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FUNCTION SPACES ON THE OL'SHANSKIĬ SEMIGROUP AND THE GEL'FAND-GINDIKIN PROGRAM

by

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0. Introduction.

For a semi-simple Lie group G, one often studies its representations by restricting to a maximal compact subgroup K. An important category are the Harish-Chandra modules, which are modules M for the Lie algebra $\mathfrak{g}$ with an admissible action of K. Analytic continuation in the parameters defining certain standard modules M leads to the problem of irreducibility of M and the problem of finding the Jordan-Hölder composition series in terms of K-types at the reducible points.

In this paper we treat in a special case the analogous problem when K is replaced by one of its non-compact real forms $H \subset G$. Admissible $(\mathfrak{g}, K)$-modules are then replaced by $(\mathfrak{g}, H)$-modules, where the H-types are infinite-dimentional irreducible discrete series (or continuations) of finite multiplicity.

Studying spectra of non-compact subgroups is common in the physics litterature, as well as in certain aspects of representation theory. Here we want to mention the works of Jakobsen and Vergne [11] for the spectrum of the scale extended Poincaré group inside $\mathrm{SU}(2,2)$ and (for unitary representations) the spectrum of $H \subset G$ when both have a Hermitian symmetric space. Also connected with the present point of view is the

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study by Kashiwara and Vergne [13] relating asymptotic K-types and a certain N-spectrum.

We consider $G^\flat = \mathrm{SU}(2,2)$, $K^\flat = \mathrm{S}\big(\mathrm{U}(2) \times \mathrm{U}(2)\big)$ and $H^\flat = \mathrm{S}\big(\mathrm{U}(1,1) \times \mathrm{U}(1,1)\big)$ and the analytic continuation of the holomorphic discrete series of $G^\flat$ induced from a character of $K^\flat$. The corresponding series $M_\lambda^\flat$ of $(\mathfrak{g}^\flat, K^\flat)$-modules is well-known, in particular the composition series, see Speh [23], at the reducible points $\lambda = 1, 0, -1, -2, \ldots$. Here $M_1^\flat$ has a bottom subquotient equals to the "smallest" irreducible unitary representation of $G^\flat$, the so-called wave representation W. Replacing $K^\flat$ by $H^\flat$ we compute explicitly the composition series for the corresponding $(\mathfrak{g}^\flat, H^\flat)$-modules M_λ. In particular we find the $H^\flat$-types occurring in W (these could also have been found by the methods of Jacobsen and Vergne [12]). In effect we realize W in a space of functions (solutions to the wave equation) on $G = \mathrm{U}(1,1)$ viewed as a Lorentz manifold locally conformally equivalent to Minkowski space, in agreement with the results in [24]. For $\lambda = 2$ we obtain a new realization of the classical Hardy space for $\mathrm{U}(2)$, *i.e.* a unitary irreducible representation of $\mathrm{SU}(2,2)$, as a space of holomorphic functions on Ol'shanskiĭ's semigroup. Contrary to the claim by Gindikin [4], p. 679, we show in section 4 that the classical Hardy space is strictly contained in the Ol'shanskiĭ Hardy space. The difference is due to a decay condition at infinity in the semigroup to ensure a removal of singularities of the holomorphic functions.

Our technique is as in [19] that of expanding the distribution $\det(1-x)^{-\lambda}$, this time on $\mathrm{U}(1,1)$, in terms of matrix coefficients of holomorphic discrete series representations, and from this read off the Hermitian invariant form on M_λ. As a final note, we compare the composition series, the harmonic analysis and the wave equation on $\mathrm{U}(1,1)$, $\mathrm{U}(2)$, and $\mathrm{Herm}(2,\mathbb{C})$ ($\sqrt{-1}$ times the Lie algebra of $\mathrm{U}(2)$, viewed as Minkowski space). The connection between the three spaces is given by Cayley transforms, and all three admit $\mathrm{SU}(2,2)$ as the conformal group of (local) transformations.

This paper is organized as follows:

1. Cayley transform and geometry of Γ^0.

Let $G^\flat = \mathrm{SU}(2,2)$, then $G^\flat_{\mathbb{C}} = \mathrm{SL}(4,\mathbb{C})$. Let us consider on $\mathbb{C}^4$ the Hermitian form $\beta^\flat$ defined by

$$\beta^\flat(\xi,\eta) = \xi_1\overline{\eta}_1 + \xi_2\overline{\eta}_2 - \xi_3\overline{\eta}_3 - \xi_4\overline{\eta}_4, \qquad (\xi,\eta \in \mathbb{C}^4).$$

For $X \in \mathcal{M}(4,\mathbb{C})$, let $X^\sharp$ be its adjoint with respect to $\beta^\flat$

$$\beta^\flat(X\xi,\eta) = \beta^\flat(\xi, X^\sharp\eta),$$

i.e.

$$X^\sharp = J^\flat X^* J^\flat, \quad J^\flat = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $X^* = \overline{X}^t$.

We realize then $\mathrm{SU}(2,2)$ and $\mathfrak{su}(2,2)$, its Lie algebra, as

$$\mathrm{SU}(2,2) = \{g \in \mathrm{SL}(4,\mathbb{C});\ g^\sharp = g^{-1}\},$$

$$\mathfrak{su}(2,2) = \{X \in \mathcal{M}(4,\mathbb{C});\ X^\sharp = -X,\ \mathrm{tr}(X) = 0\}.$$

We remark that $g \in \mathrm{SU}(2,2)$ if g is of determinant 1 and satisfies

$$g^* J^\flat g = J^\flat.$$

Let $G = \mathrm{U}(1,1)$, then $G_{\mathbb{C}} = \mathrm{GL}(2,\mathbb{C})$ and consider on $\mathbb{C}^2$ the Hermitian form β defined by

$$\beta(\xi,\eta) = -\xi_1\overline{\eta}_1 + \xi_2\overline{\eta}_2, \qquad (\xi,\eta \in \mathbb{C}^2).$$

For $X \in \mathcal{M}(2,\mathbb{C})$, let $X^\natural$ be its adjoint with respect to β

$$\beta(X\xi,\eta) = \beta(\xi, X^\natural\eta),$$

i.e.

$$X^\natural = J X^* J, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We realize then $\mathrm{U}(1,1)$ and $\mathfrak{u}(1,1)$, its Lie algebra, as

$$U(1,1) = \{g \in \mathrm{GL}(2,\mathbb{C});\ g^\natural = g^{-1}\},$$

$$\mathfrak{u}(1,1) = \{X \in \mathcal{M}(2,\mathbb{C});\ X^\natural = -X\}.$$

We remark that $g \in \mathrm{U}(1,1)$ satisfies

$$g^* J g = J,$$

and that if $X \in i\mathfrak{u}(1,1)$ then $\beta(X\xi,\xi) \in \mathbb{R}$ for all $\xi \in \mathbb{C}^2$. Let C be the cone in $i\mathfrak{u}(1,1)$ defined by

$$C = \{X \in i\mathfrak{u}(1,1);\ \beta(X\xi,\xi) \leq 0,\ \forall\xi \in \mathbb{C}^2\}.$$

It is clear that C a is closed convex cone which is pointed (*i.e.* $C \cap -C = \{0\}$), generating (*i.e.* $C - C = i\mathfrak{u}(1,1)$ or equivalently $C^0 \neq \emptyset$) and $\mathrm{Ad}(G)$-invariant. Then (*cf.* [8]) $\Gamma = G\exp(C)$ is a closed semigroup contained in $\mathrm{GL}(2,\mathbb{C})$. The decomposition $\Gamma = G\exp(C)$ is the so-called *Ol'shanskiĭ decomposition.* In this paper, except in the last section, we will not use the Ol'shanskiĭ coordinates. In fact Γ can be simply described as follows:

$$\begin{aligned}\Gamma &= \{\gamma \in \mathrm{GL}(2,\mathbb{C});\ \beta(\gamma\xi,\gamma\xi) \leq \beta(\xi,\xi),\ \forall \xi \in \mathbb{C}^2\} \\ &= \{\gamma \in \mathrm{GL}(2,\mathbb{C});\ J - \gamma^* J \gamma \geq 0\}.\end{aligned}$$

The group $G^\flat$ acts on the generalized unit disc

$$\mathcal{D} = \{Z \in \mathcal{M}(2,\mathbb{C});\ I - Z^*Z > 0\},$$

via

$$g \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The scalar holomorphic discrete series representations are

$$\big(U_\lambda(g)f\big)(Z) = \det(CZ+D)^{-\lambda} f(g^{-1}\cdot Z), \quad g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

for $\lambda = 4, 5, 6, \ldots$ which all are unitary and irreducible in the Hilbert space $\mathcal{H}_\lambda(\mathcal{D})$ of holomorphic functions on $\mathcal{D}$, square integrable with respect to Lebesgue measure times the density

$$\det(I - Z^*Z)^{\lambda-4}.$$

The Hilbert space $\mathcal{H}_\lambda(\mathcal{D})$ has the reproducing kernel (up to a constant N_λ depending on λ, see [19])

$$K_\lambda^{\mathcal{D}}(Z,W) = N_\lambda \det(I - W^*Z)^{-\lambda}. \tag{2}$$

The infinitesimal Harish-Chandra module $M_\lambda^\flat$ consists of all holomorphic polynomials in $\mathcal{D}$ and admits a $\mathfrak{g}^\flat$-action corresponding to (1) for all λ. Furthermore, we have the "binomial formula" [19]

$$\begin{cases} \det(I - W^*Z)^{-\lambda} \\ \qquad = \sum a_{m,j}(\lambda)\overline{(\det W)^m D^j_{q,q'}(W)}(\det Z)^m D^j_{q,q'}(Z), \\ a_{m,j}(\lambda) \\ \qquad = (2j+1)\dfrac{\Gamma(m+\lambda-1)\Gamma(m+2j+\lambda)}{\Gamma(m+1)\Gamma(m+2j+2)\Gamma(\lambda-1)\Gamma(\lambda)}, \end{cases} \tag{3}$$

where the sum is over $m = 0, 1, 2, \ldots$ and all matrix coefficients $D^j_{q,q'}(Z)$ of the spin j representation of SU(2) *i.e.* $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and q, $q' = -j, -j+1, \ldots, j-1, j$. Here $\Gamma(x)$ is the usual continuation of $(x-1)!$ for any x.

We now wish to realize the representation U_λ over a different domain, namely the interior Γ^0 of the Ol'shanskiĭ semigroup Γ, *via* a certain Cayley transform and study the analogues of (1), (2) and (3).

