0, f_\gamma(y))$, f_γ is a germ of diffeomorphism of $\mathbf{C}$ which has $y_0 \in \mathbf{C}$ as a fixed point. Then $f_\gamma \in \mathcal{H}(F,p)$. It follows that $f_\gamma(y) = y + \int_{\gamma_1} \psi dx$ and that

(1.9)
$$f'_\gamma(y_0) = \exp \int_{\gamma_1} \psi_y dx.$$
These expressions are written under the assumption that γ avoids L_∞ and $(\psi)_\infty$, which is granted if we deform slightly γ inside its homotopy class in $\pi_1(F,p)$. Let C^* be the invariant algebraic curve C deprived of the singularities of $\mathcal{F}$ which belong to it. From (1.9) it follows that we have in fact a group homomorphism
$$L_{\mathcal{F},C}\colon H_1(C^*, \mathbf{R}) \to \mathbf{C}^*$$
$$\gamma \mapsto \exp \int_{\gamma_1} \psi_y dx$$

called the *linear holonomy group of* C^*, and $L_{\mathcal{F},C}(\gamma)$ is the linear holonomy associated to γ.

Attention has been placed upon the situation where γ is a small loop around a singularity $q \in C$ of the foliation $\mathcal{F}$. The number $\operatorname{ind}_q(\mathcal{F}, C) := (2i\pi)^{-1} \int_\gamma \psi_y dx$ is called the *index of* $\mathcal{F}$ *relatively to* C *at* q (see [1]), and $L_{\mathcal{F},C}(\gamma) = \exp 2i\pi \operatorname{ind}_q(\mathcal{F}, C)$. Our interest in this paper consists in finding examples where singularities do not contribute to the linear holonomy group, but $L_{\mathcal{F},C}$ is not trivial due to the topology of C; furthermore, we estimate the degree of the examples as a function of the degree of C.

It is interesting to regard the situation of a Darboux foliation, defined by a closed 1-form

$$\omega = \sum_{j=1}^{n} \lambda_j \frac{dP_j}{P_j} = 0 \tag{1.10}$$

where $\lambda_j \in \mathbf{C}$, P_j are polynomials, $1 \leq j \leq n$. All curves $P_j = 0$, $1 \leq j \leq n$ are invariant; we take $P_1 = 0$ which is assumed for simplicity to be smooth and transversal to $P_2 = 0, \ldots, P_n = 0$. The points of intersection of $P_1 = 0$ with $P_j = 0$ are singularities, and the linear holonomies at these singularities are $\ell_j = \exp 2i\pi\lambda_j\lambda_1^{-1}$. If $\gamma \in H_1((P_1 = 0)^*, \mathbf{R})$, there exist integer numbers $m_2, \ldots, m_n$ (depending on γ) such that $L_{\mathcal{F},C}(\gamma) = \prod_{j=2}^{n} \ell_j^{m_j}$, as an easy computation shows. Therefore, the linear holonomies associated to curves in $H_1((P_1 = 0)^*, \mathbf{R})$ turn out to be trivial once we know that the linear holonomies at the singularities are trivial.

Since we will work in the situation where the singularities of $\mathcal{F}$ along C have linear holonomies equal to 1, the group homomorphism $L_{\mathcal{F},C}: H_1(C^*, \mathbf{R}) \to \mathbf{C}^*$ may be thought as defined on $H_1(C, \mathbf{R})$; this will be assumed from now on, unless explicitly stated on the contrary. Our result is:

THEOREM. — *Let* C *be a smooth algebraic curve of* $\mathbf{C}P(2)$ *of degree* $d \in \mathbf{N}$, *and* $\phi: H_1(C, \mathbf{R}) \to \mathbf{C}^*$ *be a homomorphism of groups. There exists a foliation* $\mathcal{F}$ *of* $\mathbf{C}P(2)$ *of degree at most* $4d^2 - 3d$ *such that* C *is* $\mathcal{F}$*-invariant and* $L_{\mathcal{F},C} = \phi$.

Our construction in fact allows us, once given a priori a finite number of points along C and a set of complex numbers assigned to them, to produce a foliation which has those points as singularities with those numbers as their linear holonomies. But we can not avoid the appearance of other singularities with trivial linear holonomies; the estimate involving degrees will change to account for this situation. In order to avoid complications in the exposition, we prefer to restrict to the simplest case. Finally, we do not know whether our estimation for the degree is the best possible or not (although the construction we make seems to be the simplest possible).

