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MAXIMALLY DEGENERATE LAPLACIANS

by **Steven ZELDITCH(*)**

Introduction.

Let M be a compact manifold and let $\mathcal{M}_1$ denote the space of C^∞ Riemannian metrics of volume 1 on M. For each $g \in \mathcal{M}$ we will denote by $0 = \lambda_0(g) < \lambda_1(g) < \lambda_2(g) < \cdots \uparrow \infty$ the sequence of *distinct* eigenvalues of the Laplacian Δ_g and by $m_k(g) :=$ multiplicity of $\lambda_k(g)$ the eigenvalue multiplicity function. From work of Colin de Verdière [CV2] and others, one knows that if $\dim M \geq 3$, then $m_k(g)$ is unbounded as g varies in $\mathcal{M}_1$ for each fixed $k > 0$. Nevertheless, there is a precise sense in which certain metrics g have maximal multiplicity functions $m_k(g)$. Following physics terminology, we will call their Laplacians *maximally degenerate* (see Definition 1).

The canonical examples are the CROSSes (compact rank one symmetric spaces): (S^d, can), $(\mathbb{RP}^d, \mathrm{can})$, $(\mathbb{CP}^n, \mathrm{can})$, $(\mathbb{HP}^n, \mathrm{can})$, $(\mathbb{C}a\mathbb{P}^2, \mathrm{can})$. They have the property that $m_k(g) = ak^{d-1} + O(k^{d-2})$ for certain $a > 0$; here, $d = \dim(M)$. The exponent $d-1$ and coefficient a are maximal, as one can see from Weyl's law; in fact the lower order terms are also maximal, as will be clear from the fact that CROSS Laplacians are maximally degenerate. The maximal growth of m_k characterizes CROSSes (and their

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quotients) among homogeneous spaces. The question is, does it characterize them among all riemannian manifolds?

Before stating the question precisely, we will need a precise definition of maximal degeneracy. It is based on the following

THEOREM A. — *Let (M^d, g) be a Riemannian d-manifold for which $m_k(g) = ak^{d-1} + O(k^{d-2})$, for some $a > 0$. Then g is a Zoll metric: i.e. the geodesic flow $G_g^t : S^*M \to S^*M$ is periodic.*

A Zoll metric in this sense is called a P_ℓ-manifold in the book of Besse ([Besse]). All geodesics are closed, with minimal *common* period ℓ. Possibly, some exceptionally short geodesics occur, with lengths ℓ/m for some $m \in \mathbb{N}$. The metric is called C_ℓ if no such exceptional geodesics occur, i.e. if G_g^t defines a free S^1 action on S^*M. For example, the CROSSes are C_ℓ manifolds, but the lens spaces $S^3/\mathbb{Z}_p$ are only P_ℓ. (Here, $\mathbb{Z}_p$ is a cyclic subgroup of S^3.) A metric is also called SC_ℓ if each closed geodesic is simply closed as a curve on M.

It follows that metrics with exceptionally high muliplicity functions must be Zoll, and in particular the maximally degenerate ones will be Zoll. To define this maximal degeneracy, we first recall that a P_ℓ-Laplacian can be expressed in the form

$$\Delta_g = \left(\frac{2\pi}{\ell}\right)^2 \left(A + \frac{\alpha}{4}\right)^2 + Q^\# \tag{0.1}$$

where A is a positive elliptic element of Ψ^1 with $\operatorname{Spec}(A) \subseteq \mathbb{N}$, where $Q^\# \in \Psi^0, [Q^\#, \Delta] = 0$, and where α is the common Morse index of the ℓ-periodic geodesics. Here, Ψ^m is the space of m-th order pseudodifferential operators on M. Thus, the spectrum $\operatorname{Spec}(\Delta_g)$ consists of a union of widely separated eigenvalue clusters

$$C_k = \left\{ \left(\frac{2\pi}{\ell}\right)^2 \left(k + \frac{\alpha}{4}\right)^2 + \mu_{kj} : j = 1, \ldots, d_k \right\} \tag{0.2}$$

where $d_k = \# C_k$ and where $|\mu_{kj}| \leq M$ for some $M > 0$. (See [W], [CV] or §2, Lemma 2 for more precise statements and references.) The "multiplicity" d_k of the k-th cluster has been studied in depth by Colin de Verdière, and by Boutet de Monvel-Guillemin ([CV], [BMG]). In the case of a C_ℓ-metric, $d_k = R(k + \alpha/4)$ for a certain polynomial R, which is identified in [BMG] as the Hilbert polynomial of the space $G(M, g)$ of geodesics. Thus d_k is a symplectic invariant of $G(M, g)$. In the case of P_ℓ-manifolds, d_k can acquire an additional oscillatory term [CV], but it is still an invariant of the space (now an orbifold) $G(M, g)$. We then say:

DEFINITION 1. — *A Laplacian Δ_g is maximally degenerate if it is a Zoll Laplacian with the property that there is precisely one distinct eigenvalue in each eigenvalue cluster: $m_k = d_k$.*

There is a subtle ambiguity here regarding the definition C_k for a finite number of small k. Since it will not play a role in the main part of this paper, we will not discuss it further. However, the cautious reader may wish to qualify the definition by replacing $(\forall k)$ by $(\forall k \geq k_0)$ for some $k_0 \in \mathbb{N}$.

The question raised above is therefore:

PROBLEM 1. — Characterize the Zoll metrics with maximally degenerate Laplacians. What is the relation between the geometry of g and the width of the eigenvalue clusters C_k? (See [G3] for the case of potentials.)

A concrete problem of this kind is the following:

PROBLEM 2 ([Yau], Problem 41]). — Let (M^2, g) be a surface, and suppose that $m_k(g) = m_k(S^2, \mathrm{can}) = 2k + 1$. Must $(M^2, g) = (S^2, \mathrm{can})$?