The Šilov boundary of $\mathcal{D}$ is U(2) whereas that of Γ^0 is U(1, 1), so in effect we replace harmonic analysis on U(2) by harmonic analysis on U(1, 1). Since

$$\mathrm{U}(1,1) \cong \big(\mathrm{U}(1) \times \mathrm{SU}(1,1)\big)/\mathbb{Z}_2$$

the unitary irreducible representations are those of $\mathrm{U}(1) \times \mathrm{SU}(1,1)$ that are trivial on $(-1, -I)$. We shall in fact only need the holomorphic discrete series (although the principal series mysteriously seems to turn up in an analytic continuation), namely

$$\pi_{n,j}(\mathrm{e}^{i\theta} g) = \mathrm{e}^{in\theta}\, \pi_j(g), \qquad g \in \mathrm{SU}(1,1) \tag{4}$$

where $n \in \mathbb{Z}$ and π_j are the holomorphic discrete series representations of SU(1, 1), $j = 2, 3, \ldots$ and $n + j$ must be even. Recall that

$$\big(\pi_j(g) f\big)(z) = (cz+d)^{-j} f(g^{-1} \cdot z), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5}$$

where SU(1, 1) acts on the unit disc by fractional linear transformations, and the monomials

$$e_k(z) = z^k \left(\frac{(k+j-1)!}{k!\,(j-1)!} \right)^{1/2}, \qquad k = 0, 1, 2, \ldots$$

form an orthonormal basis. The distribution character of (5) is then on the compact Cartan subgroup given by

$$\chi_j \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix} = \frac{\mathrm{e}^{-i(j-1)\theta}}{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}},$$

and correspondingly, the character of (4) is

$$\chi_{n,j}\left(\mathrm{e}^{i\theta_1} \begin{pmatrix} \mathrm{e}^{i\theta_2} & 0 \\ 0 & \mathrm{e}^{-i\theta_2} \end{pmatrix} \right) = \mathrm{e}^{in\theta_1} \frac{\mathrm{e}^{-i(j-1)\theta_2}}{\mathrm{e}^{i\theta_2} - \mathrm{e}^{-i\theta_2}}. \tag{6}$$

Our main task will be to expand the distribution $\det(1-x)^{-\lambda}$ viewed as a suitable boundary value of an holomorphic function, on U(1, 1) in terms of the characters $\chi_{n,j}(x)$.

To see how this question ties up with the representation theory, we consider the generalized Cayley transform $\mathcal{C}$ from $\mathrm{Herm}(2,\mathbb{C})$, the space of 2×2 Hermitian complex matrices, to $\mathrm{U}(1,1)$. The transform $\mathcal{C}$ is quite analogous to the usual Cayley transform, but it only compactifies certain directions in $\mathrm{Herm}(2,\mathbb{C})$ (it is in fact a partial Cayley transform in the group-theoretic sense):

$$\mathcal{C}(X) = (X - iJ)(X + iJ)^{-1}, \qquad X \in \mathrm{Herm}(2,\mathbb{C}).$$

On the complexification of $\mathrm{Herm}(2,\mathbb{C})$ (away from the singular set where the inverses are unbounded) for $\gamma = \mathcal{C}(Z)$ and $\gamma_1 = \mathcal{C}(Z_1)$ we have that

$$\begin{aligned}\frac{1}{2i}(Z - Z_1^*) &= \frac{1}{2i}\big((I-\gamma)^{-1}(I+\gamma)\,iJ + iJ\,(I+\gamma_1^*)(I-\gamma_1^*)^{-1}\big)\\ &= (I-\gamma)^{-1}\,(J - \gamma\, J\,\gamma_1^*)\,(I-\gamma_1^*)^{-1}.\end{aligned}$$

Also, since $\mathcal{C}(Z) = I - 2iJ(Z+iJ)^{-1}$, we have the relations

$$\begin{aligned}J - \gamma_1^* J\,\gamma &= 2(Z_1 + iJ)^{*-1}\Big(\frac{1}{i}(Z - Z_1^*)\Big)(Z+iJ)^{-1}\\ &= \big(J(\gamma_1 - I)\big)^*\Big(\frac{1}{2i}(Z - Z_1^*)\Big)\big(J(\gamma - I)\big).\end{aligned}$$

From this we find the image of the generalized upper half-plane

$$T = \big\{Z = X + iY \in \mathcal{M}(2,\mathbb{C});\ X^* = X,\ Y^* = Y \text{ and } Y > 0\big\}$$

under $\mathcal{C}$, indeed one verifies by these relations.

LEMMA 1.1. — *The transform $\mathcal{C}$ is an analytic diffeomorphism from an open dense subset of* $\mathrm{Herm}(2,\mathbb{C})$ (*and its complexification* $\mathcal{M}(2,\mathbb{C})$) *to an open subset of* $\mathrm{U}(1,1)$ (*and its complexification* $\mathrm{GL}(2,\mathbb{C})$); *furthermore, $\mathcal{C}$ maps an open dense subset of T biholomorphically onto the complex manifold* $\Gamma^0 = \{\gamma \in \mathrm{GL}(2,\mathbb{C});\ J - \gamma^* J\,\gamma > 0\}$, *the interior of* Γ.

In fact, modulo a set of measure zero, namely

$$\Sigma = \big\{Z \in T;\ \det(Z + iJ) = 0\big\},$$

Γ^0 is just another realization of $G^\flat/K^\flat$. To be specific, let

$$J' = \begin{pmatrix} J & 0\\ 0 & -J\end{pmatrix}$$

and realize $G^\flat$ as all matrices satisfying

$$g^* J'\, g = J', \quad \det g = 1. \tag{7}$$

Then the usual fractional linear transformations

$$g \cdot \gamma = (A\gamma + B)(C\gamma + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

defines an action of $G^\flat$ (with singularities of measure zero) on Γ^0. Indeed

$$\begin{aligned}
J - (g\cdot\gamma)^* J\, (g\cdot\gamma) \\
&= (C\gamma + D)^{*-1}\big[(C\gamma+D)^* J\,(C\gamma+D) \\
&\qquad - (A\gamma+B)^* J\,(A\gamma+B)\big](C\gamma+D)^{-1}, \\
&= (C\gamma+D)^{*^{-1}}\big[J - \gamma^* J\,\gamma\big](C\gamma+D)^{-1},
\end{aligned}$$

using (7) *i.e.*

$$\begin{aligned}
A^* J\, A - C^* J\, C &= J, \\
A^* J\, B - C^* J\, D &= 0, \\
B^* J\, B - D^* J\, D &= -J,
\end{aligned}$$

which gives that both Γ^0 and $\mathrm{U}(1,1)$ are preserved under the action (when defined). Note that

$$X \longmapsto X' = -iJ\,X$$

is a linear isomorphism from $\mathrm{Herm}(2,\mathbb{C})$ to $\mathfrak{u}(1,1)$, the Lie algebra of $\mathrm{U}(1,1)$, and that

$$\begin{aligned}
\mathcal{C}(X) &= (iJ\,X' - iJ)(iJ\,X' + iJ)^{-1} \\
&= -\,iJ\,(X' - I)(X' + I)^{-1} iJ,
\end{aligned}$$

where conjugation by iJ preserves $\mathrm{U}(1,1)$.

Remark 1.2. — This version of Cayley transform, *i.e.*

$$\mathcal{C}'(X') = (X' - I)(X' + I)^{-1}$$

has surprising and interesting properties for a large class of linear Lie groups: $\mathrm{U}(p,q)$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{O}^*(2n)$ and $\mathrm{SO}(2,n)$. It was introduced in Weyl's book [25] and has been studied by Hilgert [7] and Paneitz [21] and more generally for all hermitian symmetric spaces of non compact type by Faraut and Korányi [2] and Loos [16]. Many of our results will generalize using this.

Note that Γ^0 is not simply connected, so that non-zero analytic functions in Γ^0 do not necessarily have analytic logarithms. Concerning the geometry of Γ^0, we note the following explicit computation (some of which is known from the general theory).

LEMMA 1.3. — *If $\gamma \in \Gamma^0$, then $I \pm \gamma$ is non-singular. The semigroup Γ^0 is closed under forming adjoints; furthermore, for all $\gamma_1, \gamma_2 \in \Gamma^0$, $J - \gamma_1^* J \gamma_2$ is non-singular.*

Proof. — If $\gamma \in \Gamma^0$ then for any non-zero $\alpha \in \mathbb{C}^2$

$$\beta(\gamma\alpha, \gamma\alpha) < \beta(\alpha, \alpha). \tag{8}$$

Therfore, γ does not have eigenvalues ± 1. Suppose $\gamma_1, \gamma_2 \in \Gamma^0$, then

$$J - \gamma_2^* \gamma_1^* J \gamma_1 \gamma_2 = J - \gamma_2^* J \gamma_2 + \gamma_2^* (J - \gamma_1^* J \gamma_1) \gamma_2 > 0.$$

For the next assertion, we use the relation for $\gamma \in \Gamma^0$

$$(I - \gamma)^{-1} (J - \gamma J \gamma^*) (I - \gamma^*)^{-1} = \big(J(\gamma - I)\big)^{*-1} (J - \gamma^* J \gamma) \big(J(\gamma - I)\big)^{-1}.$$

It is easy to check that Γ^0 is both left and right invariant under $\mathrm{U}(1,1)$ (this in general follows from Ol'shanskiĭ [17]), and also from (8) that $I \pm \gamma$ in non-singular for all $\gamma \in \Gamma^0$. But then the last assertion follows immediately. □

The semigroup property gives the following characterization of Γ^0 due to Ol'shanskiĭ [17].

PROPOSITION 1.4. — *Consider the action*

$$g \cdot w = \frac{aw + b}{cw + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $G_{\mathbb{C}} = \mathrm{GL}(2, \mathbb{C})$ on the unit disc $D = \{w \in \mathbb{C};\ |w| < 1\}$. Then g belongs $\Gamma^{-1} \cap \mathrm{SL}(2, \mathbb{C}) = \{\gamma \in \mathrm{SL}(2, \mathbb{C});\ J - \gamma^* J \gamma \leq 0\}$ if and only if $g \cdot D \subset D$. □

We let $K = \mathrm{U}(1) \times \mathrm{U}(1)$ be the maximal compact subgroup in G; $K_{\mathbb{C}}$ consists of the diagonal matrices and

$$\begin{aligned} \Gamma^0 \cap K_{\mathbb{C}} &= \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix};\ |z_2|^2 < 1 < |z_1|^2 \right\} \\ &= \left\{ \begin{pmatrix} z_0 w^{-1} & 0 \\ 0 & z_0 w \end{pmatrix};\ |w|^2 < |z_0|^2 < |w|^{-2} \right\}. \end{aligned}$$

Thus the character (6) has an analytic extension to $\Gamma^0 \cap K_{\mathbb{C}}$, namely

$$\chi_{n,j}\left(z_0 \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix}\right) = z_0^n \frac{w^j}{1 - w^2}$$

and so $\chi_{n,j}$ has an analytic continuation to the open set of elliptic points in Γ^0. In fact, $\chi_{n,j}$ is analytic on all of Γ^0, but for the moment, let's also see how the matrix coefficients of $\pi_{n,j}$ in (4) behave under analytic continuation into Γ^0. These are in terms of the orthonormal monomial basis e_k for the representation space of π_j given by

$$D^{r,s}_{n,j}(g) = \big(\pi_{n,j}(g)e_r, e_s\big), \tag{10}$$

at this point with $j = 2, 3, 4, \ldots$, $n + j$ even and $r, s = 0, 1, 2, \ldots$. By virtue of Proposition 1.4 $\pi_{n,j}(\gamma)$ leaves the representation space stable, and it follows that (10) is actually analytic on Γ^0. We note for later use a few of the coefficients (10)

$$\begin{cases} D^{0,0}_{n,j}(\gamma) = (\det\gamma)^{(n+j)/2} a^{-j}, \\ D^{0,1}_{n,j}(\gamma) = (\det\gamma)^{(n+j)/2} c a^{-j-1}, \end{cases} \tag{11}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0$ ensures that

$$|a|^2 + |b|^2 - |c|^2 - |d|^2 > 0, \quad \text{and}$$

$$\big(-1 + |a|^2 - |c|^2\big)\big(1 + |b|^2 - |d|^2\big) - |\bar{a}b - \bar{c}d|^2 > 0,$$

so that a^{-j} is indeed analytic. Strictly speaking, to make $(\det\gamma)^{(n+j)/2}$ analytic for all n, we should in Γ^0 disregard the hypersurface $\det\gamma = 0$. This means those $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which $c = \lambda a$, $d = \lambda b$, $|\lambda| < 1$ and $|a|^2 - |b|^2 > (1 - |\lambda|^2)^{-1}$.