In connection with our subject we have the following works: 1) in [3], [6], [8], limit cycles are created from a certain number of vanishing cycles after perturbing a foliation of $\mathbf{C}P(2)$ which has a meromorphic first integral; the cycles lie in different leaves of the foliation; 2) in [7], the ambient space is a line bundle over a compact holomorphic curve. A finite set of points is given on the curve along with assigned complex numbers, and a foliation is constructed to realize such a data as linear holonomies of singularities. We will extend this construction in the Appendix to generate more linear holonomy due to the topology of the curve.

It should be mentioned that the problem of characterizing the groups of holomorphic diffeomorphisms which can be realized as holonomies groups of invariant algebraic curves seems to be a very hard one.

This paper is organized as follows: in §1 we use some facts from the theory of Riemann surfaces in order to choose convenient abelian differentials related to linear holonomies groups; these differentials are naturally associated to foliations. Next, in §2, we construct a foliation to which a differential given a priori is associated, and compute its degree, finishing then the proof of theorem. We close the paper with the Appendix aforementioned.

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1. PERIODS of MEROMORPHIC DIFFERENTIALS

Let us keep the notation used in the Introduction. From (1.9) we see that all information concerning linear holonomies comes from the meromorphic 1-form $\eta(\mathcal{F}) := (\psi_y dx)|_C$ of C. The periods of $\eta(\mathcal{F})$ arise from small loops around its poles and also from non-zero loops in $H_1(C, \mathbf{R})$; in this section we collect some facts related to both of them.

We start looking at the poles of $\eta(\mathcal{F})$.

Let us take the affine coordinate system in such a way as to have the tangencies of C with the vertical lines x =const. of quadratic type. The poles of $\eta(\mathcal{F})$ appear possibly at the following points:

(i) $q = (x_0, y_0) \in C$, $P(x_0, y_0) = Q(x_0, y_0) = 0$, $f_y(x_0, y_0) \neq 0$.

In this case, $q \in C$ is a singularity of $\mathcal{F}$ and

$$\mathrm{Res}_q \eta(\mathcal{F}) = \mathrm{ind}_q(\mathcal{F}, C). \tag{1.11}$$

(ii) $q = (x_0, y_0) \in C$, $f_y(x_0, y_0) = 0$.

(iii) $q \in C \cap L_\infty$.

LEMMA 1. — *In case (ii) and (iii) the equality*

$$1 + \mathrm{Res}_q \eta(\mathcal{F}) = \mathrm{ind}_q(\mathcal{F}, C) \tag{1.12}$$

holds true.

Proof. — 1) We prove first (1.12) in case (i). We may assume that $q = (x_0, y_0) = (0,0)$ and that locally C is written as $x = y^2\phi(y)$, where $\varphi \in \mathcal{O}_1$, $\phi(0) \neq 0$. Therefore

$$\begin{aligned}
&\frac{P}{Q}(y^2\phi(y), y) = 2y\phi(y) + y^2\phi'(y) \qquad \text{for } y \neq 0\\
&\Rightarrow \left(\frac{P}{Q}\right)_x (y^2\phi(y), y) - (2y\phi(y) + y^2\phi'(y)) + \left(\frac{P}{Q}\right)_y (y^2\phi(y), y)\\
&\qquad\qquad = (2y\phi(y) + y^2\phi'(y))'\\
&\Rightarrow \left(\frac{P}{Q}\right)_x (y^2\phi(y), y) - \left(\frac{Q}{P}\right)_y (y^2\phi(y), y)(2y\phi(y) + y^2\phi'(y))\\
&\qquad\qquad = \frac{(2y\phi(y) + y^2\phi'(y))'}{2y\phi(y) + y^2\phi'(y)}\\
&\Rightarrow \mathrm{ind}_q(\mathcal{F}, C) - \mathrm{Res}_q \eta(\mathcal{F}) = 1.
\end{aligned}$$

2) Let now $q \in C \cap L_\infty$. From (1.7):

$$\begin{aligned}
&u\xi(u, v) + \psi(u^{-1}, vu^{-1}) = v \Rightarrow\\
&u\xi_v + u^{-1}\psi_y(u^{-1}, vu^{-1}) = 1, \qquad \text{for } u \neq 0.
\end{aligned}$$

Therefore $\xi_v du - \eta(\mathcal{F}) = \dfrac{du}{u}$, and (1.12) follows. □

We remark that in cases (ii) and (iii), $q \in C$ may not be a singularity when $\mathrm{Res}_q \eta(\mathcal{F}) = -1$. The index $\mathrm{ind}_q(\mathcal{F}, C)$ is not powerful enough to detect the existence of a singularity, but if it does exist, its linear holonomy is trivial.