The relation between these problems is as follows: First, if $m_k(g) = 2k + 1$, then (M, g) must be a Zoll surface and hence M must be either $\mathbb{RP}^2$ or S^2 [Besse]. Let us assume $M = S^2$. In §2 we will show:

THEOREM B. — *The assumption $m_k(g) = 2k + 1$ is equivalent to the maximal degeneracy of a Laplacian Δ_g on S^2.*

Hence Problem 2 reduces to the question, is Δ_{can} the only maximally degenerate Laplacian on S^2?

In the remainder of the paper, we give a number of partial results on these problems in the cases where $M = S^2$ and $M = \mathbb{RP}^d$. The case of $M = S^2$ is perhaps the most interesting in that the moduli space of Zoll metrics is infinite dimensional, indeed of functional dimension 2 [Besse]. The case of $M = \mathbb{RP}^d$ is the simplest, in that the canonical metric is probably the only C_ℓ-metric; at least, it is known to be the only one with the property that the time to the first conjugate locus is constant[loc.cit.].

Our results on S^2 are contained in Theorems C–E. The first one is of an operator theoretic nature and prepares the way for the study of residual spectral invariants of Zoll Laplacians.

THEOREM C. — *Let g be a Zoll metric on S^2, and let A_g be the* *operator defined in (0.1). Then*

(a) A_g *commutes with an effective action* $\pi : SO(3) \to UF(S^2)$ *of* $SO(3)$ *by unitary Fourier Integral Operators.*

(b) *For any* $b \in C^\infty(S^*S^2)$ *invariant under* G^t, *there exists* $B \in \Psi^0$ *with* $\sigma_B = b$ *and with* $[B, A_g] = 0$.

(c) *For any geodesic* γ *there is a Toeplitz structure* Π_γ *on the cone thru* γ *with* $[A_g, \Pi_\gamma] = 0$.

(d) *The principal symbol of the operator* $Q^\#$ *in (0.1) is given by the non-commutative residue of the Toeplitz operator* $2\Pi_\gamma\sqrt{\Delta_g}$, *that is*

$$\sigma_{Q^\#}(\gamma) = 2 \operatorname{res} \Pi_\gamma \sqrt{\Delta_g}.$$

(e) *If* Δ_g *is maximally degenerate, then it commutes with the* $SO(3)$*-action in* (a), *the operator* B *in (b) and the operator* Π_γ *in (c).*

(f) *If* Δ_g *is maximally degenerate, then*

$$\Delta_g = \left(A + \frac{1}{2}\right)^2 - \frac{1}{4} + S$$

where S *is a smoothing operator (also a function of* Δ_g*). Hence,* $\operatorname{res} \Pi_\gamma \sqrt{\Delta_g} = 0$ *for all* γ.

Thus, the common belief that high degeneracies are caused by symmetries is justified in this case. Maximally degenerate Laplacians are evidently "maximally symmetric." Above, we use the fact that $[Q^\#, A_g] = 0$ to identify $\sigma_{Q^\#}$ with a function on the space $G(S^2, g)$ of geodesics. Also, we have normalized the area so that the period $\ell = 2\pi$. For the definition and background of the non-commutative residue of a Fourier Integral or Toeplitz operator, see the remarks at the end of the introduction.

It follows from Theorem C that the complete symbol of $Q^\#$ is zero in the maximally degenerate case. Our next result is a geometric formula for the principal symbol $\sigma_{Q^\#}$. As will be seen, the subprincipal symbol $\sigma_{\text{sub}}(Q^\#)$ is zero for any Zoll metric, so the vanishing of $\sigma_{Q^\#}$ is the condition for the clusters to have widths $O(k^{-2})$. To state the result, we will need some more notation: First, as above, $G(S^2, g)$ will denote the space of geodesics of g. The Radon or X-ray transform is then the operator

$$R : C^\infty(S^2) \to C^\infty(G(S^2, g)) \quad Rf(\gamma) := \int_\gamma f.$$

Also, the tangent space $T_\gamma(G(S^2, g))$ at a geodesic γ can be identified with the space $\mathcal{J}_\gamma^\perp$ of normal Jacobi fields along γ. We denote in particular by

Θ_s the normal Jacobi field satisfying $\Theta_s(s) = 0, \nabla\Theta_s(s) = \nu(s)$. Here, γ is understood to be parametrized by arclenght s from some basepoint m, $\nu(s)$ is the unit normal at $\gamma(s)$ and ∇ is the Riemannian connection on TS^2 corresponding to g. τ will denote the scalar curvature of g. Finally, $\{,\}$ will denote the Poisson bracket on $G(S^2, g)$, which inherits a natural quotient symplectic structure from T^*S^2, and H_f will denote the Hamilton vector field of a function f.

THEOREM D. — *Let g be a Zoll metric on S^2. Then:*

(a) *$\sigma_{Q^\#}$ is determined up to a constant by the formula*

$$\Big\{\sigma_{Q^\#} - \frac{1}{12}R(\tau), b\Big\}(\gamma) = -\frac{1}{2^3 3!}db\Big(\int_\gamma \tau_\nu(s)\overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s\, ds\Big)$$

where $b \in C^\infty(G(S^2,g))$, and where $\overline{\nabla}$ denotes any connection on $TG(S^2,g)$. The right side is independent of the choice of connection.

(b) *The clusters C_k have widths $O(k^{-2})$ if and only if*

$$H_{R(\tau)}(\gamma) = 4\int_\gamma \tau_\nu(s)\overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s\, ds.$$

Weinstein has conjectured that $\sigma_{Q^\#} = C(R(\tau) - 1/4)$ for some constant C [W]. The formula above shows that there is a second term (which does not vanish [Z1].) By employing special connections $\overline{\nabla}$, one can use it to obtain (lengthier) formulae for the symbol in terms of the curvature and Jacobi fields along γ. Similar formulae for the lower order terms in the symbol expansion of $Q^\#$ could be obtained by the same method, but as they rapidly become lengthy we do not display the results. The unfortunate complication of the second term as well as lack of knowledge of the Radon transform on a general Zoll surface has made it difficult to characterize the maximally degenerate metrics more simply; it may be that the canonical sphere is the only one with cluster widths of order $O(k^{-2})$, but we do not see how to derive this from (a).

