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ON BOUNDARY SLOPES OF IMMERSED INCOMPRESSIBLE SURFACES

by Mark D. BAKER

1. Introduction.

Let M be a compact, orientable, irreducible 3-manifold with ∂M a torus. Hatcher [H] showed that there are only finitely many slopes on ∂M realized by boundary curves of embedded incompressible, ∂ -incompressible surfaces in M .

In this paper we show that there can be infinitely many slopes on ∂M realized by the boundary curves of immersed, incompressible, ∂ -incompressible surfaces in M which are embedded in a neighborhood of ∂M .

2. Notation and statement of results.

Let T_0 be the torus with an open disk removed, pictured in Figure 1. Let D_x (resp. D_y) denote the right handed Dehn twist about the loop x (resp. the loop y).

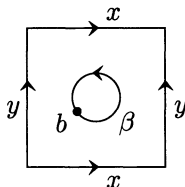


Figure 1

Consider the manifold M , a once punctured torus bundle over S^1 , given by

$$M \cong T_0 \times [0, 1] / (h(s), 0) \sim (s, 1)$$

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where $h = D_y^{-4} \circ D_x^4$. Fix a basepoint $b \in \partial T_0$. The loops $\alpha = b \times [0, 1] / \sim$, $\beta = \partial T_0$ give a coordinate system for ∂M ; the loop $\alpha^\mu \beta^\lambda$ in ∂M is represented by the pair $(\mu, \lambda) \in \mathbb{Z}^2$ and is said to have slope μ/λ .

Now, if $S \subset M$ is an immersed surface, properly embedded in a neighborhood of ∂M , then ∂S consists of parallel simple closed curves in ∂M parametrized by a coprime pair (μ, λ) .

THEOREM. — *For M as above, the coprime pair (μ, λ) is realized by the boundary curves of an immersed, incompressible, ∂ -incompressible surface provided $\mu \geq 1$ and $|\lambda| > \mu$.*

Remarks.

1) It suffices to prove the theorem for (μ, λ) with $\mu \geq 1$ and $\lambda > \mu$ since M admits an involution sending the curves α, β to $\alpha, -\beta$. Indeed, let $k: T_0 \rightarrow T_0$ be a reflection in the diagonal (see Figure 1) followed by D_y^{-4} . Then k^2 is isotopic to $D_y^{-4} \circ D_x^4$ and the map $(x, t) \mapsto (k(x), t + \frac{1}{2})$ induces the desired involution on M .

2) The immersed surfaces of the theorem are virtually embedded in M : they lift to embedded surfaces in finite covers of M (see §3) and one obtains virtually Haken manifolds by Dehn filling on M with respect to these boundary slopes.

3) There exist manifolds with torus boundary (N Seifert fibered for example) for which only finitely many slopes on ∂N are realized as the boundary curves of essential immersed surfaces.

3. Proof of theorem.

3.1. — We prove our result by constructing, for each (μ, λ) , a finite covering space $\widetilde{M} \rightarrow M$ such that:

- (i) The loop $\alpha^\mu \beta^\lambda$ lifts to loops in each of the four components of $\partial \widetilde{M}$.
- (ii) In $H_1(\widetilde{M}; \mathbb{Z})$ there exists a relation of the form $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$ where γ_i is a lift of $\alpha^\mu \beta^\lambda$ to the i -th component of $\partial \widetilde{M}$.

Property (ii) implies that \widetilde{M} contains an incompressible, ∂ -incompressible surface S' whose boundary consists of the loops $\gamma_1, \dots, \gamma_4$. Indeed, if we consider a triangulation for \widetilde{M} and simplicial homology, the fact that $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4$ is a primitive element in $H_1(\partial \widetilde{M}; \mathbb{Z})$ that is

zero in $H_1(\widetilde{M}; \mathbb{Z})$ implies that these loops bound an oriented 2-complex, K , in \widetilde{M} with property that an even number of triangles meet at each interior edge, with oriented sum equal to zero. Thus by cutting and pasting along the edges of K , and then pulling apart at vertices, we obtain an embedded surface which can then be compressed. Now by property (i) this surface projects to an immersed incompressible surface S in M with boundary consisting of four parallel copies of $\alpha^\mu \beta^\lambda$.