COROLLARY 2. — *Let $d = \deg(C)$. Then* $\sum\limits_{q\in \mathrm{sing}(\mathcal{F})} \mathrm{ind}_q(\mathcal{F}, C) = d^2$.

Proof. — 1) We may choose the affine coordinate system in order to have only points of type (ii) as singularities of $\mathcal{F}$ along C. The contribution of points (i), (ii) and (iii) to the residues of $\eta(\mathcal{F})$ are:

- points of type (i): $\sum\limits_{q\in \mathrm{sing}(\mathcal{F})} \mathrm{ind}_q(\mathcal{F}, C)$,
- points of type (ii): $-d(d-1)$,
- points of type (iii): $-d$.

Since $\sum\limits_{q} \mathrm{Res}\eta(\mathcal{F}) = 0$, the corollary follows. □

This corollary is a particular case of a theorem proved in [1].

We remark that there are no obstructions to finding meromorphic 1-forms in compact holomorphic curves with ascribed residues along a given finite set of points (except that the sum of the residues is zero), see [5].

So far we have studied the periods of the meromorphic 1-form $\eta(\mathcal{F})$ at the points of its polar set. What we want to establish now about the remaining periods is that there are no obstructions to realizing all of them, at least at the level of meromorphic differentials.

In order to be more precise, let us consider a canonical basis $a_1, \dots, a_g$, $b_1, \dots, b_g$ for $H_1(C, \mathbf{R})$, and fix representatives in each class of these elements, still denoted by $a_1, \dots, a_g$, $b_1, \dots, b_g$ ($g := \mathrm{genus}(C)$).

THEOREM 3. — *Given a collection $(\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g) \in \mathbf{C}^g$, there exists a meromorphic 1-form η such that $\lambda_i = a_i$-period of η, $\mu_i = b_i$-period of η.*

Proof. — 1) Let $\mathcal{H}_C$ be the complex vector space of holomorphic differentials of C. The map

$$\mathcal{H}_C \ni \eta \mapsto \left(\int_{a_1} \eta, \dots, \int_{a_g} \eta\right) \in \mathbf{C}^g$$

is a linear isomorphism (see [5]), so that we can find a (unique) holomorphic differential with ascribed a-periods.

2) Now we want to realize b-periods. Let us fix ℓ distinct points $p_1, \dots, p_\ell \in C$ outside the a-curves and b-curves, and define $D := p_1 + \dots + p_\ell$. Let $V_C = \{$meromorphic differentials of C with poles at $p_1, \dots, p_\ell$

of order at most 2, zero residues at these poles and zero a-periods} and $T: V_C \to \mathbf{C}^g$,

$$\eta \mapsto T(\eta) = \left(\int_{b_1} \eta, \dots, \int_{b_g} \eta \right).$$

It is easily seen that $\dim V_C = \ell$.

The following sequence is exact:

$$0 \to \mathbf{C} \xrightarrow{\text{inc}} L(D) \xrightarrow{d} V_C \xrightarrow{T} \mathbf{C}^g \tag{1.13}$$

where $L(D) = \{$meromorphic functions h of C, $(h) \geq -D\}$. ((h) stands for the divisor associated to h), and d is the usual differentiation. Our problem is to add a last arrow

$$0 \to \mathbf{C} \xrightarrow{\text{inc}} L(D) \xrightarrow{d} V_C \xrightarrow{T} \mathbf{C}^g \to 0 \tag{1.14}$$

keeping exacteness. It is enough to have $\dim L(D) = \ell - g + 1$. By the theorem of Riemann-Roch, this is implied by $\ell \geq 2g - 1$; we then pick up such an integer $\ell \in \mathbf{N}$.

It follows that adding differentials of $\mathcal{H}_C$ and V_C we may realize all a-periods and b-periods. □

Remarks.

1) Since $d(d-1) \geq 2g - 1$, where $d :=$ degree(C) and $g :=$ genus(C), all poles of the meromorphic 1-form given by Theorem 3 may be placed along the points $(x_0, y_0) \in C \setminus L_\infty$ where $f_y(x_0, y_0) = 0$.

2) Let $\mathcal{F}$ be a foliation of $\mathbf{C}P(2)$ which has C as a invariant curve; if $\eta(\mathcal{F}) \in \mathcal{H}_C + V_C$, then any singularity of $\mathcal{F}$ along C has trivial linear holonomy.

3) We may add to the meromorphic differentials in $\mathcal{H}_C + V_C$ convenient meromorphic differentials in order to realize ascribed residues along a given finite set of points.

In the next sections, we construct a foliation $\mathcal{F}$, and compute its degree, for which $\eta(\mathcal{F})$ is given a priori in $\mathcal{H}_C + V_C$.

