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THE POINCARÉ–BENDIXSON THEOREM AND ARATIONAL FOLIATIONS ON THE SPHERE

by Igor NIKOLAEV (*)

Introduction.

The Poincaré-Bendixson theorem is an elementary and important statement regarding the structure of α- and ω-limit sets of orbits of flows on the 2-sphere. It says that on the 2-sphere the long-time behavior of trajectories of any C^∞ flow with a finite number of equilibria,(1) must be regular. Namely, the α- and ω-limit set of every trajectory is necessarily a (separatrix) cycle or an isolated equilibrium point. This remarkable fact is purely topological and can be extended to a very few surfaces. (2) There are no analogies of the Poincaré-Bendixson theorem on the higher genus surfaces, since the non-trivial recurrent motions complicate the structure of the respective limit sets, see [33].

It is known also, that if one formally replaces 'trajectories' by 'leaves' and 'C^∞ flow' by 'C^∞ foliation' (for definitions see the section below), then a word-by-word analogy of the Poincaré-Bendixson theorem on the 2-sphere will fail. For warming-up examples, the paper of H. Rosenberg [36] is recommended where various smooth labyrinths in the disc are constructed. (A *labyrinth* is a singular foliation $\mathcal{L}$ in the disc with a leaf which is dense in some region U of the disc. Clearly, the disc can be compactified to a sphere with the corresponding labyrinth on it.)

Being a classical object in geometry, foliations are much akin to flows on surfaces. As in the case of flows, foliations define a partition of a surface

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(1) This claim cannot be omitted. A relevant counterexample is constructed in [4].
(2) Namely, a projective plane $\mathbb{R}P^2$ for which S^2 is a double covering surface. Aranson [2] and, independently, Markley [27] proved it to the Klein bottle.

into disjoint trajectories called *leaves*, this partition having a very simple local structure: apart from the singularities it looks like a family of parallel lines. Unlike flows, foliations may have non-orientable singularities, *i.e.* singularities which cannot be thought of as a phase portrait of a vector field. The presence of the latter in the labyrinth explains its complex behavior.

Indeed, every non-orientable singularity can be rendered orientable on an auxiliary Riemannian surface defined locally by a complex function $z \mapsto z^2$. Globally, each (non-orientable) foliation $\mathcal{F}$ can be lifted to a 2-fold covering surface, M, with a ramification set of index 2 in the points $\mathrm{Sing}\,\mathcal{F}$, see [8] for details. Usually, the genus of the covering surface is greater than the genus of the underlying one. So even for a 2-sphere, the genus of M is $g \geq 1$, and the corresponding (involutive) covering flow on M allows non-trivial recurrent orbits. (As a relevant paradigm, one can think about a 'standard' foliation on S^2 with 4 thorns which can be lifted to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with an irrational flow on it.)

Historically, different authors contributed to understanding of such kind of phenomena. Nemytskii and Stepanov were the first to mention 'strange' singularities in the book [32]. From the geometrical point of view, singular foliations on surfaces were investigated by Thurston [37], Rosenberg [36], Levitt [23], [24], [25], Levitt and Rosenberg [26], Gutiérrez [16], Langevin and Possani [22], Aranson and Zhuzhoma [5]. Structural stability of foliations was studied by Guinez [13], [14], Gutiérrez [15], Gutiérrez and Sotomayor [17], [18] and Kadyrov [20]. (Ergodic properties of foliations were taken up by J. Hubbard, A. Katok, M. Keane, H. Masur, E. Sataev, Ya. Sinai and others. Similar problems in the one-dimensional dynamics have been investigated by V. I. Arnold, M. Herman, M. Martens, W. de Melo, P. Mendes, S. van Strien, J.-C. Yoccoz, see [31] for the bibliography.)

Consider now a C^∞ foliation on the 2-sphere with a finite number of non-orientable singularities. It is unclear *a priori* whether it will have non-trivial recurrent leaves, or not. We can ask: which extra conditions on the foliation will provide its regular ('Poincaré-Bendixson') behavior?

In the present paper this particular question is considered. A foliation is supposed to have an (even) number of singular points of 'tripod', 'thorn' or 'sun-set' type. All of them were proved to be structurally stable [8]. The divergence of the foliation in the above singularities is introduced, and it is proved that the non-positive values of the divergence at each singular point will ensure 'regularity' of the foliation.

In §1 of the paper the differentiability and other preliminary facts on foliations are discussed. The main statement is formulated in §2 and is proved in §3. §4 contains necessary analytical tools. In §5 the problem of H. Rosenberg is discussed.

1. Differentiability of foliations.

This section is devoted to the basic facts of the general theory of (singular) foliations on surfaces. More extended exposition can be found in [31], see also [3], [8].

Roughly speaking, foliations can be defined as flows which admit a finite number of non-orientable singularities.(3) As it was mentioned earlier, it is impossible to produce a foliation given on one surface with the help of a flow on the same surface. Nevertheless, this goal can be achieved on some auxiliary 2-dimensional manifold, M, which covers the initial surface twice, being ramified over non-orientable singularities. In defining foliations below, an axiomatic approach is chosen.

DEFINITION 1.1. — *Let M be a compact orientable 2-dimensional manifold. A foliation $\mathcal{F}$ of differentiabilty class C^r is defined to be a triple (M, π^t, θ), where $\pi^t : M \times \mathbb{R} \to M$ is a C^r-smooth flow; $\theta : M \to M$, $\theta^2 = \mathrm{id}_M$ is an involution on M which satisfies the following axioms:*

(i) *θ preserves the orbits $O(x) = \{\pi^t(x) \mid t \in \mathbb{R}\}$ of the flow π^t, that is*

$$\forall x \in M,\ O\big(\theta(x)\big) = \theta\big(O(x)\big);$$

(ii) *θ fixes a finite even number of points $W = \{p_1, ..., p_{2k}\}$ on M,*

$$\theta(x) \cap x = \begin{cases} x, & \text{if } x \in W, \\ \emptyset, & \text{if } x \in M \backslash W; \end{cases}$$

(iii) *if $W = \emptyset$, then M consists of two connected components $N_1 \cup N_2$ so that the involution $\theta : N_1 \to N_2$ is a homeomorphism between them.*

A foliation $\mathcal{F} = (M, \pi^t, \theta)$ is called orientable, if $W = \emptyset$. Otherwise it is called non-orientable.

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(3) In fact, also a global non orientabilty (Reeb components, *e.g.*) may have place. However, we assume only a local non orientability, passing if necessary to a non-ramified double covering surface.