The cover \widetilde{M} is obtained by constructing a cover $F \rightarrow T_0$ to which $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h}: F \rightarrow F$. Then \widetilde{M} is the mapping torus of the pair (F, \tilde{h}) .

3.2. — We construct $F \rightarrow T_0$ by cutting and pasting together copies of the following two covers of T_0 :

a) The four-fold cover $X' \rightarrow T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$ defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$.

b) The two-fold cover $Y' \rightarrow T_0$ corresponding to the kernel of the map $\theta: \pi_1(T_0) \rightarrow \mathbb{Z}/2$ defined by $\theta([x]) = 0$ and $\theta([y]) = 1$.

Note that X' (resp. Y') is a torus with four (resp. two) boundary circles.

Now alter X' (resp. Y') by making two (resp. one) vertical cuts τ_1, τ_2 (resp. τ) between boundary circles as shown in Figure 2a (resp. Figure 2b). Denote the cut surfaces by X and Y .

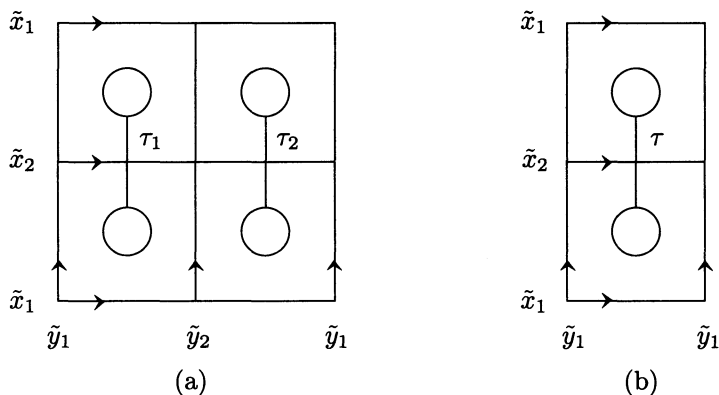


Figure 2

3.3. — Since $\lambda > \mu$, we have that $\lambda - \mu = 2r + s$ for $0 \leq s \leq 1$. Then $F \rightarrow T_0$ is obtained by gluing together $\mu + 2r$ copies of X and $2s$ copies

of Y as indicated in steps i) – v) below. The cover F for $(\mu, \lambda) = (2, 7)$, with edges glued as numbered, is illustrated in Figure 3.

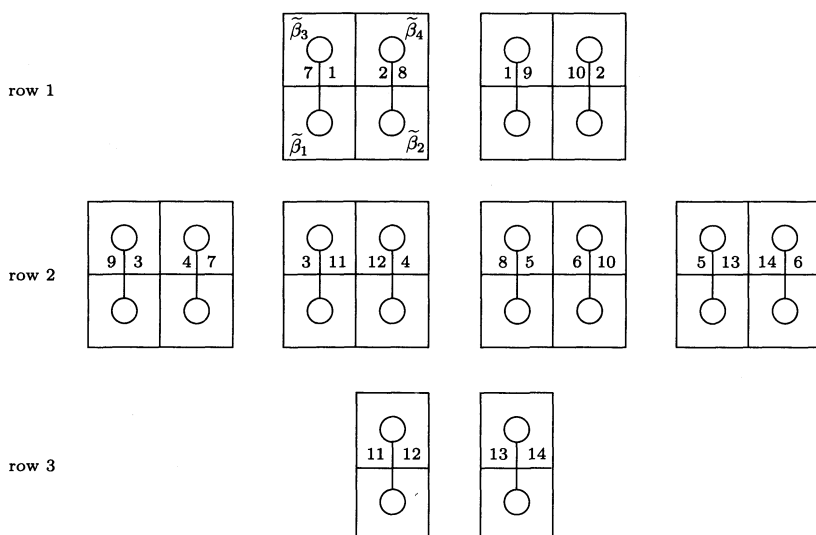


Figure 3

i) First arrange μ copies of X , indexed X_1, \dots, X_μ , in a row, followed by the remaining $2r$ copies of X , indexed X_1^2, \dots, X_{2r}^2 in a second row. The $2s$ copies of Y , indexed Y_1, \dots, Y_{2s} form the third row.