2. CONSTRUCTION OF FOLIATIONS

Let η be a meromorphic 1-form of the smooth algebraic curve $C \subset \mathbf{C}P(2)$. We consider affine coordinate systems (x, y), $(u, v) = (x^{-1}, x^{-1}y) \in \mathbf{C}^2$ which satisfy

C1) C is covered by these coordinates.

C2) C is transversal to $L_0 = \overline{\{x=0\}} \cup L_\infty = \overline{\{u=0\}}$.

C3) The tangencies of C with the vertical fibration x =const are of quadratic type.

Let $f(x,y) = 0$ be the polynomial equation for C. We will look for a rational function ψ of $\mathbf{C}P(2)$ such that the foliation $\mathcal{F}$ defined by $\frac{dy}{dx} = \psi(x,y)$ has C as an invariant curve and $\eta(\mathcal{F}) = \eta$. We will try a function ψ in the following form:

$$(1.15)\quad \psi(x,y) = -\frac{f_x(x,y)}{f_y(x,y)} + f(x,y)\psi^1(x,y) = \psi^0(x,y) + f(x,y)\psi^1(x,y)$$

where ψ^1 is a rational function to be determined. Let $\eta = \beta dx|_C$, so that β is a meromorphic function of C. From (1.15) we derive

$$(1.16)\qquad \psi_y = \psi^0_y + f_y\psi^1 + f\psi^1_y.$$

Since we demand that $\eta(\mathcal{F}) = \eta$, or $\psi_y|_C dx = \beta dx$, (1.16) implies that

$$(1.17)\qquad \psi^1|_C = \frac{\beta - \psi^0_y|_C}{f_y|_C}.$$

The function $\varphi := \frac{\beta - \psi^0_y|_C}{f_y|_C}$ is a meromorphic function of C. We want to extend it as a rational function ψ^1 of $\mathbf{C}P(2)$. This is always possible and in fact there are many extensions: once ψ^1 is one of them, we may consider $\psi^1 + f \cdot h$ as well, where h is any rational function of $\mathbf{C}P(2)$ such that $(h)_\infty \not\supset C$. The problem now is to determine the degree of the resulting foliation, what we do by counting singularities along C. The 1-form η, which will presumably be $\eta(\mathcal{F})$ at the end of the construction, does not carry all the information concerning the singularities of $\mathcal{F}$. For instance, the foliation $\mathcal{G}$ defined by $-xdy + y(x^2+y)dx = 0$ has $C = \overline{\{y=0\}}$ as an invariant algebraic curve and $(0,0) \in C$ is a singularity, but $\eta(\mathcal{G}) = xdx$ is holomorphic at $(0,0) \in C$. The conclusion is that we have to follow carefully the steps along the construction of the foliation in order to be able to compute the degree. We will use the ideas of [2]; in this paper, a meromorphic function is given along an analytic curve of $\mathbf{C}^2$, and an extension to $\mathbf{C}^2$ as a meromorphic function is shown to exist.

2.1. Poles of the function φ.

As we indicated before, *the 1-form $\eta \in \mathcal{H} + V$ may be chosen to have its poles in the set of points of $C \setminus L_\infty$ where $f_y = 0$.* The poles of the function $\varphi = \dfrac{\beta - \psi_y^0|_C}{f_y|_C}$, where $\eta = \beta dx$, are possibly:

(a) *points of $C \setminus L_\infty$ where $f_y = 0$.*

Since the computation is local, we may assume that the point under examination is $(0,0) \in \mathbf{C}^2$ and $f(x,y) = x - y^2$. Writing $\eta = a(y)dy = a(y)\dfrac{dx}{2y}$, we get $\beta(y) = \dfrac{a(y)}{2y}$. We conclude that:

- if $(0,0)$ is a pole of η of order 2, then it is a pole of β of order 3 and a pole of φ of order 4.
- if $(0,0)$ is a regular point of η (that is, $a(y)$ is holomorphic at $(0,0)$), it follows that φ has a pole of order 3 due to the fact that $\psi_y^0|_C$ has a pole of order 2.