(Prove that all points $x \in W$ are equilibria of the flow π^t. Show also that the forward orbit $\{\pi^t(x) \mid t \geq 0\}$ through a point $x \in M$ goes by θ to the backward orbit $\{\pi^t(\theta(x)) \mid t \leq 0\}$ through the point $\theta(x)$, and *vice versa*. In particular, $\alpha(\theta(x)) = \theta(\omega(x))$, where α and ω are limit sets of the corresponding orbits.)

Clearly, if a triple $\mathcal{F} = (M, \pi^t, \theta)$ is given and $W \neq \emptyset$, then factorizing with respect to θ, one obtains a non-orientable foliation $F = \pi^t/\theta$ on the surface M/θ. It is a remarkable fact that the converse is also true. Let F be a foliation on a surface N with $2k$ non-orientable singularities and q orientable singularities.

LEMMA 1.1 (Hurwitz). — *Let $W = \{p_1, ..., p_{2k}\}$ be an even number of points on a compact orientable surface N. Then there exists a unique two-fold ramified covering compact orientable surface, M, with a covering mapping $p : M \to N$ and ramification points of index 2 at W. Moreover, there exists a homeomorphism $\theta : M \to M$ which fixes points $p^{-1}(W)$ (and only them) and such that $\theta^2 = \mathrm{id}_M$. For surface N it holds that $N = M/\theta$.*

LEMMA 1.2 (Riemann-Hurwitz formula). — *Denote by g_M the genus of surface M, which covers twice a surface N of genus g_N, with ramification set $W = \{p_1, ..., p_{2k}\}$. Then the following formula is valid:*

$$g_M = 2g_N + k - 1. \tag{1}$$

LEMMA 1.3. — *Foliation F on the surface N can be written as a triple (M, π^t, θ), where flow π^t has $2(p+q)$ singularities (all of them being orientable).*

Proof. — See [19], [8], [31]. □

DEFINITION 1.2. — *Let $\mathcal{F} = (M, \pi^t, \theta)$ be a foliation of the class C^r. Then, the factor-foliation $\mathcal{F}/\theta$ of the factor-surface M/θ is said to be C^r-smooth.*

Differentiability class of a foliation $\mathcal{F}$ in the points $x \in M \setminus \operatorname{Sing} \mathcal{F}$ is well understood. This class is supposed to be C^r in the point x if there exists a C^r-diffeomorphism bringing the foliated neighborhood U of x to the box $\left\{(u,v) \mid -\frac{1}{2} \leq u \leq \frac{1}{2},\ -\frac{1}{2} \leq v \leq \frac{1}{2}\right\}$, foliated by the family of parallel lines $v = C,\ -\frac{1}{2} \leq C \leq \frac{1}{2}$.

The situation will change dramatically if one tries to extend the definition of differentiability to the set $\operatorname{Sing} \mathcal{F}$. First H. Rosenberg [36] noticed that there is a lot of freedom here. (Of course, if the singularity is orientable, *i.e.* given by a flow, then the differentiability class of $\mathcal{F}$ coincides with the differentiability class of the flow in the point of singularity. But not all singularities are orientable.) Let F be a non-orientable foliation on an underlying surface N. One can impose 'local models' at the k-prong singularity, as proposed by Levitt and Rosenberg in [26] with regard to the measured foliations. A foliation F is of the *class* C^r *in the point* p, if there exists a C^r-diffeomorphism bringing $F \cap O_\varepsilon(p)$ to the k-prong saddle, given by the level set of the complex function $|\operatorname{Re} z^{k/2}|$. Another definition of differentiability is due to Levitt [25] and uses the 'transversal structure' of foliations. A foliation F is said to be C^r*-smooth in the singularity* p if the monodromy mapping between any pair of transversal segments which hit nearby separatrices of p can be extended to a mapping of the class C^r.

In the above cited definitions, differentiability is introduced directly, via foliations $\widehat{F}$ at the underlying surface N. Being non-orientable, $\widehat{F}$ cannot carry a 'tangential structure'. From our point of view, this is what is responsible for differentiabilty of $\widehat{F}$ (at least it is so when $\widehat{F}$ is embeddable into a flow). Further the differentiabilty is defined in terms of its covering flow, as proposed by Definition 1.2.

Denote by $\mathcal{F}^r(M)$ a space of all C^r-smooth foliations (M, π^t, θ) defined on a surface M and endow it with the uniform C^r topology. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}^r(M)$ be elements of the space $\mathcal{F}^r(M)$.

DEFINITION 1.3. — *$\mathcal{F}_1 = (M, \pi_1^t, \theta)$ and $\mathcal{F}_2 = (M, \pi_2^t, \theta)$ are said to be topologically conjugate (this equivalence relation is written as $\mathcal{E} : \mathcal{F}_1 \sim \mathcal{F}_2$) if there exists a homeomorphism $h : M \to M$ which takes orbits of π_1^t into orbits of π_2^t, preserving their orientation, and such that $h \circ \theta = \theta \circ h$ for all $x \in M$. An element $\mathcal{F} \in \mathcal{F}^r(M)$ is called structurally stable if there exists a neighborhood U of $\mathcal{F}$ such that for all $\mathcal{G} \in U$, the relation $\mathcal{G} \sim \mathcal{F}$ holds.*

The germs of structurally stable orientable singularities $q \in \operatorname{Sing} \mathcal{F}$ are well investigated: it is classical that they are topologically (and, beyond resonances, even analytically) equivalent to their 1-jets, and the configuration of leaves near such points looks either like a 4-separatrix saddle, or like a node or, else, a focus. Suppose that $p \in \operatorname{Sing} \mathcal{F}$ is a non-orientable singularity. Then in the point p the germ w of vector field obeys a $\mathbb{Z}_2$-symmetry [4] with respect to an involution θ and has either of

[4] Namely, $w(\theta(x)) = -w(x), w(0) = 0$, where $\theta : x \mapsto -x$ is a rotation through the angle π.

two 'normal forms':

$$w(u,v) = \left[\alpha u^2 + (\beta-1)uv + \phi(u,v)\right]\frac{\partial}{\partial u} + \left[(\alpha-1)uv + \beta v^2 + \psi(u,v)\right]\frac{\partial}{\partial v} \tag{2}$$

if $\alpha\beta(\alpha+\beta-1) \neq 0$, and

$$w(u,v) = [\alpha u^2 + \beta uv - v^2 + \phi(u,v)]\frac{\partial}{\partial u} + [(\alpha+1)uv + \beta v^2 + \psi(u,v)]\frac{\partial}{\partial v} \tag{3}$$

if $\alpha[\beta^2 + (\alpha+1)^2] \neq 0$, where ϕ and ψ consist of higher order monomials of even degrees.

