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# SINGULARITIES OF HYPERDETERMINANTS

**by J. WEYMAN (1) and A. ZELEVINSKY (2)**

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**Contents.**



## 0. Introduction.

In this paper we continue the study of *hyperdeterminants* recently undertaken in [4], [5], [12]. The hyperdeterminants are analogs of determinants for multi-dimensional "matrices". Their study was initiated by

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Cayley [1] and Schläfli [11] but then was largely abandoned for 150 years. An immediate goal of the present paper was to prove the following statement conjectured in [4], Section 5: for matrices of dimension $\geq 3$ and format different from $2\times 2\times 2$, the singular locus $\nabla_{\mathrm{sing}}$ of the variety $\nabla$ of "degenerate" matrices has codimension 1 in $\nabla$. As shown in [4], Section 5, this gives a complete description of the matrix formats for which the hyperdeterminant can be computed by Schläfli's method of iterating discriminants. Our main result (Theorem 0.5 below) not only proves the above conjecture, but presents a classification of irreducible components of $\nabla_{\mathrm{sing}}$ for all matrix formats.

Even leaving aside applications to Schläfli's method, we believe that the study of $\nabla_{\mathrm{sing}}$ is important on its own right. To illustrate this point, we remark that the ordinary square matrices have a natural stratification according to their rank. The set $\nabla$ of degenerate matrices is the largest closed stratum of this stratification, and the next closed stratum (that is, the set of matrices of corank $\geq 2$) is exactly the singular locus $\nabla_{\mathrm{sing}}$. Thus, studying the varieties $\nabla$ and $\nabla_{\mathrm{sing}}$ for multi-dimensional matrices can be seen as a natural first step towards their meaningful classification. (This gives an alternative approach to the classification by means of the tensor rank; the relationships between these two approaches are not clear.)

Our results can be also interpreted in the context of projective algebraic geometry. For a given matrix format, the variety $\nabla$ is the cone over the projectively dual variety of the product of several projective spaces taken in its Segre embedding (see [8], Section *IV*.D for an overview of enumerative problems related to projective duality). We hope that the results and methods of the present paper can be extended to a more general class of projectively dual varieies. For example, it would be very interesting to classify irreducible components of the singular loci of the projectively dual varieties $(G/P)^\vee$, where $G$ is a complex semisimple group, $P$ is a parabolic subgroup of $G$, and $G/P$ is projectively embedded as the orbit of a highest weight vector in some irreducible representation of $G$. Our classification can be seen as the first step in this direction. The methods that we use involve an analysis of incidence varieties and their tangent spaces, and a homological approach based on the interpretation of hyperdeterminant as the determinant of a certain Koszul type complex. We hope that these methods will prove useful in more general situations.

Now we present a systematic account of our results and introduce the notation to be used throughout the paper. We fix $r \geq 3$ and positive integers $k_1, k_2, \ldots, k_r$, and denote by $M = \mathbf{C}^{(k_1+1)\times\cdots\times(k_r+1)}$ the space of all matrices $A = \|a_{i_1,\ldots,i_r}\|_{0\le i_j\le k_j}$ of format $(k_1+1)\times\cdots\times(k_r+1)$. For each $j = 1,\ldots,r$ let $V_j$ be a vector space of dimension $k_j+1$ supplied with a coordinate system $(x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j}^{(j)})$. Each $A \in M$ gives rise to a multilinear form on $V_1\times\cdots\times V_r$, that we write as

$$F(A,x) = \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}. \tag{0.1}$$

In more invariant terms, we can think of $A$ as an element of $V_1^*\otimes\cdots\otimes V_r^*$, or, more geometrically, as a section of the vector bundle $\mathcal{O}(1,\ldots,1)$ on the product of projective spaces $P(V_1)\times\cdots\times P(V_r)$. There is a natural left action of the group

$$G = GL(V_1)\times\cdots\times GL(V_r)$$

on $V_1\times\cdots\times V_r$ and a right action of $G$ on $M$ such that

$$F(Ag,x) = F(A,gx) \quad (A\in M,\, x\in V_1\times\cdots\times V_r).$$

We denote by $\partial_j^i$ the partial derivative $\dfrac{\partial}{\partial x_i^{(j)}}$. Let $Y = (V_1 - \{0\})\times\cdots\times(V_r - \{0\})$. We say that $x\in Y$ is a *critical point* of a matrix $A\in M$ if $F(A,x) = \partial_j^i F(A,x) = 0$ for all $i,j$. By definition, $A\in M$ is *degenerate* if it has at least one critical point in $Y$. In a more geometric way, let $X = P(V_1)\times\cdots\times P(V_r)$. There is a natural projection $\mathrm{pr}: Y\to X$ (so the coordinates $x_i^{(j)}$ of a point $y\in Y$ are the homogeneous coordinates of $\mathrm{pr}(y)\in X$). This projection makes $Y$ a principal fiber bundle over $X$ with the structure group $(\mathbf{C}^*)^r$. It is clear that for every $A\in M$ the set of critical points of $A$ in $Y$ is a union of fibers of the projection $\mathrm{pr}: Y\to X$. We shall say that a point $x\in X$ is a critical point of $A$ if the fiber $\mathrm{pr}^{-1}(x)\subset Y$ consists of critical points of $A$. Now we consider the *incidence variety*

$$Z = \{(A,x)\in M\times X : x \text{ is a critical point of } A\}. \tag{0.2}$$

Then the variety $\nabla\subset M$ of degenerate matrices is the image $\mathrm{pr}_1(Z)$, where $\mathrm{pr}_1$ is the projection $(A,x)\mapsto A$. This description implies at once that $\nabla$ is irreducible (the irreduciblity of $\nabla$ follows from that of $Z$, and $Z$ is irreducible, since it is a vector bundle over an irreducible variety $X$). It is known (see, e.g., [4], [5]) that $\nabla$ is a hypersurface in $M$ if and only if the matrix format satisfies the "polygon inequality"

$$k_j \le \sum_{i\ne j} k_i \quad (j = 1,\ldots,r); \tag{0.3}$$

in this case the defining equation of $\nabla$ is the *hyperdeterminant* $\mathrm{Det}(A)$. Unless specifically stated otherwise, we shall always assume that (0.3) holds. Our goal in this paper is to describe the irreducible components of the singular locus $\nabla_{\text{sing}}$ of the hypersurface $\nabla$.

We shall show that $\nabla_{\text{sing}}$ admits the decomposition

$$\nabla_{\text{sing}} = \nabla_{\text{node}} \cup \nabla_{\text{cusp}} \tag{0.4}$$

into the union of two closed subsets that we call, respectively, *node* and *cusp type singularities*. The variety $\nabla_{\text{node}}$ is the closure of the set of matrices having more than one critical point on $X$. In other words, consider an incidence variety

$$Z^{(2)} = \{(A, x, y) \in M \times X \times X : x \neq y, (A, x) \in Z, (A, y) \in Z\}, \tag{0.5}$$

and let $\nabla^0_{\text{node}} \subset M$ be the image $\text{pr}_1(Z^{(2)})$, where $\text{pr}_1$ is the projection $(A, x, y) \mapsto A$. Then $\nabla_{\text{node}}$ is the Zariski closure of $\nabla^0_{\text{node}}$ in $M$.

Informally speaking, $\nabla_{\text{cusp}}$ is the variety of matrices $A \in M$ having a critical point $x \in X$ which is not a simple quadratic singularity of the form $F(A, x)$. To be more precise, let $x^0 \in Y$ be the point with the coordinates $x_i^{(j)} = \delta_{i,0}$. The *quadratic part* of $A$ at $x^0$ is, by definition, the matrix

$$B(A) = \|\partial_j^i \partial_{j'}^{i'} F(A, x^0)\|_{1 \leq j,j' \leq r; 1 \leq i \leq k_j; 1 \leq i' \leq k_{j'}}; \tag{0.6}$$

here the pair $(i, j)$ is considered as the row index, and $(i', j')$ as the column index. We define $H(A, x^0) = \det B(A)$ and call $H(A, x^0)$ the *Hessian* of $A$ at $x^0$. Now consider the variety

$$\nabla^0_{\text{cusp}} = \{A \in M : (A, \text{pr}(x^0)) \in Z, \ H(A, x^0) = 0\} \tag{0.7}$$

and define

$$\nabla_{\text{cusp}} = \nabla^0_{\text{cusp}} \cdot G, \tag{0.8}$$

the set of matrices obtained from $\nabla^0_{\text{cusp}}$ by the action of the group $G$ on $M$. (It is easy to see that $\nabla_{\text{cusp}}$ is closed, so we do not need to take the Zariski closure here.)

We postpone the proof of (0.4) until Section 6, after we get more information on the structure of the varieties $\nabla_{\text{node}}$ and $\nabla_{\text{cusp}}$ (we will show that the inclusion $\nabla_{\text{sing}} \subseteq \nabla_{\text{node}} \cup \nabla_{\text{cusp}}$ is a special case of a general result by N.Katz [7]).

The variety $\nabla_{\text{node}}$ can be further decomposed into the union of several irreducible varieties labeled by the subsets $J \subset \{1, \ldots, r\}$ (including the empty set $J$ but excluding $J = \{1, \ldots, r\}$). Namely, we set

$$Z^{(2)}(J) = \{(A, x, y) \in Z^{(2)} : x^{(j)} = y^{(j)} \Leftrightarrow j \in J\} \tag{0.9}$$

and define $\nabla_{\text{node}}(J)$ as the Zariski closure of the image $\text{pr}_1(Z^{(2)}(J))$. The same argument as the one used above to establish the irreducibility of $\nabla$, shows that each $\nabla_{\text{node}}(J)$ is irreducible. It is clear that

$$\nabla_{\text{node}} = \bigcup_J \nabla_{\text{node}}(J). \tag{0.10}$$

In view of (0.4) and (0.10), the list of irreducible components of $\nabla_{\text{sing}}$ can be obtained as follows. We take all the irreducible components of $\nabla_{\text{cusp}}$ and all the varieties $\nabla_{\text{node}}(J)$, and then eliminate from this list all the varieties that are contained in some other varieties from the list. Our main result is a complete description of irreducible components of $\nabla_{\text{sing}}$ for all matrix formats satisfying (0.3) (Theorem 0.5 below).

Without loss of generality, we can assume that $k_1 \geq k_2 \geq \cdots \geq k_r$. Then (0.3) simply means that $k_1 \leq k_2 + \cdots + k_r$. Following [4], [5], we say that the matrix format is *boundary* if $k_1 = k_2 + \cdots + k_r$ and *interior* if $k_1 < k_2 + \cdots + k_r$. We shall see that for the boundary format the singular locus $\nabla_{\text{sing}}$ is always an irreducible hypersurface in $\nabla$. This is in sharp contrast with the case of interior format: we shall see that generically in this case $\nabla_{\text{sing}}$ has two irreducible components, both of codimension 1 in $\nabla$. The origin of this difference between the interior and boundary formats lies in the fact that for the boundary format the hyperdeterminant has another interpretation as the resultant of a system of multilinear forms ([4], Section 4).

We start our investigation with the structure of irreducible components of $\nabla_{\text{cusp}}$.

THEOREM 0.1.

(a) *If the matrix format is interior and $(k_1, \ldots, k_r) \neq (1, 1, 1)$ then $\nabla_{\text{cusp}}$ is an irreducible hypersurface in $\nabla$ (that is, an irreducible subvariety of codimension 2 in the matrix space $M$). In the exceptional case of $2 \times 2 \times 2$ matrices (i.e., when $(k_1, \ldots, k_r) = (1, 1, 1)$), the variety $\nabla_{\text{cusp}}$ has three irreducible components, each of codimension 2 in $\nabla$.*

(b) *If the matrix format is boundary then $\nabla_{\text{cusp}}$ is an irreducible subvariety of codimension 2 in $\nabla$.*

Theorem 0.1 is proved in Section 2. The main ingredient of the proof is a solution of the following linear algebra problem, that we find of independent interest. It stems from the observation that the Hessian matrix in (0.6) is a symmetric $(k_1 + \cdots + k_r) \times (k_1 + \cdots + k_r)$ matrix with zero diagonal blocks of sizes $k_1 \times k_1, \ldots, k_r \times k_r$. In Section 1 we investigate the prime factorization of the determinant of such a matrix, in particular, find for which $k_1, \ldots, k_r$ this determinant is an irreducible polynomial in matrix entries.

Turning to the node type singularities, we shall call $\nabla_{\text{node}}(\emptyset)$ the *generic* node variety, and the varieties $\nabla_{\text{node}}(J)$ with $J \neq \emptyset$ the *special* ones. It turns out that in most of the cases the special node varieties are contained either in $\nabla_{\text{node}}(\emptyset)$ or in $\nabla_{\text{cusp}}$. More precisely, in Section 3 we prove the following.

THEOREM 0.2.

(a) *Suppose that either* $\#(J) = r-1$, *or* $\#(J) = r-2$ *and* $k_j \geq \sum_{i \neq j} k_i - 1$ *for one of the two indices* $j \in \{1, \ldots, r\} - J$. *Then* $\nabla_{\text{node}}(J) \subset \nabla_{\text{cusp}}$.

(b) *Suppose that either* $\#(J) < r-2$, *or* $\#(J) = r-2$ *and* $\sum_{j \in J} k_j > 2$. *Then* $\nabla_{\text{node}}(J) \subset \nabla_{\text{node}}(\emptyset)$.

We proceed to consider the generic variety $\nabla_{\text{node}}(\emptyset)$ and its relationship with $\nabla_{\text{cusp}}$. The following theorem is proved in Section 4.

THEOREM 0.3.

(a) *If the matrix format is interior and different from* $2 \times 2 \times 2$ *and* $3 \times 3 \times 2$, *then* $\nabla_{\text{node}}(\emptyset)$ *is an irreducible hypersurface in* $\nabla$ *neither containing nor contained in* $\nabla_{\text{cusp}}$. *In the exceptional cases of* $2 \times 2 \times 2$ *and* $3 \times 3 \times 2$ *matrices we have* $\nabla_{\text{node}}(\emptyset) \subset \nabla_{\text{cusp}}$.

(b) *If the matrix format is boundary and different from* $3 \times 2 \times 2$, *then* $\nabla_{\text{node}}(\emptyset)$ *is an irreducible hypersurface in* $\nabla$ *containing* $\nabla_{\text{cusp}}$. *In the exceptional case of* $3 \times 2 \times 2$ *matrices,* $\nabla_{\text{node}}(\emptyset) \subset \nabla_{\text{cusp}}$.

An easy check shows that Theorem 0.2 covers all special node varieties $\nabla_{\text{node}}(J)$ with the following list of exceptions:

(1) The format $3 \times 2 \times 2$; $J = \{1\}$.

(2) The format $3 \times 3 \times 3$; $J = \{1\}$, $\{2\}$ or $\{3\}$.

(3) The format $m \times m \times 3$, $m > 3$; $J = \{3\}$.

(4) The format $2 \times 2 \times 2 \times 2$; $J = \{i, j\}$ $(1 \leq i < j \leq 4)$.

(5) The format $m \times m \times 2 \times 2$, $m > 2$, $J = \{3, 4\}$.

These exceptional cases are treated in Section 5. The results can be summarized as follows.

THEOREM 0.4. — *In each of the cases (1) to (5) above, $\nabla_{\text{node}}(J)$ is an irreducible hypersurface in $\nabla$, and all these hypersurfaces are different from each other. In Case (1) (i.e., for $3 \times 2 \times 2$ matrices), $\nabla_{\text{node}}(\{1\})$ contains $\nabla_{\text{cusp}}$. In the remaining cases (2) to (5), all $\nabla_{\text{node}}(J)$ are different from $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{cusp}}$.*

Putting together all the above results, we obtain our final classification of irreducible components of $\nabla_{\text{sing}}$.

MAIN THEOREM 0.5.

(a) *If the matrix format is boundary then $\nabla_{\text{sing}}$ is an irreducible hypersurface in $\nabla$. Furthermore, we have $\nabla_{\text{sing}} = \nabla_{\text{node}}(\emptyset)$ if the format is different from $3 \times 2 \times 2$; in the exceptional case of $3 \times 2 \times 2$ matrices we have $\nabla_{\text{sing}} = \nabla_{\text{node}}(\{1\})$.*

(b) *If the format is interior and does not belong to the following list of exceptions, than $\nabla_{\text{sing}}$ has two irreducible components $\nabla_{\text{cusp}}$ and $\nabla_{\text{node}}(\emptyset)$, both having codimension one in $\nabla$. The list of exceptional cases consists of the following three- and four-dimensional formats:*

(1) *For $2 \times 2 \times 2$ matrices, the singular locus $\nabla_{\text{sing}}$ coincides with $\nabla_{\text{cusp}}$ and has three irreducible components, all of codimension 2 in $\nabla$.*

(2) *For $3 \times 3 \times 2$ matrices, $\nabla_{\text{sing}}$ coincides with $\nabla_{\text{cusp}}$ and is irreducible.*

(3) *For $3 \times 3 \times 3$ matrices, the singular locus $\nabla_{\text{sing}}$ has five irreducible components $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$, $\nabla_{\text{node}}(\{1\})$, $\nabla_{\text{node}}(\{2\})$, $\nabla_{\text{node}}(\{3\})$.*

(4) *For $m \times m \times 3$ matrices with $m > 3$, the singular locus $\nabla_{\text{sing}}$ has three irreducible components $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$, $\nabla_{\text{node}}(\{3\})$.*

(5) *For $2 \times 2 \times 2 \times 2$ matrices, $\nabla_{\text{sing}}$ has eight irreducible components $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{node}}(\{i, j\})$ $(1 \le i < j \le 4)$.*

(6) *For $m \times m \times 2 \times 2$ matrices with $m > 2$, the singular locus $\nabla_{\text{sing}}$ has three irreducible components $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$, $\nabla_{\text{node}}(\{3, 4\})$.*

*In each of the cases (2) to (6), all irreducible components of $\nabla_{\text{sing}}$ have codimension 1 in $\nabla$.*

Our main tool in proving the above results is an analysis of the tangent space of each of the varieties $\nabla_{\text{cusp}}, \nabla_{\text{node}}(\emptyset)$ etc., at its generic point.

Some of the proofs could be simplified if we had explicit representatives of these varieties. Unfortunately, in most of the cases, these representatives are not known. In the concluding Section 7 we present some interesting special matrices: the multi-dimensional analogs of diagonal matrices and the Vandermonde matrix.

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## 1. Symmetric matrices with diagonal lacunae.

In this section we present a solution of the following linear algebra problem. Let $k_1, \ldots, k_r$ be some positive integers, and suppose for each $(i, j)$ with $1 \leq i < j \leq r$ we are given a generic $k_i \times k_j$ matrix $B_{i,j}$. Consider the symmetric matrix

$$B = \begin{pmatrix} 0 & B_{1,2} & B_{1,3} & \ldots & B_{1,r} \\ {}^tB_{1,2} & 0 & B_{2,3} & \ldots & B_{2,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}^tB_{1,r} & {}^tB_{2,r} & {}^tB_{3,r} & \ldots & 0 \end{pmatrix}$$

of order $k_1 + \cdots + k_r$, where ${}^tB_{i,j}$ stands for the transpose matrix. We will refer to such a matrix as a *symmetric matrix of type* $(k_1, \ldots, k_r)$. The determinant $\det(B)$ is a polynomial in the matrix entries of all the $B_{i,j}$. The problem is to decompose this polynomial into the product of irreducible factors. Without loss of generality we can and will assume that $k_1 \geq k_2 \geq \cdots \geq k_r$. Then the answer is given by the following theorem.

THEOREM 1.1.

(a) *If* $k_1 > k_2 + \cdots + k_r$ *then* $\det(B)$ *is identically equal to 0.*

(b) *If* $k_1 = k_2 + \cdots + k_r$ *then* $\det(B) = (-1)^{k_1}(\det(C))^2$, *where* $C$ *is the* $k_1 \times k_1$ *matrix* $(B_{1,2}, B_{1,3}, \ldots, B_{1,r})$. *The polynomial* $\det(C)$ *is obviously irreducible.*

(c) *If* $k_1 < k_2 + \cdots + k_r$ *then* $\det(B)$ *is always irreducible, with the only exception when* $r = 3$ *and* $(k_1, k_2, k_3) = (k, k, 1)$ *for some* $k \geq 1$.