It is also interesting to note at this point that the complexification of the non-compact Cartan subgroup H of G does not meet Γ^0. Indeed

$$H_{\mathbb{C}} = \left\{ z_0 \begin{pmatrix} \lambda + \lambda^{-1} & \lambda - \lambda^{-1} \\ \lambda - \lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix};\ z_0 \in \mathbb{C}, \lambda \in \mathbb{C} \setminus \{0\} \right\}$$

where one of the conditions to be in Γ^0 reads

$$|z_0|^2 \big(|\lambda + \lambda^{-1}|^2 + |\lambda - \lambda^{-1}|^2 - |\lambda - \lambda^{-1}|^2 - |\lambda + \lambda^{-1}|^2\big) > 0$$

which cannot be satisfied. Thus there are no hyperbolic elements in Γ^0, and analytic functions $\phi(\gamma)$ that are G-invariant in Γ^0, *i.e.* $\phi(g\gamma g^{-1}) = \phi(\gamma)$ $(g \in G)$, will be determined by their values on $\Gamma^0 \cap K_{\mathbb{C}}$.

As is apparent from the above considerations, we shall be dealing with functions analytic in a domain, except possibly along a complex hypersurface. Fortunately, typically those singularities are removable by a square integrability condition.

Let $\mathcal{O}(\Omega)$ denote the space of holomorphic functions in a domain Ω, and recall (see *e.g.* Rossi and Vergne [22]) the tube domain realization of the representation (1) of $G^\flat$:

For $\lambda = 4, 5, 6, \ldots$

$$\mathcal{H}_\lambda(T) = \Big\{ f \in \mathcal{O}(T);\ \int_T \big|f(X+iY)\big|^2 \, \mathrm{d}\mu_\lambda(X+iY) < \infty \Big\},$$

where $\mathrm{d}\mu_\lambda(X+iY) = (\det Y)^{\lambda-4}\, \mathrm{d}X\, \mathrm{d}Y$, and $\mathrm{d}X\, \mathrm{d}Y$ is Lebesgue measure in $T \subset \mathbb{C}^4$, is a Hilbert space with reproducing kernel (normalizing the measure appropriately)

$$K_\lambda^T(Z, W) = \Big(\det \frac{1}{2i}(Z - W^*)\Big)^{-\lambda},$$

and the $G^\flat$-action as in (1), this time $G^\flat$ realized *via*

$$g^* J^{\flat\prime} g = J^{\flat\prime}, \quad J^{\flat\prime} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Now we wish to realize this representation over Γ^0 using the Cayley transform $\mathcal{C}$ above; the main difficulty being that this is singular on T (but with an inverse on Γ^0) and that the group action is singular on Γ^0. Intuitively, T equals Γ^0 with a complex hypersurface added at infinity, namely the hypersurface $\Sigma \subset T$

$$\Sigma = \big\{ Z \in T;\ \det(Z + iJ) = 0 \big\}.$$

To make the correspondance between functions on T and functions on Γ^0 precise, we need the following

PROPOSITION 1.5. — *Let f be a holomorphic function on $T \setminus \Sigma$, square integrable with respect to $\mathrm{d}\mu_\lambda$, $\lambda \geq 4$. Then f is actually holomorphic on all of T.*

Proof. — The measure $\mathrm{d}\mu_\lambda$ is locally equivalent to Lebesgue measure; also $\Sigma \setminus \{-iJ\}$ is a complex submanifold since

$$\det(Z + iJ) = z_{11}z_{22} - z_{12}z_{21} + i(z_{11} - z_{22}) + 1, \quad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

from which we see that the differential $(z_{22} + i, -z_{21}, -z_{12}, z_{11} - i)$ only vanishes at $Z = -iJ$. We shall first prove that the given function f can be holomorphically extended across $\Sigma \setminus \{-iJ\}$ and then use essentially the same argument to extend it across $-iJ$. Locally around a given point of $\Sigma \setminus \{-iJ\}$ we can find complex coordinates (Z_1, Z_2, Z_3, Z_4) so that Σ is given by $Z_4 = 0$ and

$$\int \left|f(Z_1, Z_2, Z_3, Z_4)\right|^2 < \infty \tag{12}$$

locally around $(Z_1, Z_2, Z_3, Z_4) = (0, 0, 0, 0)$ with respect to Lebesgue measure in these coordinates. Consider the Laurent expression of f in these coordinates

$$f(Z_1, Z_2, Z_3, Z_4) = \sum_{n=-\infty}^{+\infty} a_n(Z_1, Z_2, Z_3)\, Z_4^n$$

where a_n is holomorphic. For $n \geq 1$ (fixing the first three variables)

$$\begin{aligned} |a_{-n}|^2 &= \left|\frac{r^n}{2\pi}\int_0^{2\pi} f(r\,\mathrm{e}^{i\theta})\,\mathrm{e}^{in\theta}\,\mathrm{d}\theta\right|^2, \quad z = r\,\mathrm{e}^{i\theta}, \\ &\leq \frac{r^{2n}}{(2\pi)^2}\int_0^{2\pi} \left|f(r\,\mathrm{e}^{i\theta})\right|^2 \mathrm{d}\theta \\ &= \frac{1}{(2\pi)^2}\, r^{2n-1} M(r) \end{aligned}$$

where $M(r) = \int_0^{2\pi} |f(r\,\mathrm{e}^{i\theta})|^2 r\,\mathrm{d}\theta$ is from (12) integrable near $r = 0$. But this implies that for all integer $k > 0$ there exists an $r_k < 1/k$ so that $r_k M(r_k) < 1/k$ and therefore $r_k M(r_k) \to 0$ as $k \to 0$. This in turn implies that $a_{-n} = 0$ for $n \geq 1$. We conclude that $f(Z_1, Z_2, Z_3, Z_4)$ is actually holomorphic in Z_4 at $Z_4 = 0$ so that f extends across any point of $\Sigma \setminus \{-iJ\}$. Finally the possible singularity at $Z = -iJ$ is removed in a similar fashion. □

Remark 1.6. — We could also have used the formula

$$\int_0^{2\pi}\int_0^1 \left|f(r\,\mathrm{e}^{i\theta})\right|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^1 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2\, r^{2n+1}\,\mathrm{d}r$$

where $f(Z) = \sum_{-\infty}^{+\infty} a_n\, Z^n$ near $Z = 0$. The integral is only finite for $a_n = 0$ for $n < 0$. Note also that the same argument using Laurent series in several variables shows that a square-integrable function, holomorphic except possibly on a complex submanifold, actually has removable singularities there. Also, other L^p-conditions will do the job only when $p \geq 2$.

Now we introduce, for $\lambda = 4, 5, \ldots$, the Hilbert space

$$\mathcal{H}_\lambda(\Gamma^0) = \left\{ F \in \mathcal{O}(\Gamma^0);\ \int_{\Gamma^0} |F(\gamma)|^2 \, \mathrm{d}\nu_\lambda(\gamma) < \infty \right\},$$

where $d\nu_\lambda$ is the restriction to Γ^0 of the Lebesgue measure of $V_{\mathbb{C}}$ times the density

$$\det(J - \gamma^* J\gamma)^{\lambda-4}.$$

The Cayley transform $\gamma = \mathcal{C}(Z)$ gives a correspondence

$$f = \mathcal{C}_\lambda(F) \quad \text{where} \quad f(Z) = \det(Z + iJ)^{-\lambda} F(\gamma) \tag{13}$$

for $\lambda = 4, 5, 6, \ldots$ between $\mathcal{O}(T \setminus \Sigma)$ and $\mathcal{O}(\Gamma^0)$.

THEOREM 1.7. — *For $\lambda = 4, 5, \ldots$ and suitable normalization of the densities, $\mathcal{C}_\lambda$ gives a unitary transformation from $\mathcal{H}_\lambda(\Gamma^0)$ to $\mathcal{H}_\lambda(T)$, and the pre-image of the reproducing kernel K^T_λ for $\mathcal{H}_\lambda(T)$ gives a reproducing kernel for $\mathcal{H}_\lambda(\Gamma^0)$, namely*

$$K_\lambda(\gamma, \gamma_1) = \det(J - \gamma_1^* J\gamma)^{-\lambda}. \tag{14}$$

Furthermore, $\mathcal{H}_\lambda(\Gamma^0)$ is for all $\lambda \geq 4$ a reproducing kernel Hilbert space with kernel (14).

Proof. — Relative to Lebesgue measure, the Jacobian of $\mathcal{C}$ is equal to a constant times $|\det(Z + iJ)|^{-8}$. Also, with $\gamma_1 = \mathcal{C}(Z_1)$ and $\gamma = \mathcal{C}(Z)$ from the relations preceeding Lemma 1.1 and in the proof of Lemma 1.3 we get that

$$\det\big(2(Z + iJ)\big)^\lambda \det\Big(\frac{1}{2i}(Z - Z_1^*)\Big)^{-\lambda} \overline{\det\big(2(Z_1 + iJ)\big)}^\lambda = \det(J - \gamma_1^* J\gamma)^{-\lambda} \tag{15}$$

as well as for $f = \mathcal{C}_\lambda(F)$, $f \in \mathcal{O}(\Gamma^0)$

$$\int_{\Gamma^0} |F(\gamma)|^2 \, \mathrm{d}\nu_\lambda(\gamma) = \int_{T\setminus\Sigma} |f(Z)|^2 \, \mathrm{d}\mu_\lambda(Z) \tag{16}$$

with suitable normalization of the measures. Indeed, the left-hand side of (16) equals (ignoring constants)

$$\begin{aligned}
\int_{T\setminus\Sigma} &\left|\det(Z + iJ)^\lambda f(Z)\right|^2 \left|\det(Z + iJ)\right|^{-2\lambda+8} (\det Y)^{\lambda-4} \\
&\qquad\qquad \times \left|\det(Z + iJ)\right|^{-8} \mathrm{d}X \, \mathrm{d}Y \\
&= \int_{T\setminus\Sigma} |f(Z)|^2 (\det Y)^{\lambda-4} \, \mathrm{d}X \, \mathrm{d}Y \\
&= \int_{T\setminus\Sigma} |f(Z)|^2 \, \mathrm{d}\mu_\lambda(Z).
\end{aligned}$$