(b) *points of $C \cap L_\infty$.*

We apply (1.7) to the foliation defined by the inclination function $\psi^0 = -\dfrac{f_x}{f_y}$ and get

$$\psi_y^0(x,y) = u - u^2\xi_v^0(u,v), \quad x = u^{-1}, \quad y = u^{-1}v. \tag{1.18}$$

We have also that $f(x,y) = \dfrac{1}{u^d}\tilde{f}(u,v)$ for a polynomial $\tilde{f}$, and $\tilde{f}_v(0,v) \neq 0$ at the points of $C \cap L_\infty$. It follows that

$$f_y(x,y) = \frac{1}{u^{d-1}}\tilde{f}_v(u,v). \tag{1.18$'$}$$

Since η is holomorphic at $C \cap L_\infty$, if we write $\eta = -\tilde{\beta}(u)du$ (where $\tilde{\beta}$ is a holomorphic function) we get

$$\beta\Big(\frac{1}{u}\Big) = u^2\tilde{\beta}(u) \tag{1.18$''$}$$

and from (1.18), (1.18$'$) and (1.18$''$) we conclude that φ has a zero of order $d \in \mathbf{N}$ at the points of $C \cap L_\infty$.

Therefore, all poles of φ belong to the set of points where $f_y = 0$.

2.2. Local extensions of the function φ.

We will construct in this section a family $\{\hat{\psi}^1_\alpha\}$ of meromorphic functions defined over a covering $\{U_\alpha\}$ of $\mathbf{C}P(2)$ with the properties:

$$(1.19)\qquad \begin{cases} \text{(i)} \quad \hat{\psi}^1_\alpha - \hat{\psi}^1_\beta \in \mathcal{O}_{U_\alpha \cap U_\beta} & \text{if} \quad U_\alpha \cap U_\beta \neq \emptyset \\ \text{(ii)} \quad \hat{\psi}^1_\alpha|_{C\cap U_\alpha} = \varphi|_{C\cap U_\alpha} & \text{if} \quad C \cap U_\alpha \neq \emptyset. \end{cases}$$

Before doing that, we have to choose the affine coordinate system in order to satisfy two extra properties (C4) and (C5), (C1, C2 and C3 were introduced in §2):

C4) if $f_y(x_0, y_0) = 0$ for some $(x_0, y_0) \in C$, then $f_y(x, y_0) \neq 0$, $\forall (x, y_0) \in C$ and $x \neq x_0$;

C5) if $(0, y_0) \in C$ then $f_y(x, y_0) \neq 0$, $\forall (x, y_0) \in C$, and if $(x_0, y_0) \in C$ and $f_y(x_0, y_0) = 0$ then $y_0 \neq 0$.

In order to construct the elements of (1.19), there are several situations to be analyzed; in the sequel, all neighborhoods are supposed to be sufficiently small.

Case 1: $(x_0, y_0) \in C \setminus L_0$, $f_y(x_0, y_0) \neq 0$, $f_y(x, y_0) \neq 0$, $\forall x \in \mathbf{C}$.

Since C can given locally by $y = h(x)$, where h is a holomorphic function such that $y_0 = h(x_0)$, we may define

$$(1.20)\qquad \hat{\psi}^1_\alpha(x, y) = \varphi(x, h(x))$$

for (x, y) in a neighborhood U_α of (x_0, y_0).

Case 2: $(x_0, y_0) \in C \setminus L_\infty$, $f_y(x_0, y_0) = 0$.

There are $d \in \mathbf{N}$ points $(x_0, y_0), \ldots, (x_{d-1}, y_0) \in C$; let $x = g_j(x)$ describe C locally at these points for $0 \leq j \leq d-1$, ($g_j(y_0) = x_j$, g_j holomorphic function). Condition C4 implies that $f_y(x_j, y_0) \neq 0$, $j = 1, \ldots, d-1$. We define

(1.21)

$$\begin{aligned} \hat{\psi}^1_\alpha(x, y) &= \frac{\varphi(g_0(y), y) f(x, y)}{(g_0(y) - x)(g_1(y) - g_0(y)) \ldots (g_{d-1}(y) - g_0(y))} \left(\frac{g_0(y)}{x}\right)^{d-2} \\ &= \frac{\varphi(g_0(y), y)(g_1(y) - x) \ldots (g_{d-1}(y) - x)}{(g_1(y) - g_0(y)) \ldots (g_{d-1}(y) - g_0(y))} \left(\frac{g_0(y)}{x}\right)^{d-2} \end{aligned}$$

for $|y - y_0|$ small, $x \in \mathbf{C}^*$.

It is easily verified that $\hat{\psi}^1_\alpha(g_0(y), y) = \varphi(g_0(y), y)$ (so $\hat{\psi}^1_\alpha$ extends φ) and $\hat{\psi}^1_\alpha(g_j(y), y) = 0$ for $1 \leq j \leq d-1$. We may also write

$$\hat{\psi}^1_\alpha(x, y) = \frac{1}{x^{d-2}} \left(\sum_{k=1}^{4} \frac{A_k(x)}{(y-y_0)^k} + r(x, y) \right) \tag{1.22}$$

where $A_k(x)$ is a polynomial of degree $d-1$ at most and $r(x, y)$ is a holomorphic function defined for $|y - y_0|$ small. Let us define $P_{y_0}(x, y) = \sum_{k=1}^{4} \frac{A_k(x)}{(y-y_0)^k}$.