(d) *In the exceptional case* $r = 3$, $(k_1, k_2, k_3) = (k, k, 1)$ *we have*

$$\det(B) = 2 \cdot (-1)^k \det(B_{1,2}) \det \begin{pmatrix} B_{1,2} & B_{1,3} \\ {}^tB_{2,3} & 0 \end{pmatrix}. \tag{1.1}$$

*If* $k \geq 2$ *then two determinants in the right side of (1.1) are distinct irreducible polynomials. Finally, if* $k = 1$ *then (1.1) takes the form*

$$\det(B) = 2B_{1,2}B_{1,3}B_{2,3},$$

*so in this case* $\det(B)$ *is the product of three distinct irreducible factors.*

*Proof.* — The statements (a) and (b) are obvious because $B$ has a $k_1 \times k_1$ block of zeros. The equality (1.1) can be seen by a direct calculation, for instance, by using the Laplace expansion in the first $k$ rows of $B$. So it remains to prove (c).

We start with the case $r = 3$. Let us put our question in a more invariant setting. Let $E_1, E_2, E_3$ be three vector spaces of dimensions $k_1, k_2, k_3$, respectively. We think of each $B_{i,j}$ as the matrix of a linear map $E_j \to E_i^*$. Using the canonical isomorphism $\mathrm{Hom}\,(E_j, E_i^*) = E_i^* \otimes E_j^*$, we identify the ring of polynomial functions in the entries of $B$ with the symmetric algebra

$$S = S_\bullet((E_1 \otimes E_2) \oplus (E_1 \otimes E_3) \oplus (E_2 \otimes E_3)).$$

The group $G = GL(E_1) \times GL(E_2) \times GL(E_3)$ has a natural representation in $S$; in the matrix form, each $g_i \in GL(E_i)$ is represented by an invertible $k_i \times k_i$ matrix, and for every $f \in S$ we have ${}^g f(B) = {}^{(g_1, g_2, g_3)} f(B) = f(B^g)$, where $(B^g)_{i,j} = {}^t g_i B_{i,j} g_j$. For every triple of non-negative integers $(\alpha, \beta, \gamma)$ we say that an element $f \in S$ is a *relative $G$-invariant of degree* $(\alpha, \beta, \gamma)$ if

$${}^g f = (\det(g_1))^\alpha (\det(g_2))^\beta (\det(g_3))^\gamma f.$$

Clearly, $\det(B)$ is a relative $G$-invariant of degree $(2, 2, 2)$. Therefore, every irreducible factor of $\det(B)$ is also a relative $G$-invariant of some degree $(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \leq 2$. So our first task will be to describe all relative $G$-invariants in low degrees. Looking at the action of the center of $G$ on $S$, we obtain the following necessary condition for an occurence of a relative $G$-invariant of given degree.

LEMMA 1.2. — *If a non-zero relative $G$-invariant of degree $(\alpha, \beta, \gamma)$ occurs in $S$ then the triple $(\alpha k_1, \beta k_2, \gamma k_3)$ is in the additive semigroup generated by* $(1, 1, 0)$, $(1, 0, 1)$, *and* $(0, 1, 1)$.

Using this lemma, we will obtain the existence and uniqueness properties of relative $G$-invariants.

LEMMA 1.3.

(a) *A non-zero relative $G$-invariant of degree $(\alpha,\beta,0)$ occurs in $S$ if and only if $\alpha=\beta$ and $k_1=k_2$; in this case it is proportional to* $\det(B_{1,2})^\alpha$. *Similarly, an invariant of degree $(\alpha,0,\gamma)$ occurs when $\alpha=\gamma$ and $k_1=k_3$, and is proportional to* $\det(B_{1,3})^\alpha$*; an invariant of degree $(0,\beta,\gamma)$ occurs when $\beta=\gamma$ and $k_2=k_3$, and is proportional to* $\det(B_{2,3})^\beta$.

(b) *A non-zero relative $G$-invariant of degree $(1,1,1)$ occurs in $S$ if and only if $k_1\le k_2+k_3$ and $k_1+k_2+k_3$ is even (recall the assumption $k_1\ge k_2\ge k_3$); in this case it is proportional to the Pfaffian of the skew-symmetric matrix*

$$B_{\mathrm{alt}}=\begin{pmatrix}0 & B_{1,2} & B_{1,3}\\ -{}^tB_{1,2} & 0 & B_{2,3}\\ -{}^tB_{1,3} & -{}^tB_{2,3} & 0\end{pmatrix}. \tag{1.2}$$

*Proof.* — The "only if" part in all the statements follows from Lemma 1.2. If $k_1=k_2$ then it is clear that $\det(B_{1,2})^\alpha$ is a non-zero relative $G$-invariant of degree $(\alpha,\alpha,0)$. It is also clear that the Pfaffian of the matrix $B_{\mathrm{alt}}$ given by (1.2) is a relative $G$-invariant of degree $(1,1,1)$. Let us denote this Pfaffian by $Q(B)$. Our next task is to show that if $k_1\le k_2+k_3$ and $k_1+k_2+k_3$ is even, then $Q(B)$ is a non-zero polynomial. To do this, it suffices to construct $B$ such that $\det(B_{\mathrm{alt}})\ne 0$.

Notice that in the category of triples of vector spaces $(E_1,E_2,E_3)$ endowed by three linear maps $B_{i,j}:E_j\to E_i^*$ $(1\le i<j\le 3)$, there is a natural notion of the direct sum. With some abuse of notation, we shall denote a triple as above by the same symbol $B$ as the corresponding symmetric matrix. We write the direct sum of such triples by $B=B'\oplus B''$. Clearly, the ranks of $B$, $B_{\mathrm{alt}}$ and of each $B_{i,j}$ are additive with respect to the direct sum.

Let $B^{(1)}$ be the triple with $E_1=0$, $\dim(E_2)=\dim(E_3)=1$ and such that $B^{(1)}_{2,3}$ is an isomorphism. We define the triples $B^{(2)}$ and $B^{(3)}$ in a similar way, so that each $B^{(i)}$ has $E_i=0$. Take

$$B=(p-k_1)B^{(1)}\oplus(p-k_2)B^{(2)}\oplus(p-k_3)B^{(3)}, \tag{1.3}$$

where $2p=k_1+k_2+k_3$. Then $B$ has type $(k_1,k_2,k_3)$. Since for each of the triples $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ the corresponding matrix $B_{\mathrm{alt}}$ has full rank, the same is true for $B$. Thus, $Q(B)\ne 0$, as desired.

To prove the uniqueness of the above invariants, we shall use the language of Schur modules (see, e.g. [3]). Using the Cauchy decomposition of the symmetric algebra of the tensor product of two vector spaces, we can decompose $S$ as follows:

$$
\begin{aligned}
(1.4)\quad S &= \bigoplus_{d_{12},d_{13},d_{23}\geq 0} (S_{d_{12}}(E_1\otimes E_2)\otimes S_{d_{13}}(E_1\otimes E_3)\otimes S_{d_{23}}(E_2\otimes E_3))\\
&= \bigoplus_{\lambda_{12},\lambda_{13},\lambda_{23}} (S_{\lambda_{12}}(E_1)\otimes S_{\lambda_{12}}(E_2))\otimes(S_{\lambda_{13}}(E_1)\otimes S_{\lambda_{13}}(E_3))\\
&\qquad\qquad \otimes (S_{\lambda_{23}}(E_2)\otimes S_{\lambda_{23}}(E_3))\\
&= \bigoplus_{\lambda_{12},\lambda_{13},\lambda_{23}} (S_{\lambda_{12}}(E_1)\otimes S_{\lambda_{13}}(E_1))\otimes(S_{\lambda_{12}}(E_2)\otimes S_{\lambda_{23}}(E_2))\\
&\qquad\qquad \otimes (S_{\lambda_{13}}(E_3)\otimes S_{\lambda_{23}}(E_3)),
\end{aligned}
$$

the summation over all triples of partitions $(\lambda_{12},\lambda_{13},\lambda_{23})$ such that $l(\lambda_{ij})\leq \min(k_i,k_j)$, where $l(\lambda)$ is the number of non-zero parts of a partition $\lambda$. Using this language, a relative $G$-invariant of degree $(\alpha,\beta,\gamma)$ is an occurence of the Schur module

$$S_{(\alpha^{k_1})}(E_1)\otimes S_{(\beta^{k_2})}(E_2)\otimes S_{(\gamma^{k_3})}(E_3)$$

in the decomposition of (1.4) into irreducible $G$-modules. Let $c^{\lambda}_{\mu\nu}$ denote the multiplicity of $S_\lambda(E)$ in the tensor product $S_\mu(E)\otimes S_\nu(E)$. The multiplicities $c^{\lambda}_{\mu\nu}$ are given by the well-known Littlewood-Richardson rule; in particular, it is known that $c^{\lambda}_{\mu\nu}$ is independent of $E$ when $\dim(E)\geq \max(\ell(\lambda),\ell(\mu),\ell(\nu))$. We see that the dimension of the space of relative $G$-invariants of degree $(\alpha,\beta,\gamma)$ in $S$ is equal to

$$(1.5)\qquad \sum_{\lambda_{12},\lambda_{13},\lambda_{23}} c^{(\alpha^{k_1})}_{\lambda_{12},\lambda_{13}}\, c^{(\beta^{k_2})}_{\lambda_{12},\lambda_{23}}\, c^{(\gamma^{k_3})}_{\lambda_{13},\lambda_{23}},$$

the sum over all triples of partitions $(\lambda_{12},\lambda_{13},\lambda_{23})$ such that $l(\lambda_{ij})\leq \min(k_i,k_j)$. In particular, this dimension is equal to 1 when $(\alpha,\beta,\gamma)=(\alpha,\alpha,0)$ and $k_1=k_2$; in this case, the only triple $(\lambda_{12},\lambda_{13},\lambda_{23})$ contributing to (1.5) is $((\alpha^{k_1}),(0),(0))$. This completes the proof of Lemma 1.3 (a).

To complete the proof of (b), we use the following well-known property of Littlewood-Richardson coefficients: $c^{\lambda}_{\mu\nu}=0$ unless $|\lambda|=|\mu|+|\nu|$ and the diagram of the partition $\lambda$ contains each of the diagrams of $\mu$ and $\nu$ (see, e.g., [9]; as usual, $|\lambda|$ stands for the sum of all the parts of $\lambda$). It follows that for $(\alpha,\beta,\gamma)=(1,1,1)$, the only non-zero contribution to (1.5) is from the triple $(\lambda_{12},\lambda_{13},\lambda_{23})=((1^{p-k_3}),(1^{p-k_2}),(1^{p-k_1}))$, where $k_1+k_2+k_3=2p$. This contribution is 1, because $c^{(1^{k+\ell})}_{(1^k),(1^\ell)}=1$ for all $k,\ell$. Lemma 1.3 is proved.

According to Lemma 1.3 (a), $S$ has no relative invariants in degrees $(0,0,1),(0,0,2),(0,1,2)$ and the degrees obtained from them by permutations of the components. This easily implies that $\det(B)$ fails to be irreducible only if it is either divisible by some $\det(B_{i,j})$ (in the case when $k_i = k_j$), or is a scalar multiple of $Q^2 = \det(B_{\mathrm{alt}})$ (for instance, the latter opportunity takes place in the case $k_1 = k_2 + k_3$.) Recall that we are in the situation of Theorem 1.1 (c), so we assume that

$$k_2 + k_3 > k_1 \geq k_2 \geq k_3 \geq 2; \tag{1.6}$$

we have to show that in this case neither $\det(B_{i,j})$ nor $\det(B_{\mathrm{alt}})$ divide $\det(B)$.

In view of Lemma 1.3, there are three cases to consider:

*Case 1.* — $(k_1, k_2, k_3) = (k, k, \ell)$ with $k \geq \ell \geq 2$. We have to show that $\det(B)$ is not divisible by $\det(B_{1,2})$. It is enough to construct a matrix $B$ of given type such that $\det(B) \neq 0$ but $\det(B_{1,2}) = 0$.

*Case 2.* — $(k_1, k_2, k_3) = (k, \ell, \ell)$ with $2\ell > k > \ell \geq 2$. We have to show that $\det(B)$ is not divisible by $\det(B_{2,3})$. It is enough to construct a matrix $B$ of given type such that $\det(B) \neq 0$ but $\det(B_{2,3}) = 0$.

*Case 3.* — $k_1 + k_2 + k_3 = 2p$ is even. We have to show that $\det(B)$ is not proportional to $\det(B_{\mathrm{alt}})$. It is enough to construct a matrix $B$ of given type such that $\det(B) \neq 0$ but $\det(B_{\mathrm{alt}}) = 0$.

In each of these cases, we construct a desired matrix as a certain direct sum in the category of triples (cf. (1.3) above). We will use the triples $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ defined above, and one more triple $B^{(0)}$ with all $E_i$ one-dimensional and such that each $B^{(0)}_{i,j}$ is an isomorphism. We can choose a basis in each $E_i$ so that the matrix representing $B^{(0)}$ has the form

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly, for each of the triples $B^{(0)}, \ldots, B^{(3)}$ the corresponding matrix $B$ has full rank. On the other hand, $B_{1,2}$ obviously does not have full rank for any triple having $B^{(1)}$ or $B^{(2)}$ as a direct summand, and similarly for $B_{1,3}$ and $B_{2,3}$. Similarly, $B_{\mathrm{alt}}$ does not have full rank for any triple having $B^{(0)}$ as a direct summand.

Now we can construct a desired matrix $B$ as follows. In Case 1 take

$$B = (\ell - 2)B^{(0)} \oplus B^{(1)} \oplus B^{(2)} \oplus (k - \ell + 1)B^{(3)}.$$

In Case 2 take

$$B = (2\ell - k)B^{(0)} \oplus (k - \ell)B^{(2)} \oplus (k - \ell)B^{(3)}.$$

Finally, in Case 3 take

$$B = (k_2 + k_3 - k_1)B^{(0)} \oplus (k_1 - k_2)B^{(2)} \oplus (k_1 - k_3)B^{(3)}.$$

This completes the proof of Theorem 1.1 (c) for $r = 3$.

Now we consider the case $r = 4$. We assume that $k_2 + k_3 + k_4 > k_1 \geq k_2 \geq k_3 \geq k_4 \geq 1$, and we want to show that in this case $\det(B)$ is an irreducible polynomial (in particular, that it is not identically equal to 0).

If we set $B_{3,4} = 0$ then our matrix $B$ becomes a symmetric matrix of type $(k_1, k_2, k_3 + k_4)$. Let us denote this specialized matrix by $B'$. Since $k_3 + k_4 \leq k_1 + k_2$, the sizes of zero blocks are such that $\det(B')$ (and hence $\det(B)$) is not identically equal to 0. Furthermore, the irreducibility of $\det(B')$ implies that of $\det(B)$. By the above analysis of the case $r = 3$, $\det(B')$ fails to be irreducible only if one of the numbers $(k_1, k_2, k_3 + k_4)$ is equal to the sum of two others, or if $(k_1, k_2, k_3 + k_4)$ is a permutation of the triple of the kind $(k, k, 1)$. This leaves us with only two cases to consider:

*Case 1.* — All the numbers $k_i$ are equal to each other, that is, $(k_1, k_2, k_3, k_4) = (k, k, k, k)$ for some $k \geq 1$.

*Case 2.* — $(k_1, k_2, k_3, k_4) = (2, 1, 1, 1)$.

First we treat Case 1. Consider the specialization

$$B_{1,4} = B_{2,4} = B_{3,4} = \mathrm{Id}, \tag{1.7}$$

the identity $k \times k$ matrix. Subtracting the first row block of $B$ from the second and the third, and doing the same thing with column blocks, we see that $\det(B)$ is equal (up to sign) to the determinant of the $2k \times 2k$ matrix

$$\begin{pmatrix} -(B_{1,2} + {}^tB_{1,2}) & B_{2,3} - B_{1,3} - {}^tB_{1,2} \\ {}^t(B_{2,3} - B_{1,3} - {}^tB_{1,2}) & -(B_{1,3} + {}^tB_{1,3}) \end{pmatrix}. \tag{1.8}$$

Clearly, the matrix in (1.8) can be made an arbitrary symmetric matrix. Hence, the specialization (1.7) makes $\det(B)$ irreducible. This means that if $\det(B)$ has several irreducible factors, then at least one of them (in fact all but one) specializes under (1.7) to a non-zero constant. Let us denote this factor by $P(B)$.

Clearly, $P(B)$ must be relative invariant with respect to the natural action of the group $G = GL_k \times GL_k \times GL_k \times GL_k$. Under this action, a matrix satisfying (1.7) can be transformed into an arbitrary matrix having all blocks $B_{i,4}$ invertible. Since such matrices form a dense set in the variety of all matrices $B$, we conclude that $P(B)$ must be one of the minors $\det(B_{1,4})$, $\det(B_{2,4})$ or $\det(B_{3,4})$. However $\det(B)$ is not divisible by either of these minors: by the obvious symmetry, it is enough to show that $\det(B)$ is not divisible by $\det(B_{3,4})$, and we have seen this already. This shows that in Case 1 $\det(B)$ is irreducible.

Case 2 can be treated in exactly the same way. We specialize

$$B_{1,4} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_{2,4} = B_{3,4} = 1,$$

and, after the calculations parallel to those in Case 1, see that this specialization makes $\det(B)$ irreducible (it becomes, up to sign, the determinant of a symmetric $3 \times 3$ matrix with the only restriction that its $(1,1)$ entry is $0$). An argument similar to that in Case 1, shows that if $\det(B)$ is reducible then it must be divisible by $B_{2,4}$ or $B_{3,4}$; we have already seen that this is not the case.

To complete the proof of Theorem 1.1, it remains to show that $\det(B)$ is irreducible in the case when $r \geq 5$ and $k_2 + \cdots + k_r > k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$. We prove this by induction on $r$ starting with the case $r = 4$ considered above. Setting $B_{r-1,r} = 0$, we obtain the matrix $B'$ of type $(k_1, \ldots, k_{r-2}, k_{r-1}+k_r)$. When $r \geq 5$, we have $k_{r-1}+k_r < k_1+\cdots+k_{r-2}$, so $\det(B')$ is irreducible (in particular, non-zero) by the inductive assumption. Hence, $\det(B)$ is also irreducible. Theorem 1.1 is proved.

We conclude this section with one more result about symmetric matrices $B$, to be used later for the analysis of cusp type singularities.

LEMMA 1.4.

(a) *Suppose $k_2 + \cdots + k_r > k_1 \geq k_2 \geq \cdots \geq k_r$. Then there exists a symmetric matrix $B$ of type $(k_1, \ldots, k_r)$ having rank $k_1 + \cdots + k_r - 1$ and such that for every $i = 1, \ldots, r$ the $i$th row block*

$$B_{i,\bullet} = (\,{}^tB_{1,i} \quad {}^tB_{2,i} \quad \cdots \quad {}^tB_{i-1,i} \quad 0 \quad B_{i,i+1} \quad \cdots \quad B_{i,r}\,)$$

*of $B$ has full rank $k_i$.*

(b) *If $r = 3$ and $(k_1, k_2, k_3) = (k, k, 1)$ for some $k \geq 1$ then there exists a symmetric matrix $B$ of type $(k_1, \ldots, k_r)$ satisfying the conditions in (a) and the additional conditions*

$$\det(B_{1,2}) = 0, \quad \det \begin{pmatrix} B_{1,2} & B_{1,3} \\ {}^tB_{2,3} & 0 \end{pmatrix} \neq 0.$$

*Proof.* — First we give another description of the types $(k_1, \ldots, k_r)$ for which $\det(B)$ is not identically equal to 0. Let $\Phi_r$ denote the semigroup of all non-negative integer vectors $\mathbf{k} = (k_1, \ldots, k_r)$ such that

$$k_i \leq \sum_{j:j\neq i} k_j \tag{1.9}$$

for all $i = 1, \ldots, r$. For every subset of indices $I \subset [1, r]$ let $\delta_I \in \mathbf{Z}_+^r$ denote the indicator vector of $I$, that is, $\delta_{I,i}$ is 1 for $i \in I$ and is 0 for $i \in [1, r] - I$.