Note that row 1 is never empty whereas either row 2 or row 3 (but not both) can be.

ii) In the first row, glue the right edge of τ_1 (resp. left edge of τ_2) in X_i to the left edge of τ_1 (resp. right edge of τ_2) in X_{i+1} for $i = 1, \dots, \mu - 1$. In the second row, perform the same gluing as in the first row, first on the X_1^2, \dots, X_r^2 for $i = 1, \dots, r - 1$ and then on $X_{r+1}^2, \dots, X_{2r}^2$ for $i = r + 1, \dots, 2r - 1$.

The gluing pattern for the remaining edges depends on the values of r and s . The three distinct cases are treated in iii) – v) below.

iii) *The case $r > 0$ and $s = 1$.* There are sixteen remaining edges to glue in this case. Glue the left edge of τ_1 in X_1 to the right edge of τ_2 in X_1^2 ; the right edge of τ_2 in X_1 to the left edge of τ_1 in X_{r+1}^2 ; the right edge of τ_1

in X_μ to the left edge of τ_1 in X_1^2 ; the left edge of τ_2 in X_μ to the right edge of τ_2 in X_{r+1}^2 .

Finally, attach Y_1 to X_r and Y_2 to X_{2r} by, in each case, gluing the left edge (resp. right edge) of τ to the right edge of τ_1 (resp. left edge of τ_2).

iv) *The case $r > 0$ and $s = 0$.* Glue the copies of X as in iii). Finish by gluing the right edge of τ_1 to the left edge of τ_2 , first in X_1^2 and then in X_{2r}^2 .

v) *The case $r = 0$ and $s = 1$.* There are eight edges to glue. Glue the left edge of τ_1 (resp. right edge of τ_2) in X_1 to the right edge of τ in Y_1 (resp. left edge of τ in Y_2).

Finally, pair the right edge of τ_1 (resp. left edge of τ_2) in X_μ with the left edge of τ in Y_1 (resp. right edge of τ in Y_2).

Note that the surface F is indeed a cover of T_0 ; some of its properties are given in:

LEMMA 1. — *The surface F is a 4λ -fold cover of T_0 . It is of genus $(2\lambda - 1)$ with four boundary components, each of which projects λ to 1 onto $\beta = \partial T_0$.*

3.4. — Now the loop x (resp. y) in T_0 is covered by loops in F which project d to 1 onto x for $d \in 1, 2, 4$ (resp. 2 to 1 onto y). Thus D_x^4 and D_y^{-4} lift to appropriate powers of Dehn twist homeomorphisms in F , whence $h = D_y^{-4} \circ D_x^4$ lifts to a homeomorphism $\tilde{h}: F \rightarrow F$, which fixes ∂F pointwise. Denote by \widetilde{M} the mapping torus of (F, \tilde{h}) :

$$\widetilde{M} = F \times [0, 1] / (\tilde{h}(s), 0) \sim (s, 1).$$

From the construction of \widetilde{M} , it is clear that we can choose on each boundary torus, T_i , of \widetilde{M} a pair of loops $\tilde{\alpha}_i, \tilde{\beta}_i$ which cover α, β in ∂M . The loop $\tilde{\alpha}_i$ (resp. $\tilde{\beta}_i$) projects 1 to 1 onto α (resp. λ to 1 onto β). We will index the four tori in all cases so that the labelling matches that of X_1 in Figure 3.

3.5. — It is clear from the preceding paragraph that the loops $\tilde{\alpha}_i^\mu \tilde{\beta}_i^\lambda$ in $\partial \widetilde{M}$ project homeomorphically to $\alpha^\mu \beta^\lambda$ in ∂M . Thus property (i) of §3.1 is verified and all that remains to show is that property (ii) holds.