The inverse transformation to $\mathcal{C}_\lambda$ is given by

$$F(\gamma) = \det(iJ\gamma - iJ)^{-\lambda} f(Z), \quad Z = (\gamma + I)(iJ\gamma - iJ)^{-1} \tag{17}$$

and this is well-defined on all of $\mathcal{O}(T)$. If the integral in (16) is finite, *i.e.* if $F \in \mathcal{H}_\lambda(\Gamma^0)$, then by Proposition 1.5 f is holomorphic across Σ so that it is actually an element of $\mathcal{H}_\lambda(T)$. Combined with (17) we thus established the unitarity of $\mathcal{C}_\lambda$, and from (15) we read off the statement about the reproducing kernels, namely (Hilbert inner products)

$$f(Z) = \big(f, K_\lambda^T(Z,\cdot)\big) = \det(Z + iJ)^{-\lambda}\big(F, K_\lambda(\gamma,\cdot)\big) = \det(Z + iJ)^{-\lambda} F(\gamma)$$

where $\gamma = \mathcal{C}(Z)$ and $f = \mathcal{C}_\lambda(F)$. Note that in particular we get that $K_\lambda(\gamma,\cdot) \in \mathcal{H}_\lambda(\Gamma^0)$ ($\gamma \in \Gamma^0$) and that these span all of $\mathcal{H}_\lambda(\Gamma^0)$. Our main goal is to study $\mathcal{H}_\lambda(\Gamma^0)$ under continuation in λ; the first step as stated in the last part of Theorem follows easily by considering evaluation at a point in Γ^0 of functions in $\mathcal{H}_\lambda(\Gamma^0)$, $\lambda \geq 4$: Let $\gamma \in \Gamma^0$ be fixed and apply Cauchy's integral formula to an $F \in \mathcal{H}_\lambda(\Gamma^0)$ on a small polydisc around γ. This gives

$$F(\gamma) = \frac{1}{(2\pi i)^4} \int \frac{F(\eta_1,\eta_2,\eta_3,\eta_4)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}\eta_3\, \mathrm{d}\eta_4}{(\eta_1-\gamma_1)(\eta_2-\gamma_2)(\eta_3-\gamma_3)(\eta_4-\gamma_4)} \tag{18}$$

where $(\eta_1,\eta_2,\eta_3,\eta_4)$ are complex coordinates and the integral is over

$$|\eta_1-\gamma_1| = |\eta_2-\gamma_2| = |\eta_3-\gamma_3| = |\eta_4-\gamma_4| = \varepsilon.$$

Integrating (18) over $[\varepsilon_1,\varepsilon_2]^4$ we get by a simple Cauchy-Schwartz inequality that $F \mapsto F(\gamma)$ is continuous on $\mathcal{H}_\lambda(\Gamma^0)$ and therefore

$$F(\gamma) = \big(F, K_\lambda(\gamma,\cdot)\big) \tag{19}$$

for some $K_\lambda(\gamma,\cdot) \in \mathcal{H}_\lambda(\Gamma^0)$. That this is the same kernel as given in (14) follows by analytic continuation on λ. □

The geometry of Γ^0 is considerably more difficult that of T or $\mathcal{D}$, the trouble being that it is incomplete as a homogeneous domain for the automorphism group $G^\flat$. This corresponds in part to the fact that the Šilov boundary $G = \mathrm{U}(1,1)$ of Γ^0 is conformally incomplete (as Minkowski space: the Šilov boundary of T). We mention here the following result which may be checked by tedious coordinate computations, or by using the remark at the end of the paper [20].

PROPOSITION 1.8. — *Consider the following basis for the Lie algebra* $\mathfrak{u}(1,1)$ *of* G:

$$e_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

and the corresponding invariant bilinear form

$$Q\Big(\sum_{i=0}^{3} x_i e_i, \sum_{i=0}^{3} x_i e_i\Big) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Then the Cayley transform $\mathcal{C}$ *is conformal from Minkowski space to* G *equipped with the invariant Lorentz structure coming from* Q.

Remark 1.9. — The positive cone (forward timelike) in $\mathfrak{u}(1,1)$ is given by

$$-iJ(x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) \geq 0;$$

and $\mathcal{C}$ is causal in the sense that it preserves the field of positive cones in $\mathrm{Herm}(2,\mathbb{C})$ (resp. G).

2. The reproducing kernel over Γ^0.

In this section we shall obtain the analogue of (3) based on an expansion of the distribution $\det(I-\gamma)^{-\lambda}$ on G in terms of characters of irreducible unitary holomorphic representations of G. The situation is quite similar to the following simple one-dimensional case:

For any real λ, $(1-\mathrm{e}^{i\theta})^{-\lambda}$ is a distribution on the unit circle as a boundary value of the holomorphic function $(1-z)^{-\lambda}$ in the unit disc. To see this, we find its Fourier series

$$(1-\mathrm{e}^{i\theta})^{-\lambda} = \sum_{n=0}^{\infty} (-1)^n \binom{-\lambda}{n} \mathrm{e}^{in\theta}$$

where the binomial symbol as usual is defined in terms of $\Gamma(s) = (s-1)!$ and as meromorphic functions

$$(-1)^n \binom{-\lambda}{n} = \binom{\lambda+n-1}{n} \tag{20}$$

for all $n = 0,1,2,\ldots$. For positive λ these coefficients exhibit at most polynomial growth since

$$\begin{aligned}\log\Big(\frac{1}{n!}\lambda(\lambda+1)\cdots(\lambda+n-1)\Big) &\leq \log\Big(\lambda\Big(1+\frac{\lambda}{2}\Big)\cdots\Big(1+\frac{\lambda}{n}\Big)\Big)\\ &\leq \log(\lambda)+\frac{\lambda}{2}+\cdots+\frac{\lambda}{n}\\ &\leq \log(\lambda)+\lambda\,\log(n)=\log(\lambda n^{\lambda})\end{aligned}$$

so that indeed we have that $(1-e^{i\theta})^{-\lambda}$ (as boundary value from inside the unit disc) is a distribution. Also, for all real λ, $(1-z)^{-\lambda}$ has a uniformly convergent Fourier series inside the unit disc.

Similary, consider the function $\det(I-\gamma)^{-\lambda}$ for λ an integer; this is analytic in Γ^0 and in analogy with the monomials in the unit disc we wish to expand it in terms of the characters

$$\chi_{n,j}(\gamma)=z_0^n\frac{w^{-j}}{1-w^{-2}},\quad \gamma=z_0\begin{pmatrix} w & 0\\ 0 & w^{-1}\end{pmatrix}. \tag{21}$$

Lemma 2.1. — *Function* (21) *is analytic in* Γ^0 *if and only if* $n+j\geq 0$.

Proof. — We have $\chi_{n,j}(g\,\gamma\,g^{-1})=\chi_{n,j}(\gamma)$ for all $g\in G$, and from (21) this is analytic on $\Gamma^0\cap K_{\mathbb{C}}$ since here $|z_0w^{-1}|<1<|z_0w|$; also, as remarked earlier, Γ^0 contains no hyperbolic elements, so all we have to check is whether $\chi_{n,j}$ is analytic on the set

$$S=\Big\{\gamma\in\Gamma^0;\ \gamma=\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\Big\}$$

where the condition to be in Γ^0 is $|d|<1<|a|$ (and possibly $d=0$). But rewriting (21) we get on such a γ that

$$\chi_{n,j}(\gamma)=\frac{a}{a-d}\,a^{(n-j)/2}\,d^{(n+j)/2}$$

which is analytic precisely when $n+j\geq 0$. □

Now let first $m=-\lambda$ be a positive integer and consider in (21) $z_0=e^{i\theta_1}$ and $w=e^{i\theta_2}$; since we want to expand $\det(I-\gamma)^{-\lambda}$ as a boundary value from Γ^0, the series must involve only negative powers of z_0w and positive powers of z_0w^{-1}, namely

$$\det(I-\gamma)^m=(-1)^m\sum a_{n,j}(-m)\chi_{n,j}(\gamma) \tag{22}$$

where the left-hand side equals

$$\begin{aligned}&\big(1-\mathrm{e}^{i(\theta_1+\theta_2)}\big)^m\big(1-\mathrm{e}^{i(\theta_1-\theta_2)}\big)^m\\&\quad=(-1)^m\,\mathrm{e}^{im(\theta_1+\theta_2)}\big(1-\mathrm{e}^{-i(\theta_1+\theta_2)}\big)^m\big(1-\mathrm{e}^{i(\theta_1-\theta_2)}\big)^m\\&\quad=(-1)^m\,\mathrm{e}^{im(\theta_1+\theta_2)}\sum_{k=0}^{\infty}(-1)^k\binom{m}{k}\,\mathrm{e}^{-ik(\theta_1+\theta_2)}\sum_{\ell=0}^{\infty}(-1)^\ell\binom{m}{\ell}\,\mathrm{e}^{i\ell(\theta_1-\theta_2)}.\end{aligned}$$

After this is multiplied by $\mathrm{e}^{i\theta_2}-\mathrm{e}^{-i\theta_2}$, $a_{n,j}(-m)$ in (22) is then found as the coefficient to $\mathrm{e}^{in\theta_1}\,\mathrm{e}^{-i(j-1)\theta_2}$ by reordering the double series. This turns out to be after some simplification

$$a_{n,j}(-m)=(-1)^{(m+j)}(1-j)\,\frac{m!\,(m+1)!}{k!\,(m-k+1)!\,\ell!\,(m+1-\ell)!} \tag{23}$$

where

$$\ell=\tfrac{1}{2}(n+j),\quad k=m-\tfrac{1}{2}(n-j).$$

Indeed, the sum in (22) only involves finitely many characters, all analytic, namely $0\le k,\ell\le m+1$. It is important at this point to notice, that $\chi_{n,j}$ for $j=1,0,-1,-2,\ldots$ is the character of the analytic continuation of discrete series representations. The formula

$$\frac{\mathrm{e}^{-i(j-1)\theta}}{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}=\frac{\mathrm{e}^{-i(j-1)\theta}-\mathrm{e}^{i(j-1)\theta}}{\mathrm{e}^{i\theta}-e^{-i\theta}}+\frac{\mathrm{e}^{i(j-1)\theta}}{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}} \tag{24}$$

reflects the composition series of the representation (5) for $j=0,-1,-2,\ldots$. Indeed the corresponding Harish-Chandra module has the representation of dimension j as invariant subspace, and π_{1-j} as quotient, $j=0,-1,-2,\ldots$. Now the decomposition (22) does possibly involve these characters of reducible Harish-Chandra modules, as schematically represented in Figure 1.

The square denotes are those characters that we are summing over in (22) not including $j=1$. As we shall see in section 4, Figure 1 also gives a picture of composition series for a Harish-Chandra module for $G^\flat=\mathrm{SU}(2,2)$. Two of the dotted lines vary with $m=-\lambda$, and we now wish to study the analytic continuation to positive λ.

To be more specific, we shall prove that (22) still is true with the right-hand side locally uniformly convergent in Γ^0 and the coefficients given by the analytic continuation of formula (23). The summation in (22) is then not over a bounded region in the (n,j) plane but rather as indicated on Figure 2. This picture too represents a certain composition series for $G^\flat$. Since we are dealing with $G=\mathrm{U}(1,1)$, only $n+j$ even is to be considered in these pictures.

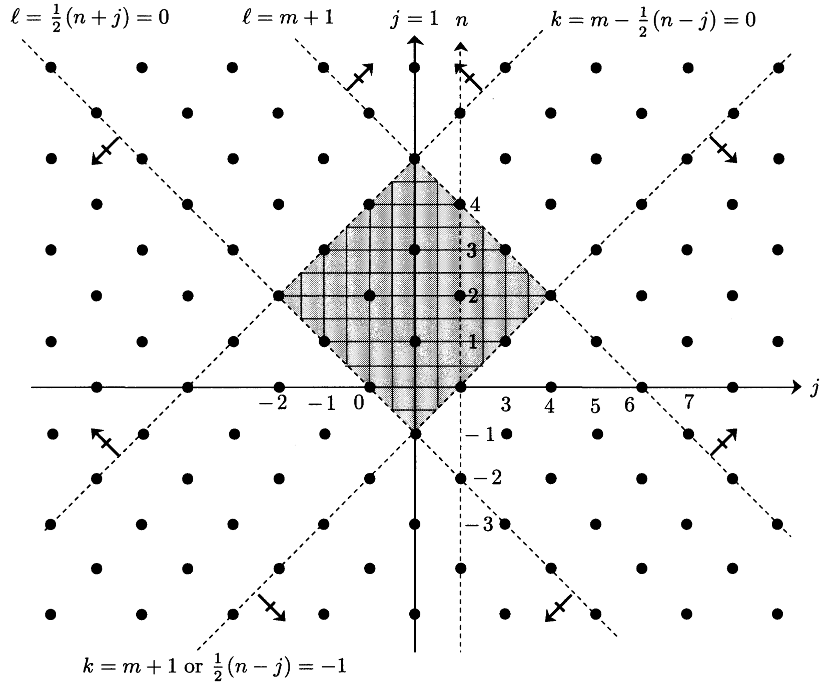

Figure 1 $(\lambda = -m = -2)$.