LEMMA 1.5. — *The semigroup $\Phi_r$ is generated by the vectors $\delta_I$ for all the subsets $I \subset [1, r]$ of cardinality 2 or 3. Furthermore, if $\mathbf{k} \in \Phi_r$ is such that all the inequalities (1.9) are strict, and $\mathbf{k}$ is not of the form $\delta_I$ for $\#(I) = 4$, then $\mathbf{k}$ can be represented as a sum $\delta_{I_1} + \cdots + \delta_{I_s}$, where each of the subsets $I_j \subset [1, r]$ has cardinality 2 or 3 and $\#(I_1) = 3$.*

*Proof of Lemma 1.5.* — Without loss of generality, we can assume that $\mathbf{k} \in \Phi_r$ has $k_1 \geq \cdots \geq k_r$. If $k_1 = k_2 + \cdots + k_r$ then we have $\mathbf{k} = \sum_{i=2}^{r} k_i \delta_{\{1,i\}}$. So let us assume that $k_1 < k_2 + \cdots + k_r$. Clearly, in this case we have $k_3 \geq 1$, so the vector $\mathbf{k}' = \mathbf{k} - \delta_{\{1,2,3\}}$ has all components non-negative. Proceeding by induction on $k_1 + \cdots + k_r$, we see that our lemma is a consequence of the following statement: if $\mathbf{k} \neq \delta_{\{1,2,3,4\}}$ then $\mathbf{k}' \in \Phi_r$. To prove this statement, we observe that the only condition in (1.9) that is not automatically fulfilled for $\mathbf{k}'$ is

$$k_4 \leq (k_1 - 1) + (k_2 - 1) + (k_3 - 1) + \sum_{i=5}^{r} k_i. \tag{1.10}$$

Rewriting (1.10) in the form

$$(k_1 - 1) + (k_2 - 1) + (k_3 - k_4) + \sum_{i=5}^{r} k_i > 0,$$

we see that it can fail only when $\mathbf{k} = \delta_{\{1,2,3,4\}}$. Lemma 1.5 is proved.

Returning to the proof of Lemma 1.4, let us put the question into a more invariant setting, just as in the proof of Theorem 1.1 above. A symmetric matrix $B$ of type $(k_1, \ldots, k_r)$ represents an $r$-tuple of vector spaces $E_1, \ldots, E_r$ of dimensions $k_1, \ldots, k_r$, endowed with a collection of linear maps $B_{i,j} : E_j \to E_i^*$ for all $i \neq j$ such that $B_{j,i} = {}^tB_{i,j}$. There is an obvious notion of the direct sum of such $r$-tuples, and the ranks in Lemma 1.4 are additive with respect to this direct sum. As above, we shall write the direct sum simply as $B \oplus B'$.

We shall construct a matrix $B$ with desired properties as the direct sum of some standard matrices. First, for each subset $I \subset [1,r]$ of cardinality 2 or 3 we consider a symmetric matrix $B^{(I)}$ of type $\delta_I$ such that all the maps $B^{(I)}_{i,j}$ for $i,j \in I$, $i \neq j$ are isomorphisms of (one-dimensional) vector spaces. If $\#(I) = 3$, $I = \{i, i', i''\}$, we also consider another matrix $B^{(I)}_0$ of type $\delta_I$ such that $B_{i,i'}$ and $B_{i,i''}$ are isomorphisms but $B_{i',i''} = 0$. Clearly, for each of these matrices the ranks in Lemma 1.4 (a) have their maximal possible value, with the exception of the matrix $B$ for $B^{(I)}_0$ that has corank 1.

Now suppose that $\mathbf{k} = (k_1, \ldots, k_r)$ is as in Lemma 1.4 (a). First we assume that $\mathbf{k} \neq (1,1,1,1)$. By Lemma 1.5, there exists a decomposition $\mathbf{k} = \delta_{I_1} + \cdots + \delta_{I_s}$, where each of the subsets $I_j \subset [1,r]$ has cardinality 2 or 3 and $\#(I_1) = 3$. Then we can choose a matrix $B$ satisfying the properties in Lemma 1.4 (a), in the following way:

$$B = B^{(I_1)}_0 \oplus B^{(I_2)} \oplus \cdots \oplus B^{(I_s)}. \tag{1.11}$$

For $\mathbf{k} = (1,1,1,1)$ an obvious check shows that we can choose

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

This completes the proof of Lemma 1.4 (a). Finally, a matrix $B$ satisfying the conditions in Lemma 1.4 (b) can be constructed by the same decomposition (1.11), where we choose the first summand $B^{(I_1)}_0$ in such a way that its component $B_{1,2}$ is an isomorphism.

## 2. Cusp type singularities.

In this section we study the cusp type variety $\nabla_{\mathrm{cusp}}$ introduced in (0.8); our goal is to prove Theorem 0.1 that provides a description of the irreducible components of $\nabla_{\mathrm{cusp}}$. We shall use the notation of the Introduction. In particular, we denote by $x^0 \in Y = (V_1 - \{0\}) \times \cdots \times (V_r - \{0\})$ the point with the coordinates $x^{(j)}_i = \delta_{i,0}$. It will be convenient for us to dehomogenize the multilinear form

$$F(A,x) = \sum a_{i_1,\ldots,i_r} x^{(1)}_{i_1} \cdots x^{(r)}_{i_r}$$

corresponding to a matrix $A$, by setting all $x^{(j)}_0$ equal to 1. More precisely, consider the subset

$$E = \{x \in Y : x^{(j)}_0 = 1 \;\; (j = 1, \ldots, r)\}.$$

Clearly, $E$ is isomorphic to the affine space

$$\mathbf{C}^{k_1+\cdots+k_r} = \mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_r}$$

with the coordinates $x_i^{(j)}$, $j \in [1, r]$, $i \in [1, k_j]$; this is an affine chart in $X = P^{k_1} \times \cdots \times P^{k_r}$ near the point $((1:0:\cdots:0), \ldots, (1:0:\cdots:0))$ which is the image of $x^0$ under the natural projection $Y \to X$.

In order to write down the form $F(A, x)$ for $x \in E$ in a more convenient way, we shall use another notation for the matrix entries of $A$. Namely, we shall write an entry $a_{i_1,\ldots,i_r}$ of $A$ as $a_{jj'\ldots j''}^{ii'\ldots i''}$, where the lower indices stand for the positions $j \in [1, r]$ for which $i_j \neq 0$, and the upper indices are these non-zero values $i_j$ in the same order. Thus, we write simply $a$ for $a_{0,\ldots,0}$, $a_j^i$ for $a_{0,\ldots,i,\ldots,0}$ with a non-zero $i$ in the $j$th place, and so on. In this notation, for $x \in E$ we have

$$F(A, x) = a + \sum a_j^i x_i^{(j)} + \sum a_{jj'}^{ii'} x_i^{(j)} x_{i'}^{(j')} + \cdots. \tag{2.1}$$

For every $x \in Y$ let $\nabla(x)$ denote the set of matrices $A \in M$ having $x$ as a critical point; equivalently, $\nabla(x) = \{A : (A, \mathrm{pr}(x)) \in Z\}$, where $Z$ is the incidence variety in (0.2). In particular, in view of (2.1), we have

$$\nabla(x^0) = \{A \in M : a = a_j^i = 0 \ \text{ for all } \ i, j\}. \tag{2.2}$$

Let $B = B(A)$ be the quadratic part of $A$ at $x^0$ (see (0.6)). Thus, $B$ has matrix entries $(a_{jj'}^{ii'})$, with $(i, j)$ the row index and $(i', j')$ the column index. In this notation, the definition (0.7) takes the form

$$\nabla_{\text{cusp}}^0 = \{A \in \nabla(x^0) : \det(B(A)) = 0\}. \tag{2.3}$$

According to (0.8), the variety $\nabla_{\text{cusp}}$ is equal to $\nabla_{\text{cusp}}^0 \cdot G$, where

$$G = GL_{k_1+1} \times \cdots \times GL_{k_r+1}.$$

In view of the equality $\nabla_{\text{cusp}} = \nabla_{\text{cusp}}^0 \cdot G$, to show that $\nabla_{\text{cusp}}$ is irreducible it is sufficient (but not necessary!) to show that $\nabla_{\text{cusp}}^0$ is irreducible. In the terminology of Section 1, the quadratic part $B$ of a matrix $A \in M$ is a symmetric matrix of type $(k_1, \ldots, k_r)$. In view of (2.1) and (2.2), Theorem 1.1 implies that $\nabla_{\text{cusp}}^0$ (and hence $\nabla_{\text{cusp}}$) is irreducible for all the interior and boundary formats except $(k_1, \ldots, k_r) = (k, k, 1)$ for some $k \geq 1$.

In our treatment of the exceptional case $(k_1, \ldots, k_r) = (k, k, 1)$ and in a subsequent computation of the dimension of $\nabla_{\text{cusp}}$, we shall replace $G$ by the subgroup $\mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_r}$, where each factor $\mathbf{C}^{k_j}$ is the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ x & I_{k_j} \end{pmatrix}$$

in $GL_{k_j+1}$ (here $x \in \mathbf{C}^{k_j}$, and $I_{k_j}$ is the identity $k_j \times k_j$ matrix). Clearly, the subgroup $\mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_r}$ acts simply transitively by translations on the affine space $E$. With some abuse of notation, we identify this subgroup with $E$: thus, for every $x = (x^{(1)}, \ldots, x^{(r)}) \in E$ we denote by the same symbol $x$ the element

$$(g_1, \ldots, g_r) \in \mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_r} \subset G,$$

where each $g_j$ has the first column $x^{(j)}$. An easy calculation shows that the right action of $E$ on the matrix space $M$ is given by the following formula:

$$(A \cdot x)^{ii'\ldots i''}_{jj'\ldots j''} = \partial^i_j \partial^{i'}_{j'} \cdots \partial^{i''}_{j''} F(A, x). \tag{2.4}$$

Returning to the cusp type singularities, it is clear from the definitions that for $g \in G$ the variety $\nabla^0_{\text{cusp}} \cdot g$ depends only on the point $g^{-1}x^0$. It follows that

$$\nabla_{\text{cusp}} = \overline{\nabla^0_{\text{cusp}} \cdot E}, \tag{2.5}$$

the bar meaning Zariski closure.

Now let us consider the exceptional case $(k_1, \ldots, k_r) = (k, k, 1)$ for some $k \geq 2$. By Theorem 1.1 (d), in this case $\nabla^0_{\text{cusp}}$ has two irreducible components. To show that $\nabla_{\text{cusp}}$ is still irreducible, we shall prove that one of the components of $\nabla^0_{\text{cusp}}$ is contained in the closure of the image of another one under the action of $E$.

To keep up with the notation of Section 1, we shall represent a matrix $A \in \nabla(x^0)$ as a pair of ordinary $(k+1) \times (k+1)$ matrices written as follows:

$$A_{\bullet\bullet 0} = \begin{pmatrix} 0 & 0 \\ 0 & B_{1,2} \end{pmatrix}, \quad A_{\bullet\bullet 1} = \begin{pmatrix} 0 & {}^tB_{2,3} \\ B_{1,3} & B' \end{pmatrix}. \tag{2.6}$$

Here $B_{1,2}$ and $B'$ are $k \times k$ matrices, and $B_{1,3}$ and $B_{2,3}$ are columns of length $k$. By Theorem 1.1 (d), two irreducible components of $\nabla^0_{\text{cusp}}$ are given by

$$\begin{aligned} \nabla' &= \{A \in \nabla(x^0) : \det(B_{1,2}) = 0\}, \\ \nabla'' &= \left\{A \in \nabla(x^0) : \det \begin{pmatrix} B_{1,2} & B_{1,3} \\ {}^tB_{2,3} & 0 \end{pmatrix} = 0\right\}. \end{aligned} \tag{2.7}$$

The irreducibility of $\nabla_{\text{cusp}}$ becomes a consequence of the following lemma.

LEMMA 2.1. — *The set of matrices $A \in \nabla'$ such that $A \cdot x \in \nabla''$ for some $x \in E$, is dense in $\nabla'$. Therefore, $\nabla_{\text{cusp}} = \overline{\nabla'' \cdot E}$.*

*Proof.* — The group $E$ has the natural direct sum decomposition $E = E_1 \oplus E_2 \oplus \mathbf{C}$, where $\dim(E_1) = \dim(E_2) = k$. We can think of $B_{1,2}$ and

$B'$ as representing linear maps $E_2 \to E_1^*$, or, equivalently, bilinear forms on $E_1 \times E_2$; we can think of $B_{1,3}$ and $B_{2,3}$ as lying in $E_1^*$ and $E_2^*$, respectively. We write $x \in E$ as a triple $(u, v, z)$ with $u \in E_1$, $v \in E_2$, $z \in \mathbf{C}$. Using (2.4), we see that the matrix $A \cdot x$ has the following form:

$$(A\cdot x)_{\bullet\bullet 0} = \begin{pmatrix} B_{1,2}(u,v)+z(B'(u,v)+B_{1,3}(u)+B_{2,3}(v)) & {}^tB_{1,2}(u)+z\,{}^tB'(u)+zB_{2,3} \\ B_{1,2}(v)+zB'(v)+zB_{1,3} & B_{1,2}+zB' \end{pmatrix},$$

$$(A\cdot x)_{\bullet\bullet 1} = \begin{pmatrix} B'(u,v)+B_{1,3}(u)+B_{2,3}(v) & {}^tB'(u)+B_{2,3} \\ B'(v)+B_{1,3} & B' \end{pmatrix}.$$

The matrix $A \cdot x$ lies in $\nabla(x^0)$ if all the components of $(A \cdot x)_{\bullet\bullet 0}$ except $B_{1,2} + zB'$ are equal to 0. These conditions can be rewritten as follows:

$$ {}^t(B_{1,2} + zB')(u) = -zB_{2,3}, \tag{2.8}$$

$$ (B_{1,2} + zB')(v) = -zB_{1,3}, \tag{2.9}$$

$$ (B_{1,2} + zB')(u, v) = 0. \tag{2.10}$$

Now let us assume that $B_{1,2} + zB'$ is invertible for some $z \neq 0$. Then (2.8) and (2.9) imply

$$u = -z\,{}^t(B_{1,2} + zB')^{-1}(B_{2,3}), \quad v = -z(B_{1,2} + zB')^{-1}(B_{1,3}).$$

Substituting this to (2.10), we get

$$ B_{2,3}\big((B_{1,2} + zB')^{-1}(B_{1,3})\big) = 0. \tag{2.11}$$

If one of $B_{1,3}$ and $B_{2,3}$ is equal to 0 then (2.11) is trivially true. If both $B_{1,3}$ and $B_{2,3}$ are non-zero, then (by changing bases in $E_1$ and $E_2$) we can assume that both of them are represented by a column vector $(1, 0, 0, \ldots, 0)$. Then (2.11) means simply that the cofactor of the matrix $B_{1,2} + zB'$ corresponding to its $(1, 1)$ entry is equal to 0. Let us denote this cofactor by $\det'(B_{1,2} + zB')$. Thus, if $\det'(B_{1,2} + zB') = 0$ then $A \cdot x$ belongs to $\nabla^0_{\mathrm{cusp}}$ but not to $\nabla'$ since $B_{1,2} + zB'$ is assumed to be invertible. Therefore, $A \cdot x \in \nabla''$. To complete the proof of the lemma it remains to establish the following elementary statement:

(*) every pair $(B_{1,2}, B')$ of $k \times k$ matrices such that $\det(B_{1,2}) = 0$, lies in the closure of the set

$$\begin{aligned} \{(B_{1,2}, B') : \det(B_{1,2}) = 0, \quad \det{}'(B_{1,2} + zB') = 0, \\ \det(B_{1,2} + zB') \neq 0 \quad \text{for some} \quad z \in \mathbf{C}\}. \end{aligned} \tag{2.12}$$

To prove (*), we construct explicitly a polynomial function $R(B_{1,2}, B')$ such that the set (2.12) contains the set

$$\{(B_{1,2}, B') : \det(B_{1,2}) = 0, \quad R(B_{1,2}, B') \neq 0\}.$$

If $k \geq 3$ then we define $R(B_{1,2}, B')$ to be the resultant of two polynomials of degree $\leq (k-1)$ in one variable $z$ given by $z \mapsto \dfrac{\det(B_{1,2} + zB')}{z}$ and $z \mapsto \det'(B_{1,2} + zB')$ (the first map is indeed a polynomial since we assume $\det(B_{1,2}) = 0$). For $k = 2$, two above polynomials are linear functions of $z$; if they are written as $az+b$ and $cz+d$ then we define $R(B_{1,2}, B') = ad - bc$. The function $R(B_{1,2}, B')$ has the desired property by the definition of the resultant. It remains to prove that $R(B_{1,2}, B')$ is not identically equal to 0. We can take $B' = -I_k$ (where $I_k$ is the identity $k \times k$ matrix). Then the roots of $\dfrac{\det(B_{1,2} + zB')}{z}$ are the non-zero eigenvalues of $B_{1,2}$, and the roots of $\det'(B_{1,2} + zB')$ are the eigenvalues of the submatrix $B'_{1,2}$ obtained from $B_{1,2}$ by taking off the first row and the first column. Adding to $B_{1,2}$ a scalar matrix $\alpha I_k$, we shift all the eigenvalues of $B_{1,2}$ and $B'_{1,2}$ by $\alpha$. So it is enough to exhibit a $k \times k$ matrix $C$ having $k$ distinct non-zero eigenvalues and such that the truncated matrix $C'$ has all eigenvalues distinct from those of $C$. We can choose $C$ to be the *companion* matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1} \end{pmatrix}$$

with generic coefficients $a_0, \ldots, a_{k-1}$. Then the characteristic polynomials of $C$ and $C'$ are

$$P(\lambda) = \det(\lambda I_k - C) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_0$$

and

$$Q(\lambda) = \det(\lambda I_k - C') = \lambda^{k-1} - a_{k-1}\lambda^{k-2} - \cdots - a_1.$$

They are related by $P(\lambda) = \lambda Q(\lambda) - a_0$, hence have no common roots if $a_0 \neq 0$. This completes the proof of the lemma (we are grateful to A. Lascoux and M. Nazarov for suggesting the use of companion matrices in the above argument).

We leave the exceptional case when $(k_1, \ldots, k_r) = (1,1,1)$ until the end of the section, and turn to the computation of $\dim(\nabla_{\mathrm{cusp}})$. Our strategy will be to find the tangent space $T_A \nabla_{\mathrm{cusp}}$ at a generic point $A$ of $\nabla^0_{\mathrm{cusp}}$. This tangent space is a subspace in $T_A M$; we shall use the differentials $da, da^i_j, \ldots$ as the coordinates in $T_A M$.

PROPOSITION 2.2. — *If the matrix format is interior and $(k_1, \ldots, k_r) \neq (1,1,1)$ then the tangent space $T_A \nabla_{\mathrm{cusp}}$ at a generic point $A$ of $\nabla^0_{\mathrm{cusp}}$*

*has codimension 2 in $T_A M$. One linear equation defining $T_A \nabla_{\mathrm{cusp}}$ in $T_A M$ is $da = 0$, and another one has the form*

$$\sum_{i,j} c_i^j \, da_j^i = 0$$

*for some coefficients $c_i^j$.*

*Proof.* — The first part of the argument makes sense for arbitrary interior or boundary format. Let

$$\tilde{Z} = \{(A, x) \in M \times E : A \cdot x \in \nabla(x^0)\}. \tag{2.13}$$

In view of (2.2) and (2.4), for every $A \in \nabla(x^0)$ the tangent space $T_{(A,x^0)}\tilde{Z}$ is given by the equations

$$da = 0, \quad da_j^i + \sum_{i',j'} a_{jj'}^{ii'} dx_{i'}^{(j')} = 0 \;\text{ for all }\; i, j, \tag{2.14}$$

where the differentials $da, da_j^i, \ldots$ and the $dx_i^{(j)}$ are the usual coordinates in $T_A M$ and $T_{x^0} E$, respectively.