3.6. LEMMA 2. — *Let γ_i denote the loop $\tilde{\alpha}_i^\mu \tilde{\beta}_i$. Then, in $H_1(\widetilde{M}; \mathbb{Z})$,*

$$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0.$$

Hence there is a properly embedded incompressible surface $S' \subset \widetilde{M}$ with $\partial S'$ the collection of boundary curves $\tilde{\alpha}_i^\mu \tilde{\beta}_i$ which projects to an immersed incompressible surface S in M with boundary consisting of four parallel copies of $\alpha^\mu \beta^\lambda$.

Proof. — It suffices to show that in $H_1(\widetilde{M}; \mathbb{Z})$ the following relation holds:

$$(*) \quad (\mu \tilde{\alpha}_1 + \tilde{\beta}_1) - (\mu \tilde{\alpha}_2 + \tilde{\beta}_2) + (\mu \tilde{\alpha}_3 + \tilde{\beta}_3) - (\mu \tilde{\alpha}_4 + \tilde{\beta}_4) = 0.$$

Note that

- (a) $\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3 + \tilde{\beta}_4 = 0$ the relation being given by the fiber surface F , and
- (b) $\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{\beta}_1 + \tilde{\beta}_3) = 0$.

Relations (a) and (b) imply (*).

One sees (b) as follows: Consider the loops \tilde{y}_1, \tilde{y}_2 on X , pictured in Figure 2, and let $\tilde{y}_{1,i}$ and $\tilde{y}_{2,i}$ denote the corresponding loops in $X_i \subset F$, $i = 1, \dots, \mu$ (see §3.3). Now on each of these X_i , let $\gamma_i \subset X_i$ (resp. $\delta_i \subset X_i$) denote a horizontal, properly embedded arc between $\tilde{\beta}_1$ and $\tilde{\beta}_2$ crossing $\tilde{y}_{2,i}$ (resp. between $\tilde{\beta}_3$ and $\tilde{\beta}_4$ crossing $\tilde{y}_{1,i}$). Then the disks $\gamma_i \times I \subset F \times I$ and $\delta_i \times I \subset F \times I$ provide the relations

$$\begin{aligned} \tilde{\alpha}_1 - \tilde{\alpha}_2 + (\tilde{h}(\gamma_i) * \gamma_i^{-1}) &= 0, \\ \tilde{\alpha}_3 - \tilde{\alpha}_4 + (\tilde{h}(\delta_i) * \delta_i^{-1}) &= 0, \end{aligned}$$

where $\tilde{h}(\gamma_i) * \gamma_i^{-1}$ (resp. $\tilde{h}(\delta_i) * \delta_i^{-1}$) denotes path composition and is equal to $2\tilde{y}_{2,i}$ (resp. $-2\tilde{y}_{2,i}$).

Combining the 2μ relations thus obtained gives:

$$\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{y}_2, 1 + \dots + \tilde{y}_2, \mu) - 2(\tilde{y}_{1,1} + \dots + \tilde{y}_{1,\mu}) = 0$$

and hence

$$\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{\beta}_1 + \tilde{\beta}_3) = 0$$

since the sum of loops about the boundary circles contributing to $\tilde{\beta}_1$ and $\tilde{\beta}_3$ in the X_i^2 and Y_i are all homologous to zero. \square

4. Concluding remarks.

1) A once-punctured torus bundle will have infinitely many slopes realized by immersed incompressible surfaces if its characteristic homeomorphism is of the form $D_x^{r_1} \circ D_y^{s_1} \circ \cdots \circ D_x^{r_k} \circ D_y^{s_k}$ where $2 \mid s_i$ and $4 \mid r_i$, $i = 1, \dots, k$, provided that $s_1 + \cdots + s_k \neq 0$. Hence the same will be true for any once-punctured torus bundle whose monodromy (in $\mathrm{SL}_2(\mathbb{Z})$) has a power that is conjugate to the monodromy of one of the above bundles.

2) The cut and paste techniques in §3 can also be used to produce families of once-punctured surface bundles over S^1 of any given genus $g > 1$ having infinitely many slopes realized by immersed incompressible surfaces.

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