The analogous calculations for a Schrödinger operator $-\Delta + V$ are much simpler and do show that a maximally degenerate Schrödinger operator has a constant potential. They also simplify in the case of a Zoll metric of revolution. From results of Bruning, Guillemin, Widom we have:

THEOREM E.

(a) *If $\Delta_{\text{can}} + V$ is a Schrödinger operator on S^2 with $m_k(V) = 2k+1$ $(\forall k)$, then $V \equiv$ const.*

(b) *If g is a metric on S^2 carrying an effective S^1-action by isometries, and $m_k(g) = 2k+1$ $(\forall k)$, then $g = \mathrm{can}$.*

So far, we have been concentrating on the case where $M = S^2$. As mentioned above, in the case $M = \mathbb{RP}^2$, the situation is much simpler, since can is the only SC_π-metric on $\mathbb{RP}^2$ (by Green's Wiedersehenfläche Theorem (cf. [Besse]). We will prove:

THEOREM F(a). — *Suppose g is a metric on $\mathbb{RP}^2$ for which $m_k(g) = m_k(\mathrm{can})$. Then $g = \mathrm{can}$, i.e. $(\mathbb{RP}^2, \mathrm{can})$ is determined by its multiplicities.*

In the case of higher dimensional $\mathbb{RP}^d$'s, we will prove a slightly weaker result (as a consequence of the Berger-Kazdan Wiedersehenraum Theorem [Besse]):

THEOREM F(b). — *Suppose g is a metric on $\mathbb{RP}^d (d \geq 2)$ such that $m_k(g) = m_k(\mathrm{can})$, and such that additionally $(\ell(g), \mathrm{vol}(\mathbb{RP}^d, g)) = (\ell(\mathrm{can}), \mathrm{vol}(\mathbb{RP}^d, \mathrm{can})$. Then $g = \mathrm{can}$.*

Here, $\ell(g)$ is the minimal period of the geodesic flow G_g^t. By rescaling g, we may of course assume with no loss of generality that $\mathrm{vol}(\mathbb{RP}^d, g) = \mathrm{vol}(\mathbb{RP}^d, \mathrm{can})$. The assumption is then that the periods of G_g^t and G_{can}^t are equal. This equality follows from Weinstein's integrality theorem (cf. [Besse]) if g is C_ℓ metric. Hence, the extra assumption in (c) is ony needed if g is a Pp_ℓ-metric on $\mathbb{RP}^d$ with exceptionally short closed geodesics. No such metrics are known, of course, but the proof of the Berger-Kazdan Wiedersehenraum theorem does not extend to them. It is natural to conjecture that the only P_π-metric on $\mathbb{RP}^d$ is can, which would immediately imply that $(\mathbb{RP}^d, \mathrm{can})$ is determined by its multiplicities. In fact, this statement would follow from the much weaker conjecture that the Weinstein integrality theorem holds for P_ℓ-metrics on $\mathbb{RP}^d$. Modulo the period-assumption, (c) shows for example that the Casimir operator on $SO(3)$ is determined by its multiplicities.

Remarks.

(a) Since Theorem C(f) only determines Δ_g spectrally up to a smoothing operator, we are compelled to study residual spectral invariants of a Zoll Laplacian and of the various Fourier Integral or Toeplitz operators associated to it. The main problem is then to calculate these invariants in sufficiently explicit geometric terms that they may be used to determine the metric. By residual we mean that the invariants are defined as non-commutative residues of certain Fourier Integral operators V. We recall that the residue is defined as follows:

$$\mathrm{res}(V) := \operatorname*{Res}_{s=0} \zeta(s, V, P)$$
$$(\zeta(s, V, P) := TrVP^{-s} \text{ for } \quad \mathrm{Re}s \gg 0)$$

with P a positive elliptic pseudodifferential operator of order 1. Just as in the case of pseudodifferential operators, $\mathrm{res}(V)$ is independent of the choice of P and is a conjugacy invariant of V. In Theorems C–D, the relevant FIO is of course the Toeplitz operator $\Pi_\gamma\sqrt{\Delta}\Pi_\gamma$. In Theorem E, it is essentially the composition $\Pi_k U_t \Pi_k$, with Π_k the (coisotropic) projection to the space of weight-k vectors for the rotation action, and with U_t the wave group. Note that the residues of the wave operator and its time derivatives at $t = 0$ give no useful information in dimension 2, and those at other periods are equivalent to the calculation of the symbol of $Q^{\#}$.

In the case of pseudodifferential operators, there is a unified local formula for the non-commutative residue due to Wodzicki and Guillemim: namely, the residue of a ΨDO A is given by the "symplectic residue" of its complete symbol (cf. [K] for terminology). No such unified formula is known for Fourier Integral or Toeplitz operators, and perhaps none can be expected. A naive analogue for a Toeplitz operator A of the local formula for ΨDO's might give $\mathrm{res}(A)$ as the symplectic residue of the complete symbol of A along the symplectic cone associated to A. In the case of $\mathrm{res}\,\Pi_\gamma\sqrt{\Delta_g}$, this naive analogue coincides with the conjecture of Weinstein mentioned above. Unfortunately, it seem to be incorrect; we hope to take up the study of the correct formula in the future. The case of the coisotropic operator $\Pi_k U_t \Pi_k$ on a general Zoll surface is yet more complicated because Π_k involves the unitary group $\exp it L$ generated by a ΨDO L which is not generally the root of a partial differential operator: namely, $\exp it L$ is the conjugate under a unitary Fourier Integral operator of a one parameter group of isometries of the standard metric (see the proof of Theorem C). This leads to a lack of locality in the computations except in the case of surfaces of revolution where L is a vector field. In this case the wave (actually, the essentially equivalent) heat expansions have already received a detailed study by Bruning [Bru], and simple qualitative properties of this expansion already suffice for applications to the inverse problem here (Theorem E). It would be of interest to understand the nature of the non-commutative residue in the general case as well.

