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TRANSVERSAL CRYSTALS OF FINITE LEVEL

by B. LE STUM and A. QUIRÓS (*)

CONTENTS

INTRODUCTION

In [Q] the second author studies families of strongly divisible filtered F-crystals in relation with Griffiths transversality. In his book [O2] Ogus

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introduces the notion of T-crystal (T for *transversal*), which provides an excellent context to study this kind of questions. He uses it to prove a version of Mazur's theorem on the relation between the action of Frobenius and the Hodge filtration on crystalline cohomology which is valid for cohomology with coefficients in an F-crystal. As applications, he gets results about Newton and Hodge polygons (Katz conjecture) and degeneration of the Hodge spectral sequence. One of his key results shows that there is an equivalence between F-spans and T-crystals, provided we restrict to objects of width less than p.

In his letter to Illusie [B3], Berthelot developes the theory of crystals of level m. We use this new theory to extend Ogus' theorem to objects of width less than p^{m+1}: after defining T-m-crystals and F-m-spans, we show that one can identify T-m-crystals of width less than p^{m+1} with a full subcategory of F-m-spans.

More precisely: let S be a torsion free p-adic formal scheme, S_0 its reduction mod p and X a smooth S_0-scheme. A T-m-crystal on X/S is a crystal E of level m with a filtration Fil by submodules which after saturation (see Definition 1.1.6), behaves like a filtration by subcrystals. If $F : X \to X'$ is the relative Frobenius of X/S_0, an F-m-span is a p-isogeny $\Phi : F^{m+1^*}E \to E'$ of p-torsion free m-crystals. We prove (Theorem 4.3.6) that if (E, Fil) is a p-torsion free T-m-crystal on X/S such that $\mathrm{Fil}^{p^{m+1}} \subset pE$, then there exists a unique F-m-span $\Phi : F^{m+1^*}E \to E'$ such that, up to saturation, $F^{m+1^*}\mathrm{Fil}$ coincides with the filtration M defined by $M^k := \Phi^{-1}(p^kE')$. This construction is functorial in (E, Fil) and the functor is fully faithful.

In order to prove this theorem, we consider a lifted situation: X is a smooth formal S-scheme, F_0 is the relative Frobenius of X_0 over S_0, $F : X \to X'$ is a lifting of F_0 and we assume that there are coordinates $t_1, \dots, t_d$ on X and X' such that $F(t_i) = t_i^p$. Then T-m-crystals correspond to Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules that are also transversal to the m-PD-ideal (p) and F-m-spans correspond to p-isogenies of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules. We prove the theorem in this local situation (Theorem 2.3.3 and Corollary 3.3.5).

Let us briefly describe the structure of this paper: in the first part, we recall Ogus' notion of transversality and Berthelot's notion of partial divided power structures as well as some properties of p-isogenies in this context. In the second part, we first recall Berthelot's theory of differential operators of finite level, we define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and we build the local version of our functor. In the third part, we define and study p-m-curvature for $\mathcal{D}^{(m)}$-modules in characteristic p and we use this notion to prove the fullfaithfulness of our functor in a local situation. In the fourth part, we recall Berthelot's theory of m-crystals, we define T-m-crystals and F-m-spans and we deduce our main theorem from its local version. In the fifth and last part, we study the behavior of T-m-crystals and F-m-spans when m varies and use it to show that our results provide some improvement on Ogus' theory.

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Conventions. — We let p be a non zero prime and $m \in \mathbb{N}$. All formal schemes are p-adic formal schemes. All schemes are locally killed by some power of p and might hence be considered as formal schemes. Also, all PD-structures are compatible with p. We will use the subindex 0 to indicate reduction mod p. We will adopt the standard multiindex notation, and if $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we will write $|\underline{k}| = k_1 + \cdots + k_d$.

1. PRELIMINARIES

1.1. Transversal filtrations.

We briefly recall the notion of a transversal module from [O2]. We call transversal what Ogus calls G-transversal and almost transversal what he calls G'-transversal. Let us first fix some terminology and notations:

1.1.1. Definition. — *Let A be a ring (in a topos). A module filtration* Fil *on an A-module M is a decreasing filtration by submodules* Fil^k *such that there exists an integer a such that* $\mathrm{Fil}^a = M$. *It is called effective if we can take $a = 0$. In general, if we set* $\mathrm{Fil}[r]^k := \mathrm{Fil}^{k+r}$, *we see that* $\mathrm{Fil}[a]$ *is an effective filtration on M. If $\varphi : (\mathcal{T}, A') \to (\mathcal{T}, A)$ is a morphism of ringed sites, (M, Fil) is a filtered A-module and* Fil^k_φ *denotes the image of* $\varphi^* \mathrm{Fil}^k$ *in* $\varphi^* M$, *then* $\varphi^*(M, \mathrm{Fil}) := (\varphi^* M, \mathrm{Fil}\,\varphi)$ *is called the inverse image of* (M, Fil).

In this article, in order to simplify the notations, we will only consider effective filtrations.

1.1.2. DEFINITION. — *A ring filtration on a ring A is a module filtration $I^{(*)}$ such that $I^{(k)}I^{(\ell)} \subset I^{(k+\ell)}$. If $(A, I^{(*)})$ is a filtered ring, we set $I := I^{(1)}$ and we say that a filtered module (M, Fil) has width at most w (with respect to I) if there exists an integer a such that $\mathrm{Fil}^a = M$ and $\mathrm{Fil}^{a+w+1} \subset IM$. A filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is a site endowed with a filtered ring. A morphism of filtered ringed sites*

$$\varphi : (\mathcal{T}', A', I'^{(*)}) \longrightarrow (\mathcal{T}, A, I^{(*)})$$

is a morphism of ringed sites such that $\varphi^* I^{(k)}$ maps into $I'^{(k)}$ for all k.

1.1.3. DEFINITION. — *A filtered module (M, Fil) in a filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is transversal (a T-module for short) if it satisfies*

$$IM \cap \mathrm{Fil}^k = I\,\mathrm{Fil}^{k-1} + I^{(2)}\,\mathrm{Fil}^{k-2} + I^{(3)}\,\mathrm{Fil}^{k-3} + \cdots$$

for all k. It is almost transversal if

$$IM \cap \mathrm{Fil}^k \subset I\,\mathrm{Fil}^{k-1} + I^{(2)}\,\mathrm{Fil}^{k-2} + I^{(3)}\,\mathrm{Fil}^{k-3} + \cdots$$

for all k and saturated if $I^{(k)}\,\mathrm{Fil}^{\ell} \subset \mathrm{Fil}^{\ell+k}$ for all k, ℓ.

Since there will sometimes be several ring filtrations involved, we will, if necessary, say (almost) transversal to $I^{(*)}$ and saturated with respect to $I^{(*)}$. If $I^{(k)} = I^k$ for all k, we will just say (almost) transversal to I and saturated with respect to I.

1.1.4. *Example.* — A filtered module (M, Fil) in a ringed site $(\mathcal{T}, A)$ is transversal to an ideal I of A if and only if it satisfies $IM \cap \mathrm{Fil}^k = I\,\mathrm{Fil}^{k-1}$ for all k.

1.1.5. *Remark.* — A filtered module is transversal if and only if it is almost transversal and saturated.

Starting from any almost transversal filtration, there exists a natural process that turns it into a transversal one:

1.1.6. DEFINITION. — *If (M, Fil) is a filtered module on a filtered ringed site $(\mathcal{T}, A, I^{(*)})$, we set*

$$\overline{\mathrm{Fil}}^k = \mathrm{Fil}^k + I\,\mathrm{Fil}^{k-1} + I^{(2)}\,\mathrm{Fil}^{k-2} + I^{(3)}\,\mathrm{Fil}^{k-3} + \cdots.$$

We call $(M, \overline{\mathrm{Fil}})$ the saturation of (M, Fil).

1.1.7. PROPOSITION (see [O2], 2.3.1).

(i) *The filtration* $\overline{\mathrm{Fil}}$ *is the finest filtration on* M *that is saturated and coarser than the given one.*

(ii) *If* (M, Fil) *is almost transversal, then its saturation is transversal.*

This saturation process is specially useful in view of the following result:

1.1.8. PROPOSITION (see [O2], 2.2.1). — *Let*

$$\varphi : (\mathcal{T}', A', I'^{(*)}) \longrightarrow (\mathcal{T}, A, I^{(*)})$$

be a morphism of filtered ringed sites such that the natural map $\varphi^{-1}A/I \to A'/I'$ *is flat. If* (M, Fil) *is an almost transversal module, then so is* $\varphi^*(M, \mathrm{Fil})$.

1.2. p-isogenies.

We introduce the m-PD-filtration $(p, \{\ \})$ and we describe transversality with respect to this filtration in terms of p-isogenies.

1.2.1. DEFINITION. — *If* A *is a* $\mathbb{Z}_{(p)}$*-algebra and* M, M' *two* p*-torsion free* A*- modules, a* p*-isogeny* $\Phi : M \to M'$ *of width at most* w *is an injective homomorphism* $\Phi : M \to M' \otimes \mathbb{Q}$ *of* A*-modules such that there exists an integer* a *such that* $p^{a+w+1}M' \subset \Phi(M) \subset p^a M'$. *It is called effective if one can take* $a = 0$. *In general, if we set* $\Phi[r] = p^{-r}\Phi$, *we see that* $\Phi[a]$ *is effective.*

As we do for filtrations, we will only consider effective p-isogenies.

Transversality with respect to p, meaning to the ideal (p), has a very nice interpretation in terms of p-isogenies:

1.2.2. PROPOSITION (see [O2], 5.1.2). — *The functor* $\Phi \mapsto (M, \mathrm{Fil})$, *where* $\mathrm{Fil}^k = \Phi^{-1}(p^k M')$, *is an equivalence from the category of* p*-isogenies of width at most* w *onto the category of filtered modules transversal to* p *of width at most* w.

Actually, the filtration that will naturally appear in the sequel is not $(p)^k$ but the m-PD-filtration defined below (and generalized in Definition 1.3.4).