Using relation (20) we easily get that the analytic continuation of $a_{n,j}$ is given by

$$a_{n,j}(\lambda) = (j-1)\frac{(\lambda+k-2)!\,(\lambda+\ell-2)!}{k!\,\ell!\,(\lambda-1)!\,(\lambda-2)!}. \tag{25}$$

THEOREM 2.2. — *On* Γ^0, *we have the following locally uniformly convergent expansion, valid for* $\lambda = 2, 3, 4, \ldots$

$$\det(I-\gamma)^{-\lambda} = (-1)^{\lambda} \sum a_{n,j}(\lambda)\, \chi_{n,j}(\gamma)$$

where the summation is over $n+j$ *even,* $n+j \geq 0$, $n-j \leq -2\lambda$ *and* $a_{n,j}(\lambda)$ *is given by* (25).

Proof. — Just like in the estimate of the growth of (20), the coefficients (25) are dominated by a polynomial in k and l (for fixed λ). Since $\chi_{n,j}$ on the diagonal matrices in Γ^0 is given as in the proof of Lemma 2.1 and Γ^0 contains no hyperbolic elements of $G_{\mathbb{C}}$, we deduce that the series

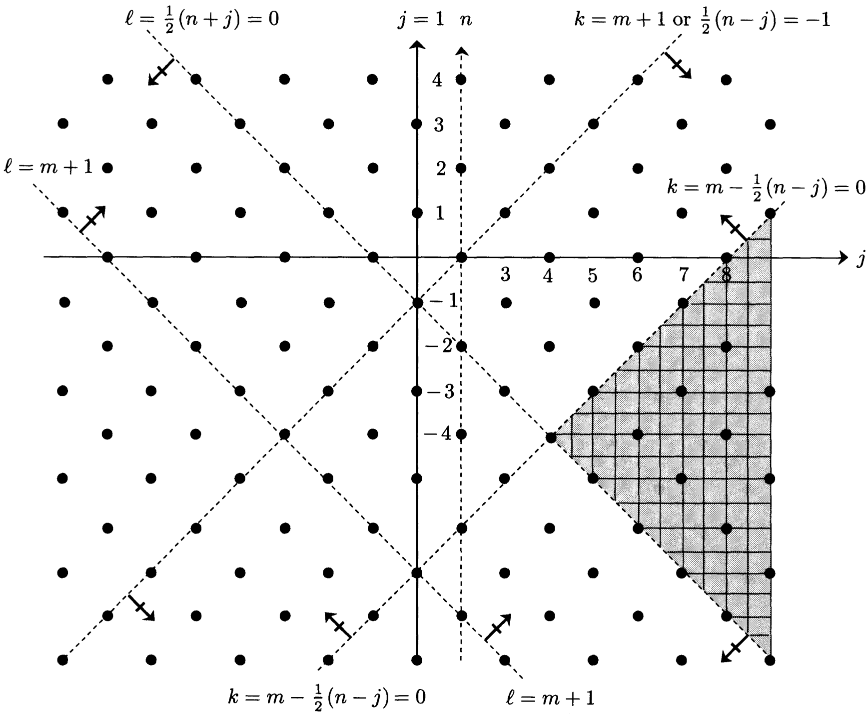

Figure 2 ($\lambda = -m = 4$ *"dual to* $\lambda = -m = -2$*").*

in question is locally uniformly convergent. Finally the sum is easily found on $K_{\mathbb{C}}$ using the binomial formula. □

Remark 2.3. — One case of particular interest is $\lambda = 2$ where $a_{n,j} = j - 1$, which is exactly the formal dimension of $\pi_{n,j}$ as a discrete series representation of G. This case will be studied later in more details.

Now the reproducing kernel itself (14) can expanded using Theorem 2.2, namely

$$\begin{aligned}
K_\lambda(\gamma_1, \gamma_2) &= \det(J - \gamma_2^* J \gamma_1)^{-\lambda} \\
&= (-1)^\lambda \det(I - J\gamma_2^* J \gamma_1)^{-\lambda} \\
&= \sum a_{n,j}(\lambda)\, \chi_{n,j}(J\gamma_2^* J\gamma_1)
\end{aligned}$$

(same summation region as in Theorem 2.2 and Figure 2). On the other hand, $\chi_{n,j}$ can be computed directly as a trace on Γ^0 by the remarks following (10). This means that

$$\chi_{n,j}(\gamma) = \sum_{r=0}^{\infty} D_{n,j}^{r,r}(\gamma)$$

where the sum is locally uniformly convergent in Γ^0 (*cf.* the character formulas in [6]) and

$$\chi_{n,j}(J\gamma_2^* J\gamma_1) = \sum_{r,s=0}^{\infty} D_{n,j}^{r,s}(J\gamma_2^* J) D_{n,j}^{s,r}(\gamma_1).$$

Now the operation $\gamma \mapsto J\gamma^* J$ is the analytic continuation of the inverse on G; therefore

$$D_{n,j}^{r,s}(J\gamma_2^* J) = \overline{D_{n,j}^{s,r}(\gamma_2)}$$

so that we finally get

THEOREM 2.4. — *The kernel K_λ has locally uniformly convergent expansion valid for $\lambda = 2, 3, 4, \ldots$*

$$K_\lambda(\gamma_1, \gamma_2) = \sum a_{n,j}(\lambda) D_{n,j}^{s,r}(\gamma_1) \overline{D_{n,j}^{s,r}(\gamma_2)} \tag{26}$$

where the summation is over all matrix coefficients of $\pi_{n,j}$, $n + j \geq 0$, $n - j \leq -2\lambda$ and $a_{n,j}(\lambda)$ is given by (25).

This is the analogue that we wanted to (3), note the close connection between the two formulas, in particular that the (formal) dimension appears as a factor, and that by replacing in (3) $m+1$ by ℓ and $m+2j+1$ by k one obtains the coefficients (25). This suggests a deeper connection between the finite-dimensional representations of U(2) and the infinite-dimensional representations of U(1,1) *via* analytic continuation.

In the formulation of Theorem 2.4 we tacitly included the representations U_λ for $\lambda = 2, 3$; although not discrete, they still have the reproducing kernel K_λ as is well-known from the usual realization of U_λ. Let $\mathcal{H}_\lambda(\Gamma^0)$, be the corresponding Hilbert space. In particular we have $\mathcal{H}_\lambda(\Gamma^0) = \mathcal{C}_\lambda^{-1}(\mathcal{H}_\lambda(\mathbb{T}))$ (at those points $\lambda = 2, 3$), and Theorem 1.7 is also valid for $\lambda = 2, 3$.

PROPOSITION 2.5. — *For fixed $\lambda = 2, 3, 4, \ldots$ $\mathcal{H}_\lambda(\Gamma^0)$ has as an orthonormal basis the functions*

$$\varphi_\sigma^\lambda(\gamma) = a_{n,j}(\lambda)^{1/2} D_{n,j}^{s,r}(\gamma) \tag{27}$$

with even $n + j \geq 0$, $n - j \leq -2\lambda$, $s, r = 0, 1, \ldots$ and σ a multi-index (n, j, s, r).

Proof. — We proved the expansion of the reproducing kernel

$$K_\lambda(\gamma_1,\gamma_2)=\sum_\sigma \varphi_\sigma^\lambda(\gamma_1)\,\overline{\varphi_\sigma^\lambda(\gamma_2)}$$

which is the general expression given an orthonormal basis for the Hilbert space. The functions $\varphi_\sigma^\lambda(\gamma)$ occurring in the sum actually all belong to $\mathcal{H}_\lambda(\Gamma^0)$, a fact which is based on the following two lemmas:

LEMMA 2.6. — *The domain Γ^0 is given explicitly as all complex 2×2 matrices $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ satisfying*

$$\begin{gathered}|a|^2+|b|^2>|c|^2+|d|^2\\ \left(-1+|a|^2-|c|^2\right)\left(1+|b|^2-|d|^2\right)>|\bar{a}b-\bar{c}d|^2\end{gathered}\tag{28}$$

and for all $\gamma\in\Gamma^0$, $|b|,|c|$ and $|d|$ is bounded by a polynomial in $|a|$.

Proof. — From the first of the inequalities (28) (which just express that $J-\gamma^*J\gamma$ has a positive trace and determinant) it is clear that it suffices to find a polynomial $p(|a|)$ in $|a|$ (independent of c and d) so that $|b|\le p(|a|)$ everywhere in Γ^0. Since $|a|^2>1+|c|^2$ is immediate we only need to find a polynomial bound for b independent of d for each fixed a and c. The second inequality implies that (using $|\bar{a}b-\bar{c}d|^2>|ab|^2+|cd|^2-2|abcd|$)

$$-1+|a|^2-|c|^2>|b|^2(1+|c|^2)-2|abcd|-|d|^2+|ad|^2.$$

If we let $x=|b|$, $y=|d|$, $\alpha=1+|c|^2$, $\beta=|ac|$, $\delta=|a|^2-1$ and $\xi=-1+|a|^2-|c|^2$, then $\delta,\xi>0$ and the inequality reads

$$\alpha\,x^2-2\beta\,xy+\delta\,y^2<\xi$$

which is interior of an ellipse in the (x,y)-plane since the discriminant

$$\begin{aligned}D=-\alpha\delta+\beta^2&=\left(1+|c|^2\right)\left(1-|a|^2\right)+|a|^2|c|^2\\ &=1-|a|^2+|c|^2<0.\end{aligned}$$

But the size of this ellipse is polynomial in $|a|$ and $|c|$, and therefore the size of Γ^0 is polynomial in $|a|$.

LEMMA 2.7. — *For a fixed $\lambda=4,5,\ldots$, the functions $D_{n,j}^{0,0}$ in (11) will belong to $\mathcal{H}_\lambda(\Gamma^0)$ for j sufficiently large and $n+j=0$.*

Proof. — The integral to be proved finite is

$$\begin{aligned}\|D^{0,0}_{n,j}\|^2_\lambda &= \int_{\Gamma^0} |a^{-j}|^2 \det(J-\gamma^* J\gamma)^{\lambda-4}\,\mathrm{d}\gamma\\ &= \int_{\Gamma^0} |a|^{-2j}\Big\{(-1+|a|^2-|c|^2)(1+|b|^2-|d|^2)\\ &\qquad\qquad -|\bar a d-\bar c d|^2\Big\}^{\lambda-4}\,\mathrm{d}\gamma\end{aligned}$$

where $\mathrm{d}\gamma$ is Lebesgue measure. Integrating first b, c and d for fixed a gives at a polynomial in $|a|$ (of degree depending on λ) times $|a|^{-2j}$. Therefore the total integral (having $|a|>1$) is finite for j sufficiently large. □

Since $\mathcal{H}_\lambda(\Gamma^0)$ carries a unitary representation of $G^\flat$, the restriction to $H^\flat = \mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(1,1))$ will again be unitary; but for different (n,j) the matrix coefficients (27) span inequivalent representations of $H^\flat$. Thus we have that $\{\varphi^\lambda_\sigma\}\cap\mathcal{H}_\lambda(\Gamma^0)$ is an orthogonal system in $\mathcal{H}_\lambda(\Gamma^0)$, which we know is a reproducing kernel Hilbert space of holomorphic functions. It remains to show, that all the functions φ^λ_σ occurring in the expansion formula for $K_\lambda(\gamma_1,\gamma_2)$ belongs to $\mathcal{H}_\lambda(\Gamma^0)$, in order to conclude that they form an orthonormal basis. This is exactly what we shall establish in more general form in the next section, where in particular a single $D^{0,0}_{n,j}$ with $n+j=0$ and j large will be shown to be cyclic for the action of the Lie algebra $\mathfrak{g}^\flat$ of $G^\flat$.