DEFINITION 1.4. — *Let $p \in \operatorname{Sing}\mathcal{F}$ be a non-orientable singularity of foliation (M, π^t, θ) in projection to the surface M/θ. Denote by H, P and E, respectively, hyperbolic, parabolic and elliptic sectors of p. The following terminology is adopted*:

Sectors	Singularity	Index	Form	Conditions
HHH	*tripod*	$-\frac{1}{2}$	(2)	$\alpha\beta(\alpha+\beta-1) < 0$, $(\alpha-1)(\beta-1)(\alpha+\beta) > 0$
HP	*sun-set*	$+\frac{1}{2}$	(2)	$\alpha\beta(\alpha+\beta-1) > 0$
H	*thorn*	$+\frac{1}{2}$	(3)	$\alpha < 0$
EP	*apple*	$+\frac{3}{2}$	(2)	$\alpha\beta(\alpha+\beta-1) < 0$, $(\alpha-1)(\beta-1)(\alpha+\beta) < 0$
E			(3)	$\alpha > 0$

The proposition below yields a complete list of structurally stable singularities, being the main result of the local theory of surface foliations.

THEOREM 1.1 (Local stability). — *Let $\mathcal{F}^r$ be a space of C^r smooth foliations on a surface M. The list of generic non-orientable singularities of foliations $\mathcal{F} \in \mathcal{F}^r$ is exhausted by those mentioned in Definition* 1.4. *Moreover, they and only they are structurally stable.*

As well as in the case of flows, local structural stability of foliation does not imply its global structural stability (the converse being certainly true). The following statement proved in [8] is a generalization of the well-known stability theorem [1], [34] to the case of foliations.

THEOREM 1.2 (Andronov-Pontryagin-Peixoto). — *Let M be an orientable surface. In the space $\mathcal{F}^r(M)$ of all C^r-smooth foliations on M there exists an open and dense subspace $\mathcal{F}_0^r$ which consists of structurally stable foliations. These foliations are called Morse-Smale, and admit the following geometrical description:*

(i) *foliations from $\mathcal{F}_0^r$ has but finitely many orientable singularities and closed orbits, they all being hyperbolic (i.e. saddles, nodes or foci, or else hyperbolic cycles);*

(ii) *the set of non-orientable singularities is also finite and is exhausted by five structurally stable singularities mentioned in Definition* 1.4;

(iii) *every leaf of $\mathcal{F} \in \mathcal{F}_0^r$ distinct from critical elements of items* (i) *and* (ii) *tends to the critical elements;*

(iv) *there are no separatrix connections between critical elements (i.e. there are no leaves tending to singular points p and q, being separatrix for both p and q; the case $p = q$ is not excluded).*

2. Main result.

Below we are interested in the case $N = S^2$. (The 2-dimensional sphere as an underlying surface.) Moreover, the set $\operatorname{Sing} F$ of a C^∞ foliations on N is fixed by an even number of non-orientable singularities of the thorn, sun-set or tripod type, described by Theorem 1.1.

DEFINITION 2.1. — *If a Morse-Smale foliaton $F \in \mathcal{F}_0^r$ has no cycles (closed 1-leaves), it is called a regular*[5] *foliation. (In other words, all leaves of F are compact and tend from a singularity to a singularity.)*

DEFINITION 2.2. — *Foliation F is called arational if it has no compact 1-leaves.*

We introduce a class, an $\mathcal{A}$-class, of foliations on the 2-sphere which are produced from a 'standard foliation with 4 thorns on S^2' by a finite number of homotopy operations of an 'opening of a leaf' and 'deformation of a thorn into a sun-set'.

• *Standard foliation of S^2.* — Consider a 2-dimensional torus T^2 as a covering surface M of a sphere with the ramification set in 4 (distinct) points

(5) Or, gradient like, see [29]

(apply the Riemann-Hurwitz formula (1) for $k = 2$). Take an irrational flow $\pi^t : T^2 \times \mathbb{R} \to T^2$ on T^2 such that chosen points lie on the different recurrent orbits. These points are declared the equilibria (two-separatrix saddles) of the flow π^t. Clearly, the flow π^t on T^2 covers a (non-orientable) foliation, F_4, on S^2 such that Sing F_4 consists of 4 thorns. All 1-leaves of F_4 are recurrent and we call it a 'standard foliation with 4 thorns'. (Further we deal with a covering representation of F_4 which is given by the triple $\mathcal{F}_4 = (T^2, \pi^t, \theta)$.)

• *Opening of a leaf.* — Take a recurrent leaf, γ, which is not a separatrix leaf of a foliation produced from F_4. Making an infinite cut along γ with the banks γ_+ and γ_-, we 'open' γ and then we glue-up between γ_+ and γ_- an infinite narrow stripe F_γ. Then one attaches sun-set and tripod singularities to γ_+ and γ_-, foliating F_γ as it is shown in Fig. 1.

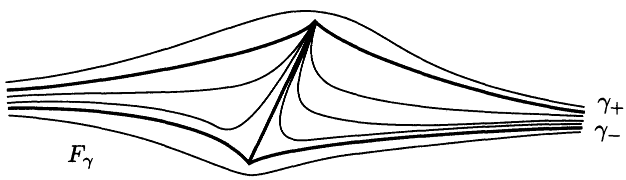

Figure 1

• *Deformation of a thorn into a sun-set.* — Take a thorn singularity of F_4 and produce an infinite cut along it separatrix γ. One 'opens' γ and then one glues-in between the banks of the cut a parabolic (nodal) sector. Clearly, one obtains a foliation on the sphere with an extra sun-set singularity instead of a thorn.

Fix a natural metric on S^2. The 'tongues' which appear as a result of the above homotopy operations can follow two different patterns at the infinity. They either 'expand', covering more and more area on S^2, so that each separatrix of an $\mathcal{A}$-foliation is 'catched' by a parabolic (nodal) sector of a sun-set. Or else, the tongues can be squeezed sufficiently fast, so that no one separatrix of an $\mathcal{A}$-foliation goes to a sun-set. (Note that by the irrationality of F_4 $\mathcal{A}$-foliations admit no cycles.)

COROLLARY 2.1. — *$\mathcal{A}$-foliations are either arational or regular.*

(Class $\mathcal{A}$ is an important class, which includes the Rosenberg labyrinths in the disc and annulus [36], the (generalized) measured foliations [37] and the Plykin attractors on the sphere [35], *etc.*)

Let F be a C^∞ $\mathcal{A}$-foliation on the 2-sphere. Let $\mathcal{F} = (M, \pi, \theta)$ be a covering representation of F on the surface M of genus $g = \frac{1}{2}|\operatorname{Sing} F| - 1$. For every $p \in \operatorname{Sing} \mathcal{F}$ one defines a divergence of $\mathcal{F}$ in p as follows.