Now consider the variety

$$\tilde{Z}_{\mathrm{cusp}} = \{(A, x) \in M \times E : A \cdot x \in \nabla^0_{\mathrm{cusp}}\}. \tag{2.15}$$

In view of (2.3), we have

$$\tilde{Z}_{\mathrm{cusp}} = \{(A, x) \in \tilde{Z} : \det(B(A \cdot x)) = 0\}. \tag{2.16}$$

At this point we would like to use the assumption that the matrix format is interior. Let us also assume that $(k_1, \ldots, k_r) \neq (k, k, 1)$. According to Theorem 1.1 (c), the polynomial $\det(B(A))$ in (2.3) is irreducible. By (2.16), $T_{(A,x^0)}\tilde{Z}_{\mathrm{cusp}}$ is given by the equations (2.14) combined with the additional equation

$$d(\det(B(A \cdot x))) = 0. \tag{2.17}$$

Let $\widehat{B}_{ii'}^{jj'}$ denote the cofactor of the quadratic part $B = B(A)$ corresponding to the entry $a_{jj'}^{ii'}$. Then the equation (2.17) evaluated at $x = x^0$ can be written as

$$\sum_{i,j,i',j'} \widehat{B}_{ii'}^{jj'} \Big( da_{jj'}^{ii'} + \sum_{i'',j''} a_{jj'j''}^{ii'i''} dx_{i''}^{(j'')} \Big) = 0. \tag{2.18}$$

In view of (2.5), the tangent space $T_A \nabla_{\mathrm{cusp}}$ at a generic point $A$ of $\nabla^0_{\mathrm{cusp}}$ is the image of $T_{(A,x^0)}\tilde{Z}_{\mathrm{cusp}}$ under the map $\mathrm{pr}_{1,*}$ induced by the first projection. In other words, $T_A \nabla_{\mathrm{cusp}}$ is defined by all the linear relations between the coordinates $da, da^{ii'}_{jj'}, \ldots$, that are the consequences of (2.14) and (2.18), that is, by the linear combinations of the equations in (2.14) and (2.18) not containing the terms with the $dx_i^{(j)}$. One of these defining equations for $T_A \nabla_{\text{cusp}}$ is, of course, the equation $da = 0$ in (2.14). According to Lemma 1.4 (a), for $A$ generic we can assume that $B$ has corank 1. This implies that the second group of equations in (2.14) produces exactly one linear relation between the coordinates $da^i_j$. This relation has the form

$$\sum_{i,j} c_i^j \, da^i_j = 0, \tag{2.19}$$

where the $c_i^j$ are the coefficients of the linear dependency between the rows of $B$.

To complete the proof of Proposition 2.2 in our case, it remains to show that we can choose $A$ so that the linear form in the variables $dx_i^{(j)}$ appearing in (2.18), is not a linear combination of the rows of $B$, and so, does not produce additional equations for $T_A \nabla_{\text{cusp}}$. Since $B$ is symmetric, its columns satisfy the same linear relation as its rows. So it is enough to produce a matrix $A$ satisfying the single inequality

$$\sum_{i,j,i',j',i'',j''} \widehat{B}^{jj'}_{ii'} c^{j''}_{i''} a^{ii'i''}_{jj'j''} \neq 0. \tag{2.20}$$

Since $B$ has corank 1, its cofactor matrix $\widehat{B}$ has rank 1, and we can choose any non-zero row of $\widehat{B}$ as the vector of the coefficients $c_i^j$ in (2.19). Let $(i'', j'')$ be the index of this row, so we have

$$c_i^j = \widehat{B}^{jj''}_{ii''} \tag{2.21}$$

for all $i, j$. To prove (2.20), we observe that a generic matrix $A$ also satisfies the last condition in Lemma 1.4 (a). This means that there exist at least two pairs of indices $(i, j)$ and $(i', j')$ such that $j \neq j'$ and $c_i^j \neq 0$, $c_{i'}^{j'} \neq 0$. In view of (2.21), this can be written as

$$\widehat{B}^{jj''}_{ii''} \neq 0, \quad \widehat{B}^{j'j''}_{i'i''} \neq 0. \tag{2.22}$$

Now, if we choose $(i'', j'')$ according to (2.21) and choose $(i, j), (i', j')$ satisfying (2.22), then we see that the coefficient of $a^{ii'i''}_{jj'j''}$ in (2.20) is equal to

$$\widehat{B}^{jj'}_{ii'} c^{j''}_{i''} + \widehat{B}^{jj''}_{ii''} c^{j'}_{i'} + \widehat{B}^{j'j''}_{i'i''} c^{j}_{i} = 2\widehat{B}^{jj''}_{ii''} \widehat{B}^{j'j''}_{i'i''},$$

and so is non-zero. Since we do not have any restrictions on the matrix entries of type $a^{ii'i''}_{jj'j''}$, the last statement allows us to satisfy (2.20). This completes the proof of Proposition 2.2 under the assumption that $(k_1, \ldots, k_r) \neq (k, k, 1)$.

Now let us assume that $(k_1, \ldots, k_r) = (k, k, 1)$ for some $k > 1$. In this case the argument is practically the same as above, with the following modification. Now, in accordance with Theorem 1.1 (d), the polynomial $\det(B(A \cdot x))$ in (2.16) is the product of two irreducible factors $H'(A, x)$ and $H''(A, x)$, . Since we have

$$d(\det(B(A \cdot x))) = H'dH'' + H''dH',$$

the equation (2.17) evaluated at $x = 0$ is equivalent to $dH''(A, x) = 0$ provided a matrix $A$ is chosen in such a way that its quadratic part $B$ satisfies the conditions in Lemma 1.4 (b). For such a matrix $A$ the above analysis shows that $T_A \nabla_{\text{cusp}}$ is given by the same equations $da = 0$ and (2.19) as above. Proposition 2.2 is proved.

PROPOSITION 2.3. — *If the matrix format is boundary then the tangent space $T_A \nabla_{\text{cusp}}$ at a generic point $A$ of $\nabla^0_{\text{cusp}}$ has codimension 3 in $T_A M$.*

*Proof.* — The proof is quite similar to that of Proposition 2.2. In the case of the boundary format, the matrix $B = B(A)$ has the form

$$B = \begin{pmatrix} 0 & C(A) \\ {}^tC(A) & B' \end{pmatrix},$$

where $C(A)$ and $B'$ are $k_1 \times k_1$ matrices, and $B'$ is a symmetric matrix of type $(k_2, \ldots, k_r)$. Thus, $\det(B(A)) = (-1)^{k_1}(\det(C(A)))^2$, and so, the equation (2.17) is replaced by

$$d(\det(C(A \cdot x))) = 0. \tag{2.23}$$

Computing the differential in (2.23) at $x = x^0$, we see that (2.18) is now replaced by

$$\sum_{i,i',j'} \widehat{C}^{1j'}_{ii'} \Big(da^{ii'}_{1j'} + \sum_{i'',j''} a^{ii'i''}_{1j'j''} dx^{(j'')}_{i''}\Big) = 0, \tag{2.24}$$

where the terms $\widehat{C}^{1j'}_{ii'}$ are the cofactors of $C = C(A)$.

It is easy to see that if $C$ is a generic matrix of corank 1, and $B'$ is a generic symmetric matrix of type $(k_2, \ldots, k_r)$ then $B$ has corank 1, and the unique linear dependency between rows of $B$ involves only the first $k_1$ rows. (This is the place where the condition $r \geq 3$ is needed; if $r = 2$ then $B'$ $= 0$, and $B$ has generically corank 2.) As in Proposition 2.2, the equations (2.14) give rise to two linear equations for the subspace $T_A \nabla_{\text{cusp}} \subset T_A M$. One of them is again $da = 0$, and another one has the form (2.19), where now all the coefficients $c^j_i$ with $j \neq 1$ are equal to 0.

The condition that (2.24) does not give rise to an additional equation for $T_A\nabla_{\text{cusp}}$, is equivalent to the fact that it is possible to choose a matrix $A$ satisfying the following analog of (2.20):

$$\sum_{j,i',j',i'',j''} \widehat{C}^{1j'}_{ii'} c^{j''}_{i''} a^{ii'i''}_{1j'j''} \neq 0. \tag{2.25}$$

However, since $c^{j''}_{i''} = 0$ for $j'' \neq 1$, the left hand side of (2.25) is always equal to 0. We conclude that there is a linear combination of (2.24) and the second group of equations in (2.14) that produces one more linear equation for $T_A\nabla_{\text{cusp}} \subset T_AM$. Hence, $T_A\nabla_{\text{cusp}}$ has codimension 3 in $T_AM$. Proposition 2.3 is proved.

To complete the proof of Theorem 0.1, it remains to treat the case $(k_1, \ldots, k_r) = (1, 1, 1)$, that is, the $2 \times 2 \times 2$ matrix format. In this case it is known and easy to check (see [4], [5]) that the group $G = GL_2 \times GL_2 \times GL_2$ has six orbits on the set $M - \{0\}$ of non-zero matrices. The variety $\nabla_{\text{cusp}} - \{0\}$ is the union of four $G$-orbits that we shall denote $\Omega_1$, $\Omega_2$, $\Omega_3$ and $\Omega_0$; their representatives are, respectively, $e_{110}+e_{101}$, $e_{110}+e_{011}$, $e_{101}+e_{011}$ and $e_{111}$, where the $e_{ijk}$ are the standard basis ("matrix units") in $M$. Each of the orbits $\Omega_1$, $\Omega_2$, $\Omega_3$ has dimension 5, i.e., is of codimension 3 in $M$, and contains $\Omega_0$ in its closure. Thus, $\nabla_{\text{cusp}}$ has three irreducible components, namely, the closures $\overline{\Omega_1}$, $\overline{\Omega_2}$, $\overline{\Omega_3}$. This completes the proof of Theorem 0.1.

Before leaving the case of $2 \times 2 \times 2$ matrices, let us make two more remarks. First, it is easy to see that $\overline{\Omega_1}$ is the set of all matrices $A$ such that the "flattened" matrix

$$\begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix}$$

has rank $\leq 1$. The components $\overline{\Omega_2}$ and $\overline{\Omega_3}$ have similar descriptions, using flattenings in two other directions. Thus, each of the components is isomorphic to the determinantal variety of $2 \times 4$ matrices of rank $\leq 1$.

Finally, the remaining two $G$-orbits whose union is $M - \nabla_{\text{cusp}}$, are the 8-dimensional orbit $M - \nabla$ (with a representative $e_{000} + e_{111}$) and the 7-dimensional orbit $\nabla - \nabla_{\text{cusp}}$ (with a representative $e_{110} + e_{101} + e_{011}$). Since all the points in $\nabla - \nabla_{\text{cusp}}$ are $G$-conjugate, it follows that all of them are smooth points of $\nabla$. Thus, we conclude that

$$\nabla_{\text{sing}} = \nabla_{\text{cusp}}. \tag{2.26}$$

## 3. Eliminating special node components.

In this section we prove Theorem 0.2. We need some notation. We denote by $[1,r]$ the set $\{1,\dots,r\}$, and for every subset $J \subset [1,r]$ let $\bar{J}$ be the complement $[1,r]-J$. Let $x(J) \in Y$ be the point with components $x^{(j)} = (1,0,\dots,0)$ for $j \in J$ and $x^{(j)} = (0,0,\dots,0,1)$ for $j \in \bar{J}$ (in particular, $x([1,r])$ is the point previously denoted by $x^0$). Recall from Section 2 that for every $x \in Y$ we denote by $\nabla(x)$ the set of matrices having $x$ as a critical point. Clearly, $\nabla(x)$ is a vector subspace in $M$ of codimension $k_1+\cdots+k_r+1$. In particular, for $x = x(J)$ this subspace has the following description. Following [4], we call the *star* of a multi-index $(i_1,\dots,i_r)$ the set of all multi-indices $(i'_1,\dots,i'_r)$ that differ from $(i_1,\dots,i_r)$ in at most one position. We denote by $\mathbf{i}(J)$ the multi-index $(i_1,\dots,i_r)$ having $i_j = 0$ for $j \in J$ and $i_j = k_j$ for $j \in \bar{J}$. Then we have

(3.1)
$$\nabla(x(J)) = \{A{\in}M{:}a_{i_1,\dots,i_r}{=}0 \text{ for all } (i_1,\dots,i_r) \text{ in the star of } \mathbf{i}(J)\}.$$

(In particular, when $J = [1,r]$, (3.1) amounts to (2.2).) Now the definition of the varieties $\nabla_{\text{node}}(J)$ given in the Introduction can be rewritten as follows:

$$\nabla_{\text{node}}(J) = \overline{(\nabla(x([1,r])) \cap \nabla(x(J))) \cdot G}, \tag{3.2}$$

where $G$ is the group $GL_{k_1+1} \times \cdots \times GL_{k_r+1}$ naturally acting on $M$, and the bar stands for the Zariski closure.

Now we are ready for the proof of Theorem 0.2 (a). In view of (3.2), to show that $\nabla_{\text{node}}(J) \subset \nabla_{\text{cusp}}$ it is enough to prove that for every $A \in \nabla(x([1,r])) \cap \nabla(x(J))$ the quadratic part $B(A)$ is degenerate. We recall from Section 2 that $B = B(A)$ is a symmetric square matrix of order $k_1+\cdots+k_r$ with entries $a_{jj'}^{ii'}$ $(j,j' \in [1,r],\ i \in [1,k_j],\ i' \in [1,k_{j'}])$, where $(i,j)$ is the row index and $(i',j')$ the column index. Now suppose that $\#(J) = r-1$, and let $\bar{J} = \{j\}$. By (3.1), we have $a_{jj'}^{k_j i'} = 0$ for all $i',j'$ as above. This means that the row of $B$ indexed by $(k_j,j)$ is equal to 0, hence $\det(B) = 0$, as required.

The second statement in Theorem 0.2 (a) is only slightly more complicated. So suppose that $\#(J) = r-2$, $\bar{J} = \{j,j'\}$, and $k_j \geq \sum_{i \neq j} k_i - 1$. It follows from (3.1) that $a_{jj'}^{k_j i'} = a_{jj'}^{i k_{j'}} = 0$ for all $i,i'$. Hence the submatrix of $B$ with the set of row and column indices $\{(i,j): 1 \leq i \leq k_j\} \cup \{(k_{j'},j')\}$ is identically zero. Since this submatrix is of size $(k_j+1) \times (k_j+1)$, and

$2(k_j + 1) > k_1 + \cdots + k_r$, we conclude that in this case we have again $\det(B) = 0$. This completes the proof of Theorem 0.2 (a).

Let us turn to the proof of Theorem 0.2 (b). We start with an observation that the stars of the multi-indices $\mathbf{i}([1,r]) = (0,0,\ldots,0)$ and $\mathbf{i}(\emptyset) = (k_1, k_2, \ldots, k_r)$ do not meet each other (this is where the assumption $r \geq 3$ comes into play). It follows that the vector space $\nabla(x([1,r])) \cap \nabla(x(\emptyset))$ has codimension $2(k_1 + \cdots + k_r + 1)$ in $M$. Using the action of the group $G$ on $M$, we see that $\nabla(x) \cap \nabla(y)$ has codimension $2(k_1 + \cdots + k_r + 1)$ in $M$ whenever $\nabla(x) \cap \nabla(y) \subset \nabla_{\mathrm{node}}(\emptyset)$. The same argument with the stars as above shows that $\nabla(x([1,r])) \cap \nabla(x(J))$ also has codimension $2(k_1 + \cdots + k_r + 1)$ in $M$ whenever $\#(J) \leq r - 3$. If $\#(J) = r - 2$ and $\bar{J} = \{j', j''\}$ then the intersection of the stars of $\mathbf{i}([1,r])$ and $\mathbf{i}(J)$ consists of two multi-indices $\mathbf{i}(J \cup \{j'\})$ and $\mathbf{i}(J \cup \{j''\})$; therefore, in this case $\nabla(x([1,r])) \cap \nabla(x(J))$ has codimension $2(k_1 + \cdots + k_r)$ in $M$.

For every $J$ and every $t \neq 0$ we define the point $x(J,t) \in Y$ as follows: its component $x^{(j)}$ is equal to $(1,0,\ldots,0,t)$ for $j \in J$ and is equal to $(1,0,\ldots,0,t^{-1})$ for $j \in \bar{J}$. The above arguments show that $\nabla(x([1,r])) \cap \nabla(x(J,t))$ is a vector subspace contained in $\nabla_{\mathrm{node}}(\emptyset)$ and having codimension $2(k_1 + \cdots + k_r + 1)$ in $M$.

LEMMA 3.1.

(a) *If* $\#(J) < r - 2$ *then*

$$\lim_{t \to 0} (\nabla(x([1,r])) \cap \nabla(x(J,t))) = \nabla(x([1,r])) \cap \nabla(x(J)), \tag{3.3}$$

*the limit taken in the Grassmannian of subspaces of codimension* $2(k_1 + \cdots + k_r + 1)$ *in* $M$.

(b) *If* $\#(J) = r-2$ *and* $\bar{J} = \{j', j''\}$ *then* $\lim_{t \to 0}(\nabla(x([1,r])) \cap \nabla(x(J,t)))$ *exists and is equal to the subspace in* $\nabla(x([1,r])) \cap \nabla(x(J))$ *given by two additional equations*

$$\sum_{j \in J} a_{jj'}^{k_j k_{j'}} = \sum_{j \in J} a_{jj''}^{k_j k_{j''}} = 0. \tag{3.4}$$

*Proof.* — In both cases (a) and (b) we shall exhibit a system of linear forms $\varphi_{i,t}$ ($i = 1, \ldots, 2(k_1 + \cdots + k_r + 1)$) on $M$ defining the subspace $\nabla(x([1,r])) \cap \nabla(x(J,t))$ and having the following property:

$(*)$ Each $\varphi_{i,t}$ is a polynomial function of $t$, and the linear forms $\varphi_{i,0} = \varphi_{i,t}|_{t=0}$ are linearly independent.

Clearly, $(*)$ implies the existence of $\lim_{t\to 0}(\nabla(x([1,r]))\cap\nabla(x(J,t)))$, and the limit subspace is defined by the linear forms $\varphi_{i,0}$ $(i=1,\ldots,2(k_1+\cdots+k_r+1))$.

The first $(k_1+\cdots+k_r+1)$ forms $\varphi_{i,t}$ will be independent of $t$: these are simply the forms $a$ and $a^i_j$ defining $\nabla(x([1,r]))$. As for the remaining forms, in case (a) we take the forms $A\mapsto F(A,x(J,t))$ and $A\mapsto\partial^i_j F(A,x(J,t))$ defining $\nabla(x(J,t))$ and multiply each of them by an appropriate power of $t$ (here we take $i=1,\ldots,k_j$ for $j\in J$ and $i=0,\ldots,k_j-1$ for $j\in\bar{J}$). For instance, we have

$$F(A,x(J,t))=\sum_{I\subset[1,r]} t^{\#(\bar{I}\cap J)-\#(\bar{I}\cap\bar{J})}a_{\mathbf{i}(I)}. \tag{3.5}$$

As $t\to 0$, the leading term in (3.5) corresponds to $I=J$ and is equal to $t^{-\#(\bar{J})}a_{\mathbf{i}(J)}$; so, taking the corresponding form $\varphi_{i,t}$ to be $A\mapsto t^{\#(\bar{J})}F(A,x(J,t))$, we see that $\varphi_{i,0}=a_{\mathbf{i}(J)}$. All the forms $A\mapsto\partial^i_j F(A,x(J,t))$ are treated in a similar way, and we obtain from them in the limit $t\to 0$ the forms $a_{i_1,\ldots,i_r}$ for all $(i_1,\ldots,i_r)$ in the star of $\mathbf{i}(J)$. This proves (3.3), since these forms $a_{i_1,\ldots,i_r}$ together with $a_{\mathbf{i}(J)}$ define $\nabla(x(J))$ (see (3.1)).