1.2.3. DEFINITION. — *For* $k = qp^m + r$ *with* $0 \leq r < p^m$, *we let* $p^{\{k\}} := p^k/q!\cdot$ *The m-PD-filtration* $(p)^{\{k\}}$ *on a* $\mathbb{Z}_p$*-algebra* A *is the finest* *ring filtration such that* $p^{\{k\}} \in (p)^{\{k\}}$. *We will also write* $(p, \{\ \})$ *for this filtration.*

In the sequel, we will also need the notion of modified binomial coefficients. Let us recall what they are:

1.2.4. Definition. — *If* $\underline{k}'$ *and* $\underline{k}'' \in \mathbb{N}^d$, *and*

$$\begin{aligned} \underline{k}' &= \underline{q}'pm + \underline{r}', & 0 \le \underline{r}' < pm, \\ \underline{k}'' &= \underline{q}''pm + \underline{r}'', & 0 \le \underline{r}'' < pm, \\ \underline{k} &= \underline{k}' + \underline{k}'' = \underline{q}\,pm + \underline{r}, & 0 \le \underline{r} < pm, \end{aligned}$$

one sets:

$$\left\{ \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\} := \frac{\underline{q}\,!}{\underline{q}'\,!\,\underline{q}''\,!} \in \mathbb{N} \quad \text{and} \quad \left\langle \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\rangle := \binom{\underline{k}}{\underline{k}'} \left\{ \begin{matrix} \underline{k} \\ \underline{k}' \end{matrix} \right\}^{-1} \in \mathbb{Z}_p.$$

Proposition 1.2.2 is still valid for the m-PD-filtration under some assumptions on the width:

1.2.5. Proposition (see [O2], 2.3.5). — *The functor «saturation with respect to* $(p, \{\ \})$*» from the category of filtered modules transversal to* p *to the category of filtered modules transversal to* $(p, \{\ \})$ *is an equivalence of categories when restricted to objects of width less than* p^{m+1}.

1.2.6. Corollary. — *The functor* $\Phi \mapsto (M, \mathrm{Fil})$ *where* Fil^k *is the saturation of* $\Phi^{-1}(p^k M')$ *with respect to* $(p, \{\ \})$ *is an equivalence from the category of* p*-isogenies of width less than* p^{m+1} *onto the category of filtered modules transversal to* $(p, \{\ \})$ *of width less than* p^{m+1}.

1.3. m-PD-structures.

We recall Berthelot's theory of partial divided powers from [B4] which generalizes the usual divided power structures in [B1].

1.3.1. Definition. — *Let* Y *be a formal scheme. An* m*-PD-structure on a coherent ideal* I *in* $\mathcal{O}_Y$ *is the data of a PD-ideal* $(J, [\ ])$ *in* I *such that* $I^{(p^m)} + pI \subset J$ *(where* $I^{(p^m)}$ *is the ideal locally generated by* f^{p^m} *with* $f \in I$*). We say that* I *is an* m*-PD-ideal or that* (Y, I, J) *is a formal* m*-PD-scheme. We will drop* J*, or even* I*, from the notations when no confusion should arise. If* $f \in I$ *and* $k = q\,p^m + r$ *with* $0 \le r < p^m$, *we write*

$$f^{\{k\}} := f^r \left(f^{p^m}\right)^{[q]}.$$

1.3.2. Definition. — *Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. The m-PD-structure on $\mathfrak{a}$ extends to a formal S-scheme X if the PD-structure on $\mathfrak{b}$ extends to a PD-structure on X (compatible with p). An m-PD-structure (I, J) on a formal S-scheme Y is said to be compatible with $(S, \mathfrak{a}, \mathfrak{b})$ if the m-PD-structure on $\mathfrak{a}$ extends to Y, the PD-structure on $J + (p)$ is compatible with the PD-structure on $\mathfrak{b} + (p)$ and $I \cap (\mathfrak{b}\mathcal{O}_Y + (p))$ is a sub PD-ideal of $\mathfrak{b}\mathcal{O}_Y + (p)$. We then say that (Y, I, J) is a formal m-PD-S-scheme.*

1.3.3. Definition. — *Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. A morphism of formal m-PD-S-schemes is a morphism of formal schemes $\varphi : Y' \to Y$ such that $\varphi^{-1}(I) \subset I'$ and $(Y', J') \to (Y, J)$ is a morphism of formal PD-schemes. If (Y, I, J) is a formal m-PD-S-scheme and X is the closed formal subscheme of Y defined by I, we say that $X \hookrightarrow Y$ is an m-PD-immersion.*

The following generalizes Definition 1.2.3 and agrees with Berthelot's new definition that replaces [B4] 1.3.8 and 1.3.7.

1.3.4. Proposition and Definition (see [B5]). — *If (Y, I, J) is a formal m-PD-S-scheme, then there exists a finest ring filtration $(I, \{\ \}) := I^{\{*\}}$ on $\mathcal{O}_Y$ such that*

(i) $I^{\{1\}} = I$,

(ii) $I^{\{n\}} \cap (J + \mathfrak{b}\mathcal{O}_Y + p\mathcal{O}_Y)$ *is a sub PD-ideal of* $J + \mathfrak{b}\mathcal{O}_Y + p\mathcal{O}_Y$,

(iii) $x^{\{h\}} \in I^{\{nh\}}$ *whenever* $x \in I^{\{n\}}$.

It is called the m-PD-filtration on $\mathcal{O}_Y$ with respect to (I, J). Then $(Y, \mathcal{O}_Y, I^{\{n\}})$ is a filtered ringed site. Moreover, any morphism of formal m-PD-S-schemes induces a morphism of the corresponding filtered ringed sites.

Universal m-PD-immersions do exist:

1.3.5. Proposition and Definition (see [B4], 2.1.1). — *Let S be a formal m-PD-scheme, X a formal S-scheme to which the m-PD-structure of S extends and $i : X \hookrightarrow Y$ an immersion into a formal S-scheme. Then i factors as an m-PD-S-immersion $X \hookrightarrow P^n_{X/S(m)}(Y)$ followed by a morphism $\varphi : P^n_{X/S(m)}(Y) \to Y$ having the following universal property: any morphism $Y' \to Y$ inducing $X' \to X$, where $X' \hookrightarrow Y'$ is an m-PD-S-immersion whose ideal satisfies $I^{\{n+1\}} = 0$, factors uniquely through φ.*

We say that $P^n_{X/S(m)}(Y)$ *is the* n*-th* m*-PD-neighborhood of* X *in* Y *and we write* $\mathcal{P}^n_{X/S(m)}(Y)$ *for its structural sheaf.*

1.3.6. *Remark.* — If $X \hookrightarrow Y$ is an immersion of schemes (locally killed by a power of p) then there exists an m-PD-S-immersion $X \hookrightarrow P_{X/S(m)}(Y)$ with the same universal property but without nilpotency condition on I. We call $P_{X/S(m)}(Y)$ the *m-PD-neighborhood of X in Y*, and write $\mathcal{P}^n_{X/S(m)}(Y)$ for its structural sheaf.

1.3.7. Definition. — *If i is the diagonal immersion*

$$X \hookrightarrow Y := X \times_S X,$$

then we drop Y from the notations in 1.3.5 *and* 1.3.6 *and we call* $\mathcal{P}^n_{X/S(m)}$ *the sheaf of* m*-th principal parts of order at most* n.

2. DIFFERENTIAL OPERATORS OF LEVEL m AND GRIFFITHS TRANSVERSALITY

2.1. Differential operators of level m.

We will now recall from [B4] Berthelot's theory of differential operators of finite level.

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. We consider $\mathcal{P}^n_{X/S(m)}$ as an $\mathcal{O}_X$-module using the first projection $X \times_S X \to X$ and we note $\theta : \mathcal{O}_X \to \mathcal{P}^n_{X/S(m)}$ the map induced by the second projection. We first recall the definition of differential operators of level m:

2.1.1. Definition. — *The* $\mathcal{O}_X$*-dual* $\mathcal{D}^{(m)}_{X/S\,n}$ *to* $\mathcal{P}^n_{X/S(m)}$ *is called the sheaf of differential operators of level* m *and order at most* n*. The natural maps* $\mathcal{P}^{n'}_{X/S(m)} \to \mathcal{P}^n_{X/S(m)}$ *for* $n \leq n'$ *induce injections* $\mathcal{D}^{(m)}_{X/S\,n} \hookrightarrow \mathcal{D}^{(m)}_{X/S\,n'}$ *and we set*

$$\mathcal{D}^{(m)}_{X/S} = \bigcup_n \mathcal{D}^{(m)}_{X/S\,n}.$$

Moreover, the natural maps

$$\mathcal{P}^{n+n'}_{X/S(m)} \longrightarrow \mathcal{P}^n_{X/S(m)} \otimes \mathcal{P}^{n'}_{X/S(m)}$$

induce bilinear maps

$$\mathcal{D}^{(m)}_{X/S\,n} \times \mathcal{D}^{(m)}_{X/S\,n'} \longrightarrow \mathcal{D}^{(m)}_{X/S\,n+n'}$$

which make $\mathcal{D}^{(m)}_{X/S}$ *into a ring called the ring of differential operators of level* m*. Its* p*-adic completion will be denoted by* $\widehat{\mathcal{D}}^{(m)}_{X/S}$.

2.1.2. *Remark.* — If $t_1, \dots, t_d$ are local coordinates on X and

$$\tau_i: = \theta(t_i) - t_i \quad \text{for all } i,$$

then $\mathcal{P}^n_{X/S(m)}$ is a free $\mathcal{O}_X$-module on the $\underline{\tau}^{\{\underline{k}\}}$ with $|\underline{k}| \leq n$.

We let $\{\underline{\partial}^{\langle \underline{k} \rangle}\}$ be the dual basis to $\{\underline{\tau}^{\{\underline{k}\}}\}$ in $\mathcal{D}^{(m)}_{X/S\,n}$.

If $\underline{k} = \underline{q}p^m + \underline{r} < p^{m+1}$, we set

$$\underline{\partial}^{[\underline{k}]}: = \underline{\partial}^{\langle \underline{k} \rangle}/q!.$$

If $n < p^{m+1}$, then the $\underline{\tau}^{\underline{k}}$ with $|\underline{k}| \leq n$ form a basis for $\mathcal{P}^n_{X/S(m)}$ and the $\underline{\partial}^{[\underline{k}]}$ form the dual basis in $\mathcal{D}^{(m)}_{X/S\,n}$. Note that $\mathcal{D}^{(m)}_{X/S}$ is generated as an $\mathcal{O}_X$-algebra by the $\partial_i^{[p^j]} = \partial_i^{\langle p^j \rangle}$ for $j \leq m$.