DEFINITION 2.3. — *Denote by E_1 and E_2 the flow boxes of a thorn and a sun-set which has been obtained by a deformation of a thorn. Denote by E_3 a (cylindric) flow box which contains a tripod and a sun-set, obtained by a homotopy of the opening of a leaf. (All of them are shown in Fig. 2 next page. Shaded regions correspond to the orbits which enter or emerge from the nodal sectors of the respective singularities.) The divergence in the above singularities is given by the table.*

Singularity	Flow box	Normal form	Divergence
thorn	E_1	(3)	$\operatorname{div}_p \mathcal{F} = 0$
sun-set	E_2	(2)	$\operatorname{div}_p \mathcal{F} = \lvert\alpha - \beta\rvert$
tripod	E_3	(2)	$\operatorname{div}_p \mathcal{F} = \beta - \alpha$
sun-set	E_3	(2)	$\operatorname{div}_p \mathcal{F} = \beta - \alpha$

The 'eigenvalues' α, β of the singularities in the flow box E_3 are chosen such that the in-going (into E_3) separatrix always corresponds to an α-eigenvalue, while out-going separatrix corresponds to a β-eigenvalue.

(The above defined divergence of an $\mathcal{A}$-foliation at the singular points has the same meaning as those of a Cherry flow on torus, considered in [28].)

THEOREM 2.1. — *Let $\mathcal{F}$ be a C^∞ $\mathcal{A}$-foliation on the 2-sphere with at least one sun-set singularity. If in every point $p \in \operatorname{Sing} \mathcal{F}$ the inequality $\operatorname{div}_p \mathcal{F} \leq 0$ holds, then all leaves of $\mathcal{F}$ are compact.*

3. Proof.

Proof will consist of two parts. First we will show that for $\mathcal{A}$-foliations there exists a global cross-section such that the corresponding return map does not admit interval exchange transformations (IET), see [21] for the definition. Next we will prove that under the restrictions, imposed on the divergence of the foliation in singular points, there exists an ergodic measure, invariant under the return map (or, what is the same, there are no 'wandering intervals'). So far, the theorem will follow.

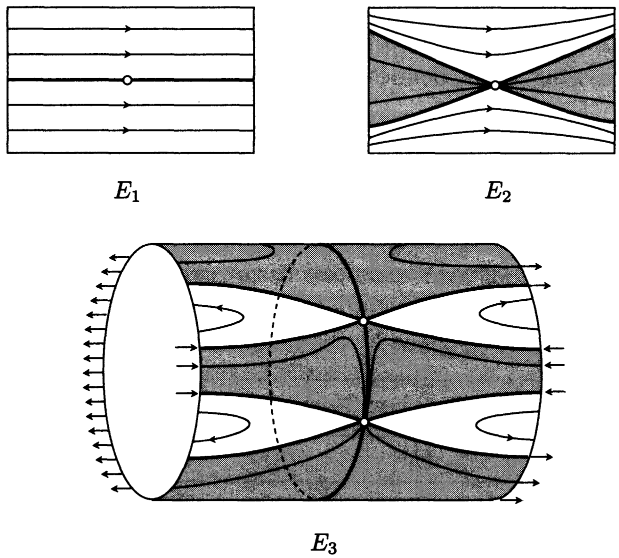

Figure 2

Let $\mathcal{F}$ be an $\mathcal{A}$-foliation on the 2-sphere.

LEMMA 3.1. — *There exists a simple closed curve S^1 which is transversal to $\mathcal{F}$ and each orbit of $\mathcal{F}$ (except for the singularities and the compact separatrices) hits S^1. Moreover, the return mapping induced on S^1 by the orbits of $\mathcal{F}$ preserves the order of the intervals of continuity.*

Proof. — By the definition, $\mathcal{A}$-foliations can be obtained from $\mathcal{F}_4 = (T^2, \pi^t, \theta)$ by a finite number of homotopies.

(i) Since $\pi^t : T^2 \times \mathbb{R} \to T^2$ is an irrational flow on torus, one can easily construct a global cross-section, S^1, of π^t (as such one of the meridians of T^2 can be chosen). The return mapping $\phi : S^1 \to S^1$, induced by the orbits of π^t, is defined everywhere except for 4 distinct points $P = \{p_i \mid i = 1, \dots, 4\}$ in which the separatrices of the 2-saddles hit S^1. The set P splits S^1 into 4 disjoint intervals. However, the permutation $i \mapsto \pi(i)$ defined by ϕ is trivial (*i.e.* it keeps the orientation and the cyclic order of the intervals on S^1).

(ii) Let now a homotopy of the deformation of a thorn into a sun-set be applied to one of the singularities of $\mathcal{F}_4$. It means that in the above mentioned set P a point $p \in P$ must be replaced by a closed segment $A_p \in S^1$, on which the mapping ϕ is not defined. (It corresponds to the substitution $E_1 \to E_2$ of the flow boxes, shown in Fig. 2.) Clearly, the cross-section S^1 remains the same. Moreover, by the arguments of (i), ϕ does not permute the continuity intervals on S^1.

(iii) Let, finally, the homotopy of the opening of a leaf be applied to an $\mathcal{F}_4$ foliation. (The general case of an arbitrary $\mathcal{A}$-foliation is treated similarly and can be included, with slight modifications, in the below scheme.) Let γ be a recurrent orbit of a C^∞ irrational flow π^t different from the separatrices of the 2-saddles. Let us 'open' γ to a narrow stripe $\gamma_\varepsilon = \gamma \times \varepsilon$. Notice that the cross-section S^1 for a such flow will remain the same, as well as the continuity intevals at S^1 will preserve their order with regard to the ϕ-actions.

Now split γ_ε into two parts, $\gamma_\varepsilon^+ \cup \gamma_\varepsilon^-$, and move aside the banks of the cut. This procedure yields us a torus with a hole, D, which breaks the stripe γ_ε.

Let $\theta : T^2 \to T^2$ be an involution, defined by the triple $\mathcal{F}_4 = (T^2, \pi^t, \theta)$. Denote by $\widetilde{\gamma}$ a recurrent orbit, which stays in the involution with γ. Likewise, let $\widetilde{\gamma}_\varepsilon, \widetilde{\gamma}_\varepsilon^+$ and $\widetilde{\gamma}_\varepsilon^-$ be the respective stripes on the torus. Denote by $\widetilde{D}$ a hole which breaks the stripe $\widetilde{\gamma}_\varepsilon$. (All the above construction for the case of torus is pictured in the Fig. 3. Clearly, it is also valid for an arbitrary compact orientable surface with an involution.)

