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TRANSVERSAL CRYSTALS OF FINITE LEVEL

by B. LE STUM and A. QUIRÓS (*)

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INTRODUCTION


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Key words: Griffiths transversality - Isogeny - Divided power of finite level - Differential operator - Frobenius structure.
introduces the notion of \( T \)-crystal (\( T \) for *transversal*), which provides an excellent context to study this kind of questions. He uses it to prove a version of Mazur's theorem on the relation between the action of Frobenius and the Hodge filtration on crystalline cohomology which is valid for cohomology with coefficients in an \( F \)-crystal. As applications, he gets results about Newton and Hodge polygons (Katz conjecture) and degeneration of the Hodge spectral sequence. One of his key results shows that there is an equivalence between \( F \)-spans and \( T \)-crystals, provided we restrict to objects of width less than \( p \).

In his letter to Illusie [B3], Berthelot develops the theory of crystals of level \( m \). We use this new theory to extend Ogus' theorem to objects of width less than \( p^{m+1} \): after defining \( T \)-\( m \)-crystals and \( F \)-\( m \)-spans, we show that one can identify \( T \)-\( m \)-crystals of width less than \( p^{m+1} \) with a full subcategory of \( F \)-\( m \)-spans.

More precisely: let \( S \) be a torsion free \( p \)-adic formal scheme, \( S_0 \) its reduction mod \( p \) and \( X \) a smooth \( S_0 \)-scheme. A \( T \)-\( m \)-crystal on \( X/S \) is a crystal \( E \) of level \( m \) with a filtration \( \mathrm{Fil} \) by submodules which after saturation (see Definition 1.1.6), behaves like a filtration by subcrystals. If \( F: X \to X' \) is the relative Frobenius of \( X/S_0 \), an \( F \)-\( m \)-span is a \( p \)-isogeny \( \Phi: F^{m+1} E \to E' \) of \( p \)-torsion free \( m \)-crystals. We prove (Theorem 4.3.6) that if \( (E, \mathrm{Fil}) \) is a \( p \)-torsion free \( T \)-\( m \)-crystal on \( X/S \) such that \( \mathrm{Fil}^{p^{m+1}} \subset pE \), then there exists a unique \( F \)-\( m \)-span \( \Phi: F^{m+1} E \to E' \) such that, up to saturation, \( F^{m+1} \mathrm{Fil} \) coincides with the filtration \( M \) defined by \( M^k := \Phi^{-1}(p^k E') \). This construction is functorial in \( (E, \mathrm{Fil}) \) and the functor is fully faithful.

In order to prove this theorem, we consider a lifted situation: \( X \) is a smooth formal \( S \)-scheme, \( F_0 \) is the relative Frobenius of \( X_0 \) over \( S_0 \), \( F: X \to X' \) is a lifting of \( F_0 \) and we assume that there are coordinates \( t_1, \ldots, t_d \) on \( X \) and \( X' \) such that \( F(t_i) = t_i^p \). Then \( T \)-\( m \)-crystals correspond to Griffiths transversal \( \hat{\mathcal{D}}^{(m)}_{X/S} \)-modules that are also transversal to the \( m \)-PD-ideal \( (p) \) and \( F \)-\( m \)-spans correspond to \( p \)-isogenies of \( \hat{\mathcal{D}}^{(m)}_{X/S} \)-modules. We prove the theorem in this local situation (Theorem 2.3.3 and Corollary 3.3.5).

Let us briefly describe the structure of this paper: in the first part, we recall Ogus' notion of transversality and Berthelot's notion of partial divided power structures as well as some properties of \( p \)-isogenies in this context. In the second part, we first recall Berthelot's theory of differential operators of finite level, we define Griffiths transversality for \( \mathcal{D}^{(m)} \)-modules
and we build the local version of our functor. In the third part, we define and study $p$-$m$-curvature for $\mathcal{D}(m)$-modules in characteristic $p$ and we use this notion to prove the full faithfulness of our functor in a local situation. In the fourth part, we recall Berthelot’s theory of $m$-crystals, we define $T$-$m$-crystals and $F$-$m$-spans and we deduce our main theorem from its local version. In the fifth and last part, we study the behavior of $T$-$m$-crystals and $F$-$m$-spans when $m$ varies and use it to show that our results provide some improvement on Ogus’ theory.

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Conventions. — We let $p$ be a non zero prime and $m \in \mathbb{N}$. All formal schemes are $p$-adic formal schemes. All schemes are locally killed by some power of $p$ and might hence be considered as formal schemes. Also, all PD-structures are compatible with $p$. We will use the subindex 0 to indicate reduction mod $p$. We will adopt the standard multiindex notation, and if $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we will write $|k| = k_1 + \cdots + k_d$.

1. PRELIMINARIES

1.1. Transversal filtrations.

We briefly recall the notion of a transversal module from [O2]. We call transversal what Ogus calls $G$-transversal and almost transversal what he calls $G'$-transversal. Let us first fix some terminology and notations:

1.1.1. Definition. — Let $A$ be a ring (in a topos). A module filtration $\text{Fil}$ on an $A$-module $M$ is a decreasing filtration by submodules $\text{Fil}^k$ such that there exists an integer $a$ such that $\text{Fil}^a = M$. It is called effective if we can take $a = 0$. In general, if we set $\text{Fil}[r]^k := \text{Fil}^{k+r}$, we see that $\text{Fil}[a]$ is an effective filtration on $M$. If $\varphi : (\mathcal{T}, A') \to (\mathcal{T}, A)$ is a morphism of ringed sites, $(M, \text{Fil})$ is a filtered $A$-module and $\text{Fil}^k_\varphi$ denotes the image of $\varphi^*\text{Fil}^k$ in $\varphi^*M$, then $\varphi^*(M, \text{Fil}) := (\varphi^*M, \text{Fil} \varphi)$ is called the inverse image of $(M, \text{Fil})$. 

In this article, in order to simplify the notations, we will only consider effective filtrations.

1.1.2. Definition. — A ring filtration on a ring $A$ is a module filtration $I(\ast)$ such that $I^{(k)}I^{(\ell)} \subseteq I^{(k+\ell)}$. If $(A, I(\ast))$ is a filtered ring, we set $I := I^{(1)}$ and we say that a filtered module $(M, \text{Fil})$ has width at most $w$ (with respect to $I$) if there exists an integer $a$ such that $\text{Fil}^a = M$ and $\text{Fil}^{a+w+1} \subseteq IM$. A filtered ringed site $(\mathcal{T}, A, I(\ast))$ is a site endowed with a filtered ring. A morphism of filtered ringed sites $$\varphi : (\mathcal{T}', A', I'(\ast)) \longrightarrow (\mathcal{T}, A, I(\ast))$$ is a morphism of ringed sites such that $\varphi^*I^{(k)}$ maps into $I'^{(k)}$ for all $k$.

1.1.3. Definition. — A filtered module $(M, \text{Fil})$ in a filtered ringed site $(\mathcal{T}, A, I(\ast))$ is transversal (a $T$-module for short) if it satisfies
$$IM \cap \text{Fil}^k = I^{(1)} \text{Fil}^{k-1} + I^{(2)} \text{Fil}^{k-2} + I^{(3)} \text{Fil}^{k-3} + \cdots$$
for all $k$. It is almost transversal if
$$IM \cap \text{Fil}^k \subseteq I^{(1)} \text{Fil}^{k-1} + I^{(2)} \text{Fil}^{k-2} + I^{(3)} \text{Fil}^{k-3} + \cdots$$
for all $k$ and saturated if $I^{(k)} \text{Fil}^\ell \subseteq \text{Fil}^{\ell+k}$ for all $k, \ell$.

Since there will sometimes be several ring filtrations involved, we will, if necessary, say (almost) transversal to $I(\ast)$ and saturated with respect to $I(\ast)$. If $I^{(k)} = I^k$ for all $k$, we will just say (almost) transversal to $I$ and saturated with respect to $I$.

1.1.4. Example. — A filtered module $(M, \text{Fil})$ in a ringed site $(\mathcal{T}, A)$ is transversal to an ideal $I$ of $A$ if and only if it satisfies $IM \cap \text{Fil}^k = I^{(1)} \text{Fil}^{k-1}$ for all $k$.

1.1.5. Remark. — A filtered module is transversal if and only if it is almost transversal and saturated.

Starting from any almost transversal filtration, there exists a natural process that turns it into a transversal one:

1.1.6. Definition. — If $(M, \text{Fil})$ is a filtered module on a filtered ringed site $(\mathcal{T}, A, I(\ast))$, we set
$$\overline{\text{Fil}}^k = \text{Fil}^k + I^{(1)} \text{Fil}^{k-1} + I^{(2)} \text{Fil}^{k-2} + I^{(3)} \text{Fil}^{k-3} + \cdots.$$ 
We call $(M, \overline{\text{Fil}})$ the saturation of $(M, \text{Fil})$. 
1.1.7. **Proposition** (see [O2], 2.3.1).

(i) The filtration $\overline{\text{Fil}}$ is the finest filtration on $M$ that is saturated and coarser than the given one.

(ii) If $(M, \text{Fil})$ is almost transversal, then its saturation is transversal.

This saturation process is specially useful in view of the following result:

1.1.8. **Proposition** (see [O2], 2.2.1). — Let $(\mathcal{T}', A', I'(\ast))$ be a morphism of filtered ringed sites such that the natural map $\varphi^{-1}A/I \rightarrow A'/I'$ is flat. If $(M, \text{Fil})$ is an almost transversal module, then so is $\varphi^*(M, \text{Fil})$.

1.2. **$p$-isogenies.**

We introduce the $m$-PD-filtration $(p, \{ \})$ and we describe transversality with respect to this filtration in terms of $p$-isogenies.

1.2.1. **Definition.** — If $A$ is a $\mathbb{Z}(p)$-algebra and $M, M'$ two $p$-torsion free $A$-modules, a $p$-isogeny $\Phi: M \rightarrow M'$ of width at most $w$ is an injective homomorphism $\Phi: M \rightarrow M' \otimes \mathbb{Q}$ of $A$-modules such that there exists an integer $a$ such that $p^a + w + 1 M' \subset \Phi(M) \subset p^a M'$. It is called effective if one can take $a = 0$. In general, if we set $\Phi[r] = p^{-r} \Phi$, we see that $\Phi[a]$ is effective.

As we do for filtrations, we will only consider effective $p$-isogenies.

Transversality with respect to $p$, meaning to the ideal $(p)$, has a very nice interpretation in terms of $p$-isogenies:

1.2.2. **Proposition** (see [O2], 5.1.2). — The functor $\Phi \mapsto (M, \text{Fil})$, where $\text{Fil}^k = \Phi^{-1}(p^k M')$, is an equivalence from the category of $p$-isogenies of width at most $w$ onto the category of filtered modules transversal to $p$ of width at most $w$.

Actually, the filtration that will naturally appear in the sequel is not $(p)^k$ but the $m$-PD-filtration defined below (and generalized in Definition 1.3.4).

1.2.3. **Definition.** — For $k = qp^m + r$ with $0 \leq r < p^m$, we let $p^{(k)} := p^k/q!$. The $m$-PD-filtration $(p)^{(k)}$ on a $\mathbb{Z}_p$-algebra $A$ is the finest...
ring filtration such that $p^{(k)} \in (p)^{(k)}$. We will also write $(p, \{ \})$ for this filtration.