3. Composition series.

We have realized the holomorphic discrete series representation (1) of $G^\flat$ over Γ^0, *i.e.* in $\mathcal{H}_\lambda(\Gamma^0)$ the action is

$$(U'_\lambda(g)f)(\gamma) = \det(C\gamma+D)^{-\lambda} f(g^{-1}\cdot\gamma),\quad g^{-1} = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \tag{29}$$

where $G^\flat$ is realized as in (7) (in particular the subgroup $H^\flat$ has $B=C=0$). Note that even when the action of a $g\in G^\flat$ is singular on Γ^0, the determinant factor in (29) ensures that the whole expression is again holomorphic in Γ^0; this again follows from Proposition 1.5.

Now we wish to compute explicitly the differential of the action (29) on our basis vectors (11) to get:

(a) the composition series in terms of these for the analytic continuation in λ, and

(b) the Hermitian invariant inner product on the corresponding $(\mathfrak{g}^\flat, H^\flat)$-modules.

Consider again the functions on Γ^0

$$D_{n,j}^{r,s}(\gamma) = \big(\pi_{n,j}(\gamma)e_r, e_s\big)$$

this time including $j = 1, 0, -1, -2, \dots$ (so that *e.g.* $D_{n,-1}^{0,0}$ span a 4-dimensional representation of $H^\flat$) and the infinitesimal action on these by the complexification of $\mathfrak{g}^\flat$. As a basis for the latter we choose

$$\Upsilon_i = \begin{pmatrix} 0 & Y_i \\ 0 & 0 \end{pmatrix}, \quad \overline{\Upsilon}_i = \Upsilon_i^t \qquad \text{(transpose)}$$

where $i = 1, 2, 3, 4$ and

$$Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

together with a basis for $\mathfrak{h}^\flat$, the Lie algebra of $H^\flat$. In every unitary representation of $G^\flat$, $\overline{\Upsilon}_i$ will be the skew-adjoint of the operator of Υ_i $(i = 1, 2, 3, 4)$.

A short calculation with the representation π_j of $\mathrm{SU}(1,1)$ shows that

$$\begin{aligned} D_{n,j}^{1,1}(\gamma) &= \frac{j+1}{j} D_{n,j+2}^{0,0}(\gamma) - d\, D_{n-1,j+1}^{0,0}(\gamma), \\ D_{n,j}^{1,0}(\gamma) &= -b\, D_{n-1,j+1}^{0,0}(\gamma) \end{aligned}$$

together with earlier noted

$$D_{n,j}^{0,1}(\gamma) = c\, D_{n-1,j+1}^{0,0}(\gamma), \quad D_{n,j}^{0,0}(\gamma) = (\det \gamma)^{(n+j)/2} a^{-j}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now under the action of the universal enveloping algebra of $\mathfrak{h}^\flat$, (n, j) is fixed and $D_{n,j}^{0,0}$ carried to any other $D_{n,j}^{r,s}$ where $0 \le r, s \le -j$ when $j \le 0$ and $0 \le r, s < \infty$ otherwise. On the other hand, the operators Υ_i and $\overline{\Upsilon}_i$ will permute the lowest $\mathfrak{h}^\flat$-types above:

LEMMA 3.1. — *For every λ, the differential of* (29) *gives the following actions of Υ_i and $\overline{\Upsilon}_i$:*

$$\Upsilon_1 \cdot D^{0,0}_{n,j} = -\Big(\frac{n+j}{2}\frac{j}{j-1} - j\Big) D^{0,0}_{n-1,j+1} + \frac{n+j}{2} D^{1,1}_{n-1,j-1},$$

$$\overline{\Upsilon}_1 \cdot D^{0,0}_{n,j} = \Big(\lambda + \frac{n-j}{2}\Big) D^{0,0}_{n+1,j-1},$$

$$\Upsilon_2 \cdot D^{0,0}_{n,j} = \frac{n+j}{2} D^{0,1}_{n-1,j-1},$$

$$\overline{\Upsilon}_2 \cdot D^{0,0}_{n,j} = -\Big(\lambda + \frac{n-j}{2}\Big) D^{1,0}_{n+1,j-1},$$

$$\Upsilon_3 \cdot D^{0,0}_{n,j} = -\frac{n+j}{2} D^{0,0}_{n-1,j-1},$$

$$\overline{\Upsilon}_3 \cdot D^{0,0}_{n,j} = \Big(\Big(\lambda + \frac{n-j}{2}\Big)\frac{j}{j-1} + j\Big) D^{0,0}_{n+1,j+1} - \Big(\lambda + \frac{n-j}{2}\Big) D^{1,1}_{n+1,j-1},$$

$$\Upsilon_4 \cdot D^{0,0}_{n,j} = -\frac{n+j}{2} D^{0,0}_{n-1,j-1},$$

$$\overline{\Upsilon}_4 \cdot D^{0,0}_{n,j} = \Big(\lambda + \frac{n-j}{2}\Big) D^{1,1}_{n+1,j-1}.$$

Proof. — Straightforward differentiation along one-parameter subgroups in $G^\flat$, *e.g.* for the fourth formula,

$$\begin{aligned}\overline{\Upsilon}_2 \cdot D^{0,0}_{n,j}(\gamma) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\lambda tb) D^{0,0}_{n,j}\begin{pmatrix} a+tba & b+tb^2 \\ c+tda & d+tdb\end{pmatrix}\right]_{t=0} \\ &= \big(\lambda + \tfrac{1}{2}(n-j)\big)\, b\, D^{0,0}_{n,j}(\gamma) = -\big(\lambda + \tfrac{1}{2}(n-j)\big) D^{1,0}_{n+1,j-1}(\gamma). \quad \square\end{aligned}$$

Note that $\mathfrak{h}^\flat_{\mathbb{C}}$ has a basis that acts diagonally on the $D^{r,s}_{n,j}$, resp. raises and lowers the indices r and s by 1 according to the representation π_j. By a simple application of the Poincaré-Birkhoff-Witt Theorem we can now conclude using Lemma 3.1 and the above remark:

THEOREM 3.2. — *Let $2 \le \lambda$ be an integer and consider the action of $\mathfrak{g}^\flat$ given by the differential of* (29); *then the algebraic span M_λ of* $\{D^{r,s}_{n,j};\ 0 \le n+j$ even, $n-j \le -2\lambda\}$ *is irreducible.*

Proof. — Every linear combination of the $D^{r,s}_{n,j}$ in question can by successive applications of operators from $\mathfrak{g}^\flat$ be mapped to $D^{0,0}_{n,j}$ with $n+j=0$ and $n-j=-2\lambda$ (see Figure 2) and this vector is cyclic. On the other hand, M_λ is invariant because if we let for example

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & Y_2 \end{pmatrix} \in \mathfrak{h}^\flat_{\mathbb{C}}$$

then Z lowers the r-index:

$$Z \cdot D_{n,j}^{r,s} = (\text{const} \times r) D_{n,j}^{r-1,s},$$

so that

$$[Z, \Upsilon_1] \cdot D_{n,j}^{0,0} = (\text{const}) D_{n-1,j-1}^{0,1}.$$

Similarly we can find a basis for $\mathfrak{h}^\flat$ that permute the Υ_i's and $\overline{\Upsilon}_i$'s: in fact $\mathfrak{h}^\flat$ are the fixpoints of an involutive automorphism of $\mathfrak{g}^\flat$, and $\mathfrak{g}^\flat = \mathfrak{h}^\flat \oplus \mathfrak{q}^\flat$ is a direct decomposition into the $+1$ and -1 eigenspace, where $\mathfrak{q}^\flat$ is spanned by the Υ_i's and $\overline{\Upsilon}_i$'s. In particular $[\mathfrak{h}^\flat, \mathfrak{q}^\flat] \subset \mathfrak{q}^\flat$ so that if $\varphi = D_{n,j}^{0,0}$, $Y \in \mathfrak{q}^\flat$ and $Z_1, \dots, Z_k \in \mathfrak{h}^\flat$ then

$$\big(Y Z_1 \dots Z_k\big) \cdot \varphi = \big([Y, Z_1] Z_2 \dots Z_k\big) \cdot \varphi + \big(Z_1 Y Z_2 \dots Z_k\big) \cdot \varphi$$

and we get by induction on k that

$$\mathfrak{q}^\flat \cdot M_\lambda \subset M_\lambda.$$

Here the induction starts due to the relations in Lemma 3.1. But then we also have for the universal envelopping algebra that

$$U(\mathfrak{g}_{\mathbb{C}}^\flat) \cdot D_{n,j}^{0,0} = M_\lambda$$

for any (n, j) in the considered range, in particular for the lowest $n + j = 0$ and $n - j = -2\lambda$. Here we used the fact that any $D_{n,j}^{r,s}$ is of the form $Z_1 \cdots Z_k D_{n,j}^{0,0}$ with $Z_1, \dots, Z_k \in \mathfrak{h}_{\mathbb{C}}^\flat$. □

Proof of Proposition 2.5 (last part). — We saw that at least our matrix coefficient $D_{n,j}^{0,0}$ (with $0 \leq n+j$ even and $n - j \leq -2\lambda$) belonged to $\mathcal{H}_\lambda(\Gamma^0)$ for j sufficiently large. $G^\flat$ acts unitarily *via* (29) on $\mathcal{H}_\lambda(\Gamma^0)$ so in particular (this being a reproducing kernel Hilbert space of holomorphic functions) $G^\flat$ will map $D_{n,j}^{0,0}$ to elements of $\mathcal{H}_\lambda(\Gamma^0)$. Therefore, all of M_λ is contained in the Hilbert space, and since

$$K_\lambda(\gamma_1, \gamma_2) = \sum_\sigma \overline{\varphi_\sigma^\lambda(\gamma_2)}\, \varphi_\sigma^\lambda(\gamma_1)$$

as in section 3, we get that the sum is over an orthonormal basis:

$$\sigma \in \big\{(n, j);\ 0 \leq n + j \text{ even},\ n - j \leq -2\lambda\big\}.$$

From now on we will work with both $D_{n,j}^{r,s}$ and the normalized matrix coefficients φ_σ^λ as in (27).

It is worth noting that in fact we have proved that the matrix coefficients $D_{n,j}^{r,s}$ have specific decay properties in Γ^0 given by

COROLLARY 3.3. — *For any* $\lambda = 4, 5, \dots$ *and* $n + j$ *even we have that*

$$\int_{\Gamma^0} \left|D_{n,j}^{r,s}(\gamma)\right|^2 \mathrm{d}\nu_\lambda(\gamma) < \infty$$

if and only if $0 \le n + j$ *and* $n - j \le -2\lambda$. *In particular, finiteness of the integral is independent of* r *and* s. *Here again* $\mathrm{d}\nu_\lambda$ *is Lebesgue measure times the density* $\det(J - \gamma^* J\gamma)^{\lambda-4}$.