Remark. — $A_4(x) \equiv 0$ when the pole of φ at (x_0, y_0) has order 3.

Case 3: points $(x_1, y_0), \ldots, (x_{d-1}, y_0) \in C \setminus L_0$ such that $f_y(x_0, y_0) = 0$ and $(x_0, y_0) \in C$.

We consider in the neighborhood of these points the expression given by (1.20), and add the one in (1.21). Clearly we get an extension of φ.

Case 4: $(x_0, y_0) \notin C \cup L_0$, $f_y(x, y_0) \neq 0$, $\forall x \in C$.

We just put $\hat{\psi}^1_\alpha(x, y) \equiv 0$ in a neighborhood U_α of the point (x_0, y_0).

Case 5: $(x_0, y_0) \notin C \cup L_0$, but $f_y(\overline{x}_0, y_0) = 0$ for some $\overline{x}_0 \in \mathbf{C}$ and $(\overline{x}_0, y_0) \in \mathbf{C}$.

We have already associated to $(\overline{x}_0, y_0) \in C$ the expression in (1.21), and we keep it in this case.

Case 6: points in $L_\infty \setminus L_0$.

Let us start with the point $u = 0$, $v = 0$ (remember that $u = x^{-1}$, $v = x^{-1}y$), which we may assume to be not in C. The points studied in Case 2 were associated to expressions like (1.21) which give rise to horizontal polar lines $y = y_0^{(j)}$, $1 \leq j \leq d(d-1)$, meeting together at $u = 0$, $v = 0$. We define

$$\hat{\psi}^1_\alpha(u, v) = \sum_{j=1}^{d(d-1)} u^{d-2} P_{y_0^{(j)}}(u^{-1}, vu^{-1}) \tag{1.23}$$

where $P_{y_0^{(j)}}(x, y)$ was defined when we treated Case 2 points. The polar divisor of the function in (1.23) is exactly $\bigcup_{j=1}^{d(d-1)} \{v - y_0^{(j)} u = 0\}$.

At the points $u = 0$, $v \neq 0$, we proceed as in Cases 1 (points of $C \cap L_\infty$) or 4.

Case 7: points in L_0.

We go back to the expression (1.21), associated to the horizontal polar line $y = y_0^{(j)}$, $1 \leq j \leq d(d-1)$. Each of them may be written as

$$\hat{\psi}^1_\alpha(x,y) = \frac{f(x,y)}{x^{d-2}} \sum_{k=-4}^{\infty} a_k(x)(y - y_0^{(j)})^k,$$

where $a_k(x)$ is a rational function on $x - x_0$. We define then

$$V(x,y) = \frac{f(x,y)}{x^{d-2}} \sum_{j=1}^{d(d-1)} \sum_{k=-4}^{-1} a_k(x)(y - y_0^{(j)})^k. \tag{1.24}$$

It is easy to check that $a_k(x) = \dfrac{b_k(x-x_0)}{c_k(x-x_0)}$ for b_k, c_k polynomials of degrees $k+4$, $k+5$, respectively ($-4 \leq k \leq -1$). We observe that V vanishes along C, so that:

- if $(0, y_0) \in C$, we define $\hat{\psi}^1_\alpha(x,y)$ by adding $V(x,y)$ to the expression (1.20);
- if $(0, y_0) \notin C$, we put $\hat{\psi}^1_\alpha(x,y) = V(x,y)$;
- at the point in the intersection $L_0 \cap L_\infty$, we take $\hat{\psi}^1_\alpha(\xi,\eta) = V(\eta\xi^{-1}, \xi^{-1})$, ($\eta = xy^{-1}$, $\xi = y^{-1}$ are affine coordinates that cover $L_0 \cap L_\infty$). A simple computation shows that this last expression is holomorphic along L_∞.

The properties listed in (1.19) are immediately verified. We may then consider the additive Cousin data $\left\{\dfrac{\hat{\psi}^1_\alpha - \hat{\psi}^1_\beta}{f}\right\}_{\alpha,\beta}$; the covering $\mathcal{U}$ of $\mathbf{C}P(2)$ by the open sets introduced along the analysis of the several Cases 1 to 7 is assumed to be a Leray covering, see [5], p. 46.