In the sequel, we will also need the notion of modified binomial coefficients. Let us recall what they are:

1.2.4. **Definition.** — If $k'$ and $k'' \in \mathbb{N}^d$, and

$$
k' = q' p m + r', \quad 0 \leq r' < pm,
$$

$$
k'' = q'' p m + r'', \quad 0 \leq r'' < pm,
$$

$$
k = k' + k'' = q p m + r, \quad 0 \leq r < pm,
$$

one sets:

$$\binom{k}{k'} := \frac{q !}{q' ! q'' !} \in \mathbb{N} \quad \text{and} \quad \langle \frac{k}{k'} \rangle := \left( \binom{k}{k'} \right)^{-1} \in \mathbb{Z}_p.$$

Proposition 1.2.2 is still valid for the $m$-PD-filtration under some assumptions on the width:

1.2.5. **Proposition** (see [O2], 2.3.5). — The functor «saturation with respect to $(p, \{ \})$» from the category of filtered modules transversal to $p$ to the category of filtered modules transversal to $(p, \{ \})$ is an equivalence of categories when restricted to objects of width less than $p^{m+1}$.

1.2.6. **Corollary.** — The functor $\Phi \mapsto (M, \text{Fil})$ where $\text{Fil}^k$ is the saturation of $\Phi^{-1}(p^k M')$ with respect to $(p, \{ \})$ is an equivalence from the category of $p$-isogenies of width less than $p^{m+1}$ onto the category of filtered modules transversal to $(p, \{ \})$ of width less than $p^{m+1}$.

1.3. **$m$-PD-structures.**

We recall Berthelot’s theory of partial divided powers from [B4] which generalizes the usual divided power structures in [B1].

1.3.1. **Definition.** — Let $Y$ be a formal scheme. An $m$-PD-structure on a coherent ideal $I$ in $\mathcal{O}_Y$ is the data of a PD-ideal $(J, [\ ])$ in $I$ such that $I^{(p^m)} + pI \subset J$ (where $I^{(p^m)}$ is the ideal locally generated by $f^{p^m}$ with $f \in I$). We say that $I$ is an $m$-PD-ideal or that $(Y, I, J)$ is a formal $m$-PD-scheme. We will drop $J$, or even $I$, from the notations when no confusion should arise. If $f \in I$ and $k = q p m + r$ with $0 \leq r < p^m$, we write

$$f^{(k)} := f^r (fp^m)^{[q]}.$$
1.3.2. **Definition.** — Let $(S, a, b)$ be a formal $m$-PD-scheme. The $m$-PD-structure on $a$ extends to a formal $S$-scheme $X$ if the PD-structure on $b$ extends to a PD-structure on $X$ (compatible with $p$). An $m$-PD-structure $(I, J)$ on a formal $S$-scheme $Y$ is said to be compatible with $(S, a, b)$ if the $m$-PD-structure on $a$ extends to $Y$, the PD-structure on $J + (p)$ is compatible with the PD-structure on $b + (p)$ and $I \cap (b \mathcal{O}_Y + (p))$ is a sub PD-ideal of $b \mathcal{O}_Y + (p)$. We then say that $(Y, I, J)$ is a formal $m$-PD-$S$-scheme.

1.3.3. **Definition.** — Let $(S, a, b)$ be a formal $m$-PD-scheme. A morphism of formal $m$-PD-$S$-schemes is a morphism of formal schemes $\varphi: Y' \to Y$ such that $\varphi^{-1}(I) \subset I'$ and $(Y', J') \to (Y, J)$ is a morphism of formal PD-schemes. If $(Y, I, J)$ is a formal $m$-PD-$S$-scheme and $X$ is the closed formal subscheme of $Y$ defined by $I$, we say that $X \hookrightarrow Y$ is an $m$-PD-immersion.

The following generalizes Definition 1.2.3 and agrees with Berthelot’s new definition that replaces [B4] 1.3.8 and 1.3.7.

1.3.4. **Proposition and Definition** (see [B5]). — If $(Y, I, J)$ is a formal $m$-PD-$S$-scheme, then there exists a finest ring filtration $(I, \{ \}) := I^{(\ast)}$ on $\mathcal{O}_Y$ such that

(i) $I^{(1)} = I,$

(ii) $I^{(n)} \cap (J + b \mathcal{O}_Y + p \mathcal{O}_Y)$ is a sub PD-ideal of $J + b \mathcal{O}_Y + p \mathcal{O}_Y,$

(iii) $x^{(h)} \in I^{(nh)}$ whenever $x \in I^{(n)}.$

It is called the $m$-PD-filtration on $\mathcal{O}_Y$ with respect to $(I, J).$ Then $(Y, \mathcal{O}_Y, I^{(n)})$ is a filtered ringed site. Moreover, any morphism of formal $m$-PD-$S$-schemes induces a morphism of the corresponding filtered ringed sites.

Universal $m$-PD-immersions do exist:

1.3.5. **Proposition and Definition** (see [B4], 2.1.1). — Let $S$ be a formal $m$-PD-scheme, $X$ a formal $S$-scheme to which the $m$-PD-structure of $S$ extends and $i: X \hookrightarrow Y$ an immersion into a formal $S$-scheme. Then $i$ factors as an $m$-PD-$S$-immersion $X \hookrightarrow P^n_{X/S(m)}(Y)$ followed by a morphism $\varphi: P^n_{X/S(m)}(Y) \to Y$ having the following universal property: any morphism $Y' \to Y$ inducing $X' \to X$, where $X' \hookrightarrow Y'$ is an $m$-PD-$S$-immersion whose ideal satisfies $I^{(n+1)} = 0$, factors uniquely through $\varphi.$
We say that $P_{X/S(m)}^n(Y)$ is the $n$-th $m$-PD-neighborhood of $X$ in $Y$ and we write $\mathcal{P}_{X/S(m)}^n(Y)$ for its structural sheaf.

1.3.6. Remark. — If $X \hookrightarrow Y$ is an immersion of schemes (locally killed by a power of $p$) then there exists an $m$-PD-$S$-immersion $X \hookrightarrow P_{X/S(m)}(Y)$ with the same universal property but without nilpotency condition on $I$. We call $P_{X/S(m)}(Y)$ the $m$-PD-neighborhood of $X$ in $Y$, and write $\mathcal{P}_{X/S(m)}^n(Y)$ for its structural sheaf.

1.3.7. Definition. — If $i$ is the diagonal immersion $X \hookrightarrow Y := X \times_S X$, then we drop $Y$ from the notations in 1.3.5 and 1.3.6 and we call $\mathcal{P}_{X/S}^n(m)$ the sheaf of $m$-th principal parts of order at most $n$.

2. DIFFERENTIAL OPERATORS OF LEVEL $m$ AND GRIFFITHS TRANSVERSALITY

2.1. Differential operators of level $m$.

We will now recall from [B4] Berthelot's theory of differential operators of finite level.

Let $(S, a, b)$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme to which the $m$-PD-structure of $S$ extends. We consider $\mathcal{P}_{X/S(m)}^n$ as an $\mathcal{O}_X$-module using the first projection $X \times_S X \rightarrow X$ and we note $\theta: \mathcal{O}_X \rightarrow \mathcal{P}_{X/S(m)}^n$ the map induced by the second projection. We first recall the definition of differential operators of level $m$:

2.1.1. Definition. — The $\mathcal{O}_X$-dual $\mathcal{D}_{X/S}^{(m)}$ to $\mathcal{P}_{X/S(m)}^n$ is called the sheaf of differential operators of level $m$ and order at most $n$. The natural maps $\mathcal{P}_{X/S(m)}^{n'} \rightarrow \mathcal{P}_{X/S(m)}^n$ for $n \leq n'$ induce injections $\mathcal{D}_{X/S}^{(m)} \hookrightarrow \mathcal{D}_{X/S}^{(m)}$ and we set

$$\mathcal{D}_{X/S}^{(m)} = \bigcup_n \mathcal{D}_{X/S}^{(m)}.$$ 

Moreover, the natural maps

$$\mathcal{P}_{X/S(m)}^{n+n'} \rightarrow \mathcal{P}_{X/S(m)}^n \otimes \mathcal{P}_{X/S(m)}^{n'}$$

induce bilinear maps

$$\mathcal{D}_{X/S}^{(m)} \times \mathcal{D}_{X/S}^{(m)} \rightarrow \mathcal{D}_{X/S}^{(m+n+n')}$$

which make $\mathcal{D}_{X/S}^{(m)}$ into a ring called the ring of differential operators of level $m$. Its $p$-adic completion will be denoted by $\widehat{\mathcal{D}}_{X/S}^{(m)}$. 
2.1.2. Remark. — If $t_1, \ldots, t_d$ are local coordinates on $X$ and
\[ \tau_i := \theta(t_i) - t_i \quad \text{for all } i, \]
then $\mathcal{P}_{X/S(m)}^n$ is a free $\mathcal{O}_X$-module on the $\{\tau_i^k\}$ with $|k| \leq n$.

We let $\{\partial^{(k)}\}$ be the dual basis to $\{\tau_i^k\}$ in $\mathcal{D}_{X/S}^{(m)}$.

If $k = q^p^n + r < p^{m+1}$, we set
\[ \partial^{[k]} = \partial^{(k)}/q^r. \]
If $n < p^{m+1}$, then the $\tau_i^k$ with $|k| \leq n$ form a basis for $\mathcal{P}_{X/S(m)}^n$ and the $\partial^{[k]}$ form the dual basis in $\mathcal{D}_{X/S}^{(m)}$. Note that $\mathcal{D}_{X/S}^{(m)}$ is generated as an $\mathcal{O}_X$-algebra by the $\partial_i^{[p]} = \partial_i^{(p)}$ for $j \leq m$.

2.1.3. Remark. — If $\varphi : Y \to X$ is a morphism of smooth formal $S$-schemes and $\mathcal{F}$ is a $\mathcal{D}_{X/S}^{(m)}$-module then $\varphi^*\mathcal{F}$ has a natural structure of $\mathcal{D}_{Y/S}^n$-module that can be described locally as follows. Let $t_1, \ldots, t_d$ be local coordinates on $X$, $t'_1, \ldots, t'_d$ be local coordinates on $Y$ and $\{\tau_i\}$ and $\{\tau'_i\}$ be the corresponding sections of $\mathcal{P}_{X/S(m)}^n$ and $\mathcal{P}_{Y/S(m)}^n$.

If $\varphi^*(\tau_i^{(j)}) = \sum f^{k,\ell}_{i,j} \tau'_k^{(\ell)}$ and $s$ is a section of $\mathcal{F}$, we have
\[ \partial^{(j)}(\varphi^*(s)) = \sum f^{k,\ell}_{i,j} \varphi^*(\partial^{(\ell)}(s)). \]

As in the classical case, $\mathcal{D}^{(m)}$-modules have an interpretation in terms of stratifications:

2.1.4. Proposition (see [B4], 2.3.2). — If $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of $\mathcal{D}_{X/S}^{(m)}$-module or an $m$-PD-stratification (defined in the obvious way).