2.1.3. *Remark.* — If $\varphi: Y \to X$ is a morphism of smooth formal S-schemes and $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module then $\varphi^*\mathcal{F}$ has a natural structure of $\mathcal{D}^{(m)}_{Y/S}$-module that can be described locally as follows. Let $t_1, \dots, t_d$ be local coordinates on X, $t'_1, \dots, t'_d$ be local coordinates on Y and $\{\tau_i\}$ and $\{\tau'_k\}$ be the corresponding sections of $\mathcal{P}^n_{X/S(m)}$ and $\mathcal{P}^n_{Y/S(m)}$. If $\varphi^*(\tau_i^{\{j\}}) = \sum f^{i,j}_{k,\ell} \tau_k'^{\{\ell\}}$ and s is a section of $\mathcal{F}$, we have

$$\partial_i^{\langle j \rangle}(\varphi^*(s)) = \sum f^{k,\ell}_{i,j}\, \varphi^*(\partial_k^{\langle \ell \rangle}(s)).$$

As in the classical case, $\mathcal{D}^{(m)}$-modules have an interpretation in terms of stratifications:

2.1.4. Proposition (see [B4], 2.3.2). — *If $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of $\mathcal{D}^{(m)}_{X/S}$-module or an m-PD-stratification (defined in the obvious way).*

2.1.5. Definition. — *A $\mathcal{D}^{(m)}_{X/S}$-module (or $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module) is locally (topologically) quasi-nilpotent if locally, given any section s, we have $\partial_i^{\langle N \rangle}(s) \to 0$ as $N \to \infty$ for any index i.*

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.1.6. Proposition (generalization of [B1], II. 4.1.3). — *If X is a smooth S-scheme (with p locally nilpotent) and $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of locally quasi-nilpotent $\mathcal{D}^{(m)}_{X/S}$-module or an m-HPD-stratification (defined in the obvious way).*

We will also have to consider formal S-schemes that are not necessarily smooth. In order to deal with this situation we need to introduce the following terminology (see also [B4], 2.3.4 and 2.3.5):

2.1.7. Definition. — *Let X be an S-scheme and $X \hookrightarrow Y$ a closed immersion into a smooth formal S-scheme. It follows from Proposition 4.1.5 below that $\mathcal{P}_{X/S(m)}(Y)$ has a natural structure of $\mathcal{D}^{(m)}_{Y/S}$-module. A $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-module is a $\mathcal{D}^{(m)}_{Y/S}$-module $\mathcal{F}$ with a structure of $\mathcal{P}_{X/S(m)}(Y)$-module such that, locally, given any sections f of $\mathcal{P}_{X/S(m)}(Y)$ and s of $\mathcal{F}$, we have*

$$\underline{\partial}^{\langle \underline{k} \rangle}(fs) = \sum \left\{ \begin{matrix} \underline{k} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\langle \underline{j} \rangle}(f)\, \underline{\partial}^{\langle \underline{k}-\underline{j} \rangle}(s).$$

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.2. Griffiths transversality for $\mathcal{D}^{(m)}$-modules.

We define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and interpret it in terms of stratifications.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme. The following generalizes the usual notion of Griffiths transversality:

2.2.1. Definition. — *A filtered $\mathcal{D}^{(m)}_{X/S}$-module $(\mathcal{F}, \mathrm{Fil})$ is a $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{F}$ together with a filtration by sub $\mathcal{O}_X$-modules. We say that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal if whenever $P \in \mathcal{D}^{(m)}_{X/S\,n}$, we have $P(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-n}$ and that it is horizontal if the Fil^k are $\mathcal{D}^{(m)}_{X/S}$-submodules. A filtered $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module $(\mathcal{F}, \mathrm{Fil})$ is a complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module $\mathcal{F}$ together with a filtration by complete sub $\mathcal{O}_X$-modules. We say that it is Griffiths transversal or horizontal if it is so $\mathrm{mod}\ p^n$ for all n.*

2.2.2. *Remarks.*

(i) What we call Griffiths transversal corresponds to what is simply called a filtration on a $\mathcal{D}$-module in the classical situation.

(ii) Assume we have local coordinates $t_1, \dots, t_d$. In order to show that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal it is sufficient to check that $\partial_i^{[p^j]} \mathrm{Fil}^k \subset \mathrm{Fil}^{k-p^j}$ for $j \leq m$ and all i.

Here is the interpretation of Griffiths transversality in terms of stratifications:

2.2.3. Definition. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{O}_X$-module with an m-PD-stratification $\{\varepsilon_n : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}\}$. We call the stratification transversal if ε_n induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$ for all n.*

2.2.4. Proposition. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module and Fil^k a filtration on $\mathcal{F}$ by sub $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is Griffiths transversal if and only if the corresponding m-PD-stratification is transversal.*

Proof. — Let $\mathcal{I}$ be the ideal of X in $P^n_{X/S(m)}$, $p_1, p_2 : P^n_{X/S(m)} \to X$ the projections, $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ the n-th Taylor isomorphism of $\mathcal{F}$ and

$$\begin{aligned}\theta : \mathcal{F} &\longrightarrow p_1^*\mathcal{F},\\ e &\longmapsto \varepsilon(1 \otimes e)\end{aligned}$$

the n-th Taylor map. Assume first the m-PD-stratification to be transversal. Since ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$, then

$$\begin{aligned}\theta\,\mathrm{Fil}^k \subset \overline{\mathrm{Fil}}^k_{p_1} &= \mathrm{Fil}^k_{p_1} + \mathcal{I}\,\mathrm{Fil}^{k-1}_{p_1} + \mathcal{I}^{\{2\}}\,\mathrm{Fil}^{k-2}_{p_1} + \cdots + \mathcal{I}^{\{n\}}\,\mathrm{Fil}^{k-n}_{p_1}\\ &\subset \mathrm{Fil}^{k-n}_{p_1}.\end{aligned}$$

If $P : \mathcal{P}^n_{X/S(m)} \to \mathcal{O}_X$ is a differential operator of level m and order less than n, then P acts on $\mathcal{F}$ as the composite of θ and $p_1^*(P)$ (*i.e.* $P(e) = (P \otimes \mathrm{Id})(\theta(e))$ so that $P\,\mathrm{Fil}^k \subset \mathrm{Fil}^{k-n}$. Thus, we see that $\mathcal{F}$ is Griffiths transversal. Conversely, assume that $\mathcal{F}$ is Griffiths transversal. We want to check that ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$ and we may assume that we have local coordinates $t_1, \dots, t_d$ on X. Thanks to the cocycle condition, it is sufficient to show that $\theta(\mathrm{Fil}^k) \subset \overline{\mathrm{Fil}}^k_{p_1}$. But if $e \in \mathrm{Fil}^k$ then

$$\theta(e) = \sum \partial^{\langle j\rangle}(e)\tau^{\{j\}} \in \sum \mathcal{I}^{\{j\}}\,\mathrm{Fil}^{k-j}_{p_1} = \overline{\mathrm{Fil}}^k_{p_1}. \qquad \square$$

The same is true for hyperstratifications. Let S be an m-PD-scheme and X a smooth S-scheme.

2.2.5. Definition. — *If $(\mathcal{F}, \mathrm{Fil})$ is a filtered $\mathcal{O}_X$-module, we call an m-HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ on $\mathcal{F}$ transversal if ε induces an isomorphism between $\overline{\mathrm{Fil}}^k_{p_2}$ and $\overline{\mathrm{Fil}}^k_{p_1}$.*

2.2.6. Proposition. — *An m-HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ on a filtered $\mathcal{O}_X$-module $(\mathcal{F}, \mathrm{Fil})$ is transversal if and only if $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal.*

Proof. — Same as Proposition 2.2.4. $\square$

2.3. Griffiths transversality and p-isogenies.

We are going to build the local version of the functor of our main theorem.

Let S be a formal m-PD-scheme, X a formal S-scheme, F_0 the relative Frobenius of X_0 over S_0 and $F: X \to X'$ a lifting of F_0. We assume that there are local coordinates $t_1, \ldots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

We will write $X_0^{(m+1)}$ for the pull back of X_0 by the $m+1$ iterate of F_0, and, with the usual slight abuse of notation, we will call

$$F_0^{m+1} : X_0 \longrightarrow X_0^{(m+1)}$$

this $m+1$ iterate of F_0 and $F^{m+1} : X \to X^{(m+1)}$ a lifting obtained by iterating the above process.

2.3.1. Lemma. — *If s is a section of a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module $\mathcal{E}$, then for $\underline{k} < p^{m+1}$, we have, with $a_{\underline{j},\underline{k}} \in \mathbb{Z}$,*

$$\underline{\partial}^{[\underline{k}]}\left(F^{m+1^*}(s)\right) = \sum p^{\underline{j}} a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}}\, F^{m+1^*}\left(\underline{\partial}^{[\underline{j}]}(s)\right).$$

Proof. — For $n = p^{m+1} - 1$, we have in $\mathcal{P}^n_{X^{(m+1)}/S(m)}$

$$\begin{aligned} F^{m+1^*}(\tau_i) &= (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} \\ &= \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k \\ &= \sum_{k=1}^{p^{m+1}-1} p\, c_{i,k}\, t_i^{p^{m+1}-k} \tau_i^k \end{aligned}$$

with $c_{i,k} \in \mathbb{Z}$. Thus we can write

$$F^{m+1^*}(\underline{\tau}^{\underline{j}}) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}}\, \underline{\tau}^{\underline{k}}$$

with $a_{\underline{j},\underline{k}} \in \mathbb{Z}$. Therefore, if s is a section of $\mathcal{E}$, we have

$$\underline{\partial}^{[\underline{k}]}\left(F^{m+1^*}(s)\right) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\left(\underline{\partial}^{[\underline{j}]}(s)\right). \qquad \square$$

This lemma allows us to show that Frobenius pulls back transversal modules to horizontal modules:

2.3.2. PROPOSITION. — *If $(\mathcal{E}, \mathrm{Fil})$ is a Griffiths transversal $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module (or $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module) on $X^{(m+1)}$ which is saturated with respect to $(p, \{\ \})$, then $F^{m+1^*}(\mathcal{E}, \mathrm{Fil})$ is horizontal.*

Proof. — We have seen that if s is a section of $\mathcal{E}$, then for $\underline{k} < p^{m+1}$, we have

$$\underline{\partial}^{[\underline{k}]}\big(F^{m+1^*}(s)\big) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\partial^{[\underline{j}]}(s)\big).$$

Since $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal, we know that if $s \in \mathrm{Fil}^{\ell}$, we have $(\underline{\partial}^{[\underline{j}]}(s)) \in \mathrm{Fil}^{\ell-[\underline{j}]}$. It follows that $F^{m+1^*}(\partial^{[\underline{j}]}(s)) \in \mathrm{Fil}^{\ell-[\underline{j}]}$ so that

$$p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big) \in p^{\underline{j}}\, \mathrm{Fil}^{\ell-[\underline{j}]}.$$