In case (b) we proceed in the same manner, with the following modification. When we normalize as above the form $A\mapsto\partial^0_{j''}F(A,x(J,t))$, that is, multiply it by the appropriate power of $t$, we obtain the form $\varphi'_t$ whose two first leading terms are

$$a^{k_{j'}}_{j'}+t\Big(a+\sum_{j\in J}a^{k_j k_{j'}}_{jj'}\Big).$$

Since the forms $a^{k_{j'}}_{j'}$ and $a$ lie in our set of forms defining $\nabla(x([1,r]))$, we can replace $\varphi'_t$ by

$$\varphi_t=t^{-1}(\varphi'_t-a^{k_{j'}}_{j'}-ta).$$

Then we have

$$\varphi_0=\sum_{j\in J}a^{k_j k_{j'}}_{jj'}.$$

In a similar way, starting from the form $A\mapsto\partial^0_{j'}F(A,x(J,t))$ we obtain a form whose leading term at $t=0$ is the second sum in (3.4). Now Lemma 3.1 (b) follows from the obvious fact that the two forms in (3.4) and all the forms $a_{i_1,\ldots,i_r}$ for $(i_1,\ldots,i_r)$ in the stars of $\mathbf{i}(J)$ and $\mathbf{i}(\emptyset)$ are linearly independent.

Theorem 0.2 (b) in the case when $\#(J)<r-2$ follows from Lemma 3.1 (a). Now assume that we are in the situation of Lemma 3.1 (b), and that

$\sum_{j \in J} k_j > 2$. To complete the proof of Theorem 0.2, we have only to show that there exists a dense Zariski open set $U \subset \nabla(x([1,r])) \cap \nabla(x(J))$ with the following property: for every matrix $A \in U$ there exists $g \in G$ such that $Ag$ also belongs to $\nabla(x([1,r])) \cap \nabla(x(J))$ and in addition satisfies (3.4). Let us recall from [4], [5] that the action of each $GL_{k_j+1}$ on $M$ can be described as follows. We call the $i$th slice of $A$ in direction $j$ and denote by $A^i_j$ the set of all matrix entries $a_{i_1,\ldots,i_r}$ with $i_j = i$. Then $GL_{k_j+1}$ acts by linear transformations of the slices of $A$ in direction $j$. We are going to choose an element $g = (g_1, \ldots, g_r) \in G$ with the following properties:

(1) The components $g_{j'}$ and $g_{j''}$ of $g$ are identity transformations.

(2) Each $g_j$ for $j \in J$ leaves unchanged the slices $A^i_j$ with $i < k_j$ and transforms $A^{k_j}_j$ into a linear combination of the slices $A^i_j$ with $1 \le i \le k_j$, that is,

$$(Ag)^{k_j}_j = \sum_{i=1}^{k_j} c_{ij} A^i_j$$

for some constants $c_{ij}$ with $c_{k_j,j} \ne 0$.

Clearly, every $g \in G$ satisfying (1) and (2) preserves the subspace $\nabla(x([1,r])) \cap \nabla(x(J)) \subset M$. For a given $A$ the conditions that $Ag$ satisfies (3.4), are two linear equations on the coefficients $c_{ij}$. Since the number of unknowns is $\sum_{j \in J} k_j \ge 3$, it is easy to see that the set of matrices $A \in \nabla(x([1,r])) \cap \nabla(x(J))$ for which these two equations have a solution with all $c_{k_j,j} \ne 0$, is indeed Zariski open and dense in $\nabla(x([1,r])) \cap \nabla(x(J))$. This completes the proof of Theorem 0.2.

*Remark 3.2.* — The above arguments actually show that, under the conditions in Theorem 0.2 (b), the incidence variety $Z^{(2)}(J)$ (see (0.9)) lies in the closure of $Z^{(2)}(\emptyset)$. A simple dimension count shows that if $\#(J) = r - 2$ and $\sum_{j \in J} k_j \le 2$ then $\dim(Z^{(2)}(J)) \ge \dim(Z^{(2)}(\emptyset))$, and so, $Z^{(2)}(J)$ cannot lie in the closure of $Z^{(2)}(\emptyset)$. Unfortunately, this does not yet prove that in the latter case $\nabla_{\text{node}}(J)$ is not contained in $\nabla_{\text{node}}(\emptyset)$; the proof that this is indeed the case will be given in the next section.

## 4. The generic node component.

In this section we study the generic node variety $\nabla_{\text{node}}(\emptyset)$ and, in particular, prove Theorem 0.3 from the Introduction. Specializing (3.2), we see that

$$\nabla_{\text{node}}(\emptyset) = \overline{(\nabla(x^0) \cap \nabla(x')) \cdot G}, \tag{4.1}$$

where $x^0 = x([1,r])$ has the coordinates $x_i^{(j)} = \delta_{i,0}$, and $x' = x(\emptyset)$ is the "opposite" point with the coordinates $x_i^{(j)} = \delta_{i,k_j}$. As in Section 2, we shall study the tangent space $T_A \nabla_{\text{node}}(\emptyset)$ for a generic $A \in \nabla(x^0) \cap \nabla(x')$.

PROPOSITION 4.1.

(a) *Suppose the matrix format is such that* $k_2 + \cdots + k_r \geq k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$, *and it is different from* $2 \times 2 \times 2$, $3 \times 2 \times 2$ *or* $3 \times 3 \times 2$ *(that is,* $(k_1, \ldots, k_r) \neq (1,1,1), (2,1,1), (2,2,1)$*). Then, for a generic* $A \in \nabla(x^0) \cap \nabla(x')$ *the tangent space* $T_A \nabla_{\text{node}}(\emptyset)$ *is the subspace of codimension* 2 *in* $T_A M$ *given by the equations*

$$da_{0,0,\ldots,0} = da_{k_1,k_2,\ldots,k_r} = 0. \tag{4.2}$$

(b) *In each of the exceptional cases of* $2 \times 2 \times 2$, $3 \times 2 \times 2$ *or* $3 \times 3 \times 2$ *matrices we have* $\nabla_{\text{node}}(\emptyset) \subset \nabla_{\text{cusp}}$.

*Proof.* — Let $w = (w_1, \ldots, w_r) \in G$ be the element such that for $j = 1, \ldots, r$ the component $w_j \in GL_{k_j+1}$ of $w$ is the permutation matrix interchanging the first and last coordinate. Clearly, $w$ is an involution interchanging $x^0$ and $x'$. Let $E \subset Y$ be the affine subspace in $Y$ given by the equations $x_0^{(j)} = 1$ $(j = 1, \ldots, r)$ (see Section 2); then $wE \subset Y$ is the affine subspace in $Y$ given by the equations $x_{k_j}^{(j)} = 1$ $(j = 1, \ldots, r)$. The natural coordinates in $E$ are the $x_i^{(j)}$ for $j = 1, \ldots, r$, $i \in [1, k_j]$; for a point $y \in wE$ we take as its coordinates the $(wy)_i^{(j)}$, where $j$ and $i$ are as above.

Consider the variety

$$\tilde{Z}^{(2)} = \{(A, x, y) \in M \times E \times wE : (A, \mathrm{pr}\,(x), \mathrm{pr}\,(y)) \in Z^{(2)}(\emptyset)\} \tag{4.3}$$

(see (0.5), (0.9)). In other words, a point $(A, x, y) \in M \times E \times wE$ belongs to $\tilde{Z}^{(2)}$ if and only if $A \in \nabla(x) \cap \nabla(y)$ and the components $x^{(j)}$, $y^{(j)}$ of $x$ and $y$ are not proportional to each other for $j = 1, \ldots, r$. Since both $E$ and $wE$ are affine charts for $X = P^{k_1} \times \cdots \times P^{k_r}$, it follows that

$$\nabla_{\text{node}}(\emptyset) = \overline{\mathrm{pr}_1 \tilde{Z}^{(2)}}. \tag{4.4}$$

Therefore, for a generic $A \in \nabla(x^0) \cap \nabla(x')$ the tangent space $T_A \nabla_{\text{node}}(\emptyset)$ is the image of $T_{(A,x^0,x')} \tilde{Z}^{(2)}$ under the map $\mathrm{pr}_{1,*}$ induced by the first projection.

Let $B = B(A) = \|a_{jj'}^{ii'}\|$ be the quadratic part of $A$ at $x^0$ (see Section 2). We shall also consider the matrix $C = B(A \cdot w)$; it is natural to call $C$ the quadratic part of $A$ at $x'$. We write the entries of $C$ as ${}^w a_{jj'}^{ii'}$ ($j, j' = 1, \ldots, r$; $i \in [1, k_j]$, $i' \in [1, k_{j'}]$); for instance, we have ${}^w a_{12}^{ii'} = a_{i,i',k_3,\ldots,k_r}$ if $i < k_1, i' < k_2$, and ${}^w a_{12}^{k_1 k_2} = a_{0,0,k_3,\ldots,k_r}$.

Using this notation, we see that $T_{(A,x^0,x')}\tilde{Z}^{(2)}$ is the subspace in $T_{(A,x^0,x')}(M \times E \times wE)$ given by the equations (2.14) and similar equations obtained from (2.14) by replacing $(A, x)$ with $(A \cdot w, y)$. More precisely, $T_{(A,x^0,x')}\tilde{Z}^{(2)}$ is given by two equations (4.2) and the equations

$$da_j^i + \sum_{i',j'} a_{jj'}^{ii'} dx_{i'}^{(j')} = 0, \quad d^w a_j^i + \sum_{i',j'} {}^w a_{jj'}^{ii'} d(wy)_{i'}^{(j')} = 0 \tag{4.5}$$

for all $j = 1, \ldots, r$ and $i \in [1, k_j]$.

As in Section 2, we see that $\nabla_{\mathrm{node}}(\emptyset)$ has codimension 2 in $M$ if and only if its tangent space at a generic $A \in \nabla(x^0) \cap \nabla(x')$ is given by (4.2), and this happens if and only if the equations (4.5) do not produce additional relations between the $da_j^i$ and $d^w a_j^i$. This happens exactly when both matrices $B(A)$ and $B(A \cdot w)$ are invertible. Furthermore, if $B(A)$ is not invertible, then by definition, $A \in \nabla_{\mathrm{cusp}}$. Thus, Proposition 4.1 becomes a consequence of the following lemma.

LEMMA 4.2.

(a) *Under the conditions of Proposition 4.1 (a), for a generic $A \in \nabla(x^0) \cap \nabla(x')$ both matrices $B(A)$ and $B(A \cdot w)$ are invertible.*

(b) *In each of the exceptional cases in Proposition 4.1 (b), $\det(B(A)) = 0$ for every $A \in \nabla(x^0) \cap \nabla(x')$.*

*Proof.* — The condition that $A \in \nabla(x^0) \cap \nabla(x')$ means that $a_{i_1,\ldots,i_r} = 0$ whenever the multi-index $(i_1, \ldots, i_r)$ differs in at most one position from either $(0, \ldots, 0)$ or $(k_1, \ldots, k_r)$. On the other hand, the entries of $B(A)$ are those $a_{i_1,\ldots,i_r}$ for which $(i_1, \ldots, i_r)$ differs from $(0, \ldots, 0)$ in exactly two positions. Similarly, the entries of $B(A \cdot w)$ are those $a_{i_1,\ldots,i_r}$ for which $(i_1, \ldots, i_r)$ differs from $(k_1, \ldots, k_r)$ in exactly two positions. We see that if $r \geq 5$, then the condition that $A \in \nabla(x^0) \cap \nabla(x')$ imposes no restrictions on the matrix entries of $B(A)$ and $B(A \cdot w)$. Thus, for $r \geq 5$ the statement (a) is obvious.

If $r = 4$ then the condition that $A \in \nabla(x^0) \cap \nabla(x')$ forces $B(A)$ and $B(A \cdot w)$ to have some common entries. Namely, we have the relation

$$a_{12}^{k_1 k_2} = a_{k_1,k_2,0,0} = {}^w a_{34}^{k_3 k_4}$$

and all the relations obtained from it by a permutation of indices $1,2,3,4$. Thus, in this case our statement is a consequence of the following lemma (where we use the terminology and notation of Section 1).

LEMMA 4.3. — *Suppose $k_2+k_3+k_4 \geq k_1 \geq k_2 \geq k_3 \geq k_4 \geq 1$. Then there exist two invertible symmetric matrices $B$ and $C$ of type $(k_1,k_2,k_3,k_4)$ with the following property: whenever $\{\alpha,\beta,\gamma,\delta\} = \{1,2,3,4\}$, the $(k_\alpha,k_\beta)$ entry of the component $B_{\alpha\beta}$ of $B$ is equal to the $(k_\gamma,k_\delta)$ entry of the component $C_{\gamma\delta}$ of $C$.*

We prefer to prove Lemma 4.3 after treating the most complicated case $r=3$ in Lemma 4.2. In this case the condition that $A \in \nabla(x^0) \cap \nabla(x')$ imposes the restrictions of two kinds on the matrix entires of $B(A)$ and $B(A\cdot w)$. First, we have the relation

$$a_{12}^{k_1 i} = a_{k_1,i,0} = {}^w a_{23}^{ik_3}$$

for $0 < i < k_2$, and all the relations obtained from it by a permutation of indices $1,2,3$. Second, we have

$$a_{12}^{k_1 k_2} = a_{k_1,k_2,0} = 0, \quad {}^w a_{12}^{k_1 k_2} = a_{0,0,k_3} = 0$$

and again, all the relations obtained from them by a permutation of indices $1,2,3$. As in Section 1, we shall write the matrices $B = B(A)$ and $C = B(A\cdot w)$ in the block form

$$B = \begin{pmatrix} 0 & B_{1,2} & B_{1,3} \\ {}^tB_{1,2} & 0 & B_{2,3} \\ {}^tB_{1,3} & {}^tB_{2,3} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & C_{1,2} & C_{1,3} \\ {}^tC_{1,2} & 0 & C_{2,3} \\ {}^tC_{1,3} & {}^tC_{2,3} & 0 \end{pmatrix}.$$

Then the above conditions can be formulated as follows.

(1) The last row of $B_{1,2}$ (respectively, of $B_{1,3}, B_{2,3}, {}^tB_{1,2}, {}^tB_{1,3}, {}^tB_{2,3}$) coincides with the last row of ${}^tC_{2,3}$ (respectively, of $C_{2,3}, C_{1,3}, {}^tC_{1,3}, {}^tC_{1,2}, C_{1,2}$).

(2) Each of the matrices $B_{i,j}, C_{i,j}$ has the entry in the last row and the last column equal to 0.

Let us call a triple $(k_1,k_2,k_3)$ *roomy* if there exist two invertible symmetric matrices $B,C$ of type $(k_1,k_2,k_3)$ satisfying (1) and (2). Lemma 4.2 in the case $r=3$ can now be reformulated as follows.

LEMMA 4.4. — *A triple $(k_1,k_2,k_3)$ with $k_2+k_3 \geq k_1 \geq k_2 \geq k_3 \geq 1$ is roomy if and only if it is different from $(1,1,1),(2,1,1)$ and $(2,2,1)$.*

*Proof of Lemma 4.4.* — First, let us show that in each of the three exceptional cases, a matrix $B$ satisfying (2) has $\det(B) = 0$. Indeed, if

$(k_1, k_2, k_3) = (1,1,1)$ then $B = 0$; if $(k_1, k_2, k_3) = (2,1,1)$ then the second row of $B$ is 0; finally, if $(k_1, k_2, k_3) = (2,2,1)$ then $B$ has the $3 \times 3$ block of zeros in the rows and columns $2,4,5$.

Clearly, the exceptional cases are exactly those for which $k_1 + k_2 + k_3 < 6$. Our next step is to show that $(k_1, k_2, k_3)$ is roomy if $k_1 + k_2 + k_3$ is equal to 6 or 7. There are four such triples $(k_1, k_2, k_3)$, namely, $(3,2,1), (2,2,2), (3,3,1)$ and $(3,2,2)$. In each of these cases we just exhibit invertible $B$ and $C$ satisfying (1) and (2).

When $(k_1, k_2, k_3) = (3,2,1)$, we set

$$B = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&1\\ 0&1&0&0&0&0\\ 1&0&0&1&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&1&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&1&0&0 \end{pmatrix}.$$

When $(k_1, k_2, k_3) = (2,2,2)$, we set

$$B = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0 \end{pmatrix}.$$

When $(k_1, k_2, k_3) = (3,3,1)$, we set

$$B = \begin{pmatrix} 0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&1&0&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0 \end{pmatrix}.$$

Finally, when $(k_1, k_2, k_3) = (3,2,2)$, we set

$$B = \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&1&1&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0\\ 0&1&0&0&1&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&1&0&0&0 \end{pmatrix}.$$

It is immediately seen by inspection that all the above pairs $(B, C)$ satisfy (1) and (2). The check that all these matrices are invertible, is also easy. For instance, the matrix $B$ in the case $(k_1, k_2, k_3) = (3, 2, 2)$ has only one non-zero entry in each of the rows and columns $1, 3, 4, 7$; the remaining submatrix lying in the rows and columns $2, 5, 6$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

hence is invertible.

To complete the proof of Lemma 4.4, it remains to show that all the triples $(k_1, k_2, k_3)$ with $k_1 + k_2 + k_3 > 7$ are also roomy. Let us recall from Section 1 the notation $\Phi_3$ for the semigroup of triples $\mathbf{k} = (k_1, k_2, k_3) \in \mathbf{Z}_+^3$ satisfying the "triangle" inequality. Let

$$\Psi_3 = \{\mathbf{k} \in \Phi_3 : k_1 \geq k_2 \geq k_3 \geq 1\}.$$

We claim that the following statement holds:

(A) If $\mathbf{k} \in \Psi_3$ can be represented as a sum $\mathbf{k}' + \mathbf{k}''$ in such a way that $\mathbf{k}' \in \Phi_3$, $\mathbf{k}'' \in \Psi_3$ and $\mathbf{k}''$ is roomy, then $\mathbf{k}$ is roomy.

To prove (A), we choose $B'$ and $C'$ to be invertible symmetric matrices of type $\mathbf{k}'$, and $B''$ and $C''$ to be invertible symmetric matrices of type $\mathbf{k}''$ satisfying (1) and (2). Then we form the direct sums $B = B' \oplus B''$ and $C = C' \oplus C''$ (as defined in Section 1). Clearly, both $B$ and $C$ are invertible symmetric matrices of type $\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$. To complete the proof of (A), it remains to observe that the conditions (1) and (2) for $B$ and $C$ depend only on the last rows of all the components of $B$ and $C$, and so, follow from the corresponding conditions for $B''$ and $C''$.

We shall also use the following almost obvious statement:

(B) If $\mathbf{k}$ belongs to $\Psi_3$ and is different from $(1, 1, 1)$ and $(2, 1, 1)$ then at least one of the triples $\mathbf{k} - (1, 1, 0)$, $\mathbf{k} - (1, 0, 1)$ and $\mathbf{k} - (0, 1, 1)$ also belongs to $\Psi_3$.

Now the statement that all the triples $(k_1, k_2, k_3) \in \Psi_3$ with $k_1 + k_2 + k_3 \geq 6$ are roomy, follows by induction on $k_1 + k_2 + k_3$, where the base of induction is provided by the cases $k_1 + k_2 + k_3 = 6, 7$ treated above, and the induction step follows from (A) and (B). This completes the proof of Lemma 4.4.

*Proof of Lemma 4.3.* — If the triple $(k_1, k_2, k_3 + k_4)$ is roomy, then we can take as $B$ and $C$ invertible matrices of type $(k_1, k_2, k_3 + k_4)$ satisfying (1) and (2); the condition (2) ensures that all the matrix entries that we have to worry about, are equal to 0. This leaves us only two cases to consider: $(k_1, k_2, k_3, k_4)$ equals $(1,1,1,1)$ or $(2,1,1,1)$. In the first case we can take

$$B = C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and in the second case we take

$$B = C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

It is easy to see that both matrices are invertible; since all the entries appearing in Lemma 4.3, are now equal to 1, we are done. This completes the proof of Lemma 4.3, Lemma 4.2 and Prcposition 4.1.