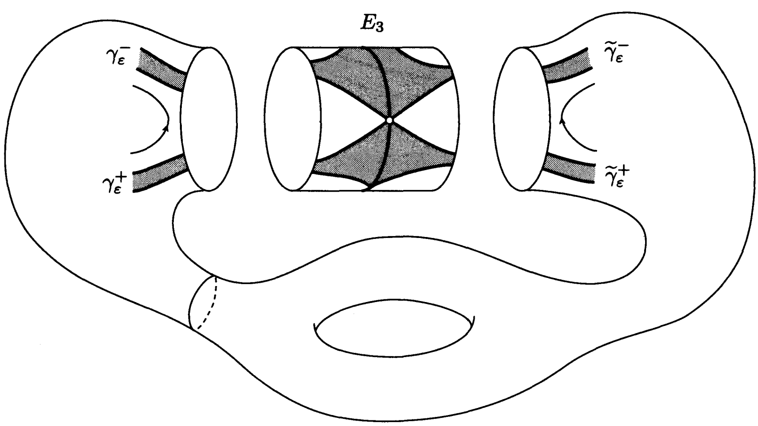

Figure 3

What remains to be done is to glue-up a 'handle' into T^2 which connects D and $\widetilde{D}$. The handle must contain a pair of singularities covering the tripod and the sun-set and equivalent to the flow box E_3 shown in Fig. 2. We claim that the above surface, $T^2 \# T^2$, is a double covering representation surface for a 'standard' foliation on S^2 with one leaf opening. (In Fig. 3 it is shown how to glue-up the relevant handle. Shaded regions must be pasted together.)

Indeed, all the orbits in the stripe γ_ε^+ enter the stable parabolic sector of the singularity in the flow box E_3. All orbits in the stripe γ_ε^- emerge from the unstable sector of the same singularity. Clearly, the above operations does not change the position of the global cross-section S^1. And, as it was noticed at the top of this item, the continuity intervals on S^1 will preserve their cyclic order under the ϕ-actions of the flow on $T^2 \# T^2$. Lemma is proved. □

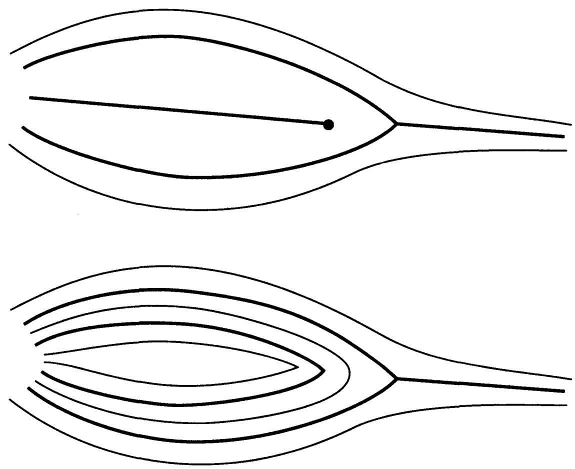

Figure 4

Remark 3.1. — It is tempting to consider also the 'leaf openings' shown in the Fig. 4. However, if we consider the representation of these leaf openings at the covering surface M, it will yield, in general, the interval exchange mappings over the global cross-section S^1. In this case Lemma 3.1 fails. (Problem: find all 'admissible openings' of the standard foliation F_4.)

Let $\phi : S^1 \to S^1$ be a return mapping defined by an $\mathcal{A}$-foliation on a cross-section S^1. As it was mentioned earlier ϕ is defined on $S^1 \setminus \bigcup_{k\in K} A_k$, where A_k are closed sets in S^1. The set A_k is either an isolated point or a segment which corresponds to the points where orbits that go to (go out) a sink (source) section of a saddle-node type singularity representing the sun-set, hit S^1. Our nearest purpose is to extend ϕ to the set $\bigcup_{k\in K} A_k$.

(i) Suppose a separatrix of a 2-saddle hits S^1, as given by the flow-box E_1 shown in Fig. 2. According to Lemma 4.1, the transition map ϕ_0 in the neighborhood of 2-saddle is $y \approx \mathrm{e}^{\pi\beta/\alpha}x + \cdots$. Since $\phi = \phi_R \circ \phi_0$, where ϕ_R is a diffeomorphism along 'regular' part of the flow, the diffeomorphism ϕ near $A_k = p$ has the form $x \mapsto a_0\, \mathrm{e}^{\pi\beta/\alpha}x + \cdots$. We extend ϕ in an evident way:

$$\phi(p) = 0. \tag{4}$$

(ii) Suppose that A_k is a segment in S^1, where 'blind' orbits of the flow in the flow-box E_2 hit S^1. The transition mapping ϕ_0 near the saddle-node type singularity is proven (see §4) to have the form $y \approx |x|^{\alpha/\beta} + \cdots$. By the condition $\mathrm{div}_p\, \mathcal{F} \le 0$ one gets immediately $\alpha = \beta$. Taking advantage of the regular component, the diffeomorphism ϕ near the ends of A_k is given by $|x| \mapsto a_0|x| + \cdots$. Note that by the symmetry of the flow a_0 has the same value for the 'right' and 'left' ends of A_k. Therefore we extend ϕ to A_k linearly:

$$\phi = a_0 x, \quad x \in A_k. \tag{5}$$

(iii) Finally, let us consider the flow-box E_3. Suppose that the transition map near the 6-saddle p_1 is $\phi_0^{(r)}$ and near the saddle-node p_2 is $\phi_0^{(\ell)}$. According to Lemma 4.1, they have the form

$$\phi_0^{(r)}(x) = |x|^{\alpha_1/\beta_1} + \cdots \quad \text{and} \quad \phi_0^{(\ell)} = |x|^{\alpha_2/\beta_2} + \cdots,$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are the local values in the corresponding points. On account of the regular components a_0, b_0, we prolong ϕ to $A_k = [0,1]$ as follows:

$$\phi = \begin{cases} a_0|x-1|^{\alpha_1/\beta_1}, & x \ge 1, \\ -a_0|x-1|^{\alpha_1/\beta_1}, & 1-\varepsilon_1 \le x < 1, \\ \phi \in C^\infty \text{ and monotone}, & \varepsilon_2 < x < 1-\varepsilon_1, \\ b_0|x|^{\alpha_2/\beta_2}, & 0 < x \le \varepsilon_2, \\ -b_0|x|^{\alpha_2/\beta_2}, & x \le 0. \end{cases} \tag{6}$$

By now, we have ϕ extended to the entire S^1. Note that by Lemma 3.1 ϕ is a homeomorphism and by the condition $\mathrm{div}\, \mathcal{F} \le 0$ it is also a diffeomorphism of the circle.