Since $(\mathcal{E}, \mathrm{Fil})$ is saturated with respect to $(p, \{\ \})$, so is $F^{m+1^*}(\mathcal{E}, \mathrm{Fil})$ and therefore

$$\begin{aligned}\underline{\partial}^{[\underline{j}]}\big(F^{m+1^*}(s)\big) &= \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, t^{\underline{j}p^{m+1}-\underline{k}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big)\\ &\in \sum p^{\underline{j}}\, \mathrm{Fil}^{\ell-[\underline{j}]} = \sum p^{\{\underline{j}\}}\, \mathrm{Fil}^{\ell-[\underline{j}]} \subset \mathrm{Fil}^{\ell}. \qquad \square\end{aligned}$$

2.3.3. THEOREM. — *Assume S has no p-torsion. Let $(\mathcal{E}, \mathrm{Fil})$ be a p-torsion free Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module of width less than p^{m+1} which is transversal to $(p, \{\ \})$. Then there exists a unique p-isogeny $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules such that $F^{m+1^*}\mathrm{Fil}^k$ is the saturation of $\Phi^{-1}(p^k\mathcal{F})$ with respect to $(p, \{\ \})$.*

Proof. — Follows from Corollary 1.2.6 and Proposition 2.3.2. □

2.3.4. DEFINITION. — *Given any lifting $F : X \to X'$ of the relative Frobenius of X_0 over S_0, an F^{m+1}-p-isogeny on X/S will be a p-isogeny of the form $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ is a $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module and $\mathcal{F}$ is a $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module.*

2.3.5. — Theorem 2.3.3 gives a functor μ from the category of p-torsion free Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module of width less than p^{m+1} that are transversal to $(p, \{\ \})$ to the category of F^{m+1}-p-isogenies of width less than p^{m+1} on X/S. We will show in section 3.3 that this functor is fully faithful.

3. $\mathcal{D}^{(m)}$-MODULES IN CHARACTERISTIC p AND GRIFFITHS TRANSVERSALITY

3.1. p-m-curvature of a $\mathcal{D}^{(m)}$-module.

We define p-m-curvature for $\mathcal{D}^{(m)}$-modules in characteristic p and study the relation between it being zero and horizontal sections.

Let S be a scheme of characteristic p and X a smooth S-scheme. We let

- $\mathcal{D}^{(m)+}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{O}_X$;
- $\mathcal{K}^{(m)}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{E}nd(\mathcal{O}_X)$.

3.1.1. Definitions. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. The sheaf $\mathcal{F}^{\nabla}$ of horizontal sections of $\mathcal{F}$ is the part of $\mathcal{F}$ on which $\mathcal{D}^{(m)+}_{X/S}$ acts as zero. The p-m-curvature of $\mathcal{F}$ is the restriction to $\mathcal{K}^{(m)}_{X/S}$ of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{E}nd(\mathcal{F})$.*

3.1.2. *Remark.* — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. Then it follows from [B4], 2.2.6, that $\mathcal{F}$ has zero p-m-curvature if, locally on X, we have for all i, $\partial_i^{\langle p^{m+1}\rangle}(s) = 0$ for any $s \in \mathcal{F}$. In particular, in case $m = 0$, zero p-m-curvature is the same as zero p-curvature.

Let $F : X \to X'$ be the relative Frobenius of X over S.

3.1.3. Lemma. — *If $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module, then $\mathcal{D}^{(m)+}_{X/S}$ acts as zero on sections of the form $F^{m+1^*}(s)$ with $s \in \mathcal{E}$.*

Proof. — This is a local question. We have

$$\begin{aligned} F^{m+1^*}(\tau_i) &= (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} \\ &= \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k \\ &= \tau_i^{p^{m+1}} = p!\, \tau_i^{\{p^{m+1}\}} = 0. \end{aligned}$$

It follows that, if $0 < j < p^{m+1}$, then $F^{m+1^*}(\tau^j) = 0$, so that, for any section s of $\mathcal{E}$, we have $\partial^{[j]}(F^{m+1^*}(s)) = 0$. □

3.1.4. Proposition. — *The trivial $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{O}_X$ has zero p-m-curvature and the canonical map $\mathcal{O}_{X^{(m+1)}} \to F^{m+1}_* \mathcal{O}_X^{\nabla}$ is bijective.*

Proof. — The first assertion is an obvious consequence of the definition. The second one is local and we may therefore choose local coordinates $t_1, \dots, t_d$. These coordinates define an étale map from X to $\mathbb{A}^d_S$. The relative Frobenius being cartesian with respect to étale morphisms and to base change, this map provides us with an isomorphism

$$F_*^{m+1}\mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes_{\mathbb{F}_p[t_1,\dots,t_d]} \mathbb{F}_p[t_1,\dots,t_d]^{(m+1)}$$

where $\mathbb{F}_p[t_1,\dots,t_d]^{(m+1)}$ is $\mathbb{F}_p[t_1,\dots,t_d]$ seen as a module over itself via the $(m+1)$-st power of Frobenius. If $\mathbb{F}_p[t_1,\dots,t_d]_{<p^{(m+1)}}$ denotes the space of polynomials of degree stricly less than p^{m+1} in each variable, the canonical map

$$\mathbb{F}_p[t_1,\dots,t_d] \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1,\dots,t_d]^{(m+1)}_{<p^{(m+1)}} \longrightarrow \mathbb{F}_p[t_1,\dots,t_d]^{(m+1)}$$

is bijective and therefore

$$F_*^{m+1}\mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1,\dots,t_d]_{<p^{(m+1)}}.$$

Since $F_*^{m+1}\mathcal{D}^{(m)+}_{X/S}$ acts as zero on $\mathcal{O}_{X^{(m+1)}}$, we are reduced to showing that if $f \in \mathbb{F}_p[t_1,\dots,t_d]_{<p^{(m+1)}}$ and $\mathcal{D}^{(m)+}_{X/S}$ acts as zero on f, then $f \in \mathbb{F}_p$. One may first prove that if A is an $\mathbb{F}_p$-algebra and $f \in A[t^{p^j}]$ is such that $\partial^{\langle p^j\rangle}(f) = 0$, then $f \in A[t^{p^{j+1}}]$ and then use induction on d. The details are left to the reader. □

3.1.5. PROPOSITION

(i) *If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module then $F_*^{m+1}\mathcal{F}^\nabla$ is a sub $\mathcal{O}_{X^{(m+1)}}$-module of $F_*^{m+1}\mathcal{F}$.*

(ii) *If $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module then $F^{m+1^*}\mathcal{E}$ has zero p-m-curvature.*

Proof. — Again, these are local questions. For the first assertion, we have to show that if s is a section of $\mathcal{F}^\nabla$ and f is a section of $\mathcal{O}_{X^{(m+1)}}$ then $\partial^{\langle \underline{k}\rangle}((F^{m+1^*}(f)s) = 0$ for $\underline{k} \neq 0$. For the second one, we have to show that if s is a section of $\mathcal{F}$ and f is a section of $\mathcal{O}_X$, then $\partial_i^{\langle p^{m+1}\rangle}(fF^{m+1^*}(s)) = 0$. Using the formula

$$\underline{\partial}^{\langle \underline{k}\rangle}(fs) = \sum \left\{ \begin{matrix} \underline{k} \\ \underline{j} \end{matrix} \right\} \underline{\partial}^{\langle \underline{j}\rangle}(f)\, \underline{\partial}^{\langle \underline{k}-\underline{j}\rangle}(s),$$

both statements are easy consequences of Lemma 3.1.3 and Proposition 3.1.4. □

3.2. Cartier's theorem for $\mathcal{D}^{(m)}$-modules.

We generalize Cartier's theorem (see [K], 5.1) to $\mathcal{D}^{(m)}_{X/S}$-modules. We let S, X and $F: X \to X'$ be as in section 3.1.

3.2.1. LEMMA. — *Let $t_1, \ldots, t_d$ be local coordinates on X and*

$$P := \sum_{\underline{k}<p^{m+1}} (-\underline{t})^{\underline{k}} \underline{\partial}^{[\underline{k}]}.$$

If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature, then P is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$.

Proof. — We follow the first part of the proof of Proposition 5.1 in [K]. Since $\mathcal{F}$ has zero p-m-curvature, we have $\partial_i^{\langle j\rangle}(s) = 0$ for $j \geq p^{m+1}$. There should therefore be no confusion if we write $\underline{\partial}^{[\underline{j}]}(s) = 0$ for $\underline{j}$ such that $\max(j_i) \geq p^{m+1}$. If $s \in \mathcal{F}$, we have

$$\begin{aligned}
\underline{\partial}^{[\underline{j}]}(P(s)) &= \underline{\partial}^{[\underline{j}]}\Big(\sum (-\underline{t})^{\underline{k}}\, \underline{\partial}^{[\underline{k}]}(s)\Big) \\
&= \sum\sum \underline{\partial}^{[\underline{i}]}((-\underline{t})^{\underline{k}})\,(\underline{\partial}^{[\underline{j}-\underline{i}]}\,\underline{\partial}^{[\underline{k}]})(s) \\
&= \sum\sum (-1)^{\underline{i}} \binom{\underline{k}}{\underline{i}} (-\underline{t})^{\underline{k}-\underline{i}} \binom{\underline{k}+\underline{j}-\underline{i}}{\underline{k}}\, \underline{\partial}^{[\underline{k}+\underline{j}-\underline{i}]}(s) \\
&= \sum\sum (-1)^{\underline{i}} \binom{\underline{\ell}+\underline{i}}{\underline{i}} (-\underline{t})^{\underline{\ell}} \binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}}\, \underline{\partial}^{[\underline{\ell}+\underline{j}]}(s) \\
&= \sum \Big(\sum (-1)^{\underline{i}} \binom{\underline{\ell}+\underline{i}}{\underline{i}} \binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}}\Big) (-\underline{t})^{\underline{\ell}}\, \underline{\partial}^{[\underline{\ell}+\underline{j}]}(s)
\end{aligned}$$

and, if $\underline{j} \neq 0$, we have

$$\sum (-1)^{\underline{i}} \binom{\underline{\ell}+\underline{i}}{\underline{i}} \binom{\underline{\ell}+\underline{j}}{\underline{\ell}+\underline{i}} = \binom{\underline{\ell}+\underline{j}}{\underline{\ell}} \sum (-1)^{\underline{i}} \binom{\underline{j}}{\underline{i}} = 0.$$