Our next goal is to prove the first statement in Theorem 0.3 (a). So we assume that the matrix format is interior and different from $2 \times 2 \times 2$ and $3 \times 3 \times 2$. We know already that in this case both $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{cusp}}$ are irreducible hypersurfaces in $\nabla$ (Propositions 4.1, 2.2). So it is enough to show that $\nabla_{\text{node}}(\emptyset) \neq \nabla_{\text{cusp}}$.

We will show that the conormal bundles of $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{cusp}}$ in the matrix space $M$ cannot coincide because they have different generic fibers. To do this, we put the question in a more invariant setting. Recall that $M$ is identified with $V_1^* \otimes \cdots \otimes V_r^*$, where each $V_j$ is a vector space of dimension $k_j + 1$. The coordinates $x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j}^{(j)}$ form a basis in $V_j^*$ for $j = 1, \ldots, r$. The cotangent space of $M$ at every point $A$ is canonically identified with $M$; under this identification, a covector $da_{i_1,\ldots,i_r}$ corresponds to a decomposable tensor $x_{i_1}^{(1)} \otimes \cdots \otimes x_{i_r}^{(r)}$.

In this notation, Proposition 4.1 says that for a generic $A \in \nabla(x^0) \cap \nabla(x')$, the conormal space $T^*_{A,\nabla_{\text{node}}(\emptyset)}M$ is the 2-dimensional subspace in $M$ spanned by $x_0^{(1)} \otimes \cdots \otimes x_0^{(r)}$ and $x_{k_1}^{(1)} \otimes \cdots \otimes x_{k_r}^{(r)}$. Using (4.1), we conclude that the generic fiber of the conormal bundle $T^*_{\nabla_{\text{node}}(\emptyset)}M$ is a 2-dimensional subspace in $M$ spanned by two decomposable tensors $u^{(1)} \otimes \cdots \otimes u^{(r)}$ and $v^{(1)} \otimes \cdots \otimes v^{(r)}$, where $u^{(j)}, v^{(j)} \in V_j^*$ are linearly independent for $j = 1, \ldots, r$.

Reformulating in the same way Proposition 2.2, we see that the generic fiber of the conormal bundle $T^*_{\nabla_{\text{cusp}}}M$ is a 2-dimensional subspace of a vector space of the form

$$W = \sum_{j=1}^{r} w^{(1)} \otimes \cdots \otimes w^{(j-1)} \otimes V_j^* \otimes w^{(j+1)} \otimes \cdots \otimes w^{(r)}, \tag{4.6}$$

where $0 \neq w^{(j)} \in V_j^*$ for $j = 1, \ldots, r$. An easy check shows that if $W$ contains a decomposable tensor $u^{(1)} \otimes \cdots \otimes u^{(r)}$, then $u^{(j)}$ is proportional to $w^{(j)}$ for all values of $j$ except, maybe, one. Therefore, for any two decomposable tensors $u^{(1)} \otimes \cdots \otimes u^{(r)}$ and $v^{(1)} \otimes \cdots \otimes v^{(r)}$ contained in $W$, there is an index $j$ such that $u^{(j)}$ is proportional to $v^{(j)}$. Hence, $W$ cannot contain a generic fiber of $T^*_{\nabla_{\text{node}}(\emptyset)}M$. This shows that $\nabla_{\text{node}}(\emptyset) \neq \nabla_{\text{cusp}}$, and so completes the proof of Theorem 0.3 (a).

Let us turn to the proof of Theorem 0.3 (b). The case of the format $3 \times 2 \times 2$ is covered by Proposition 4.1 (b). So it remains to prove that $\nabla_{\text{cusp}} \subset \nabla_{\text{node}}(\emptyset)$ if the matrix format is boundary and different from $3 \times 2 \times 2$. We shall deduce this from the following statement, that holds for arbitary boundary format (including $3 \times 2 \times 2$).

LEMMA 4.5. — *In the case of boundary format we have* $\nabla_{\text{cusp}} \subset \nabla_{\text{node}}(\emptyset) \cup \nabla_{\text{node}}(\{1\})$.

If our boundary format is different from $3 \times 2 \times 2$, then $\nabla_{\text{node}}(\{1\}) \subset \nabla_{\text{node}}(\emptyset)$ by Theorem 0.2 (b); so in this case Lemma 4.5 implies that $\nabla_{\text{cusp}} \subset \nabla_{\text{node}}(\emptyset)$, as required.

*Proof of Lemma 4.5.* — First we show that in the case of boundary format, the variety $\nabla_{\text{node}}(\emptyset) \cup \nabla_{\text{node}}(\{1\})$ admits another interpretation, more convenient for our purposes. We shall say that $x \in Y$ is a *semi-critical point* of a matrix $A \in M$ if $F(A, x) = \partial_1^i F(A, x) = 0$ for all $i = 0, 1, \ldots, k_1$ (cf. the definition of a critical point in the Introduction). The following property is crucial for us.

LEMMA 4.6. — *Suppose $A$ is a matrix of boundary format. A point $x \in Y$ is a semi-critical point of $A$ if and only if there exists a critical point $y$ of $A$ such that all the components $y(j)$ of $y$ except, maybe, $y^{(1)}$, coincide with the corresponding components of $x$.*

*Proof.* — The "if" part is obvious. The "only if" part (that is, the statement that every semi-critical point can be made critical by changing its first component) is the "Cayley trick" proved in [4], Section 3 (see also [6], [13]).

Lemma 4.6 implies at once the following description of $\nabla_{\text{node}}(\emptyset) \cup \nabla_{\text{node}}(\{1\})$.

LEMMA 4.7. — *In the case of boundary format, $\nabla_{\text{node}}(\emptyset) \cup \nabla_{\text{node}}(\{1\})$ is the closure of the set of matrices having two semi-critical points $x, y \in Y$ such that for $j = 2, \ldots, r$ the components $x^{(j)}$ and $y^{(j)}$ are not proportional to each other.*

Now everything is ready for the proof of Lemma 4.5. Our method will be similar to that used in the proof of Theorem 0.2 (b) in Section 3. For every $t \neq 0$ we define the point $y(t) \in Y$ as follows: its component $y(t)^{(j)}$ is equal to $(1, 0, \ldots, 0, t)$ for $j = 2, \ldots, r$ and is equal to $(1, 0, \ldots, 0)$ for $j = 1$. Let $W(t) \subset M$ denote the space of matrices having $x^0$ as a critical point and $y(t)$ as a semi-critical point. By Lemma 4.7, $W(t) \subset \nabla_{\text{node}}(\emptyset) \cup \nabla_{\text{node}}(\{1\})$ for all $t \neq 0$. By definition, $W(t)$ is a vector subspace in $M$ defined by the equations

$$a = a_j^i = 0 \quad (j = 1, \ldots, r;\ i = 1, \ldots, k_j); \tag{4.7}$$

$$a + t \sum_{j \neq 1} a_j^{k_j} + t^2 \sum_{j, j' \neq 1} a_{jj'}^{k_j k_{j'}} + \cdots = 0; \tag{4.8}$$

$$t \sum_{j \neq 1} a_{1j}^{ik_j} + t^2 \sum_{j, j' \neq 1} a_{1jj'}^{ik_j k_{j'}} + \cdots = 0 \quad (i = 1, \ldots, k_1). \tag{4.9}$$

Clearly, all these equations are linearly independent, so the codimension of $W(t)$ in $M$ is

$$(k_1 + k_2 + \cdots + k_r + 1) + (k_1 + 1) = 3k_1 + 2.$$

Now let us pass to the limit $t \to 0$. The same argument as in Lemma 3.1 shows that $\lim_{t \to 0} W(t)$ exists in the Grassmannian of subspaces of codimension $3k_1 + 2$ in $M$, and that the limit subspace $W(0)$ is given by the equations (4.7) and

$$\sum_{j, j' \neq 1} a_{jj'}^{k_j k_{j'}} = 0; \tag{4.10}$$

$$\sum_{j \neq 1} a_{1j}^{ik_j} = 0 \quad (i = 1, \ldots, k_1). \tag{4.11}$$

(As $t \to 0$, (4.8) becomes (4.10), since, in view of (4.7), the first two terms in (4.8) are 0.)

It remains to show that for almost all matrices $A \in \nabla_{\text{cusp}}$ (that is, lying in some dense Zariski open set $U \subset \nabla_{\text{cusp}}$) there exists $g \in G$ such that $Ag \in W(0)$. In view of (0.8), we can assume that $A \in \nabla^0_{\text{cusp}}$. This means that $A$ satisfies (4.7), and the matrix $C = \|a^{ii'}_{1j'}\|$ (having $i = 1, \ldots, k_1$ as the row index, and $(i', j')$ with $j' = 2, \ldots, r$; $i' = 1, \ldots, k_{j'}$ as the column index) is degenerate. Since we are only interested in a dense set of matrices, we can assume that $C$ has corank 1, and that a non-zero vector $(x^{(2)}, \ldots, x^{(r)})$ from the kernel of $C$ has all components $x^{(j)}$ non-zero. By the action of the subgroup $GL_{k_2+1} \times \cdots \times GL_{k_r+1} \subset G$, we can transform each $x^{(j)}$ to the vector $(0, \ldots, 0, 1)$, hence transform $A$ into a matrix satisfying (4.11). Furthermore, acting on $A$ by the subgroup $GL_{k_1+1}$, we can transform the kernel of ${}^tC$ to $\mathbf{C}(0, \ldots, 0, 1)$, and so assume that $A$ also satisfies

$$a^{k_1 i}_{1j} = 0 \quad \text{for all} \quad i, j. \tag{4.12}$$

To complete the proof, it remains to observe that every matrix $A$ satisfying (4.7), (4.11) and (4.12) can also be made to satisfy (4.10) by a linear transformation

$$A^0_1 \mapsto aA^0_1 + bA^{k_1}_1, \quad A^{k_1}_1 \mapsto cA^0_1 + dA^{k_1}_1$$

of two slices $A^0_1$ and $A^{k_1}_1$ in the first direction (for the definition of slices see the proof of Theorem 0.2 (b) in Section 3). This proves Lemma 4.5, and hence completes the proof of Theorem 0.3.

## 5. Exceptional cases: the zoo of three- and four-dimensional matrices.

In this section we investigate the variety $\nabla_{\text{node}}(J)$ in the exceptional cases (1)–(5) listed in the introduction; in particular, we shall prove Theorem 0.4.

In fact, rearranging if necessary the indices $1, 2, \ldots, r$, we reduce the five exceptional cases to the following two:

(A) The format $(k+1) \times (k+1) \times 3$, $k \geq 1$; $J = \{3\}$.

(B) The format $(k+1) \times (k+1) \times 2 \times 2$, $k \geq 1$, $J = \{3, 4\}$.

Our first goal is to give a unified description of $\nabla_{\text{node}}(J)$ in both cases. In case (A) we shall represent a matrix $A$ of format $(k+1) \times (k+1) \times 3$ as an $(k+1) \times (k+1)$ matrix $A(y)$ whose entries are linear forms on $\mathbf{C}^3$. In an explicit matrix form, if $A = \|a_{\alpha\beta\gamma}\|_{0\le\alpha,\beta\le k;0\le\gamma\le 2}$, and $y = (y_0, y_1, y_2) \in \mathbf{C}^3$, then the matrix entries of $A(y)$ are

$$a_{\alpha\beta}(y) = \sum_{\gamma=0}^{2} a_{\alpha\beta\gamma} y_\gamma.$$

For future use, we set $Y_0 = \mathbf{C}^3 - \{0\}$.

In a similar way, in case (B) we represent a matrix $A$ of format $(k+1) \times (k+1) \times 2 \times 2$ as an $(k+1) \times (k+1)$ matrix $A(y)$ whose entries are linear forms on $\mathbf{C}^2 \otimes \mathbf{C}^2$. In an explicit matrix form, if $A = \|a_{\alpha\beta\gamma\delta}\|_{0\le\alpha,\beta\le k;0\le\gamma,\delta\le 1}$, and $y = (y_{\gamma\delta})_{\gamma,\delta=0,1} \in \mathbf{C}^2 \otimes \mathbf{C}^2$, then the matrix entries of $A(y)$ are

$$a_{\alpha\beta}(y) = \sum_{\gamma,\delta=0,1} a_{\alpha\beta\gamma\delta} y_{\gamma\delta}.$$

In this case we denote by $Y_0 \subset \mathbf{C}^2 \otimes \mathbf{C}^2$ the set of non-zero decomposable tensors; thus,

$$Y_0 = \{y = (y_{\gamma\delta}) \ne 0 : y_{00}y_{11} - y_{01}y_{10} = 0\}.$$

PROPOSITION 5.1. — *In each of the cases (A) and (B), we have*

$$\nabla_{\text{node}}(J) = \{A \in M : \ \operatorname{corank} A(y) \ge 2 \ \text{ for some } \ y \in Y_0\}. \tag{5.1}$$

*Proof.* — Let us temporarily denote by $\nabla'$ the variety in the right hand side of (5.1). It is easy to see that $\nabla'$ is Zariski closed in $M$; this follows from the fact that $Y_0 \cup \{0\}$ is the cone over a projective variety $S$ (in case (A) we have $S = P^2$, and in case (B) $S = P^1 \times P^1$ in the Segre embedding). Clearly, $\nabla'$ is invariant under the action of $G$, where $G = GL_{k+1} \times GL_{k+1} \times GL_3$ in case (A), and $G = GL_{k+1} \times GL_{k+1} \times GL_2 \times GL_2$ in case (B).

In view of (3.2), to prove that $\nabla_{\text{node}}(J) \subset \nabla'$, it is enough to show that $\nabla(x^0) \cap \nabla(x(J)) \subset \nabla'$. But this follows at once from the definitions: if we set $y^0 \in Y_0$ to be $(1,0,0)$ in case (A) and $(1,0) \otimes (1,0)$ in case (B), then for $A \in \nabla(x^0) \cap \nabla(x(J))$ the matrix $A(y^0)$ has the first and the last row and column equal to 0.

To prove the reverse inclusion, we denote by $W \subset M$ the subspace of matrices $A$ such that $A(y^0)$ has the first and the last row and column equal to 0. Clearly, $\nabla' = W \cdot G$. Therefore, it is enough to show that there exists a dense Zariski open subset $U \subset W$ such that $U \subset (\nabla(x^0) \cap \nabla(x(J))) \cdot G$. By definition, the subspace $\nabla(x^0) \cap \nabla(x(J))$ in $W$ is given by the equations

$$a_{00\gamma} = a_{kk\gamma} = 0 \ \text{ for } \gamma = 1, 2$$

in case (A), and by the equations

$$a_{00\gamma\delta} = a_{kk\gamma\delta} = 0 \quad \text{for} \quad (\gamma, \delta) = (1,0),\ (0,1)$$

in case (B). Thus, in both cases it is enough to prove the following statement about $2 \times 2 \times 2$ matrices.

LEMMA 5.2. — *Let $G'$ denote the subgroup $GL_2 \times GL_2 \times \{e\}$ of the group $G = GL_2 \times GL_2 \times GL_2$ acting on the space $\mathbf{C}^{2\times2\times2}$ of $2 \times 2 \times 2$ matrices. Then there exists a dense subset $U \subset \mathbf{C}^{2\times2\times2}$ such that every $A \in U$ can be transformed by the action of $G'$ into a matrix having*

$$(5.2) \qquad a_{000} = a_{001} = a_{110} = a_{111} = 0.$$

*Proof of the lemma.* — Since the conditions (5.2) are preserved by the action of the subgroup $\{e\} \times \{e\} \times GL_2 \subset G$, it is enough to prove that almost every $A \in \mathbf{C}^{2\times2\times2}$ is $G$-conjugate to a matrix satisfying (5.2). But it is known (see [4], Section 5, or the end of Section 2 above) that $G$ has a dense orbit $U \subset \mathbf{C}^{2\times2\times2}$, whose representative can be chosen to be the matrix with the only non-zero entries $a_{100} = a_{011} = 1$. Since this representative satisfies (5.2), we are done. Lemma 5.2 and hence Proposition 5.1 are proved.

The fact that in each of the cases (A) and (B), $\nabla_{\text{node}}(J)$ is an irreducible hypersurface in $\nabla$, is a consequence of the following.

PROPOSITION 5.3. — *In each of the cases (A) and (B), the tangent space $T_A\nabla_{\text{node}}(J)$ at a generic $A \in \nabla(x^0) \cap \nabla(x(J))$ has codimension 2 in $T_AM$. In case (A), it is given by the equations*

$$(5.3) \qquad da_{000} = da_{kk0} = 0;$$

*in case (B) it is given by*

$$(5.4) \qquad da_{0000} = da_{kk00} = 0.$$

*Proof.* — We shall treat only case (A), the case (B) being completely similar. The argument is basically the same as that in the proof of Proposition 4.1, so we shall be brief. First, it is easy to see that $T_A\nabla_{\text{node}}(J)$ at a generic $A \in \nabla(x^0) \cap \nabla(x(J))$ has the following description. This is the image under the first projection of the tangent space $T_{(A,x^0,x(J))}Z'$, where $Z'$ is the variety of triples $(A', x, y)$ such that $x, y \in Y$ are vectors of the form

$$x = ((1, x_1^{(1)}, \ldots, x_k^{(1)}), (1, x_1^{(2)}, \ldots, x_k^{(2)}), (1, z_1, z_2)),$$

$$y = ((y_0^{(1)}, \ldots, y_{k-1}^{(1)}, 1), (y_0^{(2)}, \ldots, y_{k-1}^{(2)}, 1), (1, z_1, z_2)),$$

and $A' \in \nabla(x) \cap \nabla(y)$. The space $T_{(A,x^0,x(J))}Z'$ is given by (5.3) and the following set of equations:

$$-da_{00\gamma} = \sum_{\alpha=1}^{k} a_{\alpha 0\gamma} dx_\alpha^{(1)} + \sum_{\beta=1}^{k} a_{0\beta\gamma} dx_\beta^{(2)} \quad (\gamma = 1, 2), \tag{5.5}$$

$$-da_{0\beta 0} = \sum_{\alpha=1}^{k-1} a_{\alpha\beta 0} dx_\alpha^{(1)} + a_{0\beta 1} dz_1 + a_{0\beta 2} dz_2 \quad (\beta = 1, \ldots, k-1), \tag{5.6}$$

$$-da_{\alpha 00} = \sum_{\beta=1}^{k-1} a_{\alpha\beta 0} dx_\beta^{(2)} + a_{\alpha 01} dz_1 + a_{\alpha 02} dz_2 \quad (\alpha = 1, \ldots, k-1), \tag{5.7}$$

$$-da_{kk\gamma} = \sum_{\alpha=0}^{k-1} a_{\alpha k\gamma} dy_\alpha^{(1)} + \sum_{\beta=0}^{k-1} a_{k\beta\gamma} dy_\beta^{(2)} \quad (\gamma = 1, 2), \tag{5.8}$$

$$-da_{k\beta 0} = \sum_{\alpha=1}^{k-1} a_{\alpha\beta 0} dy_\alpha^{(1)} + a_{k\beta 1} dz_1 + a_{k\beta 2} dz_2 \quad (\beta = 1, \ldots, k-1), \tag{5.9}$$

$$-da_{\alpha k 0} = \sum_{\beta=1}^{k-1} a_{\alpha\beta 0} dy_\beta^{(2)} + a_{\alpha k 1} dz_1 + a_{\alpha k 2} dz_2 \quad (\alpha = 1, \ldots, k-1), \tag{5.10}$$

$$-da_{k00} = a_{k01} dz_1 + a_{k02} dz_2, \tag{5.11}$$

$$-da_{0k0} = a_{0k1} dz_1 + a_{0k2} dz_2. \tag{5.12}$$

It remains to show that for a generic $A \in \nabla(x^0) \cap \nabla(x(J))$, the equations (5.5)–(5.12) do not produce additional relations between the $da_{\alpha\beta\gamma}$. Clearly, the condition that $A \in \nabla(x^0) \cap \nabla(x(J))$, does not impose any restrictions on the coefficients of the linear forms that appear in the right hand sides of (5.5)–(5.12). So we need only to show that the $(4k+2) \times (4k+2)$ matrix of coefficients of these linear forms is generically non-degenerate. However, this matrix becomes block-triangular if we arrange the variables in the following order:

$$dx_k^{(1)}, dx_k^{(2)}, dx_\alpha^{(1)}\ (\alpha = 1, \ldots, k-1), dx_\beta^{(2)}\ (\beta = 1, \ldots, k-1),$$

$$dy_0^{(1)}, dy_0^{(2)}, dy_\alpha^{(1)}\ (\alpha = 1, \ldots, k-1), dy_\beta^{(2)}\ (\beta = 1, \ldots, k-1), dz_1, dz_2.$$

There are four diagonal blocks of size $(k-1)\times(k-1)$: two of them are equal to the matrix $C_1 = \|a_{\alpha\beta 0}\|_{\alpha,\beta=1,\dots,k-1}$, and two equal to ${}^tC_1$. There are also three $2\times 2$ blocks: the matrix

$$C_2 = \begin{pmatrix} a_{k01} & a_{k02} \\ a_{0k1} & a_{0k2} \end{pmatrix},$$

and two other blocks obtained from $C_2$ by the transposition or a permutation of columns. Therefore, if we choose $A$ so that both $C_1$ and $C_2$ be non-degenerate, then $T_A\nabla_{\mathrm{node}}(J)$ wiil be given by (5.3). Proposition 5.3 is proved.