(b) As H. Donnelly and C. Gordon have pointed out to the author, Theorem (D.2) has already been proved by M. Engman [Engman], by a different method.

(c) The calculation of $\sigma_{Q^\#}$ is an outgrowth of joint work with A. Uribe. We used a very different method (quasi-modes and Toeplitz operators) to determine this symbol. The method presented here leads more rapidly to simpler formulae for the symbol, although they are still not as transparent as we would like. In [Z1] we present yet another method of calculating the symbol, based on constructing the cluster projections as semi-classical Lagrangean distributions. Residual invariants for many other operators can be calculated in a similar way. Our attention to the Zoll case is in large part because it provides the simplest model for the local calculation of residues.

1. High multiplicities implies Zoll: proof of Theorem A.

Assume the distinct eigenvalues $0 < \lambda_1(g) < \lambda_1(g) < \cdots$ of (M^d, g) satisfy the hypothesis:

$$(1.1) \qquad \text{(HM)} \quad m_k(g) = ak^{d-1} + O(k^{d-2}) \ \text{(for some } a > 0).$$

We wish to show that the geodesic flow G^t on S^*M, i.e. the Hamilton flow of the norm function $|\xi|$, is periodic.

The proof is a variation on Ivrii's estimate of the remainder term $R(\lambda)$ in the Weyl law, $N(\lambda) = \frac{\text{vol}(S^*M)}{(2\pi)^n}\lambda^n + R(\lambda)$. His estimate is that $R(\lambda) = o(\lambda^{d-1})$ if and only if the closed geodesics form a set of measure 0. We refer in particular to the exposition in [HoIV], XXIX, which we will closely follow

Let Π^* be the microlocal period function on $T^*M\backslash 0$, i.e.

$$\Pi^*(x, \xi) = \begin{cases} \inf\{T > 0 : G^T(x, \xi) = (x, \xi)\} \\ 0 \ \text{ if no such } T \text{ exists.} \end{cases}$$

For any $T > 0$, set

$$\Gamma_T = \{(x, \xi) : \Pi^*(x, \xi) > T\}.$$

Since Π^* is lower semi-continuous, Γ_T is an open cone in $T^*M\backslash 0$.

Now assume for purposes of contradiction, that G^t has a non-periodic point. Then $\Gamma_T \neq \phi$ for any $T > 0$. Fix a large T (to be determined later) and let $\Psi^0_T = \{A \in \Psi^0 : WF(A) \subset \Gamma_T\}$, i.e. Ψ^0_T is the space of zero-th order ψDO's with microsupport in Γ_T. For $B \in \Psi^0_T$, set:

$$N(\lambda, B^*B) := \sum_{\sqrt{\lambda_k}\leq\lambda}^{*} \operatorname{Tr} B^*B\restriction_{E_k}$$

where $\sum^*$ denotes the sum over *distinct* eigenvalues, where E_k is the Δ-eigenspace of eigenvalue λ_k. The function $N(\lambda, B^*B)$ satisfies a modified Weyl law, given in [HoIV], Theorem 29.1.5. Before stating it, we make some simplifying observations. First we may assume that the sub-principal symbol b^s of B^*B equals 0. This will be the case if, for instance, B is chosen to be self-adjoint. Second, we note that the term

$$\partial_\lambda \int_{|\xi|<\lambda} \Big(|\xi|^s b + \frac{1}{2} i\{b, |\xi|\}\Big) dx\, d\xi$$

in the remainder estimate in [HoIV], Theorem 29.1.5 is automatically zero in our case, since $|\xi|^s = 0$ and since the integral of $\{b, |\xi|\}$ vanishes for any b. It follows that

$$N(\lambda, B^*B) = (2\pi)^{-d} \iint_{|\xi|<\lambda} b\, dx\, d\xi + R(\lambda, B^*B) \tag{1.2}$$

where b is the principal symbol of B^*B, and where the remainder term satisfies

$$\overline{\lim_{\lambda\to\infty}} \lambda^{-(d-1)} |R(\lambda, B^*B)| \leq \frac{C_d}{T} \Big|\iint_{|\xi|<1} b\, dx\, d\xi\Big|, \tag{1.3}$$

where C_d is a constant depending only on the dimension. We refer to [HoIV], loc. cit. for the proof of (1.3).

Next, we observe that

$$\operatorname{Tr} B^*B\restriction_{E_k} = N(\sqrt{\lambda_k} + 0, B^*B) - N(\sqrt{\lambda_k} - 0, B^*B) \tag{1.4}$$

where, as usual, $f(x \pm 0) = \lim_{\varepsilon\downarrow 0} f(x \pm \varepsilon)$. Since the principal term $(2\pi)^{-d} \iint_{|\xi|<\lambda} b\, dx\, d\xi$ in the Weyl law is continuous, we get

$$\operatorname{Tr} B^*B\restriction E_k = R(\sqrt{\lambda_k} + 0, B^*B) - R(\sqrt{\lambda_k} - 0, B^*B). \tag{1.4b}$$

It follows from (1.3) that

$$|R(\lambda, B^*B)| \leq \frac{C_d + 1}{T} \Big|\iint_{|\xi|<1} b\, dx\, d\xi\Big| \lambda^{d-1} \tag{1.5}$$

for sufficiently large λ, say for $\lambda \geq \lambda_0(T)$. Hence, we have

$$\operatorname{Tr} B^*B\restriction_{E_k} \leq 2\frac{C_d + 1}{T} \Big|\iint_{|\xi|<1|} b\, dx\, d\xi\Big| \lambda_k^{\frac{d-1}{2}} \tag{1.6}$$

for $\sqrt{\lambda_k} \geq \lambda_0(T)$. Hence

$$N(\lambda, B^*B) \leq 2\frac{C_d+1}{T}\bar{b} \sum_{\sqrt{\lambda_k}\leq\lambda}{}^{*} \lambda_k^{\frac{d-1}{2}} + O_T(1) \tag{1.7}$$

where $\bar{b} = \iint_{|\xi|<1} b\, dx\, d\xi$.