Thus we see that P maps f into $\mathcal{F}^{\nabla}$. Since P restricts to the identity on $\mathcal{F}^{\nabla}$, it is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$. $\square$

3.2.2. PROPOSITION. — *Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature. Then the canonical map $F^{m+1^*} F^{m+1}_{*}\mathcal{F}^{\nabla} \to \mathcal{F}$ is an isomorphism.*

Proof. — We follow the end of the proof of Proposition 5.1 in [K]. The question is local on X and we may therefore assume that we have local coordinates $t_1, \dots, t_d$. We have seen in Lemma 3.2.1 that P is a projector from $\mathcal{F}$ onto $\mathcal{F}^\nabla$. It follows that the map

$$\begin{aligned} T : \mathcal{F} &\longrightarrow F^{m+1^*} F_*^{m+1} \mathcal{F}^\nabla, \\ s &\longmapsto \sum_{\underline{k} < p^{m+1}} \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(s) \end{aligned}$$

is well defined. Let us show that T is a right inverse to the canonical map $U : F^{m+1^*} F_*^{m+1} \mathcal{F}^\nabla \to \mathcal{F}$. If $s \in \mathcal{F}$, then

$$\begin{aligned} (U \circ T)(s) &= \sum \underline{t}^{\underline{k}} P \underline{\partial}^{[\underline{k}]}(s) \\ &= \sum \underline{t}^{\underline{k}} \sum (-\underline{t})^{\underline{\ell}} \, \underline{\partial}^{[\underline{\ell}]} \, \underline{\partial}^{[\underline{k}]}(s) \\ &= \sum \sum (-1)^{\underline{\ell}} \underline{t}^{\underline{k}+\underline{\ell}} \binom{\underline{k}+\underline{\ell}}{\underline{\ell}} \underline{\partial}^{[\underline{k}+\underline{\ell}]}(s) \\ &= \sum \Big(\sum (-1)^{\underline{\ell}} \binom{\underline{j}}{\underline{\ell}} \Big) \underline{t}^{\underline{j}} \underline{\partial}^{[\underline{j}]}(s) = s. \end{aligned}$$

We have seen that $F_*^{m+1} \mathcal{O}_X^\nabla = \mathcal{O}_{X^{(m+1)}}$ and it follows that U is a bijection in the case $\mathcal{F} = \mathcal{O}_X$. Hence, T is also a left inverse to U in this case, which implies that for any $f \in \mathcal{O}_X$, we have $T(f) = f \otimes 1$. In general, we have for $f \in \mathcal{O}_X$ and $s \in \mathcal{F}^\nabla$,

$$\begin{aligned} (T \circ U)(f \otimes s) = T(fs) &= \sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(fs) \\ &= \sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(f) s \\ &= \Big(\sum \underline{t}^{\underline{k}} \otimes P \underline{\partial}^{[\underline{k}]}(f) \Big) (1 \otimes s) \\ &= T(f)(1 \otimes s) = (f \otimes 1)(1 \otimes s) = f \otimes s. \qquad \square \end{aligned}$$

3.2.3. Proposition. — *Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module, $\mathcal{F} = F^{m+1^*}\mathcal{E}$ (as $\mathcal{D}^{(m)}_{X/S}$-module) and $\eta : \mathcal{E} \to F_*^{m+1}\mathcal{F}$ be the adjunction map. Then*

(i) *The map η induces a natural isomorphism $\mathcal{E} \cong F_*^{m+1}\mathcal{F}^\nabla$ of $\mathcal{O}_{X^{(m+1)}}$-modules.*

(ii) *In the situation of Lemma 3.2.1, the action of P on $F_*^{m+1}\mathcal{F}$ factors through η.*

(iii) *If $\mathcal{F}'$ is a sub-$\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{F}$, then the natural map $F^{m+1^*} F_*^{m+1} \mathcal{F} \to \mathcal{F}$ induces an isomorphism $F^{m+1^*}(\eta^{-1}(F_*^{m+1}\mathcal{F}')) \cong \mathcal{F}'$.*

Proof. — We know from Proposition 3.1.5 (ii) that $\mathcal{F}$ has zero p-m-curvature. It follows from Proposition 3.2.2 that

$$F^{m+1^*}\mathcal{E} \cong F^{m+1^*}F_*^{m+1}\mathcal{F}^\nabla$$

and we use the faithful flatness of F to obtain assertion (i).

In order to prove assertion (ii), we recall from Lemma 3.2.1 that the image of P acting on $\mathcal{F}$ is (contained in) $\mathcal{F}^\nabla$. It therefore follows from (i) that the action of P on $F_*^{m+1}\mathcal{F}$ factors through

$$\eta : \mathcal{E} \cong F_*^{m+1}\mathcal{F}^\nabla \longrightarrow F_*^{m+1}\mathcal{F}.$$

Finally, for (iii), since $\mathcal{F}$ has zero p-m-curvature, so does $\mathcal{F}'$. The map η being functorial, it follows from (i) that it induces $\mathcal{F}'^\nabla \cong F_*^{m+1}\mathcal{F}'$ so that

$$\mathcal{F}' \cong F^{m+1^*}\mathcal{F}'^\nabla \cong F^{m+1^*}\left(\eta^{-1}(F_*^{m+1}\mathcal{F}')\right). \qquad \square$$

3.2.4. COROLLARY (Cartier's theorem). — *The functors $\mathcal{E} \mapsto F^{m+1^*}\mathcal{E}$ and $\mathcal{F} \mapsto F_*^{m+1}\mathcal{F}^\nabla$ give an equivalence between the category of $\mathcal{O}_{X^{(m+1)}}$-modules and the category of $\mathcal{D}^{(m)}_{X/S}$-modules with zero p-m-curvature.* $\square$

3.3. F^{m+1}-p-isogenies and Griffiths transversality.

We have built in section 2.3 a functor μ that associates F^{m+1}-p-isogenies to some filtered $\widehat{\mathcal{D}}^{(m)}$-modules. We are now going to define a functor α from F^{m+1}-p-isogenies to filtered $\widehat{\mathcal{D}}^{(m)}$-modules that will allow us to prove that μ is fully faithful.

The setting is as in section 2.3: S is a p-torsion free formal scheme, X is a smooth formal S-scheme, F_0 is the relative Frobenius of X_0 over S_0 and $F : X \to X'$ is a lifting of F_0. We also assume that there are local coordinates $t_1, \ldots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

If $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ is an F^{m+1}-p-isogeny on X/S, we consider the filtration M on $F^{m+1^*}\mathcal{E}$ given by

$$M^k : = \Phi^{-1}(p^k\mathcal{F})$$

and the filtration Fil on $\mathcal{E}$ given by

$$\mathrm{Fil}^k : = \eta^{-1}(F_*^{m+1}M^k),$$

where $\eta : \mathcal{E} \to F_*^{m+1}F^{m+1^*}\mathcal{E}$ is the adjunction map. We will write $\overline{\mathrm{Fil}}$ for the saturation of Fil with respect to $(p, \{\ \})$. This way, we get a functor

$$\alpha : (\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}) \longmapsto (\mathcal{E}, \overline{\mathrm{Fil}})$$

with values in the category of filtered $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules transversal to $(p, \{\ \})$.

3.3.1. Lemma. — *If*

$$P := \sum_{\underline{k}<p^{m+1}} (-\underline{t})^{\underline{k}} \underline{\partial}^{[\underline{k}]},$$

then there exists Q, *reducing to* 1 mod p, *such that*

$$P\big(F^{m+1^*}(s)\big) = F^{m+1^*}\big(Q(s)\big)$$

for any section s *of a* $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$*-module* $\mathcal{E}$.

Proof. — From Lemma 2.3.1, we deduce that

$$\underline{t}^{\underline{k}} \underline{\partial}^{[\underline{k}]}\big(F^{m+1^*}(s)\big) = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, \underline{t}^{\underline{j}p^{m+1}} F^{m+1^*}\big(\underline{\partial}^{[\underline{j}]}(s)\big) = F^{m+1^*}\big(Q_{\underline{k}}(s)\big)$$

where $Q_{\underline{k}} : = \sum p^{\underline{j}}\, a_{\underline{j},\underline{k}}\, \underline{t}^{\underline{j}} \underline{\partial}^{[\underline{j}]}$ and we let

$$Q = \sum_{\underline{k}<p^{m+1}} (-1)^{\underline{k}} Q_{\underline{k}}. \qquad \square$$

The following result is of technical nature and is needed in the next proposition:

3.3.2. Lemma. — *Let* $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ *be an* F^{m+1}*-p-isogeny on* X/S *and* M, Fil *and* η *as above. Then* $\eta_0 : \mathcal{E}_0 \to F^{m+1}_{0*}F^{m+1^*}_0\mathcal{E}_0$ *is strictly compatible with the induced filtrations (i.e. we have* $\mathrm{Fil}^k_0 = \eta_0^{-1}(F^{m+1}_{0*}M^k_0)$*).*

Proof. — We follow the proof of Theorem 2.2 of [O1]. The map is clearly compatible with the induced filtrations and we are left with proving the strictness. Let $s_0 \in \mathcal{E}_0$ be such that $\eta_0(s_0) \in F^{m+1}_{0*}M^k_0$. We want to prove that there exists a lifting $s \in \mathcal{E}$ of s_0 such that $\Phi(\eta(s)) = p^k s'$. It is clearly sufficient to show that for any i there exists a lifting $s \in \mathcal{E}$ of s_0, and u such that $\Phi(\eta(s) + p^i u) = p^k s'$ and then take $i = k$. We prove this by induction on i, the case $i = 1$ being just our assumption.