2.3. Solution of the Cousin problem.

Since $H^1(\mathbf{C}P(2), \mathcal{O}) = 0$ (see [4]) and $\mathcal{U}$ is a Leray covering, we can find a meromorphic function G of $\mathbf{C}P(2)$ such that

$$G - \frac{\hat{\psi}^1_\alpha}{f} \in \mathcal{O}_{U_\alpha} \quad \forall U_\alpha \in \mathcal{U}. \tag{1.25}$$

We define

$$\psi^1 := fG. \tag{1.26}$$

Clearly ψ^1 extends φ, and it is the function we employ to define ψ in (1.15).

As for the polar divisor $(\psi^1)_\infty$, it certainly coincides with $(\hat{\psi}^1_\alpha)_\infty$ in every open set $\pi^{-1}(U_\alpha)$ when $U_\alpha \subset \mathbf{C}P(2) \setminus L_\infty$. When $U_\alpha \cap L_\infty \neq \emptyset$, (1.25) and (1.26) imply that L_∞ is a line of poles of ψ^1 of order $d \in \mathbf{N}$ at most.

2.4. Computation of the degree.

We have at last a foliation $\mathcal{F}$ defined by

$$\frac{dy}{dx} = -\frac{f_x}{f_y} + f\psi^1 \tag{1.27}$$

which has $C \subset \mathbf{C}P(2)$ as an invariant curve whose linear holonomy group is given *a priori*; ψ^1 comes from the last section.

Let Z be the meromorphic vector field defined by (1.4). The restriction $Z|_C$ will have zeroes and poles, which we proceed now to describe; they coincide with the singularities of $\mathcal{F}$ along C (the singularities of $\mathcal{F}$ lying in the set $C \cap L_\infty$ will be poles of $Z|_C$).

The singularities of $\mathcal{F}$ along C, computed with multiplicities, are as follows:

1) *points in* $\{x = 0\} \cap C$

   Both $\{x = 0\}$ and $C \subset \mathbf{C}P(2)$ are invariant curves of $\mathcal{F}$, so that all d points of their intersection are singularities. Since C is transversal to $\{x = 0\}$, each of them has multiplicity at most $d - 2 \in \mathbf{N}$ (according to the order of $\{x = 0\}$ as a polar line of ψ^1).

2) *points* $(x_0, y_0) \in C$ *such that* $f_y(x_0, y_0) \neq 0$ *but* $f_y(x, y_0) = 0$ *for some* $x \in \mathbf{C}$.

   We have seen that the line of poles $y = y_0$ has order 4 or 3 (§2.1). Again, since C is transversal to any such line, we conclude that the points of their intersection are singularities of $\mathcal{F}$ (with multiplicities 4 or 3).

   There are $2g - 1$ lines of poles of order 4, and $d(d-1) - (2g-1)$ lines of poles of order 3 (at most). It follows that there exists $(d-1)(2g-1)$ singularities of order 4 and $[d(d-1) - (2g-1)](d-1)$ singularities of order 3 (of the type described above).

3) *points* $(x_0, y_0) \in C$ *where* $f_y(x_0, y_0) = 0$

   There are $(2g-1)$ poles of φ in C of order 4; an easy computation shows that each of these points has multiplicity 3 as a singularity of $\mathcal{F}$. Analogously, there are $[d(d-1)-(2g-1)]$ singularities of $\mathcal{F}$ of multiplicity 2 (due to poles of φ in C of order 3).

4) points in $C \cap L_\infty$ (in number of $d \in \mathbf{N}$ points).

   We write (1.27) in the coordinates (u, v), $u = x^{-1}$, $v = x^{-1}y$ using (1.7):

$$\xi = \frac{v - \psi_\infty}{u} = \frac{v - [\psi_\infty^0 + f(u^{-1}, u^{-1}v)\psi^1(u^{-1}, u^{-1}v)]}{u}$$

   where $\psi_\infty^0 = \psi^0(u^{-1}, u^{-1}v)$ and $\psi^0 = -\dfrac{f_x}{f_y}$ (see (1.15)). Since $\xi_\infty^0 = \dfrac{d\tilde{f} - u\tilde{f}_u}{\tilde{f}_v}$, where $f(u^{-1}, u^{-1}v) = u^{-d}\tilde{f}(u, v)$ for some polynomial $\tilde{f}$, it follows that

$$\xi = \frac{d\tilde{f} - u\tilde{f}_u}{u\tilde{f}_v} + \frac{1}{u^{d+1}}\tilde{f} \cdot \psi_\infty^1. \tag{1.28}$$

   Now ψ_∞^1 has a vertical line of poles $\overline{\{u=0\}}$ of order $d \in \mathbf{N}$ at most (§2.3); we conclude then that any point of $C \cap L_\infty$ has multiplicity $2d+1$ at most as a singularity of $\mathcal{F}$, and $(\deg(\mathcal{F})-1)-(2d+1)$ at most as order of the pole of $Z|_C$. Applying the Poincaré-Hopf theorem to $Z|_C$, we find at once that degree $(\mathcal{F}) < 4d^2 - 3d$. Our theorem is proved.