Now let us use the multiplicity assumption to show

(1.8) CLAIM.

$$\sum_{\sqrt{\lambda_k}\leq\lambda}{}^{*} \lambda_k^{d-1/2} = C_1\lambda^d + O(\lambda^{d-1})$$

for a certain (computable) $C_1 > 0$.

Proof. — Indeed, by the usual Weyl law,

$$N(\lambda) = \sum_{\sqrt{\lambda_k}\leq\lambda}{}^{*} \operatorname{mult}(\lambda_k) = (2\pi)^{-d}\operatorname{vol}(S^*M)\lambda^d + O(\lambda^{d-1}).$$

Now let

$$N^*(\lambda) = \sum_{\sqrt{\lambda_k}\leq\lambda}{}^{*} 1$$

be the number of distinct eigenvalues $< \lambda$. Then $N^*(\sqrt{\lambda_k}) = k$, so the (HM) assumption (1.1) may be put as follows:

$$\text{(HM)}\quad m_k(g) = a(N^*(\sqrt{\lambda_k}))^{d-1} + O((N^*(\sqrt{\lambda_k}))^{d-2}). \tag{1.9}$$

Hence,

$$\begin{aligned} N(\lambda) &= \sum_{\sqrt{\lambda_k}\leq\lambda} ((N^*(\sqrt{\lambda_k}))^{d-1} + O((N^*(\sqrt{\lambda_k}))^{d-2}) \\ &= a\int_0^\lambda N^*(\lambda)^{d-1} dN^*(\lambda) + O\Big(\int_0^\lambda N^*(\lambda)^{d-2} dN^*(\lambda)\Big) \end{aligned} \tag{1.10}$$

where $dN^*(\lambda) = \sum_k^{*} \delta(\lambda - \sqrt{\lambda_k})$. Performing the integration, we get

$$\begin{aligned} N(\lambda) &= \frac{a}{d} N^*(\lambda)^d + O(N^*(\lambda)^{d-1}) \\ &= (2\pi)^{-d}\operatorname{vol}(S^*M)\lambda^d + O(\lambda^{d-1}). \end{aligned} \tag{1.11}$$

It follows that $N^*(\lambda) = C_1\lambda + O(1)$, where $C_1 = (d/a(2\pi)^{-d}\operatorname{vol}(S^*M))^{1/d}$. Therefore

$$\begin{aligned} \sum_{\sqrt{\lambda_k}\leq\lambda} \lambda_k^{\frac{d-1}{2}} &= \int_0^\lambda \lambda^{d-1} dN^*(\lambda) \\ &= \lambda^{d-1}N^*(\lambda) + (d-1)\int_0^\lambda N^*(\lambda)\lambda^{d-1} d\lambda \\ &= C_1\lambda^d + O(\lambda^{d-1}), \end{aligned} \tag{1.12}$$

proving the claim. □

Now we combine (1.7)–(1.8) to get

$$N(\lambda, B^*B) \leq \frac{D}{T}\bar{b}\lambda^d + O(\lambda^{d-1}) + O_T(1) \tag{1.13}$$

for a certain constant $D > 0$ depending only on n and vol(M). But (1.13) contradicts the Weyl law (1.2), since the pair imply

$$(2\pi)^{-d}\bar{b}\lambda^d + O(\lambda^{d-1}) \leq \frac{D}{T}\bar{b}\lambda^d + O(\lambda^{d-1}) + O_T(1)$$

which is impossible if T is chosen to be sufficiently large and if we let $\lambda \to \infty$.

The contradiction shows that all geodesics of (M, g) must be periodic. By Wadsley's theorem [Besse], (0.40), the closed geodesics must have a common period $\ell < \infty$, i.e. (M, g) must be a Zoll (i.e. P_ℓ-) manifold. □

2. Laplacians on S^2: proof of Theorem B.

In this section we assume (M^2, g) is an orientable P_ℓ-surface, i.e. that $G^\ell = \text{id}$. Necessarily $M^2 = S^2$ since $\pi_1(M^2)$ is finite [Besse], 4.3. We will normalize g so that area $(S^2, g) = 4\pi$. Below, α will denote the common Morse index of all closed geodesics of period ℓ.

Proof of Theorem B. — We begin with the standard Lemma 2.1 ([CV], Theorem 1.1).

LEMMA 2.1. — *Let (M, g) be a P_ℓ-manifold. Then there exists a positive elliptic $A \in \Psi^1$ and a $Q_{-1} \in \Psi^{-1}$ such that:*

(i) $\text{Spec}(A) \subseteq \mathbb{N}$

(ii) $\sqrt{\Delta} = \frac{2\pi}{\ell}\left(A + \frac{\alpha}{4}\right) + Q_{-1}$

(iii) *A and Q_{-1} are functions of Δ.*

Proof. — It follows from [DG], Theorem 3.1, that $\exp(-i\ell(\sqrt{\Delta} - \alpha/4)) = \text{Id} + C$, with $C \in \Psi^{-1}$. We then wish to define $Q_{-1} \in \Psi^{-1}$ by: $\exp(i\ell Q_{-1}) = (\text{Id} + C)^{-1}$, i.e. by $Q_{-1} = -\frac{1}{i\ell}\text{Log}(\text{Id} + C)$. The question is how to determine the branch of Log.