DEFINITION 3.1. — *A diffeomorphism $\phi : S^1 \to S^1$ is said to have a critical point $x \in S^1$ if $D\phi(x) = 0$.*

Clearly, there are no critical points of ϕ in A_k defined by (4) and (5), and there are exactly two critical points $x(p_1) = 0$ and $x(p_2) = 1$ of ϕ defined by (6).

DEFINITION 3.2. — *Let $I \subset S^1$ be an interval in S^1. The interval I is called a wandering interval if $\phi^n(I) \cap \phi^m(I) = \emptyset$, for all $n \neq m$.*

LEMMA 3.2. — *Let $\mathcal{F}$ be a C^∞-smooth $\mathcal{A}$-foliation with $\operatorname{div}_p \mathcal{F} \leq 0$ in each $p \in \operatorname{Sing} \mathcal{F}$. Let $\phi : S^1 \to S^1$ be the corresponding return map on the global cross-section S^1, prolonged according to (4)–(6). Then ϕ has no wandering intervals.*

Proof. — For the proof below we use the Denjoy arguments [11] refined by the technique of Yoccoz [38]. Let $I \subset S^1$ be an interval in S^1 with the ends a and b and let $|I|$ be the length of I. Suppose $\phi : S^1 \to S^1$ be an orientation preserving C^1-homeomorphism with a finite set of critical points K. Along with [38] we consider the function

$$M(\phi, I) = \begin{cases} \frac{|\phi(I)|}{|I|} \big[D\phi(a) D\phi(b)\big]^{-1/2} & a, b \notin K, \\ \infty & a, b \in K. \end{cases}$$

The function M is multiplicative, $M(\phi \circ \psi, I) = M(\phi, \psi(I)) M(\phi, I)$. The strategy one should follow now is to estimate $M(\phi, I)$ from below. It will be done step by step as follows.

(i) Suppose that I does not lie in the vicinity of K. By the Mean Value Theorem, one has

$$M(\phi, I) = D\phi(\xi) \big[D\phi(a) D\phi(b)\big]^{-1/2}, \quad \xi \in I.$$

By assumption, ϕ is $C^{1+\varepsilon}$ and therefore the variation of $D\phi$ on I is finite

$$M(\phi, I) \geq \exp\big(-\tfrac{1}{2} \operatorname{Var}_{\{I\}} D\phi\big).$$

(ii) Let $x_i \in K$ be a critical point of ϕ. Suppose $I \subseteq (x_i - \varepsilon, x_i)$ or $I \subseteq (x_i, x_i + \varepsilon)$.

(a) Let, in addition, $x_i \notin \bar{I}$. In the vicinity of x_i the diffeomorphism ϕ is given by (6):

$$\phi = \pm A|x|^\delta, \quad \text{where} \quad A \in \{a_0^{(i)}, b_0^{(i)}\}_{i=1}^K, \quad \delta \in \{\alpha_i/\beta_i\}_{i=1}^K.$$

Yoccoz [38] demands $(D\phi)^{-1/2}$ to be convex in I (or, what is the same, $S\phi \leq 0$ on I, where $S\phi = \phi'''/\phi' - \frac{3}{2}(\phi''/\phi'))$. We leave it to the reader to check that the Schwarzian derivative S of ϕ is non-positive. Then the inequality $M(\phi, I) \geq 1$ follows [38].

We give an independent estimate. Since $D\phi = \pm A\delta|x|^{\delta-1}$, we calculate directly $M^2(\phi, I) = (\xi^2/ab)^{\delta-1}$, where $\xi \in I$ is the Mean Value Point. For the convex functions $\xi \approx \frac{1}{2}(a+b)$ and therefore $M^2(\phi, I) \geq 1$ (recall that $\delta \geq 1$ by the assumption of the theorem). Hence, $M(\phi, I) \geq 1$.

(b) Suppose, finally, that $x_i \in \bar{I}$. In this case [38] one has $0 \leq x_i - a \leq b - x_i$. We denote $I_1 = [x_i, b]$ and, in view of (6), $D\phi(a)D\phi(b) \leq A^2\delta^2|b - x_i|^{2(\delta-1)}$ and $|\phi(I_1)| = \int_{x_i}^{b} D\phi(t)\,\mathrm{d}t = |b - x_i|^\delta$. Now it follows that

$$M(\phi, I) \geq \frac{|\phi(I_1)|}{2|I_1|}\left[D\phi(a)D\phi(b)\right]^{-1/2} \geq \frac{1}{2A\delta}.$$

The last step to prove Lemma 3.2 is to establish the

PROPOSITION 3.1. — *Let ϕ be a homeomorphism of S^1 specified in lemma 3.2. Then there exists a positive constant C such that for all $N \geq 1$, all $0 \leq k \leq N$ and all $I \subset S^1$, not degenerate to a point, the inequality $M(\phi^k, I) \geq C$ holds.*

Proof follows from the estimates above; we refer the reader to [38]. □

To finish the proof of Lemma 3.2, let us suppose the contrary: there exists a wandering interval $I \subset S^1$ with the endpoints a and b. In particular, it yields that $\lim_{k\to\infty} |\phi^k(I)| = 0$. On account of the order of points a_k, b_k on S^1, it means that $M(\phi^k, I) \to 0$ as $k \to \infty$. This is a contradiction with Proposition 3.1 which proves Lemma 3.2. □

Proof of the Theorem 2.1. — Denote by $A = \bigcup_{k\in K} A_k$ a set of all points $x \in S^1$ where the orbits going-out from the source section of a saddle node k hit S^1 first time, and by $\Omega = \bigcup_{k\in K} \Omega_k$ those points of $x \in S^1$, where the orbits going-in a sink section of a saddle-node k hit S^1 first time. Take a forward orbit $O(A) = \{\phi^{n_1}(A) \mid n_1 \in \mathbb{N}\}$ of A and a backward orbit $O(\Omega) = \{\phi^{-n_2}(\Omega) \mid n_2 \in \mathbb{N}\}$ of Ω. It follows from Lemma 3.2 that there exist a finite n_1 and n_2 such that $O(A)$ and $O(\Omega)$ cover $S^1 \setminus (A \cup \Omega)$. It means that each orbit of $\mathcal{F}$, different from the equilibria, goes from a source section of a saddle-node to a sink section of a saddle-node. In particular, all leaves of $\mathcal{F}$ are compact.