So, let us assume that $s \in \mathcal{E}$ is a lifting of s_0 such that

$$\Phi(\eta(s) + p^i u) = p^k s'.$$

Since Φ is a morphism of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules, it commutes with the operator P of the lemma. Using Lemma 3.3.1, we have

$$\begin{aligned} p^k P(s') &= P(p^k s') = P\big(\Phi(\eta(s) + p^i u)\big) \\ &= \Phi\big(P(\eta(s)) + P(p^i u)\big) = \Phi\big(\eta(Q(s)) + p^i P(u)\big). \end{aligned}$$

We have seen in Proposition 3.2.3 (ii) that the action of P on $F_0^{m+1^*}\mathcal{E}_0$ factors through $\eta_0 : \mathcal{E}_0 \to F_{0*}^{m+1}F_0^{m+1^*}\mathcal{E}_0$. We can therefore write

$$P(u) = \eta(v) + pw.$$

It follows that

$$p^k P(s') = \Phi\big(\eta(Q(s)) + p^i\eta(v) + p^{i+1}w\big) = \Phi\big(\eta(Q(s) + p^i v) + p^{i+1}w\big).$$

It just remains to observe that $Q(s) + p^i v$ is a lifting of s_0 since Q is the identity mod p. □

3.3.3. Proposition. — *Let $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be an F^{m+1}-p-isogeny on X/S, and M and* Fil *as above. Then we have $F^{m+1^*}\,\mathrm{Fil}^k = M^k$.*

Proof. — We follow the proof of Lemma 5.2.11 in [O2]. The modules $\mathcal{E}$ and $\mathcal{F}$ are p-torsion free and the filtrations Fil^k and M^k are transversal to p. From this, we deduce that the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & F^{m+1^*}\,\mathrm{Fil}^{k-1} & \xrightarrow{p} & F^{m+1^*}\,\mathrm{Fil}^k & \longrightarrow & F_0^{m+1^*}\,\mathrm{Fil}_0^k & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{k-1} & \xrightarrow{p} & M^k & \longrightarrow & M_0^k & \longrightarrow & 0
\end{array}$$

has exact rows. Hence, by induction, it is sufficient to prove that $F_0^{m+1^*}\,\mathrm{Fil}_0^k = M_0^k$ But we have seen in Proposition 3.2.3 (iii) that

$$F_0^{m+1^*}\big(\eta_0^{-1}(F_{0*}^{m+1}M_0^k)\big) = M_0^k$$

and we know from Lemma 3.3.2 that $\eta_0^{-1}(F_{0*}^{m+1}M_0^k) = \mathrm{Fil}_0^k$. □

We will show in Proposition 5.2.5 that the filtration $\overline{\mathrm{Fil}}$ in the definition of α is not always Griffiths transversal when $m > 0$. Nevertheless, for the functor μ of 2.3.5, we have the following:

3.3.4. Theorem. — *When restricted to the essential image of μ, the functor α is a quasi-inverse to μ.*

Proof. — Follows from Proposition 3.3.3. □

3.3.5. Corollary. — *The functor μ is fully faithful.* □

4. TRANSVERSAL m-CRYSTALS

4.1. m-crystals.

We recall Berthelot's theory of m-crystals from [B3].

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. If X is an S-scheme, we will always assume that the m-PD-structure of S extends to X.

4.1.1. Definition. — *If $X \hookrightarrow Y$ is an m-PD-S-immersion of S-schemes, we say that Y is an m-PD-S-thickening of X.*

4.1.2. Definition. — *Let X be an S-scheme. The m-th crystalline site of X/S is the category $\mathrm{Cris}^{(m)}(X/S)$ of m-PD-S-thickenings $U \hookrightarrow Y$ with U open in X, endowed with a suitable topology. As in the classical case, the site $\mathrm{Cris}^{(m)}(X/S)$ is functorial in X/S.*

4.1.3. *Remark.* — There exists a unique sheaf $\mathcal{I}_{X/S}^{\{n\}}$ on $\mathrm{Cris}^{(m)}(X/S)$ whose value on (Y, I, J) is $I^{\{n\}}$. We will write

$$\mathcal{O}_{X/S} : = \mathcal{I}_{X/S}^{\{0\}} \quad \text{and} \quad \mathcal{I}_{X/S} : = \mathcal{I}_{X/S}^{\{1\}}.$$

It is clear that $(\mathrm{Cris}^{(m)}(X/S), \mathcal{O}_{X/S}, \mathcal{I}_{X/S}^{\{n\}})$ is a filtered ringed site.

4.1.4. Definition. — *Let X be an S-scheme. To any sheaf E on $\mathrm{Cris}^{(m)}(X/S)$ and any object Y of $\mathrm{Cris}^{(m)}(X/S)$, one associates in the obvious way a sheaf E_Y on Y. If E is an $\mathcal{O}_{X/S}$-module, any morphism $\varphi : Y' \to Y$ of m-PD-thickenings gives a natural morphism $\varphi^* E_Y \to E_{Y'}$. We call E an m-crystal if these maps are all bijective.*

The proofs of the following statements are straightforward generalizations of those of the analogous results from [B1]. They should appear in a forthcoming article of Berthelot as announced in [B4].

4.1.5. Proposition. — *If $X \hookrightarrow Y$ is a closed immersion of S-schemes and E is an m-crystal on X, then $i_* E$ is an m-crystal on Y.*

4.1.6. Corollary. — *If $\overline{S} = \mathrm{Spec}\, \mathcal{O}_S/\mathfrak{a}$ and $\overline{X} = X \times_S \overline{S}$, then the restriction functor $\mathrm{Cris}^{(m)}(X/S) \to \mathrm{Cris}^{(m)}(\overline{X}/S)$ induces an equivalence between the categories of m-crystals on X/S and on $\overline{X}/S$.*

4.1.7. Proposition. — *Let $i : X \hookrightarrow Y$ be a closed immersion of S-schemes with Y smooth. Then the functor $E \mapsto E_Y := (i_* E)_Y$ is an equivalence of categories between m-crystals on X and locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}_{Y/S}^{(m)}$-modules.*

4.1.8. Proposition. — *Let X be a smooth formal S-scheme and let X_n denote its reduction* mod p^{n+1}. *The functor*

$$E \longmapsto E_X := \varprojlim E_{X_n}$$

is an equivalence of categories between m-crystals on X_0 and locally topologically quasi-nilpotent complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules.

4.2. T-m-Crystals.

We define T-m-crystals and relate them to differential modules. Note that we call T-m-crystals what Ogus would call proto-T-m-crystals.

Let S be a formal m-PD-scheme.

4.2.1. Proposition and Definition. — *Let $f : (U', Y') \to (U, Y)$ be a morphism of m-PD-S-thickenings such that $U' \to U$ is flat and $(\mathcal{F}, \mathrm{Fil})$ a T-module on $(Y, \mathcal{O}_Y, \mathcal{I}^{\{n\}})$. Then $Tf^*(\mathcal{F}, \mathrm{Fil}) := (f^*\mathcal{F}, \overline{\mathrm{Fil}}^k_f)$ is a T-module called the T-inverse image of $(\mathcal{F}, \mathrm{Fil})$.*

Proof. — This follows from Proposition 1.1.7 (ii) and Proposition 1.1.8. □

4.2.2. Definition. — *Let X be an S-scheme. If E is any T-module on* $\mathrm{Cris}(X/S)^{(m)}$ *and Y any object of* $\mathrm{Cris}(X/S)^{(m)}$, *then E_Y is in a natural way a T-module. If $f : Y' \to Y$ is a morphism in* $\mathrm{Cris}(X/S)^{(m)}$, *then there is a natural morphism of filtered modules $Tf^*E_Y \to E_{Y'}$. We call E a T-m-crystal if these maps are all isomorphisms of filtered modules (i.e. such that $\overline{\mathrm{Fil}}^k_f = \mathrm{Fil}^k$).*

The category of T-m-crystals is functorial with respect to flat morphisms: if $\varphi : X' \to X$ is a flat morphism and E a T-m-crystal on X/S, then

$$T\varphi^*(E, \mathrm{Fil}) : = (\varphi^* E, \overline{\mathrm{Fil}}^k_\varphi)$$

is a T-m-crystal.

4.2.3. *Example.* — The *trivial* T-m-crystal is $(\mathcal{O}_{X/S}, \mathcal{I}^{\{k\}}_{X/S})$ whose value at X is the trivial filtered module $\mathcal{O}_X = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 = 0$.

The following generalize Proposition 3.2.2 and Theorem 3.2.3 of [O2]:

4.2.4. PROPOSITION. — *If $i : X \hookrightarrow Y$ is a closed immersion into a smooth S-scheme and E a T-m-crystal on X/S, then*

$$i_*(E, \mathrm{Fil}) := (i_*E, i_* \mathrm{Fil})$$

is a T-m-crystal which is transversal to $(i_*\mathcal{I}_{X/S}, \{\ \})$.

Proof. — Same proof as [O2], 3.2.2. □

4.2.5. PROPOSITION. — *Let $i : X \hookrightarrow Y$ be a closed S-immersion into a smooth S-scheme. Then the functor $E \mapsto E_Y$ is an equivalence of categories between T-m-crystals on X and Griffiths transversal locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-modules which are transversal to the m-PD-filtration of $\mathcal{P}_{X/S(m)}(Y)$.*

Proof. — Let $p_1, p_2 : P_X(Y^2) \to P_X(Y)$ be the projections. If E is a T-m-crystal, we have an isomorphism of filtered modules

$$\varepsilon : Tp_2^* E_Y \xrightarrow{\sim} E_{Y^2} \xleftarrow{\sim} Tp_1^* E_Y\,,$$

which means that the HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ is transversal and therefore, by Proposition 2.2.6, that E_Y is Griffiths transversal.

Conversely, let $\mathcal{F}$ be a Griffiths transversal locally quasi-nilpotent $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-module which is transversal to the m-PD-filtration of $\mathcal{P}_{X/S(m)}(Y)$. There exists, by Proposition 4.1.7, a unique m-crystal E such that $E_Y = \mathcal{F}$. Let $X \hookrightarrow T$ be an m-PD-thickening. Since Y is smooth, i extends locally on T to a map $f : T \to Y$ which in turn extends to an m-PD-morphism $g : T \to P_X(Y)$. We then set

$$\mathrm{Fil}^k E_T = \overline{\mathrm{Fil}}^k_g,$$

so that $(E_T, \mathrm{Fil}) = Tg^*(\mathcal{F}, \mathrm{Fil})$. If this is well defined, it is clear that we obtain a quasi-inverse to our functor. It is actually sufficient to check that the HPD-stratification $\varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ is transversal. But this follows again from Proposition 2.2.6. □

4.2.6. COROLLARY. — *Let X be a smooth formal S-scheme. Then the functor $E \mapsto E_X$ is an equivalence of categories between T-m-crystals on X_0/S and locally topologically quasi-nilpotent Griffiths transversal complete $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules transversal to $(p, \{\ \})$.* □

4.3. T-m-crystals and F-m-spans.

We define F-m-spans and use them to describe T-m-crystals.

Let S be a formal m-PD-scheme, X a smooth S_0-scheme, and $F: X \to X'$ the relative Frobenius of X over S_0.