Since $C \in \Psi^{-1}$, its eigenvalues $C_n = \exp(-i\ell(\lambda_n - \alpha/4)) - 1$ tend to 0 as $n \to \infty$. Fix n_0 so that $|C_n| \leq 1/2$ for $n \geq n_0$, and let $\mathrm{Log}(1+C_n)$ denote the principal branch. Then choose any branch for the finite number of remaining eigenvalues. $\mathrm{Log}(\mathrm{Id} + C)$ is then defined by the spectral theorem, and its power series expansion shows that it is in Ψ^{-1}.

Set $A := \sqrt{\Delta} - \alpha/4 - Q_{-1}$. Then $\exp i\ell A = \mathrm{Id}$, and the properties (i)–(iii) are easily verified. For further details, we refer to [CV]. □

COROLLARY 2.1. — *$\Delta = (2\pi/\ell)^2(A + \alpha/4)^2 + Q^{\#}$ where $Q^{\#} \in \Psi^0$.*

Hence, $\mathrm{Spec}(\Delta_g) \subset \cup_k I_k$, where

$$I_k = \Big[\Big(\frac{2\pi}{\ell}\Big(k + \frac{\alpha}{4}\Big)\Big)^2 - M,\ \Big(\frac{2\pi}{\ell}\Big(k + \frac{\alpha}{4}\Big)\Big)^2 + M\Big],$$

with $M = \|Q^{\#}\|$. For sufficiently large k, the intervals I_k are disjoint and we can unambiguously define the k-th cluster by $C_k = \mathrm{Spec}(\Delta) \cap I_k$. The eigenvalues of Δ in C_k may be written in the form

$$\lambda_{k,r} = \Big(\frac{2\pi}{\ell}\Big)^2 \Big(k + \frac{\alpha}{4}\Big)^2 + \mu_{k,r} \quad (r = 1, \dots, d_k) \tag{2.2}$$

where $d_k = \#C_k$ and where $-M \leq \mu_{k,1} \leq \cdots \leq \mu_{k,d_k} \leq M$. Let us also denote by E_k the span of the Δ_g-eigenfunctions $\phi_{k,r}$ with $\lambda_{k,r} \in C_k$. Then $\dim E_k = d_k$, $A \restriction_{E_k} = k$, and $Q^{\#} \restriction_{E_k}$ has eigenvalues $\mu_{k,r}$.

Now let us specialize to the case of Zoll surfaces (S^2, g) with the multiplicities $m_k(g) = 2k+1$. To prove that there can only be one eigenvalue in each cluster, i.e. that $m_k = d_k$, we use a formula, due to Colin de Verdière [CV], Theorem 1.4 and Boutet de Monvel-Guillemin [BMG] for the multiplicity d_k of C_k. In the case of surfaces it reads:

$$d_k = b_1\Big(k + \frac{\alpha}{4}\Big) + \sum_{j=1}^{N} \sum_{p=1}^{m_j - 1} \omega_{j,p}^k R_{j,p} \tag{2.3}$$

where $b_1 \in \mathbb{Q}$, $\omega_{j,p} = \exp\Big(i\ell \dfrac{p}{m_j}\Big)$ and where $R_{j,p} \in \mathbb{C}$. The second, oscillatory term, is caused by the exceptional short closed geodesics, which can occur on a P_ℓ-surface. It is absent on a C_ℓ-surface, and possibly cancels on a given P_ℓ-surface.

It follows from (2.3) that

$$d_k = b_1 k + O(1).$$

We wish to prove that $b_1 = 2$; this implies Theorem B, given the rigidity of the other constants.

In the case of C_ℓ-surfaces, the proof is simple: first, $\sqrt{\lambda_{k,r}} = 2\pi/\ell k + O(1)$, so by Weyl's law,

$$\begin{aligned} N(\lambda) &= \sum_{k \leq \frac{\ell}{2\pi}\lambda} d_k + O(1) \\ &= b_1\Big(\frac{\ell}{2\pi}\Big)\lambda^2 + O(\lambda) \\ &= (2\pi)^{-1}\,\mathrm{area}(S^2, g)\lambda^2 + O(\lambda) \end{aligned}$$

or

$$b_1 = 2 \cdot \Big(\frac{2\pi}{\ell}\Big)^2. \tag{2.4}$$

If g is C_ℓ, then Weinstein's integrality theorem [Besse], Theorem 2.21 implies that

$$i(S^2, g) = \frac{\mathrm{area}(S^2, g)}{\mathrm{area}(S^2, g_0)}\Big(\frac{2\pi}{\ell}\Big)^2 = 1.$$

Hence, by our normalization of g, $2\pi/\ell = 1$ and $b_1 = 2$.

If g is only P_ℓ this proof apparently breaks down, since $i(S^2, g)$ is possibly not equal to 1. Therefore, we need an independent proof that works in both cases. The proof is contained in a couple of lemmas.

LEMMA 2.2. — *Let k_0 be any index so that C_k is well-defined for $k \geq k_0$, let d_k^* be the number of distinct eigenvalues in C_k, let $N^*(\lambda_{k_0,1})$ be the number of* distinct *eigenvalues $< \lambda_{k_0,1}$ and let $d_+^* = \overline{\lim}_{k\to\infty} d_k^*$. Then:*

(i) $d_k^* = d_+^*$ *for sufficiently large* $k \geq k_1$

(ii) $b_1 = 2(d_+^*)^2$, *and* $d_+^* = \dfrac{2\pi}{\ell}$

(iii) $d_k = 2(d_+^*)^2(k - 1 - k_0) + 2N^*(\lambda_{k_0,1})d_+^* + (d_+^*)^2$ *for* $k \geq k_1$.