2.1.5. Definition. — A $\mathcal{D}_{X/S}^{(m)}$-module (or $\mathcal{D}_{X/S}^{(m)}$-module) is locally (topologically) quasi-nilpotent if locally, given any section $s$, we have $\partial_i^{(N)}(s) \to 0$ as $N \to \infty$ for any index $i$.

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.1.6. Proposition (generalization of [B1], II. 4.1.3). — If $X$ is a smooth $S$-scheme (with $p$ locally nilpotent) and $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is equivalent to give it a structure of locally quasi-nilpotent $\mathcal{D}_{X/S}^{(m)}$-module or an $m$-HPD-stratification (defined in the obvious way).
We will also have to consider formal $S$-schemes that are not necessarily smooth. In order to deal with this situation we need to introduce the following terminology (see also [B4], 2.3.4 and 2.3.5):

**2.1.7. Definition.** — Let $X$ be an $S$-scheme and $X \hookrightarrow Y$ a closed immersion into a smooth formal $S$-scheme. It follows from Proposition 4.1.5 below that $\mathcal{P}_{X/S(m)}(Y)$ has a natural structure of $\mathcal{D}_{Y/S}^{(m)}$-module. A $\mathcal{P}_{X/S(m)}(Y)$-$\mathcal{D}_{Y/S}^{(m)}$-module is a $\mathcal{D}_{Y/S}^{(m)}$-module $\mathcal{F}$ with a structure of $\mathcal{P}_{X/S(m)}(Y)$-module such that, locally, given any sections $f$ of $\mathcal{P}_{X/S(m)}(Y)$ and $s$ of $\mathcal{F}$, we have

$$\partial^{(k)}(fs) = \sum \left\{ \binom{k}{j} \partial^{(j)}(f) \partial^{(k-j)}(s) \right\}.$$

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

**2.2. Griffiths transversality for $\mathcal{D}^{(m)}$-modules.**

We define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and interpret it in terms of stratifications.

Let $S$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme. The following generalizes the usual notion of Griffiths transversality:

**2.2.1. Definition.** — A filtered $\mathcal{D}_{X/S}^{(m)}$-module $(\mathcal{F}, \text{Fil})$ is a $\mathcal{D}_{X/S}^{(m)}$-module $\mathcal{F}$ together with a filtration by sub $\mathcal{O}_X$-modules. We say that $(\mathcal{F}, \text{Fil})$ is Griffiths transversal if whenever $P \in \mathcal{D}_{X/S}^{(m)}$, we have $P(\text{Fil}^k) \subset \text{Fil}^{k-n}$ and that it is horizontal if the $\text{Fil}^k$ are $\mathcal{D}_{X/S}^{(m)}$-submodules. A filtered $\widehat{\mathcal{D}}_{X/S}^{(m)}$-module $(\mathcal{F}, \text{Fil})$ is a complete $\widehat{\mathcal{D}}_{X/S}^{(m)}$-module $\mathcal{F}$ together with a filtration by complete sub $\mathcal{O}_X$-modules. We say that it is Griffiths transversal or horizontal if it is so mod $p^n$ for all $n$.

**2.2.2. Remarks.**

(i) What we call Griffiths transversal corresponds to what is simply called a filtration on a $\mathcal{D}$-module in the classical situation.

(ii) Assume we have local coordinates $t_1, \ldots, t_d$. In order to show that $(\mathcal{F}, \text{Fil})$ is Griffiths transversal it is sufficient to check that $\partial^{[p^j]}(\text{Fil}^k) \subset \text{Fil}^{k-p^j}$ for $j \leq m$ and all $i$.

Here is the interpretation of Griffiths transversality in terms of stratifications:
2.2.3. **Definition.** — Let \((\mathcal{F}, \text{Fil})\) be a filtered \(\mathcal{O}_X\)-module with an \(m\)-PD-stratification \(\{\varepsilon_n : p_n^2 \mathcal{F} \xrightarrow{\sim} p_n^1 \mathcal{F}\}\). We call the stratification transversal if \(\varepsilon_n\) induces an isomorphism between \(\overline{\text{Fil}}^k_{p_n^2}\) and \(\overline{\text{Fil}}^k_{p_n^1}\) for all \(n\).

2.2.4. **Proposition.** — Let \(\mathcal{F}\) be a \(\mathcal{D}^{(m)}_{X/S}\)-module and \(\text{Fil}^k\) a filtration on \(\mathcal{F}\) by sub \(\mathcal{O}_X\)-modules. Then \(\mathcal{F}\) is Griffiths transversal if and only if the corresponding \(m\)-PD-stratification is transversal.

**Proof.** — Let \(\mathcal{I}\) be the ideal of \(X\) in \(P^n_{X/S(m)}\); \(p_1, p_2 : P^n_{X/S(m)} \to X\) the projections, \(\varepsilon : p_2^* \mathcal{F} \xrightarrow{\sim} p_1^* \mathcal{F}\) the \(n\)-th Taylor isomorphism of \(\mathcal{F}\) and

\[
\theta : \mathcal{F} \to p_1^* \mathcal{F},
\]

\[e \mapsto \varepsilon(1 \otimes e)
\]

the \(n\)-th Taylor map. Assume first the \(m\)-PD-stratification to be transversal. Since \(\varepsilon\) induces an isomorphism between \(\overline{\text{Fil}}^k_{p_n^2}\) and \(\overline{\text{Fil}}^k_{p_n^1}\), then

\[
\theta \text{Fil}^k \subset \overline{\text{Fil}}^k_{p_1} = \text{Fil}^k_{p_1} + J \text{Fil}^k_{p_1} + J^{(2)} \text{Fil}^k_{p_1} + \cdots + J^{(n)} \text{Fil}^k_{p_1}
\]

\[\subset \text{Fil}^{k-n}_{p_1}.
\]

If \(P : \mathcal{F}^n_{X/S(m)} \to \mathcal{O}_X\) is a differential operator of level \(m\) and order less than \(n\), then \(P\) acts on \(\mathcal{F}\) as the composite of \(\theta\) and \(p_1^* (P)\) (i.e. \(P(e) = (P \otimes \text{Id})(\theta(e))\)) so that \(P \text{Fil}^k \subset \text{Fil}^{k-n}\). Thus, we see that \(\mathcal{F}\) is Griffiths transversal. Conversely, assume that \(\mathcal{F}\) is Griffiths transversal. We want to check that \(\varepsilon\) induces an isomorphism between \(\overline{\text{Fil}}^k_{p_2}\) and \(\overline{\text{Fil}}^k_{p_1}\), and we may assume that we have local coordinates \(t_1, \ldots, t_d\) on \(X\). Thanks to the cocycle condition, it is sufficient to show that \(\theta(\text{Fil}^k) \subset \overline{\text{Fil}}^k_{p_1}\). But if \(e \in \text{Fil}^k\) then

\[
\theta(e) = \sum \theta^{(j)}(e) \tau^{(j)} \in \sum \tau^{(j)} \text{Fil}^{k-j}_{p_1} = \overline{\text{Fil}}^k_{p_1}.
\]

The same is true for hyperstratifications. Let \(S\) be an \(m\)-PD-scheme and \(X\) a smooth \(S\)-scheme.

2.2.5. **Definition.** — If \((\mathcal{F}, \text{Fil})\) is a filtered \(\mathcal{O}_X\)-module, we call an \(m\)-HPD-stratification \(\varepsilon : p_2^* \mathcal{F} \xrightarrow{\sim} p_1^* \mathcal{F}\) on \(\mathcal{F}\) transversal if \(\varepsilon\) induces an isomorphism between \(\overline{\text{Fil}}^k_{p_2}\) and \(\overline{\text{Fil}}^k_{p_1}\).

2.2.6. **Proposition.** — An \(m\)-HPD-stratification \(\varepsilon : p_2^* \mathcal{F} \xrightarrow{\sim} p_1^* \mathcal{F}\) on a filtered \(\mathcal{O}_X\)-module \((\mathcal{F}, \text{Fil})\) is transversal if and only if \((\mathcal{F}, \text{Fil})\) is Griffiths transversal.

**Proof.** — Same as Proposition 2.2.4. \(\square\)
2.3. Griffiths transversality and $p$-isogenies.

We are going to build the local version of the functor of our main theorem.

Let $S$ be a formal $m$-PD-scheme, $X$ a formal $S$-scheme, $F_0$ the relative Frobenius of $X_0$ over $S_0$ and $F: X \to X'$ a lifting of $F_0$. We assume that there are local coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F^*(t_i) = t_i^p$.

We will write $X_0^{(m+1)}$ for the pull back of $X_0$ by the $m+1$ iterate of $F_0$, and, with the usual slight abuse of notation, we will call

$$F_0^{m+1}: X_0 \to X_0^{(m+1)}$$

this $m+1$ iterate of $F_0$ and $F^{m+1}: X \to X^{(m+1)}$ a lifting obtained by iterating the above process.

2.3.1. Lemma. — If $s$ is a section of a $D^{(m)}_{X^{(m+1)}/S}$-module $\mathcal{E}$, then for $k < p^{m+1}$, we have, with $a_{\frac{i}{k}, \frac{k}{k}} \in \mathbb{Z}$,

$$\vartheta^{[k]}(F^{m+1^*}(s)) = \sum p^k a_{\frac{i}{k}, \frac{k}{k}} t_i^p t_i^{p^{m+1}-k} F^{m+1^*}(\vartheta^{[\frac{i}{k}]}(s)).$$

Proof. — For $n = p^{m+1} - 1$, we have in $T^n_{X^{(m+1)}/S(m)}$

$$F^{m+1^*}(\tau_i) = (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} = \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k$$

$$= \sum_{k=1}^{p^{m+1}-1} p c_{i,k} t_i^{p^{m+1}-k} \tau_i^k$$

with $c_{i,k} \in \mathbb{Z}$. Thus we can write

$$F^{m+1^*}(\tau_i) = \sum p^k a_{\frac{i}{k}, \frac{k}{k}} t_i^p t_i^{p^{m+1}-k} \tau_i^k$$

with $a_{\frac{i}{k}, \frac{k}{k}} \in \mathbb{Z}$. Therefore, if $s$ is a section of $\mathcal{E}$, we have

$$\vartheta^{[k]}(F^{m+1^*}(s)) = \sum p^k a_{\frac{i}{k}, \frac{k}{k}} t_i^p t_i^{p^{m+1}-k} F^{m+1^*}(\vartheta^{[\frac{i}{k}]}(s)).$$

This lemma allows us to show that Frobenius pulls back transversal modules to horizontal modules:
2.3.2. Proposition. — If \((E, \text{Fil})\) is a Griffiths transversal \(\mathcal{D}^{(m)}_{X^{(m+1)}/S^{-}}\) module (or \(\mathcal{D}^{(m)}_{X^{(m+1)}/S^{-}}\)-module) on \(X^{(m+1)}\) which is saturated with respect to \((p, \{ \})\), then \(F^{m+1*}(E, \text{Fil})\) is horizontal.