4.3.1. Definition. — *If (E, Fil) is a filtered m-crystal where the Fil^k are not merely sub modules but sub m-crystals, then we say that (E, Fil) is horizontal.*

Note that a horizontal filtered m-crystal is not a T-m-crystal. Let us describe the saturation process:

4.3.2. Proposition

(i) *Any horizontal filtered m-crystal (E, Fil) on X/S that is almost transversal to $(p, \{\ \})$ is almost transversal to $(\mathcal{J}_{X/S}, \{\ \})$. In particular, $(E, \overline{\mathrm{Fil}})$ is a T-m-crystal.*

(ii) *The functor $(E, \mathrm{Fil}) \to (E, \overline{\mathrm{Fil}})$ from the category of horizontal filtered m-crystals on X/S that are transversal to $(p, \{\ \})$, to the category of T-m-crystals is fully faithful.*

Proof.

(i) Let $X \hookrightarrow T$ be an m-PD-immersion and I the ideal of X in T. We have to show that (E_T, Fil) is almost transversal to $(I, \{\ \})$. This question is local on T. The scheme X being smooth over S_0, it locally lifts to a smooth formal scheme Y over S. Since Y is smooth and $X \hookrightarrow T$ is nilpotent, there exists, locally on T, a map $\varphi: T \to Y$ that induces the identity on X. The m-PD-structure on T is compatible with $(p, \{\ \})$, so that the map φ is an m-PD-morphism. Since (E_Y, Fil) is almost transversal to $(p, \{\ \})$, it follows from Proposition 1.1.8 that (E_T, Fil) is almost transversal to $(I, \{\ \})$. Applying Proposition 1.1.7 (ii), we get the last assertion.

(ii) We have to show that $\overline{\mathrm{Fil}}^k$ determines Fil^k. This is a local question on X. The scheme X being smooth over S_0, it locally lifts to a smooth formal scheme Y over S. Since (E_Y, Fil) is saturated with respect to $(p, \{\ \})$, we have $\overline{\mathrm{Fil}}^k E_Y = \mathrm{Fil}^k E_Y$. It follows from Corollary 4.2.6 that $\mathrm{Fil}^k E$ is determined by $\mathrm{Fil}^k E_Y$ and hence by $\overline{\mathrm{Fil}}^k E$. □

4.3.3. Definition. — *If (E, Fil) is in the image of this last functor, we call it a horizontal T-m-crystal.*

We are now able to globalize the local results of parts 2 and 3:

4.3.4. Proposition. — *If (E, Fil) is a T-m-crystal on $X^{(m+1)}/S$, then $TF^{m+1^*}(E, \mathrm{Fil})$ is a horizontal T-m-crystal.*

Proof. — This follows from Proposition 2.3.2 and Proposition 4.3.2 (i). □

4.3.5. Definition. — *An F-m-span is a p-isogeny $\Phi : F^{m+1^*}E \to E'$ of m-crystals.*

4.3.6. Theorem. — *Assume S has no p-torsion. Let (E, Fil) be a p-torsion free T-m-crystal on $X^{(m+1)}/S$ of width less than p^{m+1}. Then there exists a unique F-m-span $\Phi : F^{m+1^*}E \to E'$ of width less than p^{m+1} such that the saturations of $F^{m+1^*}\mathrm{Fil}^k$ and $\Phi^{-1}(p^kE')$ with respect to $(\mathcal{I}_{X/S}, \{\ \})$ coincide. This construction is functorial in (E, Fil) and the functor is fully faithful.*

Proof. — Follows from Theorem 2.3.3, Proposition 4.3.2 (ii) and Corollary 3.3.5. □

5. COMPARISON OF TRANSVERSALITY PROPERTIES FOR VARIOUS LEVELS

From now on, m' will be an integer larger than m and $\{\ \}'$ will denote divided powers of level m'. We will also write $d := m' - m$.

5.1. Changing level and Griffiths transversality.

After recalling how to obtain a $\mathcal{D}^{(m)}$-module from a $\mathcal{D}^{(m')}$-module, we show that, for filtered $\mathcal{D}^{(m')}$-modules transversal to p of width at most p^{m+1}, Griffiths transversality can be checked on the corresponding filtered $\mathcal{D}^{(m)}$-module. We give a counterexample for higher width.

5.1.1. — We recall some results from [B4].

(i) If Y is a formal scheme and I is a coherent ideal in $\mathcal{O}_Y$, then any m-PD-structure $(J, [\ ])$ on I is also an m'-PD-structure on I. If $(S, \mathfrak{a}, \mathfrak{b})$ is a formal m-PD-scheme and (Y, I, J) is a formal m-PD-S-scheme, then it is also a formal m'-PD-S-scheme. We should also remark that the m'-PD-filtration is finer than the m-PD-filtration.

(ii) Let S be a formal m-PD-scheme, X a formal S-scheme to which the m-PD-structure of S extends and $i : X \hookrightarrow Y$ an immersion into a formal

S-scheme, then there are canonical maps $P^n_{X/S(m')}(Y) \to P^n_{X/S(m)}(Y)$. They are bijective for $n < p^{m+1}$.

(iii) Assume now that X is smooth over S. Then we get canonical maps

$$\mathcal{D}^{(m)}_{X/S\,n} \longrightarrow \mathcal{D}^{(m')}_{X/S\,n}$$

that are bijective for $n < p^{m+1}$. They glue to give canonical maps $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m')}_{X/S}$ and, after completion, $\widehat{\mathcal{D}}^{(m)}_{X/S} \to \widehat{\mathcal{D}}^{(m')}_{X/S}$. We can therefore consider any $\mathcal{D}^{(m')}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m')}_{X/S}$-module) as a $\mathcal{D}^{(m)}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module).

(iv) Assume moreover that S has no p-torsion. Then one easily checks that the obvious functor from $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules to $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules is faithful. It is even fully faithful when restricted to p-torsion free objects.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. If $(\mathcal{F}, \mathrm{Fil})$ is a Griffiths transversal $\mathcal{D}^{(m')}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m')}_{X/S}$-module), then it is also Griffiths transversal as a $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module (resp. $\widehat{\mathcal{D}}^{(m)}_{X/S}$-module).

The converse is true under some additional hypothesis:

5.1.2. Proposition. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{D}^{(m')}_{X/S}$-module of width at most p^{m+1} that is Griffiths transversal as a $\mathcal{D}^{(m)}_{X/S}$-module and transversal to p. Then it is also Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module.*

Proof. — We have to show that, if $P \in \mathcal{D}^{(m')}_{X/S}$ is an m'-PD-differential operator of order at most n, then $P(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-n}$. Thanks to 5.1.1 (iii), we may assume that $n \geq p^{m+1}$. We proceed by induction on k.

- If $k \leq p^{m+1}$, then $\mathrm{Fil}^{k-n} = \mathcal{F}$ and our assertion is trivial.
- If $k > p^{m+1}$, transversality to p and the condition on the width give us that $\mathrm{Fil}^k = p\,\mathrm{Fil}^{k-1}$. It follows that

$$P(\mathrm{Fil}^k) = pP(\mathrm{Fil}^{k-1}) \subset p\,\mathrm{Fil}^{k-1-n} \subset \mathrm{Fil}^{k-n}. \qquad \square$$

The bound on the width is sharp as the following shows:

5.1.3. *Example.* — We take X to be the affine line over S and we consider $(\mathcal{F}, \mathrm{Fil})$ where $\mathcal{F} = \mathcal{O}_X$ and Fil is defined as follows:

- for $0 \leq k \leq p^{m+1}$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;
- for $k > p^{m+1}$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p^{k-p^{m+1}-1}t^{p^{m+1}}$.

It is clear that $(\mathcal{F}, \mathrm{Fil})$ is a filtered $\mathcal{D}^{(m')}_{X/S}$-module of width $p^{m+1} + 1$. It is transversal to p because, for $k \leq p^{m+1}$, both $(p) \cap \mathrm{Fil}^k$ and $p\,\mathrm{Fil}^{k-1}$ are generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k-1$, together with $p\,t^k$, while $(p) \cap \mathrm{Fil}^{p^{m+1}+1}$ and $p\,\mathrm{Fil}^{p^{m+1}}$ are generated by the elements $p^{p^{m+1}+1-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p\,t^{p^{m+1}}$.

To show that $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal as a $\mathcal{D}^{(m)}_{X/S}$-module, let us remark that

$$\partial^{[r]}(p^{k-i}t^i) = \begin{cases} \binom{i}{r} p^{k-i}t^{i-r} & \text{if } r \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\partial^{[r]}(\mathrm{Fil}^k) \subset \mathrm{Fil}^{k-r}$ when $0 \leq k \leq p^{m+1}$. Moreover, when $r \leq p^m$, we have $\binom{p^{m+1}}{r} \in (p)$ and therefore

$$\partial^{[r]}(\mathrm{Fil}^{p^{m+1}+1}) \subset p\,\mathrm{Fil}^{p^{m+1}-r} \subset \mathrm{Fil}^{p^{m+1}+1-r}.$$

Nevertheless, $(\mathcal{F}, \mathrm{Fil})$ is not Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module because $t^{p^{m+1}} \in \mathrm{Fil}^{p^{m+1}+1}$ but $\partial^{[p^{m+1}]}(t^{p^{m+1}}) = 1 \notin \mathrm{Fil}^1$.

5.2. Frobenius descent and F^{m+1}-p-isogenies.

We are going to apply Berthelot's theory of Frobenius descent to F^{m+1}-p-isogenies and use it to study the question of the surjectivity of the functor μ of 2.3.5.

Let S be a formal m-PD-scheme and X a smooth formal S-scheme to which the m-PD-structure of S extends. Let F_0 be the relative Frobenius of X_0 over S_0 and $F: X \to X'$ a lifting of F_0. We briefly recall Berthelot's unpublished theory of Frobenius descent.

5.2.1. Proposition (see [B5]). — *The morphism*

$$F^d \times_S F^d : X \times_S X \longrightarrow X^{(d)} \times_S X^{(d)}$$

induces for all n, a unique morphism

$$F^d : P^n_{X/S(m')} \longrightarrow P^n_{X^{(d)}/S(m)}$$

compatible with the PD-structures (taking into account the PD-ideal of S). It is also compatible with the partial divided power filtrations.

It follows that, if $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(d)}/S}$-module, then $F^{d^*}(\mathcal{E})$ has a natural structure of $\mathcal{D}^{(m')}_{X/S}$-module.