Proof. — We have

$$d_k = \sum_{r=1}^{d_{k^*}} \mathrm{mult}(\lambda_{k,r}) = \sum_{r=1}^{d_{k^*}} (2n(k,r) + 1) \tag{2.5}$$

where the sums run over distinct eigenvalues in C_k, and where $\lambda_{k,r} = \lambda_{n(k,r)}$. Here, as always, $0 < \lambda_1 < \lambda_2 < \cdots$ denote the distinct eigenvalues of Δ_g. Observe that $n(k,r) = n(k,1) + r - 1$ and that $n(k+1,1) = n(k,1) + d_{k^*}$. Hence

$$\begin{aligned} d_k &= 2d_k^* n(k,1) + (d_k^*)^2 \\ &= 2d_k^* \sum_{k_0 \leq j \leq k-1} d_j^* + 2d_k^* n(k_0,1) + (d_k^*)^2. \end{aligned} \tag{2.6a}$$

Now split up the first term using $d_j^* = 1 + (d_j^* - 1)$, to get

$$\text{(2.6b)}\quad d_k = 2d_k^*(k-1-k_0) + 2d_k^* \sum_{k_0 \le j \le k-1} (d_j^* - 1) + 2d_k^* n(k_0, 1) + (d_k^*)^2.$$

It follows from (2.3) and (2.6b) that $2d_k^* \le b_1$, so that the last two terms are $O(1)$. Let us write the second term in the form $2d_k^* a_k^*(k-1-k_0)$, i.e. $a_k^* = \frac{1}{k-1-k_0} \sum_{k_0 \le j \le k-1} (d_j^* - 1)$. It is obvious that $a_k^* = O(1)$, that $(k-k_0)a_{k+1}^* = (k-1-k_0)a_k^* + (d_k^* - 1)$ and hence that $a_{k+1}^* - a_k^* = O(1/k)$. Moreover, we have $b_1 = 2d_k^*(1+a_k^*)$ $(\forall k \ge k_0)$. Hence, $(d_{k+1}^* - d_k^*)(1+a_k^*) = O(1/k)$. Since $a_k^* \ge 0$ and d_k^* is $\mathbb{N}$-valued, it follows that $d_{k+1}^* = d_k^*$ for sufficiently large k, proving (i).

It follows then that a_k^* must eventually be constant; and this constant is easily seen to be $d_+^* - 1$, proving the first part of (ii). The second part follows easily. Finally, (iii) follows from (i)–(ii) and (2.6b). □

COROLLARY 2.2.

(i) $\frac{2\pi}{\ell} \in \mathbb{N}$.

(ii) *The oscillatory term in* (2.3) *vanishes.*

LEMMA 2.3. — *If the oscillatory term in* (2.3) *vanishes, then* $2\pi/\ell = 1$ *and hence* $b_1 = 2$.

Proof. — The Zoll geodesic flow G^t lifts to an S^1 action $\widetilde{G}^t$ on S^3, with all orbits exactly twice as long as those of G^t. According to [Jac], any S^1 action on S^3 is (homeomorphically) conjugate to a linear action, i.e. one give by

$$\phi^t_{(k_1,k_2)} = \begin{bmatrix} r^t_{k_1} & \\ & r^t_{k_2} \end{bmatrix}$$

with

$$r^t_k = \begin{bmatrix} \cos k\ell t & \sin k\ell t \\ -\sin k\ell t & \cos k\ell t \end{bmatrix}.$$

In particular $\widetilde{G}^t$ either has two exceptionally short orbits (the case where k_1 and k_2 are relatively prime) or no short orbits (the case where $k_1 = k_2$). Since all orbits split in the same way under the double cover $S^3 \to S^*(S^2)$, it follows that either G^t is C_ℓ or else it has two exceptionally short orbits with different primitive periods. The first case was covered above, so we only need to consider the second.

We now use Corollary 2.2, (ii) to show that the second case cannot occur under the multiplicity assumption. Indeed, the oscillatory term must be of the form $\sum_{j=1,2} \sum_{p=1}^{k_j-1} \omega_{j,p}^k R_{j,p}$. Since k_1 and k_2 are relatively prime, the oscillatory term can only vanish if $R_{j,p} = 0 \quad (\forall j, p)$. Let us now evaluate $R_{j,p}$, following [CV] and [GU], §7.

Consider the sum

$$\Upsilon(e^{\mathrm{it}}) = \sum_{k=0}^{\infty} d_k e^{ikt}$$

which lies in the Hardy space $H_+(S^1)$ of distributions on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with only positive frequencies. Since $\Upsilon(e^{\mathrm{it}}) = \operatorname{Tr}\exp \mathrm{it}\, A$, $\Upsilon(e^{\mathrm{it}})$ is a Lagrangean distribution on S^1, with singularities at the points $\omega_j = \exp \mathrm{it}_j \in S^1$ such that G^{t_j} has fixed points. Thus, $\operatorname{sing\,supp} \Upsilon(e^{\mathrm{it}}) = \{\omega_0 = 1, \exp ip\frac{\ell}{k_1}, \exp iq\frac{\ell}{k_2} : p = 1, \dots, k_1 - 1;\ q = 1, \dots, k_2 - 1\}$. Let ρ_j be a cut-off function on S^1 with only ω_j in its support. Then

$$\widehat{\rho_j \Upsilon}(\mu) = \sum_{m=0}^{\infty} \alpha_j(m) \mu^{d_j - m} \quad (\mu \to \infty) \tag{2.7}$$

where d_j is the degree of the singularity at $t = \omega_j$, and where $\alpha_j(m)$ can be calculated from knowledge of A microlocally near $\operatorname{Fix}(G^{t_j})$. One then has,

$$d_k \sim \sum_{m=0}^{\infty} \sum_{\omega_j} \alpha_j(m) \omega_j^{-k} k^{d_j - m} \tag{2.8}$$

(see [GU], Lemma 7.1).