Proof. — We have seen that if \(s\) is a section of \(E\), then for \(k < p^{m+1}\), we have

\[
\partial^{[k]}(F^{m+1*}(s)) = \sum p^j a_{j,k} t^{j p^{m+1} - k} F^{m+1*}(\partial^{[j]}(s)).
\]

Since \((E, \text{Fil})\) is Griffiths transversal, we know that if \(s \in \text{Fil}^{\ell}\), we have \((\partial^{[j]}(s)) \in \text{Fil}^{\ell-\{j\}}\). It follows that \(F^{m+1*}(\partial^{[j]}(s)) \in \text{Fil}^{\ell-\{j\}}\) so that

\[
\sum p^j a_{j,k} t^{j p^{m+1} - k} F^{m+1*}(\partial^{[j]}(s)) \in \sum p^j \text{Fil}^{\ell-\{j\}} \subset \text{Fil}^{\ell}.
\]

2.3.3. Theorem. — Assume \(S\) has no \(p\)-torsion. Let \((E, \text{Fil})\) be a \(p\)-torsion free Griffiths transversal \(\mathcal{D}^{(m)}_{X^{(m+1)}/S^{-}}\)-module of width less than \(p^{m+1}\) which is transversal to \((p, \{ \})\). Then there exists a unique \(p\)-isogeny \(\Phi : F^{m+1*} E \rightarrow \mathcal{F}\) of \(\mathcal{D}^{(m)}_{X/S}\)-modules such that \(F^{m+1*} \text{Fil}^{k}\) is the saturation of \(\Phi^{-1}(p^k \mathcal{F})\) with respect to \((p, \{ \})\).

Proof. — Follows from Corollary 1.2.6 and Proposition 2.3.2.

2.3.4. Definition. — Given any lifting \(F : X \rightarrow X'\) of the relative Frobenius of \(X_0\) over \(S_0\), an \(F^{m+1}\)-isogeny on \(X/S\) will be a \(p\)-isogeny of the form \(\Phi : F^{m+1*} E \rightarrow \mathcal{F}\) where \(E\) is a \(\mathcal{D}^{(m)}_{X^{(m+1)}/S^{-}}\)-module and \(\mathcal{F}\) is a \(\mathcal{D}^{(m)}_{X/S}\)-module.

2.3.5. — Theorem 2.3.3 gives a functor \(\mu\) from the category of \(p\)-torsion free Griffiths transversal \(\mathcal{D}^{(m)}_{X^{(m+1)}/S^{-}}\)-module of width less than \(p^{m+1}\) that are transversal to \((p, \{ \})\) to the category of \(F^{m+1}\)-isogenies of width less than \(p^{m+1}\) on \(X/S\). We will show in section 3.3 that this functor is fully faithful.
3. $\mathcal{D}^{(m)}$-MODULES IN CHARACTERISTIC $p$ AND GRIFFITHS TRANSVERSALITY

3.1. $p$-m-curvature of a $\mathcal{D}^{(m)}$-module.

We define $p$-m-curvature for $\mathcal{D}^{(m)}$-modules in characteristic $p$ and study the relation between it being zero and horizontal sections.

Let $S$ be a scheme of characteristic $p$ and $X$ a smooth $S$-scheme. We let

- $\mathcal{D}^{(m)+}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \mathcal{O}_X$;
- $\mathcal{K}^{(m)}_{X/S}$ be the kernel of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \text{End}(\mathcal{O}_X)$.

3.1.1. DEFINITIONS. — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. The sheaf $\mathcal{F}^\nabla$ of horizontal sections of $\mathcal{F}$ is the part of $\mathcal{F}$ on which $\mathcal{D}^{(m)+}_{X/S}$ acts as zero. The $p$-m-curvature of $\mathcal{F}$ is the restriction to $\mathcal{K}^{(m)}_{X/S}$ of the canonical map $\mathcal{D}^{(m)}_{X/S} \to \text{End}(\mathcal{F})$.

3.1.2. Remark. — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module. Then it follows from [B4], 2.2.6, that $\mathcal{F}$ has zero $p$-m-curvature if, locally on $X$, we have for all $i$, $\partial_i^{(p^{m+1})}(s) = 0$ for any $s \in \mathcal{F}$. In particular, in case $m = 0$, zero $p$-m-curvature is the same as zero $p$-curvature.

Let $F : X \to X'$ be the relative Frobenius of $X$ over $S$.

3.1.3. LEMMA. — If $\mathcal{E}$ is a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module, then $\mathcal{D}^{(m)+}_{X/S}$ acts as zero on sections of the form $F^{m+1}*(s)$ with $s \in \mathcal{E}$.

Proof. — This is a local question. We have

$$F^{m+1}*(\tau_i) = (t_i + \tau_i)p^{m+1} - t_i^{p^{m+1}} = \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1}-k} \tau_i^k = \tau_i^{p^{m+1}} = p! \tau_i^{(p^{m+1})} = 0.$$ 

It follows that, if $0 < j < p^{m+1}$, then $F^{m+1}*(\tau_j) = 0$, so that, for any section $s$ of $\mathcal{E}$, we have $\partial^{[j]}(F^{m+1}*(s)) = 0$. \hfill $\square$

3.1.4. PROPOSITION. — The trivial $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{O}_X$ has zero $p$-m-curvature and the canonical map $\mathcal{O}_{X^{(m+1)}} \to F^{m+1}_*\mathcal{O}_X^\nabla$ is bijective.
Proof. — The first assertion is an obvious consequence of the definition. The second one is local and we may therefore choose local coordinates \( t_1, \ldots, t_d \). These coordinates define an étale map from \( X \) to \( \mathbb{A}^d \). The relative Frobenius being cartesian with respect to étale morphisms and to base change, this map provides us with an isomorphism

\[
F_{\ast}^{m+1}\mathcal{O}_X \cong \mathcal{O}_X^{(m+1)} \otimes_{\mathbb{F}_p[t_1, \ldots, t_d]} \mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)}
\]

where \( \mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)} \) is \( \mathbb{F}_p[t_1, \ldots, t_d] \) seen as a module over itself via the \( (m + 1) \)-st power of Frobenius. If \( \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)} \) denotes the space of polynomials of degree strictly less than \( p^{m+1} \) in each variable, the canonical map

\[
\mathbb{F}_p[t_1, \ldots, t_d] \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)}^{(m+1)} \to \mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)}
\]

is bijective and therefore

\[
F_{\ast}^{m+1}\mathcal{O}_X \cong \mathcal{O}_X^{(m+1)} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)}.
\]

Since \( F_{\ast}^{m+1}\mathcal{D}_{X/S}^{(m)} \) acts as zero on \( \mathcal{O}_X^{(m+1)} \), we are reduced to showing that if \( f \in \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)} \) and \( \mathcal{D}_{X/S}^{(m)} \) acts as zero on \( f \), then \( f \in \mathbb{F}_p \). One may first prove that if \( A \) is an \( \mathbb{F}_p \)-algebra and \( f \in A[t^{p^j}] \) is such that \( \partial^{(p^j)}(f) = 0 \), then \( f \in A[t^{p^j+1}] \) and then use induction on \( d \). The details are left to the reader. \( \square \)

3.1.5. Proposition

(i) If \( \mathcal{F} \) is a \( \mathcal{D}^{(m)}_{X/S} \)-module then \( F_{\ast}^{m+1}\mathcal{F} \) is a sub \( \mathcal{O}_X^{(m+1)} \)-module of \( F_{\ast}^{m+1}\mathcal{F} \).

(ii) If \( \mathcal{E} \) is a \( \mathcal{D}^{(m)}_{X^{(m+1)}/S} \)-module then \( F_{\ast}^{m+1}\mathcal{E} \) has zero \( p \)-\( m \)-curvature.

Proof. — Again, these are local questions. For the first assertion, we have to show that if \( s \) is a section of \( \mathcal{F}^{(m)} \) and \( f \) is a section of \( \mathcal{O}_X^{(m+1)} \) then \( \partial^{(k)}(F_{\ast}^{m+1}(f)s) = 0 \) for \( k \neq 0 \). For the second one, we have to show that if \( s \) is a section of \( \mathcal{F} \) and \( f \) is a section of \( \mathcal{O}_X \), then \( \partial^{(p^{m+1})}_i(fF_{\ast}^{m+1}(s)) = 0 \). Using the formula

\[
\partial^{(k)}(fs) = \sum \left\{ \binom{k}{j} \partial^{(j)}(f) \partial^{(k-j)}(s) \right\},
\]

both statements are easy consequences of Lemma 3.1.3 and Proposition 3.1.4. \( \square \)
3.2. Cartier’s theorem for $\mathcal{D}^{(m)}$-modules.

We generalize Cartier’s theorem (see [K], 5.1) to $\mathcal{D}^{(m)}_{X/S}$-modules.

We let $S$, $X$ and $F : X \to X'$ be as in section 3.1.

3.2.1. Lemma. — Let $t_1, \ldots, t_d$ be local coordinates on $X$ and

$$P := \sum_{k < p^m + 1} (-t)^k \partial^{[k]}.$$ 

If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature, then $P$ is a projector from $\mathcal{F}$ onto $\mathcal{F}^\vee$.

Proof. — We follow the first part of the proof of Proposition 5.1 in [K]. Since $\mathcal{F}$ has zero p-m-curvature, we have $\partial^{[j]}(s) = 0$ for $j \geq p^m + 1$. There should therefore be no confusion if we write $\partial^{[j]}(s) = 0$ for $j$ such that $\max(j_i) \geq p^m + 1$. If $s \in \mathcal{F}$, we have

$$\partial^{[j]}(P(s)) = \sum (-t)^k \partial^{[k]}(s)$$

$$= \sum \sum (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) \left( \begin{array}{c} k + j - i \\ k \end{array} \right) \partial^{[k + j - i]}(s)$$

$$= \sum \sum (-1)^i \left( \begin{array}{c} \ell + i \\ i \end{array} \right) \left( \begin{array}{c} \ell + j \\ \ell + i \end{array} \right) \partial^{[\ell + j]}(s)$$

$$= \sum \left( \sum (-1)^i \left( \begin{array}{c} \ell + i \\ i \end{array} \right) \left( \begin{array}{c} \ell + j \\ \ell + i \end{array} \right) \right) (-t)^\ell \partial^{[\ell + j]}(s)$$

and, if $j \neq 0$, we have

$$\sum (-1)^i \left( \begin{array}{c} \ell + i \\ i \end{array} \right) \left( \begin{array}{c} \ell + j \\ \ell + i \end{array} \right) = \left( \begin{array}{c} \ell + j \\ \ell \end{array} \right) \sum (-1)^i \left( \begin{array}{c} j \\ i \end{array} \right) = 0.$$

Thus we see that $P$ maps $f$ into $\mathcal{F}^\vee$. Since $P$ restricts to the identity on $\mathcal{F}^\vee$, it is a projector from $\mathcal{F}$ onto $\mathcal{F}^\vee$.