5.2.2. THEOREM (see [B5]). — *If S is a scheme, the functor $\mathcal{E} \mapsto F^{d^*}(\mathcal{E})$ induces an equivalence between the categories of $\mathcal{D}^{(m)}_{X^{(d)}/S}$-modules and $\mathcal{D}^{(m')}_{X/S}$-modules.*

It follows that the functor $\mathcal{E} \mapsto F^{d^*}(\mathcal{E})$ induces an equivalence between the category of complete $\widehat{\mathcal{D}}^{(m)}_{X^{(d)}/S}$-modules and the category of complete $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules. From Proposition 1.2.2, we get an equivalence between the category of p-isogenies of complete $\widehat{\mathcal{D}}^{(m)}_{X^{(d)}/S}$-modules and the category of p-isogenies of complete $\widehat{\mathcal{D}}^{(m')}_{X/S}$-modules. Thus, we get:

5.2.3. COROLLARY. — *The functor F^{d^*} makes the full subcategory of F^{m+1}-p-isogenies on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ is a $\widehat{\mathcal{D}}^{(m')}_{X^{(m'+1)}/S}$-module equivalent to the category of $F^{m'+1}$-p-isogenies on X/S.*

5.2.4. LEMMA. — *Let $(\mathcal{F}, \mathrm{Fil})$ be a filtered $\mathcal{D}^{(m)}_{X/S}$-module of width less than p^{m+1} that is transversal to p and $\overline{\mathrm{Fil}}$ the saturation of* Fil *with respect to $(p, \{\ \})$. Then $(\mathcal{F}, \mathrm{Fil})$ is Griffiths transversal if and only if $(\mathcal{F}, \overline{\mathrm{Fil}})$ is Griffiths transversal.*

Proof. — The filtrations are identical up to order $(p^{m+1} - 1)$ and, for any $k \geq 0$, we have

$$\mathrm{Fil}^{p^{m+1}-1+k} = p^k\, \mathrm{Fil}^{p^{m+1}-1} \quad \text{and} \quad \overline{\mathrm{Fil}}^{p^{m+1}-1+k} = (p)^{\{k\}}\overline{\mathrm{Fil}}^{p^{m+1}-1}. \qquad \square$$

Assume now that S is a p-torsion free formal PD-scheme and that there are local coordinates $t_1, \ldots, t_d$ on X and X' such that $F^*(t_i) = t_i^p$.

5.2.5. PROPOSITION. — *The functor μ of* 2.3.5 *is not in general an equivalence of categories for $m > 0$. However, it becomes an equivalence when restricted to objects of width at most p.*

Proof. — Let $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be an F^{m+1}-p-isogeny on X/S of width less than p^{m+1}. By Corollary 5.2.3, it corresponds to a unique F-p-isogeny $\Phi^0 : F^*\mathcal{E} \to \mathcal{F}'$ on $X^{(m)}/S$. We have shown in section 3.3 how to associate to Φ^0 a filtration Fil on $\mathcal{E}$ that is transversal to p. Thanks to

Proposition 3.3.3 and [O2], 5.2.12, the filtered module $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal as a $\widehat{\mathcal{D}}^{(0)}_{X^{(m+1)}/S}$-module. It follows from Lemma 5.2.4 and Proposition 3.3.3 that Φ will be in the essential image of μ if and only if $(\mathcal{E}, \mathrm{Fil})$ is Griffiths transversal as a $\widehat{\mathcal{D}}^{(m)}_{X^{(m+1)}/S}$-module. If the width is at most p this is always the case by Proposition 5.1.2, while Example 5.1.3 shows that this needs not happen for higher width. □

5.2.6. *Example.* — For $m > 0$, we can give an explicit F^{m+1}-p-isogeny of width less than p^{m+1} on the formal affine line X which is not in the essential image of μ. We take $\mathcal{E} = \mathcal{O}_{X^{(m+1)}}$ and we let $\mathcal{F}$ be the ideal of $\mathcal{O}_X$ generated by the elements $p^{p+1-i}t^{ip^{m+1}}$ for $0 \leq i \leq p-1$, together with $t^{p^{m+2}}$. It is a sub $\widehat{\mathcal{D}}^{(m)}$-module of $\mathcal{O}_X$ and we let the p-isogeny $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be multiplication by p^{p+1}. If we apply the functor α to this F^{m+1}-p-isogeny, we get the saturation of the following filtration:

- for $0 \leq k \leq p$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;
- for $k > p$, Fil^k is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p-1$, together with $p^{k-p-1}t^p$.

It is not Griffiths transversal because $t^p \in \mathrm{Fil}^{p+1}$ but $\partial^{[p]}(t^p) = 1$ is not in Fil^1 and we can use Lemma 5.2.4. □

5.2.7. *Remark.* — Let $\Phi : F^*\mathcal{E} \to \mathcal{F}$ and $\Phi' : F^*\mathcal{F} \to \mathcal{G}$ be two F-p-isogenies of width less than p. From [O2], 5.2.13, or Proposition 5.2.5, they are in the essential image of the fuctor μ for level 0. Assume that $\mathcal{E}$ and $\mathcal{G}$ are $\widehat{\mathcal{D}}^{(1)}$-modules and that $\Phi' \circ F^*(\Phi) : F^{2^*}\mathcal{E} \to \mathcal{G}$ is a morphism of $\widehat{\mathcal{D}}^{(1)}$-modules. Then it is an F^2-p-isogeny of width less than $(2p-1)$, and one may wonder if it is in the essential image of μ. One can show that this is true if $p = 2$, but if $p > 2$ the answer is no in general as the following example on the formal affine line shows:

We take $\mathcal{E} = \mathcal{O}$, we let $\mathcal{F}$ be the ideal of $\mathcal{O}$ generated by p^2, pt^p and t^{2p}, and $\mathcal{G}$ be the ideal of $\mathcal{O}$ generated by the elements $p^{p+1-i}t^{ip^2}$ with $0 \leq i \leq p-1$, together with t^{p^3}. The p-isogenies Φ and Φ' are multiplication by p^2 and p^{p-1}, respectively. The composition of $F^*(\Phi)$ and Φ is Example 5.2.6 in the case $m = 1$. □

5.3. Changing level for T-m-crystals and F-m-spans.

We study the behavior of the functors relating T-m-crystals and F-m-spans when the level changes and derive some consequences.

5.3.1. Lemma. — *The functor «saturation with respect to* $(p, \{\ \})$*» from the category of filtered modules transversal to* $(p, \{\ \}')$ *to the category of filtered modules transversal to* $(p, \{\ \})$ *gives an equivalence of categories when restricted to objects of width less than* p^{m+1}.

Proof. — This is an immediate consequence of Proposition 1.2.5. □

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a formal m-PD-scheme. If X is an S-scheme, it follows from 5.1.1 (i) that $\mathrm{Cris}^{(m)}(X/S)$ is a subsite of $\mathrm{Cris}^{(m')}(X/S)$. By restriction, any sheaf on $\mathrm{Cris}^{(m')}(X/S)$ defines a sheaf on $\mathrm{Cris}^{(m)}(X/S)$. The m'-PD-filtration restricts to a filtration on the structural sheaf $\mathcal{O}_{X/S}^{(m)}$ of $\mathrm{Cris}^{(m)}(X/S)$ that is finer than the m-PD-filtration.

Using restriction and then saturation with respect to the m-PD-filtration, any T-module E on $\mathrm{Cris}(X/S)^{(m')}$ defines a T-module on $\mathrm{Cris}(X/S)^{(m)}$. It is clear that this process is functorial and that, when applied to T-m'-crystals, it produces T-m-crystals.

Assume from now on that S has no p-torsion and that X is a smooth S_0-scheme.

5.3.2. Proposition. — *Consider the functor that associates a T-m-crystal to a T-m'-crystal. Restricted to p-torsion free T-m'-crystals of width less than p^{m+1}, it is fully faithful and its essential image is the full subcategory of p-torsion free T-m-crystals of width less than p^{m+1} whose underlying crystal is the restriction of an m'-crystal.*

Proof. — This is a local question and all our constructions are functorial. Using Corollary 4.2.6 and Lemma 5.3.1, the first assertion is a consequence of 5.1.1 (iv) and the second follows from Proposition 5.1.2. □

Let $F: X \to X'$ be the relative Frobenius of X over S_0. We will write $(X/S)_{\mathrm{cris}}^{(m)}$ for the crystalline topos of level m. In [B3] Berthelot shows that the morphism of crystalline topoi of level m induced by F^d factors canonically through the restriction map $(X/S)_{\mathrm{cris}}^{(m)} \to (X/S)_{\mathrm{cris}}^{(m')}$ to give a morphism

$$F^d : (X/S)_{\mathrm{cris}}^{(m')} \longrightarrow (X^{(d)}/S)_{\mathrm{cris}}^{(m)}.$$

Under the equivalence of Corollary 4.1.8, this construction is compatible with that of Proposition 5.2.1.

5.3.3. Proposition. — *The functor F^{d^*} makes the full subcategory of F-m-spans on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1^*}E \to E'$ where E is an m'-crystal on $X^{(m'+1)}/S$ equivalent to the category of F-m'-spans on X/S.*

Proof. — This is a again a local question. Using Corollary 4.2.6, the assertion reduces to Proposition 5.2.3. □

5.3.4. *Remark.* — When restricted to objects of width less than p^{m+1}, we have commutative diagrams:

$$\begin{tikzcd}
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m'\text{-}\textit{crystals}\\ \textit{on } X^{(m'+1)}/S\end{array} \arrow[r, hook] \arrow[d, hook] & F\text{-}m'\text{-}\textit{spans on } X/S \arrow[d, hook] \\
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m\text{-}\textit{crystals}\\ \textit{on } X^{(m'+1)}/S\end{array} \arrow[r, hook] & F\text{-}m\text{-}\textit{spans on } X^{(d)}/S
\end{tikzcd}$$

where the horizontal arrows come from Theorem 4.3.6 and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3; and, when S is a PD-scheme:

$$\begin{tikzcd}
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}m\text{-}\textit{crystals}\\ \textit{on } X^{(m+1)}/S\end{array} \arrow[r, hook] \arrow[d] & F\text{-}m\text{-}\textit{spans on } X/S \arrow[d, hook] \\
\begin{array}{l}\textit{p-torsion free}\\ T\text{-}0\text{-}\textit{crystals}\\ \textit{on } X^{(m+1)}/S\end{array} \arrow[r, hook] & F\text{-}0\text{-}\textit{spans on } X^{(m)}/S
\end{tikzcd}$$

where the top arrow comes from Theorem 4.3.6, the bottom one from Theorem 5.2.13 of [O2] and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3.

5.3.5. Proposition. — *The construction of* 4.3.6 *does not give an equivalence of categories in general. However, if S is a PD-scheme, it becomes an equivalence when restricted to objects of width at most p.*

Proof. — Follows from Corollary 4.2.6 and Proposition 5.2.5. □

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