Theorem is proven.

4. Appendix: The Dulac mapping.

Let X be a C^r-smooth vector field with a singularity at 0 which has a saddle section. Let p_1 and p_2 be the points in $W^s(0)$ and $W^u(0)$, respectively. Furthermore, let S_1 and S_2 be a C^2 curves through p_1 and p_2 transversal to X.

DEFINITION 4.1. — *The Dulac (or transition) mapping near the saddle section of a singularity is called a map $T: S_1 \to S_2$ defined between $S_1\backslash p_1$ and $S_2\backslash p_2$.*

4.1. C^0 normal forms.

As it was shown in [8], the Dumortier theorem [12] implies that a C^0 change of coordinates and time brings the germs (2) and (3) to their 2-jets:

$$w(u,v) = [\alpha u^2 + (\beta - 1)uv]\frac{\partial}{\partial u} + [(\alpha - 1)uv + \beta v^2]\frac{\partial}{\partial v}, \tag{7}$$

$$w(u,v) = [\alpha u^2 + \beta uv - v^2]\frac{\partial}{\partial u} + [(\alpha + 1)uv + \beta v^2]\frac{\partial}{\partial v}. \tag{8}$$

Both (7) and (8) are integrable and admit the first integrals (see also [30]) given, respectively, by

$$|u|^\beta |v|^\alpha |u - v|^{1-\alpha-\beta} = C, \tag{9}$$

$$\tfrac{1}{2}(\alpha + 1)\log\left|\frac{u^2}{v^2} + 1\right| + \beta \arctan\frac{u}{v} + \log|v| = C. \tag{10}$$

It is well-known that the germ $T(0)$ of the transition mapping $T: S_1 \to S_2$ does not depend on the particular choice of S_1 and S_2. So far, we consider $S_1 = (-1, x)$, $S_2 = (y, 1)$ for (9) and $S_1 = (-1, x)$, $S_2 = (1, y)$ for (10). The transition mapping $y = T(x)$ is implicitly given by

$$|x|^\alpha |1 + x|^{1-\alpha-\beta} = |y|^\beta |1 + y|^{1-\alpha-\beta}, \tag{11}$$

$$\begin{aligned} &\tfrac{1}{2}(\alpha + 1)\log|1 + x^2| - \beta \arctan\frac{1}{x} - \alpha \log|x| \\ &\qquad = \tfrac{1}{2}(\alpha + 1)\log|1 + y^2| + \beta \arctan\frac{1}{y} - \alpha \log|y|. \end{aligned} \tag{12}$$

It readily follows from (11) and (12) that the main part of $T(0)$ for the normal form (2) is

$$y \approx |x|^{\alpha/\beta} + \cdots, \tag{13}$$

and for the normal form (3) is

$$y \approx \mathrm{e}^{\pi\beta/\alpha}|x| + \cdots. \tag{14}$$

Of course, the normalizing homeomorphism h which brings (2) and (3) to (7) and (8) does not keep C^1 + 'bounded variation' $\simeq C^{1+\varepsilon}$-structure of the foliation. Still it is clear that the restriction of transformations to the class $C^{1+\varepsilon}$ will 'add' some higher degree monomials in the normal forms (7) and (8), and these monomials will not influence much on the transition mappings (13) and (14). This point of view will be formalized in the next section.

4.2. $C^{1+\varepsilon}$ normal forms.

LEMMA 4.1. — *Transition mappings T_1 and T_2 near the singular points defined by the normal forms* (2) *and* (3) *are $C^{1+\varepsilon}$-equivalent to the transition mappings given by* (13) *and* (14).

Proof. — Let $\mathcal{F} \in \mathcal{F}^r(M)$ and let 0 be an equilibrium of $\mathcal{F}$ given in some local charts by the normal form (2) or (3). If we look at $\phi(u,v)$, $\psi(u,v)$ as at the formal series of monomials of even degrees, then, according to the results of [9], after an appropriate change of coordinates and time, in typical case ϕ and ψ take the form

- $\phi(u,v) = 0,\ \psi(u,v) = \sum\limits_{m=2}^{\infty} C_m v^{2m}$ for (2) and
- $\phi(u,v) = \sum\limits_{m=2}^{\infty} C_m v^{2m},\ \psi(u,v) = 0$ for (3).

In fact, as it was proven in [10], the germs (2) and (3) are $C^{1+\varepsilon}$ finitely determined, so that there exists N such that (2) and (3) are $C^{1+\varepsilon}$-equivalent to

$$\begin{aligned} w(u,v) = {} & \big[\alpha u^2 + (\beta-1)uv\big]\frac{\partial}{\partial u} \\ & + \Big[(\alpha-1)uv + \beta v^2 + \sum_{m=2}^{N} C_m v^{2m}\Big]\frac{\partial}{\partial v}, \end{aligned} \tag{15}$$

$$w(u,v) = \Big[\alpha u^2 + \beta uv - v^2 + \sum_{m=2}^{N} C_m v^{2m}\Big]\frac{\partial}{\partial u} + \big[(\alpha+1)uv + \beta v^2\big]\frac{\partial}{\partial v}. \tag{16}$$

It is impossible to estimate directly the mapping T for the germs (15) and (16), because unlike (7), (8) they are not integrable. However, T can be evaluated after the blowing-up procedure [8] applied to (15) and (16), the result of which is given in the charts (u,η), (θ,v), where $\eta = u/v, \theta = v/u$, by the equations

$$\begin{cases} \dfrac{du}{dt} = \alpha u + (\beta - 1)u\eta, \\ \dfrac{d\eta}{dt} = -\eta + \eta^2 + \displaystyle\sum_{m=2}^{N} C_m u^{2m-2}\eta^{2m}, \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{d\theta}{dt} = -\theta + \theta^2 - \theta\displaystyle\sum_{m=2}^{N} C_m v^{2m-2}, \\ \dfrac{dv}{dt} = \beta v + (\alpha - 1)\theta v + \displaystyle\sum_{m=2}^{N} C_m v^{2m-1} \end{cases} \tag{18}$$

for (15) and by

$$\begin{cases} \dfrac{du}{dt} = \alpha u + \beta u\eta - u\eta^2 + \displaystyle\sum_{m=2}^{N} C_m u^{2m-1}\eta^{2m}, \\ \dfrac{d\eta}{dt} = \eta + \eta^3 - \displaystyle\sum_{m=2}^{N} C_m u^{2m-2}\eta^{2m+1}, \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{d\theta}{dt} = -1 - \theta^2 + \displaystyle\sum_{m=2}^{N} C_m v^{2m-2}, \\ \dfrac{dv}{dt} = \beta v + (\alpha + 1)\theta v \end{cases} \tag{20}$$

for (16). The basic idea now is to normalize (17), (18) and (19), (20) locally, then integrate it and glue together the T-mappings, defined piecewise in the vicinity of the blowing-up circle, see [8]. C^∞ orbital normal forms in the neighborhood of the hyperbolic saddle are well understood [7]. They are given in (x,y)-coordinates by

$$\frac{dx}{dt} = x + \varepsilon(x^{pk+1}y^{qk} + ax^{2pk+1}y^{2qk}), \quad \frac{dy}{dt} = -\omega y, \tag{21}$$

where $\varepsilon = 0$, if ω is not resonant and $\varepsilon = \pm 1$, if $\omega = p/q$ is resonant, $p, q, k \in \mathbb{N}$.