3.2.2. Proposition. — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module with zero p-m-curvature. Then the canonical map $F^{m+1}_* F^{m+1}_* \mathcal{F}^\vee \to \mathcal{F}$ is an isomorphism.
Proof. — We follow the end of the proof of Proposition 5.1 in [K]. The question is local on $X$ and we may therefore assume that we have local coordinates $t_1, \ldots, t_d$. We have seen in Lemma 3.2.1 that $P$ is a projector from $\mathcal{F}$ onto $\mathcal{F}^{\nabla}$. It follows that the map

$$T : \mathcal{F} \longrightarrow F_{*}^{m+1} F_{*}^{m+1} \mathcal{F}^{\nabla},$$

$$s \mapsto \sum_{k < p_{m+1}} t^k \otimes P \partial_{[k]}(s)$$

is well defined. Let us show that $T$ is a right inverse to the canonical map $U : F_{*}^{m+1} F_{*}^{m+1} \mathcal{F}^{\nabla} \rightarrow \mathcal{F}$. If $s \in \mathcal{F}$, then

$$(U \circ T)(s) = \sum_{k} t^k P \partial_{[k]}(s) = \sum_{k} t^k \sum_{\xi} (-1)^{\xi} \xi \partial_{[\xi]}(s) = \sum_{\xi} (-1)^{\xi} t^{k+\xi} \left( \frac{k + \ell}{\ell} \right) \partial_{[k+\xi]}(s) = \sum \left( \sum_{\xi} (-1)^{\xi} \left( \frac{j}{\ell} \right) t^j \partial_{[j]}(s) = s. \right.$$}

We have seen that $F_{*}^{m+1} \mathcal{O}_{X}^{\nabla} = \mathcal{O}_{X(m+1)}$ and it follows that $U$ is a bijection in the case $\mathcal{F} = \mathcal{O}_X$. Hence, $T$ is also a left inverse to $U$ in this case, which implies that for any $f \in \mathcal{O}_X$, we have $T(f) = f \otimes 1$. In general, we have for $f \in \mathcal{O}_X$ and $s \in \mathcal{F}^{\nabla}$,

$$(T \circ U)(f \otimes s) = T(fs) = \sum_{k} t^k \otimes P \partial_{[k]}(fs) = \sum_{k} t^k \otimes P \partial_{[k]}(f)s = \left( \sum_{k} t^k \otimes P \partial_{[k]}(f) \right)(1 \otimes s) = T(f)(1 \otimes s) = (f \otimes 1)(1 \otimes s) = f \otimes s. \quad \Box$$

3.2.3. Proposition. — Let $\mathcal{E}$ be a $\mathcal{D}_{X(m+1)/S}$-module, $\mathcal{F} = F_{*}^{m+1} \mathcal{E}$ (as $\mathcal{D}_{X/S}$-module) and $\eta : \mathcal{E} \rightarrow F_{*}^{m+1} \mathcal{F}$ be the adjunction map. Then

(i) The map $\eta$ induces a natural isomorphism $\mathcal{E} \cong F_{*}^{m+1} \mathcal{F}^{\nabla}$ of $\mathcal{O}_{X(m+1)}$-modules.

(ii) In the situation of Lemma 3.2.1, the action of $P$ on $F_{*}^{m+1} \mathcal{F}$ factors through $\eta$.

(iii) If $\mathcal{F}'$ is a sub-$\mathcal{D}_{X/S}$-module of $\mathcal{F}$, then the natural map $F_{*}^{m+1} F_{*}^{m+1} \mathcal{F} \rightarrow \mathcal{F}$ induces an isomorphism $F_{*}^{m+1} (\eta^{-1}(F_{*}^{m+1} \mathcal{F}')) \cong \mathcal{F}'$. 

Proof. — We know from Proposition 3.1.5 (ii) that $\mathcal{F}$ has zero $p$-$m$-curvature. It follows from Proposition 3.2.2 that

$$F^{m+1} \mathcal{E} \cong F^{m+1} F_*^{m+1} \mathcal{F}^\nabla$$

and we use the faithful flatness of $F$ to obtain assertion (i).

In order to prove assertion (ii), we recall from Lemma 3.2.1 that the image of $P$ acting on $\mathcal{F}$ is (contained in) $\mathcal{F}^\nabla$. It therefore follows from (i) that the action of $P$ on $F_*^{m+1} \mathcal{F}$ factors through

$$\eta: \mathcal{E} \cong F_*^{m+1} \mathcal{F}^\nabla \longrightarrow F_*^{m+1} \mathcal{F}.$$  

Finally, for (iii), since $\mathcal{F}$ has zero $p$-$m$-curvature, so does $\mathcal{F}'$. The map $\eta$ being functorial, it follows from (i) that it induces $\mathcal{F}^\nabla \cong F_*^{m+1} \mathcal{F}'$ so that

$$\mathcal{F}' \cong F_*^{m+1} \mathcal{F}^\nabla \cong F_*^{m+1} (\eta^{-1}(F_*^{m+1} \mathcal{F}')).$$

3.2.4. Corollary (Cartier’s theorem). — The functors $\mathcal{E} \mapsto F^{m+1} \mathcal{E}$ and $\mathcal{F} \mapsto F_*^{m+1} \mathcal{F}^\nabla$ give an equivalence between the category of $O_X^{(m+1)}$-modules and the category of $\mathcal{D}^{(m)}_{X/S}$-modules with zero $p$-$m$-curvature.

3.3. $F^{m+1}$-$p$-isogenies and Griffiths transversality.

We have built in section 2.3 a functor $\mu$ that associates $F^{m+1}$-$p$-isogenies to some filtered $\widehat{\mathcal{D}}^{(m)}$-modules. We are now going to define a functor $\alpha$ from $F^{m+1}$-$p$-isogenies to filtered $\widehat{\mathcal{D}}^{(m)}$-modules that will allow us to prove that $\mu$ is fully faithful.

The setting is as in section 2.3: $S$ is a $p$-torsion free formal scheme, $X$ is a smooth formal $S$-scheme, $F_0$ is the relative Frobenius of $X_0$ over $S_0$ and $F: X \rightarrow X'$ is a lifting of $F_0$. We also assume that there are local coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F^* (t_i) = t_i^p$.

If $\Phi: F^{m+1} \mathcal{E} \rightarrow \mathcal{F}$ is an $F^{m+1}$-$p$-isogeny on $X/S$, we consider the filtration $M$ on $F^{m+1} \mathcal{E}$ given by

$$M^k := \Phi^{-1} (p^k \mathcal{F})$$

and the filtration Fil on $\mathcal{E}$ given by

$$\text{Fil}^k := \eta^{-1} (F_*^{m+1} M^k),$$

where $\eta: \mathcal{E} \rightarrow F_*^{m+1} F^{m+1} \mathcal{E}$ is the adjunction map. We will write $\text{Fil}$ for the saturation of Fil with respect to $(p, \{ \})$. This way, we get a functor

$$\alpha: (\Phi: F^{m+1} \mathcal{E} \rightarrow \mathcal{F}) \mapsto (\mathcal{E}, \text{Fil})$$

with values in the category of filtered $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules transversal to $(p, \{ \})$.  


3.3.1. **Lemma.** — If

$$P := \sum_{k<p^{m+1}} (-t)^k \partial^{[k]}$$

then there exists $Q$, reducing to 1 mod $p$, such that

$$P(F^{m+1^*}(s)) = F^{m+1^*}(Q(s))$$

for any section $s$ of a $\mathcal{D}_{X^{(m+1)}/S}^{(m)}$-module $\mathcal{E}$.

**Proof.** — From Lemma 2.3.1, we deduce that

$$t^k \partial^{[k]}(F^{m+1^*}(s)) = \sum p^i a_{i,k} t^i p^{m+1} F^{m+1^*}(\partial^{[i]}(s)) = F^{m+1^*}(Q_k(s))$$

where $Q_k := \sum p^i a_{i,k} t^i \partial^{[i]}$ and we let

$$Q = \sum_{k<p^{m+1}} (-1)^k Q_k.$$

The following result is of technical nature and is needed in the next proposition:

3.3.2. **Lemma.** — Let $\Phi : F^{m+1^*}\mathcal{E} \to \mathcal{F}$ be an $F^{m+1^*}-p$-isogeny on $X/S$ and $M$, Fil and $\eta$ as above. Then $\eta_0 : \mathcal{E}_0 \to F_0^{m+1^*} F_0^{m+1^*} \mathcal{E}_0$ is strictly compatible with the induced filtrations (i.e. we have $\text{Fil}_0^k = \eta_0^{-1}(F_0^{m+1^*} M_0^k)$).

**Proof.** — We follow the proof of Theorem 2.2 of [O1]. The map is clearly compatible with the induced filtrations and we are left with proving the strictness. Let $s_0 \in \mathcal{E}_0$ be such that $\eta_0(s_0) \in F_0^{m+1^*} M_0^k$. We want to prove that there exists a lifting $s \in \mathcal{E}$ of $s_0$ such that $\Phi(\eta(s)) = p^k s'$. It is clearly sufficient to show that for any $i$ there exists a lifting $s \in \mathcal{E}$ of $s_0$, and $u$ such that $\Phi(\eta(s) + p^i u) = p^k s'$ and then take $i = k$. We prove this by induction on $i$, the case $i = 1$ being just our assumption.

So, let us assume that $s \in \mathcal{E}$ is a lifting of $s_0$ such that

$$\Phi(\eta(s) + p^i u) = p^k s'.$$

Since $\Phi$ is a morphism of $\mathcal{D}_{X/S}^{(m)}$-modules, it commutes with the operator $P$ of the lemma. Using Lemma 3.3.1, we have

$$p^k P(s') = P(p^k s') = P(\Phi(\eta(s) + p^i u)) = \Phi(P(\eta(s)) + P(p^i u)) = \Phi(\eta(Q(s)) + p^i P(u)).$$
We have seen in Proposition 3.2.3 (ii) that the action of $P$ on $F_0^{m+1} E_0$ factors through $\eta_0 : E_0 \to F_0^{m+1} E_0$. We can therefore write

$$P(u) = \eta(v) + pw.$$  

It follows that

$$p^k P(s') = \Phi(\eta(Q(s)) + p^i \eta(v) + p^{i+1} w) = \Phi(\eta(Q(s) + p^i v) + p^{i+1} w).$$

It just remains to observe that $Q(s) + p^i v$ is a lifting of $s_0$ since $Q$ is the identity mod $p$. 

3.3.3. PROPOSITION. — Let $\Phi : F^{m+1} E \to F$ be an $F^{m+1}$-isogeny on $X/S$, and $M$ and Fil as above. Then we have $F^{m+1} \operatorname{Fil}^k = M^k$.

Proof. — We follow the proof of Lemma 5.2.11 in [O2]. The modules $E$ and $F$ are $p$-torsion free and the filtrations $\operatorname{Fil}^k$ and $M^k$ are transversal to $p$. From this, we deduce that the commutative diagram

$$
\begin{array}{ccc}
0 & \to & F^{m+1} \operatorname{Fil}^{k-1} \\
\downarrow & & \downarrow \\
0 & \to & M^{k-1}
\end{array} \quad \begin{array}{ccc}
p & \to & F^{m+1} \operatorname{Fil}^k \\
\downarrow & & \downarrow \\
p & \to & M^k
\end{array} \quad \begin{array}{ccc}
\to & \to & \to \\
& & \to
\end{array} \quad 0
$$

has exact rows. Hence, by induction, it is sufficient to prove that $F_0^{m+1} \operatorname{Fil}_0^k = M_0^k$. But we have seen in Proposition 3.2.3 (iii) that

$$F_0^{m+1} (\eta_0^{-1} (F_0^{m+1} M_0^k)) = M_0^k$$

and we know from Lemma 3.3.2 that $\eta_0^{-1} (F_0^{m+1} M_0^k) = \operatorname{Fil}_0^k$. 