To complete the proof of Theorem 0.4, it remains to study how the special node components $\nabla_{\mathrm{node}}(J)$ are related to each other (for the formats such that there are several such components) and to $\nabla_{\mathrm{cusp}}$ and $\nabla_{\mathrm{node}}(\emptyset)$. We shall do this case by case.

*Case 1.* — *The format* $3\times 2\times 2$. In this case the most transparent way to describe $\nabla$ and $\nabla_{\mathrm{sing}}$ is probably the following. We represent a matrix $A$ of format $3\times 2\times 2$ as a 3-linear form

$$A(x,y,z) = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta\gamma}x_\alpha y_\beta z_\gamma$$

on $\mathbf{C}^3\times\mathbf{C}^2\times\mathbf{C}^2$.

PROPOSITION 5.4. — *For the format* $3\times2\times2$ *we have the following:*

(a) *A matrix* $A\in M$ *belongs to* $\nabla$ *if and only if there exist some non-zero* $y$ *and* $z$ *such that* $A(x,y,z)=0$ *for all* $x\in\mathbf{C}^3$.

(b) *A matrix* $A\in M$ *belongs to* $\nabla_{\mathrm{sing}}$ *if and only if there exists some non-zero* $x$ *such that* $A(x,y,z)=0$ *for all* $(y,z)\in\mathbf{C}^2\times\mathbf{C}^2$.

(c) *The singular locus* $\nabla_{\mathrm{sing}}$ *is isomorphic to the determinantal variety of* $3\times 4$ *matrices of rank* $\le 2$*; in particular, this is an irreducible hypersurface in* $\nabla$.

*Proof.* — The description of $\nabla$ in (a) is a special case of the general description for the boundary format. In view of Proposition 5.1 (applied to the format $2\times2\times3$), part (b) is simply another way to say that $\nabla_{\mathrm{sing}} = \nabla_{\mathrm{node}}(\{1\})$. So we need to show that $\nabla_{\mathrm{node}}(\{1\})$ contains both $\nabla_{\mathrm{node}}(\emptyset)$ and $\nabla_{\mathrm{cusp}}$. However, this follows at once from the inclusions

$$\nabla_{\mathrm{node}}(\emptyset)\subset\nabla_{\mathrm{cusp}}\subset\nabla_{\mathrm{node}}(\emptyset)\cup\nabla_{\mathrm{node}}(\{1\})$$

that we already proved (see Proposition 4.1 (b) and Lemma 4.5). Finally, part (c) follows at once from (b).

Before passing to other matrix formats, we want to present "typical" representatives of $M - \nabla$, $\nabla - \nabla_{\text{sing}}$ and $\nabla_{\text{sing}} - \nabla_{\text{cusp}}$. Namely, we claim that

$$e_{000} + e_{101} + e_{110} + e_{211} \in M - \nabla; \tag{5.13}$$

$$e_{000} + e_{101} + e_{211} \in \nabla - \nabla_{\text{sing}}; \tag{5.14}$$

$$e_{000} + e_{211} \in \nabla_{\text{sing}} - \nabla_{\text{cusp}}, \tag{5.15}$$

where the $e_{\alpha\beta\gamma}$ are "matrix units". We leave the proof as an (easy) exercise for the reader; note that the matrix in (5.13) is the "identity" matrix constructed in [4] for every boundary format.

*Case 2. — The format* $3 \times 3 \times 3$. We have seen that in this case there are five potential irreducible components of $\nabla$, namely $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$, $\nabla_{\text{node}}(\{1\})$, $\nabla_{\text{node}}(\{2\})$, $\nabla_{\text{node}}(\{3\})$. All of them have codimension 1 in $\nabla$: for $\nabla_{\text{cusp}}$ this was shown in Section 2, for $\nabla_{\text{node}}(\emptyset)$ in Section 4, for $\nabla_{\text{node}}(\{3\})$ this is Proposition 5.3, and for $\nabla_{\text{node}}(\{1\})$, and $\nabla_{\text{node}}(\{2\})$ follows from Proposition 5.3 by obvious symmetry.

PROPOSITION 5.5. — *In the case of* $3 \times 3 \times 3$ *matrices, the five varieties* $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$, $\nabla_{\text{node}}(\{1\})$, $\nabla_{\text{node}}(\{2\})$, $\nabla_{\text{node}}(\{3\})$ *are all different.*

*Proof.* — The statement that $\nabla_{\text{cusp}} \neq \nabla_{\text{node}}(\emptyset)$, was proved in Section 4, by showing that the conormal bundles of these two varieties have different generic fibers. It is possible to use the same method for proving that the varieties $\nabla_{\text{node}}(\{1\})$, $\nabla_{\text{node}}(\{2\})$, $\nabla_{\text{node}}(\{3\})$ are different from each other and from $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ (we shall in fact use it for the analysis of the next case). We prefer, however, to give another proof, by giving explicit matrices representing these varieties.

As in the previous case, we represent a matrix $A$ of format $3 \times 3 \times 3$ as a 3-linear form

$$A(x, y, z) = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta\gamma} x_\alpha y_\beta z_\gamma$$

on $\mathbf{C}^3 \times \mathbf{C}^3 \times \mathbf{C}^3$. Consider two matrices $A_1$ and $A_2$ represented by the forms

$$\begin{aligned} A_1(x, y, z) &= \sum_{i+j+k=3} x_i y_j z_k \\ &= x_0y_1z_2+x_0y_2z_1+x_1y_0z_2+x_1y_1z_1+x_1y_2z_0+x_2y_0z_1+x_2y_1z_0, \\ A_2(x, y, z) &= x_1y_1z_0+x_0y_1z_1+x_1y_2z_1+x_2y_0z_1+x_0y_2z_2+x_1y_0z_2+x_2y_1z_2. \end{aligned}$$

LEMMA 5.6.

(a) *The matrix* $A_1$ *belongs to* $\nabla_{\text{node}}(\emptyset)$ *and does not belong to any of the other four varieties in Proposition 5.5.*

(b) *The matrix* $A_2$ *belongs to* $\nabla_{\text{node}}(\{3\})$ *and does not belong to* $\nabla_{\text{cusp}} \cup \nabla_{\text{node}}(\{1\}) \cup \nabla_{\text{node}}(\{2\})$.

*Proof.* — Let $X(A)$ stand for the set of critical points of a matrix $A$ on $X = P^2 \times P^2 \times P^2$. In terms of the 3-linear form $A(x,y,z)$, the set $X(A)$ is the image under the projection $Y \to X$ of the set

$$(5.16) \qquad Y(A) = \{(x,y,z) \in (\mathbf{C}^3 - \{0\})^3 : \\ A(x',y,z) = A(x,y',z) = A(x,y,z') \text{ for all } x',y',z'\}.$$

We claim that

$$(5.17) \qquad X(A_1) = \{((1{:}0{:}0),(1{:}0{:}0),(1{:}0{:}0)),((0{:}0{:}1),(0{:}0{:}1),(0{:}0{:}1))\}.$$

Indeed, for $A = A_1$ the system of equations in (5.16) takes the form

$$(5.18) \qquad \begin{aligned} y_1z_2 + y_2z_1 &= y_0z_2 + y_1z_1 + y_2z_0 = y_0z_1 + y_1z_0 = x_1z_2 + x_2z_1 \\ &= x_0z_2 + x_1z_1 + x_2z_0 = x_0z_1 + x_1z_0 = x_1y_2 + x_2y_1 \\ &= x_0y_2 + x_1y_1 + x_2y_0 = x_0y_1 + x_1y_0 = 0. \end{aligned}$$

Now suppose that $(x,y,z) \in Y(A_1)$ and let $\alpha,\beta,\gamma$ be the maximal indices such that $x_\alpha, y_\beta$ and $z_\gamma$ are non-zero. Then none of the terms $x_\alpha y_\beta, x_\alpha z_\gamma$ or $y_\beta z_\gamma$ can appear in the equations (5.18). This leaves only two opportunities $(\alpha,\beta,\gamma) = (0,0,0)$ or $(\alpha,\beta,\gamma) = (2,2,2)$. If $(\alpha,\beta,\gamma) = (0,0,0)$, then $(x,y,z)$ has the form $((a,0,0),(b,0,0),(c,0,0))$, which is obviously a solution of (5.18). So we can assume that $(\alpha,\beta,\gamma) = (2,2,2)$, that is, $x_2y_2z_2 \neq 0$. We can write three of the equations (5.18) in the matrix form as

$$(5.19) \qquad \begin{pmatrix} 0 & z_2 & y_2 \\ z_2 & 0 & x_2 \\ y_2 & x_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0.$$

Since the matrix in (5.19) has determinant $2x_2y_2z_2 \neq 0$, we conclude that $x_1 = y_1 = z_1 = 0$. Substituting this to the remaining equations in (5.18) leaves us with the three equations which can be written as

$$\begin{pmatrix} 0 & z_2 & y_2 \\ z_2 & 0 & x_2 \\ y_2 & x_2 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

As above, this implies $x_0 = y_0 = z_0 = 0$. This completes the proof of (5.17).

In view of (5.17), $A_1 \in \nabla_{\text{node}}(\emptyset)$. To show that $A_1$ does not lie in $\nabla_{\text{cusp}}$, we have to prove that it has non-zero Hessian at $((1,0,0), (1,0,0), (1,0,0))$ and at $((0,0,1),(0,0,1),(0,0,1))$. The quadratic part of $A_1$ at both points (see Section 2) has the same form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and it is easy to see that this matrix is non-degenerate.

To complete the proof of (a), it remains to show that $A_1$ does not belong to $\nabla_{\text{node}}(\{1\}), \nabla_{\text{node}}(\{2\})$ or $\nabla_{\text{node}}(\{3\})$. In view of Proposition 5.1, this means that no non-trivial linear combination of the slices of $A_1$ in each direction can have rank $\leq 1$. By the obvious symmetry, the slices of $A_1$ in each direction are the same, and their linear combination with coefficients $a, b, c$ has the form

$$\begin{pmatrix} 0 & c & b \\ c & b & a \\ b & a & 0 \end{pmatrix}.$$

This matrix has three diagonal $2 \times 2$ minors equal to $-c^2, -b^2, -a^2$; so it can be of rank $\leq 1$ only when $a = b = c = 0$. This completes the proof of Lemma 5.6 (a).

The proof of part (b) is quite similar. First we show that

$$X(A_2) = \{((1{:}0{:}0),(1{:}0{:}0),(1{:}0{:}0)),\ ((0{:}0{:}1),(0{:}0{:}1),(1{:}0{:}0))\};$$

this implies that $A_2 \in \nabla_{\text{node}}(\{3\})$. Then we compute the Hessians of $A_2$ at the points $((1,0,0),(1,0,0),(1,0,0))$ and $((0,0,1),(0,0,1),(1,0,0))$, and find both of them to be non-zero; thus, $A_2 \notin \nabla_{\text{cusp}}$. Finally, by analyzing linear combinations of the slices of $A_2$ in the first and second direction, we show that $A_2$ does not belong to $\nabla_{\text{node}}(\{1\})$ or $\nabla_{\text{node}}(\{2\})$. We leave the details to the reader.

The proof of Proposition 5.5 is now completed as follows. By Lemma 5.6 (a), $\nabla_{\text{node}}(\emptyset)$ is different from the other four varieties in Proposition 5.5. By Lemma 5.6 (b), $\nabla_{\text{node}}(\{3\})$ is different from each of $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\{1\})$ or $\nabla_{\text{node}}(\{2\})$. Using symmetry, we see that $\nabla_{\text{node}}(\{1\})$ and $\nabla_{\text{node}}(\{2\})$ are also different from each other and from $\nabla_{\text{cusp}}$. Proposition 5.5 is proved.

*Case 3.* — *The format* $(k+1) \times (k+1) \times 3$, $k \geq 3$. In this case there are three potential irreducible components of $\nabla$, namely $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{node}}(\{3\})$. All of them have codimension 1 in $\nabla$.

PROPOSITION 5.7. — *In the case of* $(k+1) \times (k+1) \times 3$ *matrices with* $k \geq 3$, *the three varieties* $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ *and* $\nabla_{\text{node}}(\{3\})$ *are all different.*

*Proof.* — The fact that $\nabla_{\text{cusp}} \neq \nabla_{\text{node}}(\emptyset)$, was proved in Section 4. Using the notation introduced there, we deduce from (5.3) the following description of the generic fiber of the conormal bundle $T^*_{\nabla_{\text{node}}(\{3\})}M$. This is a 2-dimensional vector subspace $\Lambda$ in $V_1^* \otimes V_2^* \otimes V_3^*$ spanned by two decomposable tensors $u^{(1)} \otimes u^{(2)} \otimes w$ and $v^{(1)} \otimes v^{(2)} \otimes w$, where $u^{(j)}, v^{(j)} \in V_j^*$ are linearly independent for $j = 1, 2$. It is easy to see that, up to proportionality, the above two tensors are the only decomposable tensors in $\Lambda$. Comparing this with the description of the generic fibers of $T^*_{\nabla_{\text{cusp}}}M$ and $T^*_{\nabla_{\text{node}}(\emptyset)}M$ given in Section 4, we conclude that $\Lambda$ is different from each of these fibers. Proposition 5.7 is proved.

*Case 4.* — *The format* $2 \times 2 \times 2 \times 2$. In this case there are eight potential irreducible components of $\nabla$, namely $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ and six special node varieties $\nabla_{\text{node}}(\{i,j\})$, $1 \leq i < j \leq 4$. As in the previous cases, we know that all of them have codimension 1 in $\nabla$.

PROPOSITION 5.8. — *In the case of* $2 \times 2 \times 2 \times 2$ *matrices, the eight varieties* $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ *and* $\nabla_{\text{node}}(\{i,j\})$, $1 \leq i < j \leq 4$ *are all different.*

*Proof.* — Our argument will be very similar to that in Case 2 above. We represent a matrix $A$ of format $2 \times 2 \times 2 \times 2$ as a 4-linear form

$$A(x,y,z,w) = \sum_{\alpha,\beta,\gamma,\delta=0}^{1} a_{\alpha\beta\gamma\delta} x_\alpha y_\beta z_\gamma w_\delta$$

on $\mathbf{C}^2 \times \mathbf{C}^2 \times \mathbf{C}^2 \times \mathbf{C}^2$. Consider two matrices $A_1$ and $A_2$ represented by the forms

$$A_1(x,y,z,w) = \sum_{\alpha+\beta+\gamma+\delta=2} x_\alpha y_\beta z_\gamma w_\delta = x_0y_0z_1w_1 + \cdots + x_1y_1z_0w_0,$$

$$\begin{aligned} A_2(x,y,z,w) = x_0y_1z_0w_1 + x_1y_0z_0w_1 + x_0y_1z_1w_0 - x_1y_0z_1w_0 \\ + x_0y_0z_1w_1 + x_1y_1z_1w_1. \end{aligned}$$

LEMMA 5.9.

(a) *The matrix* $A_1$ *belongs to* $\nabla_{\text{node}}(\emptyset)$ *but not to any of the other seven varieties in Proposition 5.8.*

(b) *The matrix* $A_2$ *belongs to* $\nabla_{\text{node}}(\{3,4\})$ *but not to* $\nabla_{\text{cusp}}$ *or any* $\nabla_{\text{node}}(\{i,j\})$ *with* $\{i,j\} \neq \{3,4\}$.

*Proof.* — The proof of part (a) is parallel to the argument in the proof of Lemma 5.6 (a), so we shall only indicate the necessary modifications. We claim that

$$X(A_1) = \{((1{:}0),(1{:}0),(1{:}0),(1{:}0)),\ ((0{:}1),(0{:}1),(0{:}1),(0{:}1))\}. \tag{5.20}$$

This is proved in the same way as (5.17) above, with the following modification. The system (5.19) is replaced by

$$\begin{pmatrix} 0 & z_1w_1 & y_1w_1 & y_1z_1 \\ z_1w_1 & 0 & x_1w_1 & x_1z_1 \\ y_1w_1 & x_1w_1 & 0 & x_1y_1 \\ y_1z_1 & x_1z_1 & x_1y_1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix} = 0, \tag{5.21}$$

and the matrix in (5.21) has determinant $-3(x_1y_1z_1w_1)^2$.

The rest of the proof of (a) and the proof of (b) are straightforward, and we leave them to the reader. Note that in (b) one has to show that

$$X(A_2) = \{((1{:}0),(1{:}0),(1{:}0),(1{:}0)),\ ((0{:}1),(0{:}1),(1{:}0),(1{:}0))\}.$$

Proposition 5.8 is deduced from Lemma 5.9 in the same way as Proposition 5.5 from Lemma 5.6.

*Case 5. — The format* $(k+1)\times(k+1)\times 2\times 2$, $k \geq 2$. As in Case 3, there are three potential irreducible components of $\nabla$, namely $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ and $\nabla_{\text{node}}(\{3,4\})$. All of them have codimension 1 in $\nabla$. The following result is proved in exactly the same way as Proposition 5.7.

PROPOSITION 5.10. — *In the case of* $(k+1)\times(k+1)\times 2\times 2$ *matrices with* $k \geq 2$, *the three varieties* $\nabla_{\text{cusp}}$, $\nabla_{\text{node}}(\emptyset)$ *and* $\nabla_{\text{node}}(\{3,4\})$ *are all different.*

Combining all the above results, we see that Theorem 0.4 is proved.

## 6. Decomposition of the singular locus into cusp and node parts.

The purpose of this section is to prove the decomposition (0.4); more precisely, we shall show that

$$\nabla_{\text{sing}} = \nabla^0_{\text{node}} \cup \nabla_{\text{cusp}}, \tag{6.1}$$

where $\nabla^0_{\text{node}}$ is the variety of matrices having more than one critical point on $X$. We shall deduce the inclusion

$$\nabla_{\text{sing}} \subseteq \nabla^0_{\text{node}} \cup \nabla_{\text{cusp}} \tag{6.2}$$

from a general criterion due to N. Katz [7]. Let $\nabla_{\text{sm}} = \nabla - \nabla_{\text{sing}}$ be the set of smooth points of $\nabla$, and let $U$ be the complement of $\nabla^0_{\text{node}} \cup \nabla_{\text{cusp}}$ in $\nabla$. Then (6.2) is equivalent to $U \subseteq \nabla_{\text{sm}}$. By definition, a matrix $A \in \nabla$ belongs to $U$ if and only if $A$ has exactly one critical point $x \in X$, and the Hessian of $A$ at $x$ is non-zero. In other words, $U$ is the set of all points $A \in \nabla$, where the projection $\text{pr}_1 : Z \to \nabla$ (see (0.2)) is unramified. Clearly, $U$ is open in $\nabla$. If $U \neq \emptyset$ (and so, $U$ is dense in $\nabla$ since $\nabla$ is irreducible), then we say that $\text{pr}_1 : Z \to \nabla$ is *generically unramified*. It is proved in [7], Proposition 3.5 that $\text{pr}_1 : Z \to \nabla$ is generically unramified if and only if $\nabla$ has codimension 1 in $M$. In our situation, this happens exactly when the matrix format satisfies (0.3). We see that (6.2) is a consequence of the following criterion of N. Katz (which is valid for an arbitrary smooth projective variety $X$).