Comparing (17), (18) and (19), (20) with (21), one finds that by an orbital C^∞ transformation (17), (18) can be reduced to

$$(22)\quad \frac{du}{dt} = \alpha u, \qquad \frac{d\eta}{dt} = -\eta + \varepsilon u^{2m-2}\eta^{2m}, \qquad \alpha \in \left\{\frac{2m-1}{2m-2}\right\}_{m=2}^{N}$$

$$(23)\quad \frac{d\theta}{dt} = -\theta, \qquad \frac{dv}{dt} = \beta v,$$

and (19), (20) to

$$(24)\quad \frac{du}{dt} = \alpha u, \qquad \frac{d\eta}{dt} = \eta + \varepsilon u^{2m-2}\eta^{2m+1}, \qquad \alpha \in \left\{\frac{m}{1-m}\right\}_{m=2}^{N}$$

$$(25)\quad \frac{d\theta}{dt} - 1, \qquad \frac{dv}{dt} = 0.$$

Normal form (2).

• *Step 1.* α is not resonant. Then $\alpha \neq (2m-1)(2m-2)^{-1}$ and $\varepsilon = 0$ for the normal form (22). The transition mapping $T = T_\beta \circ T_t \circ T_\alpha$, where T_α and T_β are the transition mappings near the saddle sections defined by (22) and (23), while T_t is the transition mapping along the saddle connection which joins the saddle sections. Normal forms (22), (23) are easily integrable, so that for T_α and T_β one obtains $T_\alpha : |x| \mapsto |x|^\alpha, T_\beta : |x| \mapsto |x|^{1/\beta}$. Mapping T_t is the transition function in the 'standard flow-box' so that $T_t : x \mapsto a_0 x + \cdots$, where $a_0 \neq 0$. Finally, we come to $T : |x| \mapsto a_0|x|^{\alpha/\beta} + \cdots$, which differs from (13) only by a positive multiplier.

• *Step 2.* α is resonant. In this case in (22) $\alpha = (2m-1)(2m-2)^{-1}$ and $\varepsilon = \pm 1$. Normal form (22) is still integrable with the first integral given by

$$(26)\qquad -\eta^{1-2m}u^{2-2m} + 2\varepsilon(m-1)\log|u| = C.$$

It follows from (26) that the transition mapping T_α near the resonant saddle is $T_\alpha : |x| \mapsto |x|^{(2m-1)(2m-2)^{-1}}(1 + \cdots)$. The mappings T_β and T_t remain as before, so that in the resonant case one obtains

$$T : |x| \longmapsto a_0|x|^{\frac{1}{\beta}\left(\frac{2m-1}{2m-2}\right)} + \cdots.$$

It is evident now that T does not differ from (13) if we substitute $\alpha = (2m-1)(2m-2)^{-1}$.

Normal form (3).

- *Step 1.* α is not resonant. In this case for the normal form (24) one gets $\alpha \neq m(1-m)^{-1}$ and $\varepsilon = 0$. The transition mapping $T = T_{\alpha'} \circ T_t \circ T_\alpha$, where T_α and $T_{\alpha'}$ are transitions near the saddle defined by (24), while T_t is a 'standard flow-box' transition defined by (25). It is easy to see that $T_\alpha : |x| \mapsto |x|^{1/\alpha}, T_{\alpha'} : |x| \mapsto |x|^\alpha$ and $T_t : x \mapsto a_0 x + \cdots$ One comes to $T : |x| \mapsto \widetilde{a}_0 |x| + \cdots$, which coincides with (14) up to the choice of a positive constant $\widetilde{a}_0$.

- *Step 2.* $\alpha = m(1-m)^{-1}$ is resonant. This implies that in the normal form (24) $\varepsilon = \pm 1$. In this case (24) is integrable with the first integral

$$-\eta^{-2m} u^{2-2m} + 2\varepsilon(m-1)\log|u| = C. \tag{27}$$

As an easy consequence of (27) one obtains the following estimates:

$$T_{\alpha'} : |x| \longmapsto |x|^{\frac{m}{m-1}}(1+\cdots), \qquad T_\alpha : |x| \longmapsto |x|^{\frac{m-1}{m}}(1+\cdots).$$

The mapping T_t is as before and we obtain $T : |x| \mapsto \widetilde{a}_0|x| + \cdots$ Note that in the resonant case T coincides, up to a multiplier, with the expression given by (14).

Now all cases are considered and therefore Lemma 4.1 is proven. □

5. Remark on the problem of H. Rosenberg.

In the paper [36], p. 29, due to Harold Rosenberg, one finds a list of open problems concerning labyrinths in discs and annuli (these are special cases of $\mathcal{A}$-foliations on S^2, the simplest of which is an $\mathcal{A}$-foliation with 3 thorns and 1 sun-set singularity).

One of the proposed questions is to find a criterion to discern between two possible types of behavior of leaves in the labyrinth: they can all be compact or they can loose itself inside the labyrinth (being dense there). It was conjectured also that such are the rotation numbers of the labyrinth.

Rotation numbers for labyrinths were introduced by Aranson and Zhuzhoma [3] and serve well to discern between two transitive foliations in disc: two of them are topologically equivalent if and only if their rotation numbers ($\equiv$ homotopy rotation classes) coincide. However, rotation numbers are not defined when all leaves of the labyrinth are compact (no one leaf goes to the absolute [3]). So far, we propose here to use a divergence of the foliation in singular points to separate these two cases.

COROLLARY 5.1. — *Let $\mathcal{L}$ be a Rosenberg labyrinth in a disc. Then all leaves of $\mathcal{L}$ are compact if* $\operatorname{div} \mathcal{L} \leq 0$ *in all its singular points.*

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