We will show in Proposition 5.2.5 that the filtration $\operatorname{Fil}$ in the definition of $\alpha$ is not always Griffiths transversal when $m > 0$. Nevertheless, for the functor $\mu$ of 2.3.5, we have the following:

3.3.4. THEOREM. — When restricted to the essential image of $\mu$, the functor $\alpha$ is a quasi-inverse to $\mu$.

Proof. — Follows from Proposition 3.3.3.

3.3.5. COROLLARY. — The functor $\mu$ is fully faithful.

Proof. —
4. TRANSVERSAL $m$-CRYSTALS

4.1. $m$-crystals.

We recall Berthelot's theory of $m$-crystals from [B3].

Let $(S,a,b)$ be a formal $m$-PD-scheme. If $X$ is an $S$-scheme, we will always assume that the $m$-PD-structure of $S$ extends to $X$.

4.1.1. Definition. — If $X \hookrightarrow Y$ is an $m$-PD-$S$-immersion of $S$-schemes, we say that $Y$ is an $m$-PD-$S$-thickening of $X$.

4.1.2. Definition. — Let $X$ be an $S$-scheme. The $m$-th crystalline site of $X/S$ is the category $\text{Cris}^{(m)}(X/S)$ of $m$-PD-$S$-thickenings $U \hookrightarrow Y$ with $U$ open in $X$, endowed with a suitable topology. As in the classical case, the site $\text{Cris}^{(m)}(X/S)$ is functorial in $X/S$.

4.1.3. Remark. — There exists a unique sheaf $\mathcal{O}^{(n)}_{X/S}$ on $\text{Cris}^{(m)}(X/S)$ whose value on $(Y,I,J)$ is $I^{(n)}$. We will write

$$\mathcal{O}_{X/S} := \mathcal{O}^{(0)}_{X/S} \quad \text{and} \quad \mathcal{J}_{X/S} := \mathcal{J}^{(1)}_{X/S}.$$

It is clear that $(\text{Cris}^{(m)}(X/S), \mathcal{O}_{X/S}, \mathcal{J}_{X/S})$ is a filtered ringed site.

4.1.4. Definition. — Let $X$ be an $S$-scheme. To any sheaf $E$ on $\text{Cris}^{(m)}(X/S)$ and any object $Y$ of $\text{Cris}^{(m)}(X/S)$, one associates in the obvious way a sheaf $E_Y$ on $Y$. If $E$ is an $\mathcal{O}_{X/S}$-module, any morphism $\varphi : Y' \to Y$ of $m$-PD-thickenings gives a natural morphism $\varphi^* E_Y \to E_{Y'}$. We call $E$ an $m$-crystal if these maps are all bijective.

The proofs of the following statements are straightforward generalizations of those of the analogous results from [B1]. They should appear in a forthcoming article of Berthelot as announced in [B4].

4.1.5. Proposition. — If $X \hookrightarrow Y$ is a closed immersion of $S$-schemes and $E$ is an $m$-crystal on $X$, then $i_* E$ is an $m$-crystal on $Y$.

4.1.6. Corollary. — If $\overline{S} = \text{Spec} \mathcal{O}_S/a$ and $\overline{X} = X \times_S \overline{S}$, then the restriction functor $\text{Cris}^{(m)}(X/S) \to \text{Cris}^{(m)}(\overline{X}/S)$ induces an equivalence between the categories of $m$-crystals on $X/S$ and on $\overline{X}/S$.

4.1.7. Proposition. — Let $i : X \hookrightarrow Y$ be a closed immersion of $S$-schemes with $Y$ smooth. Then the functor $E \mapsto E_Y := (i_* E)_Y$ is an equivalence of categories between $m$-crystals on $X$ and locally quasi-nilpotent $\mathcal{P}_{X/S}^{(m)}(Y)$-$\mathcal{D}^{(m)}_{Y/S}$-modules.
4.1.8. **PROPOSITION.** — Let $X$ be a smooth formal $S$-scheme and let $X_n$ denote its reduction mod $p^{n+1}$. The functor

$$E \mapsto E_X := \lim_{\longrightarrow} E_{X_n}$$

is an equivalence of categories between $m$-crystals on $X_0$ and locally topologically quasi-nilpotent complete $\mathcal{D}^{(m)}_{X/S}$-modules.

4.2. **$T$-m-Crystals.**

We define $T$-m-crystals and relate them to differential modules. Note that we call $T$-m-crystals what Ogus would call proto-$T$-m-crystals.

Let $S$ be a formal $m$-PD-scheme.

4.2.1. **PROPOSITION AND DEFINITION.** — Let $f : (U', Y') \to (U, Y)$ be a morphism of $m$-PD-$S$-thickenings such that $U' \to U$ is flat and $(\mathcal{F}, \text{Fil})$ a $T$-module on $(Y, \mathcal{O}_Y, \mathcal{J}^{(n)})$. Then $Tf^\ast(\mathcal{F}, \text{Fil}) := (f^\ast \mathcal{F}, \text{Fil}^k)$ is a $T$-module called the $T$-inverse image of $(\mathcal{F}, \text{Fil})$.

**Proof.** — This follows from Proposition 1.1.7 (ii) and Proposition 1.1.8. $\square$

4.2.2. **DEFINITION.** — Let $X$ be an $S$-scheme. If $E$ is any $T$-module on $\text{Cris}(X/S)^{(m)}$ and $Y$ any object of $\text{Cris}(X/S)^{(m)}$, then $E_Y$ is in a natural way a $T$-module. If $f : Y' \to Y$ is a morphism in $\text{Cris}(X/S)^{(m)}$, then there is a natural morphism of filtered modules $Tf^\ast E_Y \to E_{Y'}$. We call $E$ a $T$-m-crystal if these maps are all isomorphisms of filtered modules (i.e. such that $\text{Fil}^k = \text{Fil}^k$).

The category of $T$-m-crystals is functorial with respect to flat morphisms: if $\varphi : X' \to X$ is a flat morphism and $E$ a $T$-m-crystal on $X/S$, then

$$T\varphi^\ast(E, \text{Fil}) := (\varphi^\ast E, \text{Fil}^k_{\varphi})$$

is a $T$-m-crystal.

4.2.3. **Example.** — The trivial $T$-m-crystal is $(\mathcal{O}_{X/S}, \mathcal{J}^{(k)}_{X/S})$ whose value at $X$ is the trivial filtered module $\mathcal{O}_X = \text{Fil}^0 \supset \text{Fil}^1 = 0$.

The following generalize Proposition 3.2.2 and Theorem 3.2.3 of [O2]:
4.2.4. **Proposition.** — If \( i : X \hookrightarrow Y \) is a closed immersion into a smooth \( S \)-scheme and \( E \) a \( T \)-\( m \)-crystal on \( X/S \), then

\[
i_*(E, \text{Fil}) := (i_*E, i_* \text{Fil})
\]

is a \( T \)-\( m \)-crystal which is transversal to \((i_*(J_{X/S}), \{ \})\).

**Proof.** — Same proof as [O2], 3.2.2. \(\square\)

4.2.5. **Proposition.** — Let \( i : X \hookrightarrow Y \) be a closed \( S \)-immersion into a smooth \( S \)-scheme. Then the functor \( E \mapsto E_Y \) is an equivalence of categories between \( T \)-\( m \)-crystals on \( X \) and Griffiths transversal locally quasi-nilpotent \( \mathcal{P}_{X/S(m)}(Y) \)-\( \mathcal{D}_{Y/S}^{(m)} \)-modules which are transversal to the \( m \)-PD-filtration of \( \mathcal{P}_{X/S(m)}(Y) \).

**Proof.** — Let \( p_1, p_2 : P_X(Y^2) \to P_X(Y) \) be the projections. If \( E \) is a \( T \)-\( m \)-crystal, we have an isomorphism of filtered modules

\[
\varepsilon : Tp_1^*E_Y \xrightarrow{\sim} E_{Y^2} \xleftarrow{\sim} Tp_1^*E_Y,
\]

which means that the HPD-stratification \( \varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F} \) is transversal and therefore, by Proposition 2.2.6, that \( E_Y \) is Griffiths transversal.

Conversely, let \( \mathcal{F} \) be a Griffiths transversal locally quasi-nilpotent \( \mathcal{P}_{X/S(m)}(Y) \)-\( \mathcal{D}_{Y/S}^{(m)} \)-module which is transversal to the \( m \)-PD-filtration of \( \mathcal{P}_{X/S(m)}(Y) \). There exists, by Proposition 4.1.7, a unique \( m \)-crystal \( E \) such that \( E_Y = \mathcal{F} \). Let \( X \hookrightarrow T \) be an \( m \)-PD-thickening. Since \( Y \) is smooth, \( i \) extends locally on \( T \) to a map \( f : T \to Y \) which in turn extends to an \( m \)-PD-morphism \( g : T \to P_X(Y) \). We then set

\[
\text{Fil}^k E_T = \overline{\text{Fil}}^k_g,
\]

so that \((E_T, \text{Fil}) = Tg^*(\mathcal{F}, \text{Fil})\). If this is well defined, it is clear that we obtain a quasi-inverse to our functor. It is actually sufficient to check that the HPD-stratification \( \varepsilon : p_2^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F} \) is transversal. But this follows again from Proposition 2.2.6. \(\square\)

4.2.6. **Corollary.** — Let \( X \) be a smooth formal \( S \)-scheme. Then the functor \( E \mapsto E_X \) is an equivalence of categories between \( T \)-\( m \)-crystals on \( X_0/S \) and locally topologically quasi-nilpotent Griffiths transversal complete \( \overline{\mathcal{D}}^{(m)}_{X/S}(\mathcal{D}) \)-modules transversal to \((p, \{ \})\). \(\square\)
4.3. T-m-crystals and F-m-spans.

We define F-m-spans and use them to describe T-m-crystals.

Let $S$ be a formal $m$-PD-scheme, $X$ a smooth $S_0$-scheme, and $F : X \to X'$ the relative Frobenius of $X$ over $S_0$.

4.3.1. DEFINITION. — If $(E, \text{Fil})$ is a filtered $m$-crystal where the $\text{Fil}^k$ are not merely sub modules but sub $m$-crystals, then we say that $(E, \text{Fil})$ is horizontal.

Note that a horizontal filtered $m$-crystal is not a T-m-crystal. Let us describe the saturation process:

4.3.2. PROPOSITION

(i) Any horizontal filtered $m$-crystal $(E', \text{Fil})$ on $X/S$ that is almost transversal to $(p, \{ \})$ is almost transversal to $(J_{X/S}, \{ \})$. In particular, $(E, \text{Fil})$ is a T-m-crystal.

(ii) The functor $(E', \text{Fil}) \to (E, \text{Fil})$ from the category of horizontal filtered $m$-crystals on $X/S$ that are transversal to $(p, \{ \})$, to the category of T-m-crystals is fully faithful.

Proof.

(i) Let $X \hookrightarrow T$ be an $m$-PD-immersion and $I$ the ideal of $X$ in $T$. We have to show that $(E_T, \text{Fil})$ is almost transversal to $(I, \{ \})$. This question is local on $T$. The scheme $X$ being smooth over $S_0$, it locally lifts to a smooth formal scheme $Y$ over $S$. Since $Y$ is smooth and $X \hookrightarrow T$ is nilpotent, there exists, locally on $T$, a map $\varphi : T \to Y$ that induces the identity on $X$. The $m$-PD-structure on $T$ is compatible with $(p, \{ \})$, so that the map $\varphi$ is an $m$-PD-morphism. Since $(E_Y, \text{Fil})$ is almost transversal to $(p, \{ \})$, it follows from Proposition 1.1.8 that $(E_T, \text{Fil})$ is almost transversal to $(I, \{ \})$. Applying Proposition 1.1.7 (ii), we get the last assertion.