APPENDIX

Let L be a complex line bundle over the compact Riemann surface C, identified with the 0-section. Let $c(L)$ denote the Chern class of L. We will consider here a holomorphic foliation $\mathcal{F}$ defined on L which leaves C invariant and has a finite number of singularities along C, all of them with trivial linear holonomy. As we explained in the introduction, we may speak of the homomorphism $L_{\mathcal{F},C}\colon H_1(C, \mathbf{R}) \to \mathbf{C}^*$, the linear holonomy group of C.

THEOREM 4. — *Suppose* $c(L) = 0$. *Given a group homomorphism* $\phi\colon H_1(C, \mathbf{R}) \to \mathbf{C}^*$, *there exists a foliation* $\mathcal{F}$ *as above such that* $L_{\mathcal{F},C} = \phi$.

This theorem is the basic step to proving the analogous result for all complex line bundles over C; it is enough to apply the machinery developed in [7].

Proof of Theorem.

1) The condition $c(L) = 0$ implies that we may define L by a cocycle of constant transition functions $\{\xi_{\alpha\beta}\} \in H^1(\mathcal{U}, \mathbf{C}^*)$ for some covering $\mathcal{U} = \{U_\alpha\}$ by open sets, see [5], p. 180.

   Let η be an abelian differential of C constructed according to §1. We define in each trivialization chart $U_\alpha \times \mathbf{C}$ of L the differential equation $\dfrac{dy}{dz_\alpha} = y\eta(z_\alpha)$. The closed 1-form $\Omega = \dfrac{dy}{y} - \eta(z_\alpha)dz_\alpha$ is then independent of the chart, so that it defines a foliation $\mathcal{F}$ in L. Clearly C is invariant, and the poles of η are its singularities; the fibers $\Sigma_1, \ldots, \Sigma_m$ over these singularities are also invariant.

2) Let us calculate the holonomy group of C. We fix a fiber Σ of L, transversal to $\mathcal{F}$ and take $\gamma \in H_1(C, \mathbf{R})$ avoiding the poles of η. Starting at $y_0 \in \Sigma$, we lift γ to a path ℓ_1 contained in the leaf of $\mathcal{F}$ through y_0, and we end up at $f_\gamma(y_0) \in \Sigma$. Join in Σ y_0 to $f_\gamma(y_0)$ by a path ℓ_2 disjoint of $\Sigma \cap C$. Therefore
$$\int_{\ell_1 * \ell_2} \Omega = \int_{\ell_1} \Omega + \int_{\ell_2} \Omega = \int_{\ell_2} \Omega.$$
   Notice that $\int_{\ell_2} \Omega = \int_{\ell_2} \frac{dy}{y} = \log \frac{f_\gamma(y_0)}{y_0} \mod 2\pi i\mathbf{Z}$. We want to compute $\int_{\ell_1 * \ell_2} \Omega$. Both 1-forms $\Omega_1 = \dfrac{dy}{y}$ and $\eta = \eta(z_\alpha)dz_\alpha$ are well-defined closed meromorphic 1-forms of L, $(\Omega_1)_\infty = C$ and $(\eta)_\infty = \Sigma_1 \cup \ldots \cup \Sigma_m$. We have that $\exp\left(\int_{\ell_1 * \ell_2} \Omega_1\right) = c(\gamma)$, where $c \colon H_1(C, \mathbf{R}) \to \mathbf{C}^*$ is a homomorphism which does not depend on η. Also, $\ell_1 * \ell_2$ is homologous to $\gamma + k_1\alpha_1 + \ldots + k_m\alpha_m$ in $L \backslash \Sigma_1 \cup \ldots \cup \Sigma_m$, where $k_j \in \mathbf{Z}$ and α_j is a small loop in C going around each pole of η; therefore $\int_{\ell_1 * \ell_2} \Omega_2 = \int_\gamma \eta$ (remember that the poles of η have zero residues). It follows that $f_\gamma(y_0) = \exp\left(\int_{\ell_1 * \ell_2} \Omega\right) = c(\gamma)\left(\exp \int_\gamma \eta\right) y_0$.

In particular, the holonomy group of C, defined in Σ, is linear (that is, in the coordinate coming from the trivialization chart) and of the type we asked for. □

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