Consider now the $(\mathfrak{g}^\flat, \mathfrak{h}^\flat)$-module

$$V_\lambda = \operatorname{span}\bigl\{\varphi_\sigma^\lambda;\ \sigma = (n, j, r, s),\ n + j \text{ even}\bigr\}$$

which contains the irreducible submodule M_λ when $\lambda = 2, 3, \dots$. When λ is not an integer, $\mathfrak{g}^\flat$ is transitive on the set of basis functions with $j \ge 2$ as well as on the set of basis functions with $j \ge 0$ and $0 \le r, s \le -j$; this is easily derived from the relations in Lemma 3.1. Whereas these two sets of basis functions span invariant subspaces, V_λ itself for general λ has the defect of "leaking" along $j = 1$ (corresponding to the limit of holomorphic discrete series for $\mathrm{SU}(1,1)$). The formulas for the action of Υ_1 and $\overline{\Upsilon}_3$ in Lemma 3.1 no longer hold when $j = 1$, and instead we have ($j = 1$)

$$(30) \qquad \begin{cases} \Upsilon_1 \cdot D_{n,1}^{0,0}(\gamma) = -\frac{1}{2}(n+1) d D_{n-2,1}^{0,0}(\gamma) + D_{n-1,2}^{0,0}(\gamma), \\ \overline{\Upsilon}_3 \cdot D_{n,1}^{0,0}(\gamma) = \bigl(\lambda + \frac{1}{2}(n-1)\bigr) d D_{n,1}^{0,0}(\gamma) + D_{n+1,2}^{0,0}(\gamma). \end{cases}$$

From these we can generate the action of any combination of the Υ_i and $\overline{\Upsilon}_i$ using

$$\Upsilon_1 \cdot d = 0 \quad \text{and} \quad \overline{\Upsilon}_3 \cdot d = (\lambda + 1) d^2$$

and the other similar relations.

Thus we are led to consider the invariant $\mathfrak{g}^\flat$-module R_λ consisting of rational functions (in the components a, b, c, d of γ) which are regular on Γ^0 and on G. We have $V_\lambda \subset R_\lambda$ non-invariantly, but we can still from studying V_λ get part of the composition series for R_λ. The most notable part occurs when $\lambda = 1$ as what we shall see explicitly in the next section corresponds to solutions of the wave equation on G.

THEOREM 3.4. — M_1 *has as an irreducible subspace*

$$W = \operatorname{span}\bigl\{D_{n,j}^{r,s};\ j - 1 = |n+1|,\ r, s \in \mathbb{N} \text{ and } n \in \mathbb{Z}\bigr\},$$

and this can be completed to carry the mass zero positive frequency wave representation of $G^\flat$ (*see* [11]).

Proof. — When $\lambda = 1$ setting $n = -1$ in (30) we get

$$\Upsilon_1 \cdot D^{0,0}_{-1,1}(\gamma) = D^{0,0}_{-2,2}(\gamma), \quad \overline{\Upsilon}_3 \cdot D^{0,0}_{-1,1}(\gamma) = D^{0,0}_{0,2}(\gamma),$$

and by inspection once again of the formulas in Lemma 3.1 we see that W is invariant. At the same time

$$K_1(\gamma_1, \gamma_2) = \sum_{\sigma} \varphi^1_\sigma(\gamma_1)\, \overline{\varphi^1_\sigma(\gamma_2)}$$

where the sum is over σ's occuring in W. Indeed, the normalization constants (25) turn out to be, when $\ell = \frac{1}{2}(n+j) = 0$ or $k = -1 - \frac{1}{2}(n-j) = 0$,

$$a_{n,j}(1) = (j-1)\frac{(k-1)!\,(-1)!}{k!\,0!\,0!\,(-1)!} = \frac{j-1}{k} = 1,$$

or

$$a_{n,j}(1) = (j-1)\frac{(-1)!\,(\ell-1)!}{0!\,\ell!\,0!\,(-1)!} = \frac{j-1}{\ell} = 1.$$

Also, in the degenerate case $j = -n = 1$ again $a_{-1,1}(1) = 1$ since (as in the calculation above) we can work with analytic continuation in λ and j.

In this way W is completed by assigning

$$\{D^{r,s}_{n,j};\ j-1 = |n+1|\ r,s \in \mathbb{N} \text{ and } n \in \mathbb{Z}\}$$

as an orthonormal basis, and on $\overline{W}$, $G^\flat$ acts unitarily and irreducibly, namely equivalently to the mass zero positive frequency representation. □

More general than the computation above is the observation that on M_λ we have an invariant Hermitian form $\langle\cdot,\cdot\rangle_\lambda$ which continues to all the $H^\flat$-types in Figure 2 as a meromorphic function of λ:

PROPOSITION 3.5. — *The Hermitian form*

$$\langle D^{r,s}_{n,j}, D^{r',s'}_{n',j'}\rangle_\lambda = \delta_{r,r'}\,\delta_{s,s'}\,\delta_{n,n'}\,\delta_{j,j'}\frac{1}{(j-1)}\,\frac{k!\,\ell!\,(\lambda-1)!\,(\lambda-2)!}{(\lambda+k-2)!\,(\lambda+\ell-2)!},$$

where $\ell = \frac{1}{2}(n+j)$, $k = -\lambda - \frac{1}{2}(n-j)$, $n+j$ even, satisfies the invariance property

$$\langle X\cdot v, w\rangle_\lambda + \langle v, X\cdot w\rangle_\lambda = 0 \qquad (X \in \mathfrak{g}^\flat) \tag{31}$$

with respect to the notation given by the differential of (29); *here v and w are arbitrary in the algebraic span of the $D^{r,s}_{n,j}$'s with $j \neq 1$ and the left hand side of* (31) *is to be viewed as a meromorphic function of λ.*

Proof. — By the Poincaré-Birkhoff-Witt Theorem it suffices to check (31) for $r = s = 0$ and X a generator inside the Lie algebra of $H^\flat$. Take for example $X = \Upsilon_3$ with skew-adjoint $\overline{\Upsilon}_3$ as earlier in this section; then

$$\langle \Upsilon_3 \cdot D^{0,0}_{n+1,j+1}, D^{0,0}_{n,j} \rangle_\lambda = \left(\tfrac{1}{2}(n+j)+1\right)\langle D^{0,0}_{n,j}, D^{0,0}_{n,j}\rangle_\lambda ,$$

$$\begin{aligned}\langle D^{0,0}_{n+1,j+1}, \overline{\Upsilon}_3 \cdot D^{0,0}_{n,j} \rangle_\lambda &= -\left(\lambda + \tfrac{1}{2}(n-j)+j-1\right) \\ &\qquad \times \frac{j}{j-1}\langle D^{0,0}_{n+1,j+1}, D^{0,0}_{n+1,j+1}\rangle_\lambda\end{aligned}$$

so that invariance in this case follows from the expression above, since $\lambda + \frac{1}{2}(n-j)+j-1 = \lambda + \ell - 1$. Similary we get the invariance for all the generators Υ_i $(i = 1,2,3,4)$. ◻

4. The Hardy space and the wave equation on G.

Our aim in this section is the study of the special cases $\lambda = 1$ and $\lambda = 2$.

As it was seen before,

$$C = \{X \in iu(1,1);\ \beta(X\xi,\xi) \le 0\}$$

is a closed, convex, pointed, generating and G-invariant cone. We can then (*cf.* [8]) define the *Ol'shanskiĭ's Hardy space* over the semigroup $\Gamma^0 = \mathrm{U}(1,1)\exp(C^0)$. Those spaces are studied abstractly in some particular cases by Ol'shanskiĭ's [18], Hilgert and Ólafsson [9] and in more generality by Hilgert and Neeb [8]. The case of $\mathrm{SL}(2,\mathbb{R})$ was treated by Gel'fand and Gindikin [3].

The *Ol'shanskiĭ's Hardy space* $H^2(\Gamma^0)$ over Γ^0 is the space of holomorphic functions F on Γ^0 such that

$$\|F\|_H^2 = \sup_{\gamma\in\Gamma^0} \int_G \left|F(g\gamma)\right|^2 \mathrm{d}g < \infty,$$

where dg is a fixed right invariant Haar measure on G.

The following "Paley-Wiener" theorem for $H^2(\Gamma^0)$ was first given by Ol'shanskiĭ's [18] (see also [8] and [9]).

THEOREM 4.1. — *The representation of the group G in $H^2(\Gamma^0)$ can be decomposed into a direct sum of the holomorphic discrete series representations of G which are C-dissipative.* □

A unitary representation π of the group G on a Hilbert space $\mathcal{H}$ is said to be C-dissipative if for all $X \in C$, $d\pi(X) \leq 0$ *i.e.* the spectrum of the selfadjoint operator $d\pi(X)$ is contained in $(-\infty, 0]$.

We saw that

$$\mathcal{H}_2(\Gamma^0) = \overline{\operatorname{span}}\{D^{r,s}_{n,j};\ j \geq 2,\ 0 \leq n+j \text{ even and } n-j \leq -4\}.$$

The condition $n - j \leq -4$ is due to a decay condition at infinity in the semigroup to ensure a removal singularities of the holomorphic functions. In effect, it suffices to check this for $F = D^{0,0}_{n,j}$ at an element γ of the form $\gamma = \begin{pmatrix} e^s & 0 \\ 0 & e^{-t} \end{pmatrix}$ with $s, t \in \mathbb{R}$ such that $e^{-t} < 1 < e^s$. Let

$$f(Z) = \det(Z + iJ)^{-2} F(\gamma)$$

be the corresponding function on T where

$$Z = \mathcal{C}^{-1}(\gamma) = i \begin{pmatrix} -\dfrac{1+e^s}{1-e^s} & 0 \\ 0 & \dfrac{1+e^{-t}}{1-e^{-t}} \end{pmatrix} \xrightarrow[s\to+\infty]{t\to+\infty} Z_0 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

which is a singular point. On the other hand,

$$D^{0,0}_{n,j}(\gamma) = e^{s(n-j)/2}\, e^{-t(n+j)/2},$$

$$\det(Z + iJ)^{-2} = \tfrac{1}{16}(1 - e^s)^2 (1 - e^{-t})^2,$$

and

$$\begin{aligned} \det(Z + iJ)^{-2} D^{0,0}_{n,j}(\gamma) &\underset{+\infty}{\sim} e^{-t(n+j)/2}\, e^{s(n-j)/2} (1 - e^s)^2 \\ &\underset{+\infty}{\sim} e^{s\left((n-j)/2+2\right)}\, e^{-t(n+j)/2}. \end{aligned}$$

Thus to remove the singularity at Z_0 we have to take $\frac{1}{2}(n-j) + 2 \leq 0$, *i.e.* $n - j \leq -4$.