(ii) We have to show that $\text{Fil}^k$ determines $\text{Fil}^k$. This is a local question on $X$. The scheme $X$ being smooth over $S_0$, it locally lifts to a smooth formal scheme $Y$ over $S$. Since $(E_Y, \text{Fil})$ is saturated with respect to $(p, \{ \})$, we have $\text{Fil}^k E_Y = \text{Fil}^k E_Y$. It follows from Corollary 4.2.6 that $\text{Fil}^k E$ is determined by $\text{Fil}^k E_Y$ and hence by $\text{Fil}^k E$.

4.3.3. DEFINITION. — If $(E, \text{Fil})$ is in the image of this last functor, we call it a horizontal T-m-crystal.

We are now able to globalize the local results of parts 2 and 3:
4.3.4. **Proposition.** — If \((E, \text{Fil})\) is a \(T\)-\(m\)-crystal on \(X^{(m+1)}/S\), then \(TF^{m+1} (E, \text{Fil})\) is a horizontal \(T\)-\(m\)-crystal.

**Proof.** — This follows from Proposition 2.3.2 and Proposition 4.3.2 (i).

\[ \square \]

4.3.5. **Definition.** — An \(F\)-\(m\)-span is a \(p\)-isogeny \(\Phi : F^{m+1} E \to E'\) of \(m\)-crystals.

4.3.6. **Theorem.** — Assume \(S\) has no \(p\)-torsion. Let \((E, \text{Fil})\) be a \(p\)-torsion free \(T\)-\(m\)-crystal on \(X^{(m+1)}/S\) of width less than \(p^m\). Then there exists a unique \(F\)-\(m\)-span \(\Phi : F^{m+1} E \to E'\) of width less than \(p^m\) such that the saturations of \(F^{m+1} \text{Fil}^k\) and \(\Phi^{-1}(p^k E')\) with respect to \((J_X/S, \{ \})\) coincide. This construction is functorial in \((E, \text{Fil})\) and the functor is fully faithful.

**Proof.** — Follows from Theorem 2.3.3, Proposition 4.3.2 (ii) and Corollary 3.3.5.

\[ \square \]

5. **COMPARISON OF TRANSVERSALITY PROPERTIES FOR VARIOUS LEVELS**

From now on, \(m'\) will be an integer larger than \(m\) and \(\{ \}\)' will denote divided powers of level \(m'\). We will also write \(d : = m' - m\).

5.1. **Changing level and Griffiths transversality.**

After recalling how to obtain a \(\mathcal{D}^{(m)}\)-module from a \(\mathcal{D}^{(m')}\)-module, we show that, for filtered \(\mathcal{D}^{(m')}\)-modules transversal to \(p\) of width at most \(p^{m+1}\), Griffiths transversality can be checked on the corresponding filtered \(\mathcal{D}^{(m)}\)-module. We give a counterexample for higher width.

5.1.1. — We recall some results from [B4].

(i) If \(Y\) is a formal scheme and \(I\) is a coherent ideal in \(\mathcal{O}_Y\), then any \(m\)-PD-structure \((J, [\ ])\) on \(I\) is also an \(m'\)-PD-structure on \(I\). If \((S, a, b)\) is a formal \(m\)-PD-scheme and \((Y, I, J)\) is a formal \(m\)-PD-\(S\)-scheme, then it is also a formal \(m'\)-PD-\(S\)-scheme. We should also remark that the \(m'\)-PD-filtration is finer than the \(m\)-PD-filtration.

(ii) Let \(S\) be a formal \(m\)-PD-scheme, \(X\) a formal \(S\)-scheme to which the \(m\)-PD-structure of \(S\) extends and \(i : X \hookrightarrow Y\) an immersion into a formal
S-scheme, then there are canonical maps \( P^n_{X/S(m')}(Y) \rightarrow P^n_{X/S(m)}(Y) \). They are bijective for \( n < p^{m+1} \).

(iii) Assume now that \( X \) is smooth over \( S \). Then we get canonical maps

\[
\mathcal{D}^{(m)}_{X/S n} \rightarrow \mathcal{D}^{(m')}_{X/S n}
\]

that are bijective for \( n < p^{m+1} \). They glue to give canonical maps \( \mathcal{D}^{(m)}_{X/S} \rightarrow \mathcal{D}^{(m')}_{X/S} \) and, after completion, \( \hat{\mathcal{D}}^{(m)}_{X/S} \rightarrow \hat{\mathcal{D}}^{(m')}_{X/S} \). We can therefore consider any \( \mathcal{D}^{(m')}_{X/S} \)-module (resp. \( \hat{\mathcal{D}}^{(m')}_{X/S} \)-module) as a \( \mathcal{D}^{(m)}_{X/S} \)-module (resp. \( \hat{\mathcal{D}}^{(m)}_{X/S} \)-module).

(iv) Assume moreover that \( S \) has no \( p \)-torsion. Then one easily checks that the obvious functor from \( \mathcal{D}^{(m')}_{X/S} \)-modules to \( \hat{\mathcal{D}}^{(m')}_{X/S} \)-modules is faithful. It is even fully faithful when restricted to \( p \)-torsion free objects.

Let \( S \) be a formal \( m \)-PD-scheme and \( X \) a smooth formal \( S \)-scheme to which the \( m \)-PD-structure of \( S \) extends. If \( (\mathcal{F}, \text{Fil}) \) is a Griffiths transversal \( \mathcal{D}^{(m')}_{X/S} \)-module (resp. \( \hat{\mathcal{D}}^{(m')}_{X/S} \)-module), then it is also Griffiths transversal as a \( \mathcal{D}^{(m)}_{X/S} \)-module (resp. \( \hat{\mathcal{D}}^{(m)}_{X/S} \)-module).

The converse is true under some additional hypothesis:

**5.1.2. Proposition.** — Let \( (\mathcal{F}, \text{Fil}) \) be a filtered \( \mathcal{D}^{(m')}_{X/S} \)-module of width at most \( p^{m+1} \) that is Griffiths transversal as a \( \mathcal{D}^{(m)}_{X/S} \)-module and transversal to \( p \). Then it is also Griffiths transversal as a \( \mathcal{D}^{(m')}_{X/S} \)-module.

**Proof.** — We have to show that, if \( P \in \mathcal{D}^{(m')}_{X/S} \) is an \( m' \)-PD-differential operator of order at most \( n \), then \( P(\text{Fil}^k) \subset \text{Fil}^{k-n} \). Thanks to 5.1.1 (iii), we may assume that \( n \geq p^{m+1} \). We proceed by induction on \( k \).

- If \( k \leq p^{m+1} \), then \( \text{Fil}^{k-n} = \mathcal{F} \) and our assertion is trivial.
- If \( k > p^{m+1} \), transversality to \( p \) and the condition on the width give us that \( \text{Fil}^k = p \text{Fil}^{k-1} \). It follows that

\[
P(\text{Fil}^k) = pP(\text{Fil}^{k-1}) \subset p\text{Fil}^{k-1-n} \subset \text{Fil}^{k-n}.
\]

The bound on the width is sharp as the following shows:
5.1.3. Example. — We take $X$ to be the affine line over $S$ and we consider $(\mathcal{F}, \Fil)$ where $\mathcal{F} = \mathcal{O}_X$ and $\Fil$ is defined as follows:

- for $0 \leq k \leq p^{m+1}$, $\Fil^k$ is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;
- for $k > p^{m+1}$, $\Fil^k$ is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p^{k-p^{m+1}-1}t^{p^{m+1}}$.

It is clear that $(\mathcal{F}, \Fil)$ is a filtered $\mathcal{D}_{X/S}^{(m')}$-module of width $p^{m+1} + 1$. It is transversal to $p$ because, for $k \leq p^{m+1}$, both $(p) \cap \Fil^k$ and $p\Fil^k$ are generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k - 1$, together with $pt^k$, while $(p) \cap \Fil^{p^{m+1}+1}$ and $p\Fil^{p^{m+1}+1}$ are generated by the elements $p^{p^{m+1}+1-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $pt^{p^{m+1}}$.

To show that $(\mathcal{F}, \Fil)$ is Griffiths transversal as a $\mathcal{D}_{X/S}^{(m)}$-module, let us remark that $\partial^r(\Fil^k) \subset \Fil^{k-r}$ when $0 \leq k \leq p^{m+1}$. Moreover, when $r \leq p^m$, we have $\binom{p^{m+1}}{r} \in (p)$ and therefore $\partial^r(\Fil^{p^{m+1}+1}) \subset p\Fil^{p^{m+1}+1-r} \subset \Fil^{p^{m+1}+1-r}$. Nevertheless, $(\mathcal{F}, \Fil)$ is not Griffiths transversal as a $\mathcal{D}_{X/S}^{(m)}$-module because $tp^{m+1} \in \Fil^{p^{m+1}+1}$ but $\partial^{p^{m+1}}(tp^{m+1}) = 1 \notin \Fil^1$.

5.2. Frobenius descent and $F^{m+1}$-isogenies.

We are going to apply Berthelot’s theory of Frobenius descent to $F^{m+1}$-isogenies and use it to study the question of the surjectivity of the functor $\mu$ of 2.3.5.

Let $S$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme to which the $m$-PD-structure of $S$ extends. Let $F_0$ be the relative Frobenius of $X_0$ over $S_0$ and $F : X \to X'$ a lifting of $F_0$. We briefly recall Berthelot’s unpublished theory of Frobenius descent.

5.2.1. Proposition (see [B5]). — The morphism

$$F^d \times_S F^d : X \times_S X \longrightarrow X^{(d)} \times_S X^{(d)}$$

induces for all $n$, a unique morphism

$$F^d : P^n_{X/S(m')} \longrightarrow P^n_{X^{(d)}/S(m)}$$

compatible with the PD-structures (taking into account the PD-ideal of $S$). It is also compatible with the partial divided power filtrations.
It follows that, if \( \mathcal{E} \) is a \( \mathcal{D}^{(m)}_{X(S)}/S \)-module, then \( F^{d^*}(\mathcal{E}) \) has a natural structure of \( \mathcal{D}^{(m')}_{X(S)} \)-module.

5.2.2. Theorem (see [B5]). — If \( S \) is a scheme, the functor \( \mathcal{E} \mapsto F^{d^*}(\mathcal{E}) \) induces an equivalence between the categories of \( \mathcal{D}^{(m)}_{X(S)}/S \)-modules and \( \mathcal{D}^{(m')}_{X(S)} \)-modules.