PROPOSITION 6.1 ([7], Proposition 3.5). — *Suppose the projection $\text{pr}_1 : Z \to \nabla$ is generically unramified. Then this projection is birational. Furthermore, $U$ consists of smooth points of $\nabla$ and is the biggest open set in $\nabla$ for which the projection $\text{pr}_1 : \text{pr}_1^{-1}(U) \to U$ is an isomorphism.*

The rest of the section is devoted to the proof of the reverse inclusion

$$\nabla_{\text{node}} \cup \nabla_{\text{cusp}} \subseteq \nabla_{\text{sing}}. \tag{6.3}$$

We will need two preparatory statements.

LEMMA 6.2. — *The variety of matrices $A$ having infinitely many critical points in $X$ is of codimension $\geq 2$ in $\nabla$.*

*Proof.* — This is a general statement valid for an arbitrary smooth projective variety $X$ such that its projectively dual is a hypersurface. The proof is done by a simple dimension count. By definition, the set of critical points in $X$ of $A \in \nabla$ is the fiber $\text{pr}_1^{-1}(A)$ of the projection $\text{pr}_1 : Z \to \nabla$. If this fiber is infinite then it has positive dimension; so the variety of matrices with infinitely many critical points can be written as

$$\nabla' = \{A \in \nabla : \dim(\text{pr}_1^{-1}(A)) \geq 1\}. \tag{6.4}$$

Obviously, $\nabla'$ is disjoint from the open set $U \subset \nabla$ considered above (see Proposition 6.1). Let $Z' = \text{pr}_1^{-1}(\nabla')$ and $\tilde{U} = \text{pr}_1^{-1}(U)$ be the inverse images of $\nabla'$ and $U$ in $Z$. Since $Z$ is irreducible, and $\tilde{U}$ is open in $Z$, it follows that $\dim(Z') < \dim(Z)$. By (6.8), we have $\dim(\nabla') < \dim(Z')$. Finally, in view of Proposition 6.1, we have $\dim(\nabla) = \dim(Z)$. Therefore,

$$\dim(\nabla') \leq \dim(Z') - 1 \leq \dim(Z) - 2 = \dim(\nabla) - 2,$$

as required.

The other statement we need is a special case of the general result of A. Dimca [2] on dual varieties (see also [10], formula 1 on p. 433).

PROPOSITION 6.3. — *Suppose a matrix $A \in \nabla$ has finitely many critical points in $X$. Let $H_A$ be the hyperplane in $P(V_1 \otimes \cdots \otimes V_r)$ defined by $A$. Then the multiplicity of $\nabla$ at $A$ is equal to*

$$\mathrm{mult}_A(\nabla) = \sum_{x \in X(A)} \mu(X \cap H_A, x), \tag{6.5}$$

*where the sum is over all critical points of $A$ in $X$, and $\mu(X \cap H_A, x)$ is the Milnor number of $X \cap H_A$ at $x$. In particular, $\mu(X \cap H_A, x) \geq 1$ for every $x \in X(A)$, and $\mu(X \cap H_A, x) = 1$ if and only if the Hessian of $A$ at $x$ is nondegenerate.*

Now we have all the necessary tools to prove (6.3). Let us look at Main Theorem 0.5, where, for a moment, we have to replace $\nabla_{\text{sing}}$ by $\nabla_{\text{node}} \cup \nabla_{\text{cusp}}$. We see that for all interior and boundary matrix formats except $2 \times 2 \times 2$, all the irreducible components of $\nabla_{\text{node}} \cup \nabla_{\text{cusp}}$ are of codimension 1 in $\nabla$. By Lemma 6.2, a generic point $A$ of every such component has finitely many critical points in $X$. By (6.5), the multiplicity of $\nabla$ at $A$ is $\geq 2$, so $A$ is a singular point of $\nabla$, and (6.3) follows. Combined with (6.2), this completes the proof of (6.1) for all the formats except $2 \times 2 \times 2$. In the exceptional case of $2 \times 2 \times 2$ matrices, the equality (6.1) takes the form

$$\nabla_{\text{node}} \subset \nabla_{\text{cusp}} = \nabla_{\text{sing}}, \tag{6.6}$$

where the first inclusion is a consequence of Theorems 0.2 and 0.3, and the last equality is (2.26).

## 7. Multi-dimensional "diagonal" matrices and the Vandermonde matrix.

In this section we exhibit multi-dimensional analogs of the diagonal matrices for all boundary and interior matrix formats, and an analog of the Vandermonde matrix for boundary format. We start with the diagonal matrices.

Recall from Section 1 that $\Phi_r$ denotes the semigroup of vectors $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbf{Z}_+^r$ satisfying the "polygon" inequality (1.9). We say that a multi-index $\mathbf{i} = (i_1, \ldots, i_r)$ is *diagonal* for $\mathbf{k}$ if both $\mathbf{i}$ and $\mathbf{k} - \mathbf{i}$ lie in $\Phi_r$. We call a matrix $A = \|a_{i_1,\ldots,i_r}\|_{0 \le i_j \le k_j}$ of format $(k_1+1) \times \cdots \times (k_r+1)$ *diagonal* if $a_{i_1,\ldots,i_r} = 0$ unless $(i_1, \ldots, i_r)$ is a diagonal multi-index for $(k_1, \ldots, k_r)$. For $r = 2$ this is the usual notion of diagonal matrices. Notice also that for boundary format when $k_1 = k_2 + \cdots + k_r$, a multi-index $(i_1, \ldots, i_r)$ with $0 \le i_j \le k_j$ is diagonal if and only if $i_1 = i_2 + \cdots + i_r$; in particular, the "identity" matrix having $a_{i_1,\ldots,i_r} = \delta_{i_1, i_2 + \cdots + i_r}$ is diagonal.

We shall see that a generic diagonal matrix $A$ is non-degenerate, i.e., has non-zero hyperdeterminant. This is a consequence of the following statement: there exists an extreme monomial

$$\prod_{\mathbf{i}=(i_1,\ldots,i_r)} a_{\mathbf{i}}^{d(\mathbf{i},\mathbf{k})} \tag{7.1}$$

appearing in $\mathrm{Det}\,(A)$ such that $d(\mathbf{i}, \mathbf{k}) > 0$ if and only if $\mathbf{i}$ is diagonal for $\mathbf{k}$. (Recall that a monomial is *extreme* if it corresponds to a vertex of the Newton polytope of $\mathrm{Det}\,(A)$.) To construct such a monomial, we recall that if $N(k_1, \ldots, k_r)$ stands for the degree of the hyperdeterminant of format $(k_1 + 1) \times \cdots \times (k_r + 1)$, then the generating function for the numbers $N(k_1, \ldots, k_r)$ is given by

$$\sum_{k_1,\ldots,k_r \ge 0} N(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} = \Big(1 - \sum_{i=2}^{r} (i-1) e_i(z_1, \ldots, z_r)\Big)^{-2}, \tag{7.2}$$

where the $e_i(z_1, \ldots, z_r)$ are elementary symmetric polynomials (see [4], (3.1) or [5], Chapter 14, (2.5)). As shown in [4], [5], any coefficient $N(k_1, \ldots, k_r)$ in (7.2) has a combinatorial expression as a sum of positive summands; furthermore, a combinatorial argument shows that $N(k_1, \ldots, k_r) > 0$ exactly when $(k_1, \ldots, k_r) \in \Phi_r$.

Now consider the square root of the generating function (7.2), i.e., the series

$$\sum_{k_1,\ldots,k_r \ge 0} M(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} = \Big(1 - \sum_{i=2}^{r} (i-1) e_i(z_1, \ldots, z_r)\Big)^{-1}. \tag{7.3}$$

For any two non-negative integer vectors $\mathbf{i} = (i_1, \ldots, i_r)$, $\mathbf{k} = (k_1, \ldots, k_r)$ we set

$$d(\mathbf{i}, \mathbf{k}) = M(i_1, \ldots, i_r)\, M(k_1 - i_1, \ldots, k_r - i_r). \tag{7.4}$$

THEOREM 7.1. — *For every interior or boundary matrix format, the monomial (7.1) with the exponents given by (7.4) is an extreme monomial in* $\mathrm{Det}\,(A)$ *appearing with the coefficient* $\pm 1$. *The exponent* $d(\mathbf{i},\mathbf{k})$ *is positive if and only if* $\mathbf{i}$ *is diagonal for* $\mathbf{k}$.

*Proof.* — The hyperdeterminant is a special case of the so-called $\mathcal{A}$-discriminant (see [5]), in the case when $\mathcal{A}$ is the set of vertices of the product of standard simplices $Q = \Delta^{k_1} \times \cdots \times \Delta^{k_r}$. Thus, the points of $\mathcal{A}$ correspond to multi-indices $\mathbf{i} = (i_1, \ldots, i_r)$ with $0 \le i_j \le k_j$ for $j = 1, \ldots, r$; with some abuse of notation, we denote a vertex by the same symbol $\mathbf{i}$ as the corresponding multi-index. We shall use the description of the extreme monomials in the $\mathcal{A}$-discriminant given in [5], Chapter 11, Theorem 3.2. According to this description, every extreme monomial corresponds to a coherent triangulation $T$ of $Q$ having all vertices in $\mathcal{A}$. If we write the monomial corresponding to $T$ in the form (7.1), then the exponents $d(\mathbf{i},\mathbf{k})$ are given by

$$d(\mathbf{i},\mathbf{k}) = \sum_{\sigma} (-1)^{\dim(Q)-\dim(\sigma)} \mathrm{Vol}\,(\sigma). \tag{7.5}$$

Here the sum is over all *massive* simplices $\sigma$ of $T$ having $\mathbf{i}$ as a vertex (recall from [5] that $\sigma$ is massive if the minimal face $\Gamma(\sigma)$ of $Q$ that contains $\sigma$, has the same dimension as $\sigma$); the volume form $\mathrm{Vol}\,(\sigma)$ in (7.5) is normalized so that an elementary simplex corresponding to the lattice spanned by $\mathcal{A} \cap \Gamma(\sigma)$ has volume 1.

It is well-known that $Q = \Delta^{k_1} \times \cdots \times \Delta^{k_r}$ has a coherent triangulation $T$ whose simplices correspond to the chains of $\mathcal{A}$ in the product order. In other words, a subset $\{\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_p\} \subset \mathcal{A}$ is the set of vertices of a simplex from $T$ if and only if $\mathbf{i}_0 < \mathbf{i}_1 < \cdots < \mathbf{i}_p$, with respect to the order relation

$$\mathbf{i} = (i_1, \ldots, i_r) \le \mathbf{i}' = (i'_1, \ldots, i'_r) \quad \Leftrightarrow \quad i_j \le i'_j \quad (j = 1, \ldots, r).$$

Let us apply (7.5) to this triangulation. First, it is easy to see that the simplex corresponding to a chain $(\mathbf{i}_0 < \mathbf{i}_1 < \cdots < \mathbf{i}_p)$ is massive if and only if for each $t = 1, \ldots, p$ the multi-index $\mathbf{i}_t$ differs from $\mathbf{i}_{t-1}$ in exactly one position. Let us call the chains with this property also massive. Second, it is easy to see that all the massive simplices in $T$ have volume 1 in the above normalization. So, we can rewrite (7.5) as follows:

$$d(\mathbf{i},\mathbf{k}) = \sum_{\mathbf{i}_0 < \mathbf{i}_1 < \cdots < \mathbf{i}_p} (-1)^{k_1 + \cdots + k_r - p}, \tag{7.6}$$

the sum over all massive chains in $[0, k_1] \times \cdots \times [0, k_r]$ containing $\mathbf{i}$.

It should not be too hard to deduce (7.4) from (7.6) in a direct way. However, the following trick leads to the goal quicker. Note that every massive chain contributing to (7.6) is the union of two massive chains, one having $\mathbf{i}$ as its maximal element and another having $\mathbf{i}$ as its minimal element; furthermore, these two chains can be chosen independently of each other. It follows that $d(\mathbf{i},\mathbf{k})$ can be factored as

$$d(\mathbf{i},\mathbf{k}) = d(\mathbf{i},\mathbf{i})\, d(\mathbf{k}-\mathbf{i},\mathbf{k}-\mathbf{i}). \tag{7.7}$$

On the other hand, the degree of the monomial (7.1) is equal to

$$\sum_{\mathbf{i}} d(\mathbf{i},\mathbf{k}) = N(k_1,\ldots,k_r). \tag{7.8}$$

Substituting (7.7) into (7.8) and translating this statement into the language of generating functions, we obtain

$$\Big(\sum_{k_1,\ldots,k_r\geq 0} d(\mathbf{k},\mathbf{k}) z_1^{k_1}\cdots z_r^{k_r}\Big)^2 = \sum_{k_1,\ldots,k_r\geq 0} N(k_1,\ldots,k_r) z_1^{k_1}\cdots z_r^{k_r}. \tag{7.9}$$

Comparing (7.9) with (7.2) and (7.3), we conclude that $d(\mathbf{k},\mathbf{k}) = M(k_1,\ldots,k_r)$ for all $\mathbf{k} = (k_1,\ldots,k_r)$. Substituting this into (7.7) yields (7.4). This proves that the monomial (7.1) with the exponents given by (7.4) is indeed an extreme monomial in $\mathrm{Det}\,(A)$. The fact that it appears in $\mathrm{Det}\,(A)$ with coefficient $\pm 1$, also follows from the general result about the $\mathcal{A}$-discriminant (see [5], Chapter 11, Theorem 3.2 (b)).

In view of (7.4), the last statement in Theorem 7.1 is equivalent to the following:

$$M(k_1,\ldots,k_r) > 0 \Leftrightarrow (k_1,\ldots,k_r) \in \Phi_r. \tag{7.10}$$

This is proved in exactly the same way as the corresponding statement for $N(k_1,\ldots,k_r)$ in [4], Section 3 (using the Gale-Ryser theorem on $(0,1)$-matrices); we leave the details to the reader. Theorem 7.1 is proved.

*Remarks 7.2.*

(a) So far, the hyperdeterminant was defined only up to sign. Theorem 7.1 gives us a natural choice of the sign, by requiring that the monomial given by (7.1), (7.4) occurs in $\mathrm{Det}\,(A)$ with coefficient 1.

(b) All the results about the degree $N(k_1,\ldots,k_r)$ obtained in [4], [5] have obvious analogs for $M(k_1,\ldots,k_r)$ and hence, for the exponents in the "diagonal" monomial (7.1). The proofs are exactly the same. We leave the formulations and proofs to the reader.

(c) If the matrix format is boundary, then the determinantal formula for Det $(A)$ given in [4], Theorem 4.3, implies that the hyperdeterminant of a diagonal matrix is just the monomial (7.1). This is no longer true for interior format. Using computer algebra system MACAULAY, we found that the hyperdeterminant of a diagonal $3 \times 3 \times 3$ matrix is given by

$$
\begin{aligned}
(7.11)\quad \mathrm{Det}\,(A) = &(a_{000}a_{222})^8(a_{110}a_{101}a_{011}a_{112}a_{121}a_{211})^2 \times (a_{000}^2 a_{111}^4 a_{222}^2 \\
&+ 8a_{000}a_{111}^2 a_{222}(a_{000}a_{112}a_{121}a_{211} + a_{222}a_{110}a_{101}a_{011}) \\
&+ 16(a_{000}a_{112}a_{121}a_{211})^2 + 16(a_{222}a_{110}a_{101}a_{011})^2 \\
&- 32a_{000}a_{110}a_{101}a_{011}a_{112}a_{121}a_{211}a_{222}).
\end{aligned}
$$

Here the diagonal monomial is the only one occuring in (7.11) with coefficient 1. There are two other extreme monomials occuring in (7.11): both of them have coefficient 16. These two monomials do not contain the variable $a_{111}$. It follows that there exists a *non-degenerate* diagonal matrix having $a_{111} = 0$. It would be interesting to investigate the hyperdeterminant for diagonal matrices of arbitrary interior format, in particular, to classify its extreme monomials.

Now we assume that the matrix format is boundary, that is, $k_1 = k_2 + \cdots + k_r$. We shall construct another special class of matrices, the analogs of the classical Vandermonde matrix. Let $\Lambda = (\lambda_{i,j})_{0 \le i \le k_1,\, 2 \le j \le r}$ be a $(k_1+1) \times (r-1)$ complex matrix. We define the *Vandermonde-type* matrix $A = A(\Lambda)$ of format $(k_1+1) \times \cdots \times (k_r+1)$ by the formula

$$
(7.12)\qquad a_{i,i_2,\ldots,i_r} = \lambda_{i,2}^{i_2}\lambda_{i,3}^{i_3}\cdots\lambda_{i,r}^{i_r}.
$$

If $r = 2$, $k_1 = k_2 = k$ then $A(\Lambda)$ is the usual $(k+1) \times (k+1)$ Vandermonde matrix

$$
\begin{pmatrix}
1 & \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^k \\
1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^k \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^k
\end{pmatrix}.
$$

PROPOSITION 7.3. — *The matrix $A(\Lambda)$ is non-degenerate if and only if for each $j = 2, \ldots, r$ the numbers $\lambda_{0,j}, \lambda_{1,j}, \ldots, \lambda_{k_1,j}$ are mutually distinct.*

*Proof.* — By Lemma 4.6, a matrix of boundary format is non-degenerate if and only if it has no semi-critical points in $Y = (V_1 - \{0\}) \times \cdots \times (V_r - \{0\})$. For $j = 2, \ldots, r$ we shall represent a point $x^{(j)} \in V_j$ by a polynomial

$$P_j(\lambda) = \sum_{i=0}^{k_j} x_i^{(j)} \lambda^i.$$

Then the conditions that $x = (x^{(1)}, x^{(2)}, \ldots, x^{(r)})$ is a semi-critical point of $A(\Lambda)$, take the form

$$P_2(\lambda_{i,2})P_3(\lambda_{i,3}) \cdots P_r(\lambda_{i,r}) = 0 \quad (i = 0, \ldots, k_1). \tag{7.13}$$

Let $I_j$ denote the set of indices $i$ such that $P_j(\lambda_{i,j}) = 0$. Clearly, (7.13) is equivalent to

$$I_2 \cup I_3 \cup \cdots \cup I_r = [0, k_1]. \tag{7.14}$$

Since

$$\#([0, k_1]) = k_1 + 1 = k_2 + \cdots + k_r + 1,$$

it follows that at least for one $j$ we have $\#(I_j) > k_j$.

Now suppose that all the numbers $\lambda_{i,j}$ for a given $j$ are distinct. We see that the polynomial $P_j(\lambda)$ has more than $k_j$ roots. Since $\deg(P_j) = k_j$, it follows that $P_j = 0$, hence $x^{(j)} = 0$. Therefore, $A(\Lambda)$ has no semi-critical points in $Y$.

Conversely, suppose for some $j$ not all the numbers $\lambda_{i,j}$ are distinct. Without loss of generality, we can assume that $\lambda_{0,2} = \lambda_{1,2}$. Then there exist non-zero polynomials $P_2, \ldots, P_r$ such that $P_2(\lambda_{i,2}) = 0$ for $0 \leq i \leq k_2$, and $P_j(\lambda_{i,j}) = 0$ for $3 \leq j \leq r$, $k_2 + \cdots + k_{j-1} < i \leq k_2 + \cdots + k_j$. Such polynomials $P_j$ satisfy (7.14), hence give rise to a semi-critical point of $A(\Lambda)$. Proposition 7.3 is proved.

*Remark 7.4.* — Suppose the conditions in Proposition 7.3 hold with the only exception that $\lambda_{0,2} = \lambda_{1,2}$. Then the above proof shows that, up to proportionality and changing the first component $x^{(1)}$, the matrix $A(\Lambda)$ has a finite number of semi-critical points.

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