THEOREM 4.2. — *The Hilbert space $\mathcal{H}_2(\Gamma^0)$ is equivalent to the classical Hardy space $H^2(T)$ and it is a proper invariant subspace of $H^2(\Gamma^0)$. More precisely $H^2(\Gamma^0)$ equals $\mathcal{H}_2(\Gamma^0)$ plus "two half lines".*

Proof. — The C-dissipative holomorphic discrete series representations of G are those $\pi_{n,j}$ ($j \geq 2$) such that $\big(\pi_{n,j}(\gamma)e_s, e_s\big) \leq 0$ for all $s \in \mathbb{N}$ and all $\gamma \in \Gamma^0$. Since this function is holomorphic and G-invariant it suffices to calculate it on $\exp(i\mathfrak{t} \cap C^0)$, where $\mathfrak{t}$ is the Cartan subalgebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} it_1 & 0 \\ 0 & -it_2 \end{pmatrix};\ t_1, t_2 \in \mathbb{R} \right\} \quad \text{of} \quad \mathfrak{u}(1,1).$$

Let $\gamma = \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \in \exp(i\mathfrak{t} \cap C^0)$, then $t_1, t_2 > 0$. We have

$$\big(\pi_{n,j}(\gamma)e_s, e_s\big) = e^{t_1\left((n-j)/2 - s\right)}\, e^{-t_2\left((n+j)/2 + s\right)},$$

so that $\pi_{n,j}$ is C-dissipative if $n - j \leq 0$ and $n + j \geq 0$ (with $n + j$ even). Then by Theorem 4.1, $H^2(\Gamma^0)$ is the sum of the corresponding $\pi_{n,j}$. The Hilbert space $\mathcal{H}_2(\Gamma^0)$ is the sum of $\pi_{n,j}$ such that $-j \leq n \leq j - 4$, $n + j$ even and $j \geq 2$. Thus

$$H^2(\Gamma^0) = \mathcal{H}_2(\Gamma^0) \oplus (\text{"two half lines"}),$$

as it is shown in Figure 3 (next page) where the gray domain corresponds to $\mathcal{H}_2(\Gamma^0)$. □

Let S be the Cauchy-Szegö kernel of $H^2(\Gamma^0)$ and $K = K_2$ the Cauchy-Szegö kernel of $\mathcal{H}_2(\Gamma^0)$. Recall that $K(\gamma_1, \gamma_2) = \det(J - \gamma_2^* J \gamma_1)^{-2}$.

PROPOSITION 4.3. — *The Cauchy-Szegö kernel of $H^2(\Gamma^0)$ is given by*

$$S(\gamma_1, \gamma_2) = K(\gamma_1, \gamma_2)\, Q(\gamma_1, \gamma_2) = K(\gamma_1, \gamma_2) + R(\gamma_1, \gamma_2),$$

where the Q and R are the G-bi-invariant positive definite functions

$$Q(\gamma_1, \gamma_2) = 1 + 2\sqrt{-\det \mathcal{C}'(\gamma_1 \gamma_2^*)},$$

$$R(\gamma_1, \gamma_2) = \big((\mathrm{tr}(\gamma_1\gamma_2^*) - 2)^3 (\mathrm{tr}(\gamma_1\gamma_2^*) + 2)\big)^{-1/2},$$

with $\mathcal{C}'(\gamma) = (\gamma - I)(\gamma + I)^{-1}$.

Proof. — When $\lambda = 2$, $a_{n,j} = j - 1$ which is exactly the formal dimension of $\pi_{n,j}$ as a discrete series representation of G and the character $\chi_{n,j}$ is given on $\gamma = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \in \Gamma^0 \cap K_{\mathbb{C}}$ ($\lambda < 1$) by $\chi_{n,j}(\gamma) = \dfrac{\lambda^j}{1 - \lambda^2}$.

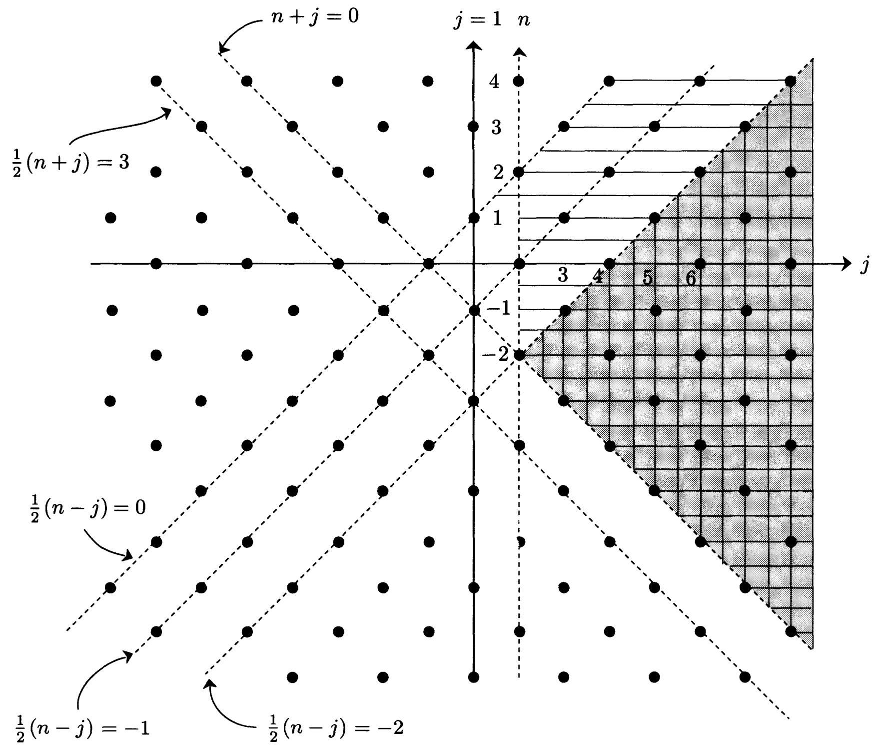

Figure 3 $(\lambda = 2, \mathcal{H}_2(\Gamma^0)$ *and* $H^2(\Gamma^0))$.

Note that $K(\gamma, I) = \dfrac{\lambda^2}{(1-\lambda)^4}$ and $\#\{n;\ n+j \text{ even},\ -j \le n \le j\} = j+1$, so that the Cauchy-Szegö kernel of $H^2(\Gamma^0)$ is given by

$$\begin{aligned}
S(\gamma, I) &= \sum_{\substack{-j\le n\le j\\ n+j \text{ even}\\ j\ge 2}} a_{n,j}(2)\,\chi_{n,j}(\gamma) = \sum_{j=2}^{\infty}\ \sum_{\substack{-j\le n\le j\\ n+j \text{ even}}} (j-1)\frac{\lambda^j}{1-\lambda^2}\\
&= \frac{1}{1-\lambda^2}\sum_{j=2}^{\infty}(j^2-1)\lambda^j = \frac{\lambda^2}{(1-\lambda)^4}\,\frac{3-\lambda}{1+\lambda}\\
&= K(\gamma, I)\,Q(\gamma, I)
\end{aligned}$$

where

$$Q(\gamma, I) = \frac{3-\lambda}{1+\lambda} = 1 + 2\sqrt{-\det\big(\mathcal{C}'(\gamma)\big)}$$

and where $\mathcal{C}'$ denote the Cayley transform $\mathcal{C}'(\gamma) = (\gamma - I)(\gamma + I)^{-1}$. We can also get an additive formula for the kernel S. In fact

$$S(\gamma, I) = K(\gamma, I) + 2\frac{\lambda^2}{1-\lambda^2}\frac{1}{(1-\lambda)^2} = K(\gamma, I) + R(\gamma, I)$$

where $R(\gamma, I)$ can be written as

$$R(\gamma, I) = 2\frac{\lambda^2}{1-\lambda^2}\frac{1}{(1-\lambda)^2} = 2\big((\mathrm{tr}(\gamma) - 2)^3(\mathrm{tr}(\gamma) + 2)\big)^{-1/2}. \qquad \square$$

Remarks 4.4.

(i) Theorem 4.2 gives a counterexample to the claim by Gindikin [3], p. 679.

(ii) Up a constant, the function R appears in Gel'fand and Gindikin's paper [3] (see also [1]) as the Cauchy-Szegö kernel of the Ol'shanskiĭ's Hardy space over the Ol'shanskiĭ's semigroup $\Gamma^{-1} \cap \mathrm{SL}(2, \mathbb{C})$.

(iii) In [3], Gel'fand and Gindikin claim that the Ol'shanskiĭ's semigroup in $\mathrm{SL}(2,\mathbb{C})$ is biholomorphically equivalent to the tube domain consisting of matrices in $\mathrm{Sym}(2,\mathbb{C})$ with positive definite imaginary part. This is not true. In fact, the tube domain is simply connected but the semigroup is not. They also claim that the Hardy space over this semigroup is equivalent to the classical Hardy space over the tube domain. We prove, in [14] and [15], that this is not true in general, in particular when $G = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{SO}^*(2r)$ and $\mathrm{U}(p,q)$. For example, for $G = \mathrm{Sp}(n,\mathbb{R})$ we prove that $H^2(G^\flat/K^\flat) \simeq H^2(\widetilde{\Gamma^0})_{\mathrm{odd}}$, where $\widetilde{\Gamma^0}$ is the double covering of the open semigroup Γ^0 having G as Šilov boundary.

Let us consider now the case when $\lambda = 1$. Let L be the operator

$$L = \Big(\frac{\partial}{\partial\theta} + i\Big)^2 + \Omega_{\mathrm{SU}(1,1)} + 1$$

in the coordinates $\mathrm{U}(1) \times \mathrm{SU}(1,1)$ where $\Omega_{\mathrm{SU}(1,1)}$ is the Casimir operator of $\mathrm{SU}(1,1)$. For all $x = (\mathrm{e}^{i\theta}, u) \in \mathrm{U}(1,1)$. We have

$$\begin{aligned} L(D^{r,s}_{n,j})(x) &= L\big(\mathrm{e}^{in\theta}\pi^{r,s}_j(u)\big) \\ &= \big((i(n+1))^2 + j(j-2) + 1\big)D^{r,s}_{n,j}(x) \\ &= \big(-(n+1)^2 + (j-1)^2\big)D^{r,s}_{n,j}(x) \end{aligned}$$

so that, if $D^{r,s}_{n,j} \in W$ then $L\,D^{r,s}_{n,j} = 0$.

Remarks 4.5.

(i) We have $\mathrm{e}^{-i\theta}\dfrac{\partial}{\partial\theta}\Big(\mathrm{e}^{i\theta}f(x)\Big) = \Big(\dfrac{\partial}{\partial\theta}+i\Big)f(x)$. This means that $L = \mathrm{e}^{-i\theta}L_0\,\mathrm{e}^{i\theta}$ where $L_0 = \dfrac{\partial^2}{\partial\theta^2}+\Omega_{\mathrm{SU}(1,1)}+1$.

(ii) L_0 is the natural conformlly invariant *Yamabe operator.*

(iii) Consider $\widetilde{f}(X) = |\det(X+iJ)|^{-1}F(x)$ where $X^* = X$ and $x = \mathcal{C}(X) = (X-iJ)(X+iJ)^{-1}$. Then the general theory ensures that $\Box\widetilde{f} = 0$ if and only if $L_0F = 0$, where $\Box$ is the wave operator in $\mathrm{Herm}(2,\mathbb{C})$ identified with the flat Minkowski space. Indeed,

$$\begin{aligned}\frac{\det(X+iJ)}{|\det(X+iJ)|} &= \sqrt{\frac{\det(X+iJ)^2}{\det(X+iJ)\det(X-iJ)}}\\ &= \sqrt{\frac{\det(X+iJ)}{\det(X-iJ)}} = \big(\sqrt{\det x}\,\big)^{-1} = \mathrm{e}^{-i\theta}.\end{aligned}$$

Hence

PROPOSITION 4.6. — *We have* $\mathrm{e}^{i\theta}\,W\,\mathrm{e}^{-i\theta}\subset\ker L_0$. $\Box$

We finally remark that the case $\lambda = 1$ corresponds to the right wave equation on $U(1,1)$ as a space-time with the metric $dx^2 = \mathrm{d}\theta^2 + \mathrm{d}u^2$ where $\mathrm{d}u^2$ is given by the Killing form on $\mathrm{SU}(1,1)$.

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