It follows that the functor \( \mathcal{E} \mapsto F^{d^*}(\mathcal{E}) \) induces an equivalence between the category of complete \( \widehat{\mathcal{D}}^{(m)}_{X(S)} \)-modules and the category of complete \( \mathcal{D}^{(m')}_{X(S)} \)-modules. From Proposition 1.2.2, we get an equivalence between the category of \( p \)-isogenies of complete \( \widehat{\mathcal{D}}^{(m)}_{X(S)} \)-modules and the category of \( p \)-isogenies of complete \( \mathcal{D}^{(m')}_{X(S)} \)-modules. Thus, we get:

5.2.3. Corollary. — The functor \( F^{d^*} \) makes the full subcategory of \( F^{m+1} \)-\( p \)-isogenies on \( X(d)/S \) consisting of those \( \Phi : F^{m+1} \mathcal{E} \to \mathcal{F} \) where \( \mathcal{E} \) is a \( \widehat{\mathcal{D}}^{(m')}_{X(m'+1)/S} \)-module equivalent to the category of \( F^{m'+1} \)-\( p \)-isogenies on \( X/S \).

5.2.4. Lemma. — Let \( (\mathcal{F}, \text{Fil}) \) be a filtered \( \mathcal{D}^{(m)}_{X(S)} \)-module of width less than \( p^{m+1} \) that is transversal to \( p \) and \( \overline{\text{Fil}} \) the saturation of \( \text{Fil} \) with respect to \( (p, \{ \}) \). Then \( (\mathcal{F}, \text{Fil}) \) is Griffiths transversal if and only if \( (\mathcal{F}, \overline{\text{Fil}}) \) is Griffiths transversal.

**Proof.** — The filtrations are identical up to order \( (p^{m+1} - 1) \) and, for any \( k \geq 0 \), we have

\[
\text{Fil}^{p^{m+1} - 1 + k} = p^k \text{Fil}^{p^{m+1} - 1} \quad \text{and} \quad \overline{\text{Fil}}^{p^{m+1} - 1 + k} = (p^k) \overline{\text{Fil}}^{p^{m+1} - 1}. \]

Assume now that \( S \) is a \( p \)-torsion free formal PD-scheme and that there are local coordinates \( t_1, \ldots, t_d \) on \( X \) and \( X' \) such that \( F^*(t_i) = t_i^p \).

5.2.5. Proposition. — The functor \( \mu \) of 2.3.5 is not in general an equivalence of categories for \( m > 0 \). However, it becomes an equivalence when restricted to objects of width at most \( p \).

**Proof.** — Let \( \Phi : F^{m+1} \mathcal{E} \to \mathcal{F} \) be an \( F^{m+1} \)-\( p \)-isogeny on \( X/S \) of width less than \( p^{m+1} \). By Corollary 5.2.3, it corresponds to a unique \( F \)-\( p \)-isogeny \( \Phi^0 : F^* \mathcal{E} \to \mathcal{F}' \) on \( X^{(m)}/S \). We have shown in section 3.3 how to associate to \( \Phi^0 \) a filtration \( \text{Fil} \) on \( \mathcal{E} \) that is transversal to \( p \). Thanks to
Proposition 3.3.3 and [O2], 5.2.12, the filtered module $(E, \text{Fil})$ is Griffiths transversal as a $\mathcal{D}^{(0)}_{X(m+1)/S}$-module. It follows from Lemma 5.2.4 and Proposition 3.3.3 that $\Phi$ will be in the essential image of $\mu$ if and only if $(E, \text{Fil})$ is Griffiths transversal as a $\mathcal{D}^{(m)}_{X(m+1)/S}$-module. If the width is at most $p$ this is always the case by Proposition 5.1.2, while Example 5.1.3 shows that this needs not happen for higher width.

5.2.6. Example. — For $m > 0$, we can give an explicit $F^{m+1}$-$p$-isogeny of width less than $p^{m+1}$ on the formal affine line $X$ which is not in the essential image of $\mu$. We take $E = \mathcal{O}_{X(m+1)}$ and we let $\mathcal{F}$ be the ideal of $\mathcal{O}_X$ generated by the elements $p^{p+1-i}t^i p^{m+1}$ for $0 \leq i \leq p - 1$, together with $t^{m+2}$. It is a sub $\mathcal{D}^{(m)}$-module of $\mathcal{O}_X$ and we let the $p$-isogeny $\Phi : F^{m+1}E \rightarrow \mathcal{F}$ be multiplication by $p^{p+1}$. If we apply the functor $\alpha$ to this $F^{m+1}$-$p$-isogeny, we get the saturation of the following filtration:

- for $0 \leq k \leq p$, $\text{Fil}^k$ is the ideal generated by the elements $p^{k-i} t^i$ for $0 \leq i \leq k$;
- for $k > p$, $\text{Fil}^k$ is the ideal generated by the elements $p^{k-i} t^i$ for $0 \leq i \leq p - 1$, together with $p^{k-p-1} t^p$.

It is not Griffiths transversal because $t^p \in \text{Fil}^{p+1}$ but $\partial[y](t^p) = 1$ is not in $\text{Fil}^1$ and we can use Lemma 5.2.4.

5.2.7. Remark. — Let $\Phi : F^*E \rightarrow \mathcal{F}$ and $\Phi' : F^*\mathcal{F} \rightarrow \mathcal{G}$ be two $F$-$p$-isogenies of width less than $p$. From [O2], 5.2.13, or Proposition 5.2.5, they are in the essential image of the functor $\mu$ for level 0. Assume that $E$ and $\mathcal{G}$ are $\mathcal{D}^{(1)}$-modules and that $\Phi' \circ F^*(\Phi) : F^2E \rightarrow \mathcal{G}$ is a morphism of $\mathcal{D}^{(1)}$-modules. Then it is an $F^2$-$p$-isogeny of width less than $(2p - 1)$, and one may wonder if it is in the essential image of $\mu$. One can show that this is true if $p = 2$, but if $p > 2$ the answer is no in general as the following example on the formal affine line shows:

We take $E = \mathcal{O}$, we let $\mathcal{F}$ be the ideal of $\mathcal{O}$ generated by $p^2$, $pt^p$ and $t^3p$, and $\mathcal{G}$ be the ideal of $\mathcal{O}$ generated by the elements $p^{p+1-i}t^i p^2$ with $0 \leq i \leq p - 1$, together with $t^{2p}$. The $p$-isogenies $\Phi$ and $\Phi'$ are multiplication by $p^2$ and $p^{p-1}$, respectively. The composition of $F^*(\Phi)$ and $\Phi$ is Example 5.2.6 in the case $m = 1$.

5.3. Changing level for $T$-$m$-crystals and $F$-$m$-spans.

We study the behavior of the functors relating $T$-$m$-crystals and $F$-$m$-spans when the level changes and derive some consequences.
5.3.1. **Lemma.** — The functor «saturation with respect to \((p, \{\})\)» from the category of filtered modules transversal to \((p, \{\})'\) to the category of filtered modules transversal to \((p, \{\})\) gives an equivalence of categories when restricted to objects of width less than \(p^{m+1}\).

**Proof.** — This is an immediate consequence of Proposition 1.2.5. □

Let \((S,a,b)\) be a formal \(m\)-PD-scheme. If \(X\) is an \(S\)-scheme, it follows from 5.1.1 (i) that \(\text{Cris}^{(m)}(X/S)\) is a subsite of \(\text{Cris}^{(m')}((X/S)'. By restriction, any sheaf on \(\text{Cris}^{(m')}(X/S)\) defines a sheaf on \(\text{Cris}^{(m)}(X/S)\). The \(m'\)-PD-filtration restricts to a filtration on the structural sheaf \(\mathcal{O}^{(m)}_{X/S}\) of \(\text{Cris}^{(m)}(X/S)\) that is finer than the \(m\)-PD-filtration.

Using restriction and then saturation with respect to the \(m\)-PD-filtration, any \(T\)-module \(E\) on \(\text{Cris}(X/S)^{(m')}\) defines a \(T\)-module on \(\text{Cris}(X/S)^{(m)}\). It is clear that this process is functorial and that, when applied to \(T\)-\(m'\)-crystals, it produces \(T\)-\(m\)-crystals.

Assume from now on that \(S\) has no \(p\)-torsion and that \(X\) is a smooth \(S_0\)-scheme.

5.3.2. **Proposition.** — Consider the functor that associates a \(T\)-\(m\)-crystal to a \(T\)-\(m'\)-crystal. Restricted to \(p\)- torsion free \(T\)-\(m'\)-crystals of width less than \(p^{m+1}\), it is fully faithful and its essential image is the full subcategory of \(p\)-torsion free \(T\)-\(m\)-crystals of width less than \(p^{m+1}\) whose underlying crystal is the restriction of an \(m'\)-crystal.

**Proof.** — This is a local question and all our constructions are functorial. Using Corollary 4.2.6 and Lemma 5.3.1, the first assertion is a consequence of 5.1.1 (iv) and the second follows from Proposition 5.1.2. □

Let \(F:X \to X'/S_0\) be the relative Frobenius of \(X\) over \(S_0\). We will write \((X/S)_{\text{cris}}^{(m)}\) for the crystalline topos of level \(m\). In [B3] Berthelot shows that the morphism of crystalline topoi of level \(m\) induced by \(F^d\) factors canonically through the restriction map \((X/S)_{\text{cris}}^{(m')} \to (X/S)_{\text{cris}}^{(m)}\) to give a morphism

\[ F^d : (X/S)_{\text{cris}}^{(m')} \to (X^{(d)}/S)_{\text{cris}}^{(m)}. \]

Under the equivalence of Corollary 4.1.8, this construction is compatible with that of Proposition 5.2.1.
5.3.3. **Proposition.** — The functor $F^{d^*}$ makes the full subcategory of $F$-$m$-spans on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1} E \to E'$ where $E$ is an $m'$-crystal on $X^{(m'+1)}/S$ equivalent to the category of $F$-$m'$-spans on $X/S$.

**Proof.** — This is a again a local question. Using Corollary 4.2.6, the assertion reduces to Proposition 5.2.3. \qed

5.3.4. **Remark.** — When restricted to objects of width less than $p^{m+1}$, we have commutative diagrams:

\[
\begin{array}{ccc}
p\text{-torsion free} & & \\
\text{T-$m'$-crystals} & \longrightarrow & \text{$F$-$m'$-spans on } X/S \\
on X^{(m'+1)}/S & \downarrow & \\
p\text{-torsion free} & \longrightarrow & \text{$F$-$m$-spans on } X^{(d)}/S \\
\text{T-$m$-crystals} & & on X^{(m'+1)}/S
\end{array}
\]

where the horizontal arrows come from Theorem 4.3.6 and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3; and, when $S$ is a PD-scheme:

\[
\begin{array}{ccc}
p\text{-torsion free} & & \\
\text{T-$m$-crystals} & \longrightarrow & \text{$F$-$m$-spans on } X/S \\
on X^{(m'+1)}/S & \downarrow & \\
p\text{-torsion free} & \longrightarrow & \text{$F$-$0$-spans on } X^{(m)}/S \\
\text{T-$0$-crystals} & & on X^{(m'+1)}/S
\end{array}
\]

where the top arrow comes from Theorem 4.3.6, the bottom one from Theorem 5.2.13 of [O2] and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3.

5.3.5. **Proposition.** — The construction of 4.3.6 does not give an equivalence of categories in general. However, if $S$ is a PD-scheme, it becomes an equivalence when restricted to objects of width at most $p$.

**Proof.** — Follows from Corollary 4.2.6 and Proposition 5.2.5. \qed
BIBLIOGRAPHY


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