We recognize the expansion (2.3), in particular that $d_0 = 1$ (resp. $d_j = 0$ if $\omega_j \neq 0$) and $\alpha_0(m) = 0$ if $m \geq 2$ (resp. $\alpha_j(m) = 0$ if $m \geq 1$ and if $\omega_j \neq 0$). The main point is that the coefficients $R_{j,p}$ in (2.3) are precisely the principal coefficients $\alpha_j(0)$ in (2.8), i.e. $R_{j,p}$ is the principal symbol of $\Upsilon(e^{\mathrm{it}})$ at $\exp\left(ip\frac{\ell}{k_j}\right)$. Hence, $R_{j,p}$ vanishes if and only if the symbol of Υ vanishes at the corresponding ω_j.

This symbol can easily be calculated, since the principal symbols of $\operatorname{Tr} e^{\mathrm{it}\sqrt{\Delta}}$ and $\operatorname{Tr} e^{\mathrm{it}\, A}$ agree at each singular point, and since the exceptional short closed geodesics are necessarily non-degenerate. If we denote by γ_1, resp. γ_2, the exceptional geodesics, we get:

$$\sigma_\Upsilon(L_{\gamma_i}) = \frac{L_{\gamma_i}^{\#} e^{\frac{i\pi}{2} m(\gamma_i)}}{|\det(I - P_{\gamma_i})|^{\frac{1}{2}}} \neq 0 \tag{2.9}$$

(see [DG] for relevant notation). This contradicts the vanishing of the $R_{j,p}$'s, shows that no exceptional geodesics occur; and completes the proof of Lemma 2.3. □

Thus we have proved: $b_1=2$. Since $\alpha=2$ for any C_ℓ-metric on S^2 [Besse] one concludes that $d_k=2k+1=m_k$. Hence Δ_g is maximally degenerate. The converse is obvious. This completes the proof of Theorem B. □

3. Maximally degenerate Laplacians on S^2: symmetries and normal forms. Proof of Theorem C.

In preparation for the proof of Theorem C, let us recall the basics of the theory of band invariants of Zoll spectra ([G1], [CV]). Band invariants arise from the asymptotics of the eigenvalue cluster measures $d\mu_k := \sum_{i=1}^{d_k} \delta(\mu_{k,i})$ on $\mathbb{R}$. The main result is

THEOREM 3.1 ([CV], [W]). — *Suppose (M,g) is a $P_{2\pi}$-manifold of dimension n. Then there exist classical symbols R of order $n-1$, resp. $R_{i,m}$ of order n_i-1, with values in $\mathcal{D}'(-M,M)$ so that for any $\rho \in C^\infty(\mathbb{R})$,*

(a) $\int_{\mathbb{R}} \rho d\mu_k = \int \rho dR\Big(k+\frac{\alpha}{4}\Big) + \sum_{i=1}^{N}\sum_{p=1}^{n_i-1} \omega_{i,p}^k \int_{\mathbb{R}} \rho dR_{i,p}(k)$, *where as* $t \to \infty$,

(b) $R(t) \sim \nu_1 t^{n-1} + \nu_3 t^{n-3} + \cdots \quad (\nu_j \in \mathcal{D}'(-M,M))$,

(c) $R_{i,p}(t) \sim \nu_{i,p,1} t^{n_i-1} + \nu_{i,p,2} t^{n_i-2} + \cdots \quad (\nu_{i,p,r} \in \mathcal{D}'(-M,M))$, $n_i \leq n-1)$.

In (b), all of the even terms $\nu_{2k} t^{n-2k}$ vanish. This is due to the fact that Δ has the transmission property [CV].

Let us apply this theorem to the case of maximally degenerate Laplacians on S^2; similar results would hold on a general Zoll manifold. By definition of maximal degeneracy, Δ and hence $Q^\#$ have just one eigenvalue in the k-th cluster eigenspace E_k. Since $2\pi/\ell = 1$ by Lemma 2.3, and since $\alpha = 2$ for any Zoll metric on S^2 [CV], [Besse] on we have that

$$\Delta\restriction_{E_k} = \left[\Big(k+\frac{1}{2}\Big)^2 + \mu_k^\#\right] \mathrm{Id} \tag{3.2}$$

where $Q^\#\restriction_{E_k} = \mu_k^\# \,\mathrm{Id}$. Hence,

$$d\mu_k = d_k \delta(\mu_k^\#) = (2k+1)\delta(\mu_k^\#), \tag{3.3}$$

at least for k sufficiently large. If we let $\rho(x) = x$ in (Theorem 3.1(a)), we get

$$(3.4)\quad (2k+1)\mu_k^{\#} = \sum_{j=0}^{\infty} c_{2j}\Big(k+\frac{1}{2}\Big)^{1-2j} + \sum_{i=1}^{N}\sum_{p=1}^{m_i-1}\sum_{j=0}^{\infty} c_{i,p,j}\omega_{i,p}^{k} k^{n_i-1-j}$$

with $c_j = \int x\, d\nu_j$, resp. $c_{ipj} = \int x\, d\nu_{ipj}$.

An immediate consequence of (3.4) is that $\mu_k^{\#} = c_0/2 + O(k^{-1})$. It follows then from part (a) of the Theorem that $d\nu_1 = 2\delta(c_0/2)$, hence that $\sigma_{Q^{\#}}$ (the principal symbol of $Q^{\#}$) is a constant.

Let us now give a preliminary normal form for maximally degenerate Laplacians on S^2 (with minor modifications, the same normal form is valid for such Laplacians on any Zoll manifold).

LEMMA 3.5. — *If Δ_g is maximally degenerate, then $\Delta_g = F(A)$ where F is a symbol (polyhomogeneous function) of order 2. Thus, there are constants c_{2j} such that $\Delta_g \sim \Big(A+\frac{1}{2}\Big)^2 - \frac{1}{4} + \sum_{j=1}^{\infty} c_{2j}\Big(A+\frac{1}{2}\Big)^{-j} \pmod{\Psi^{-\infty}}$, with asymptotics in the sense of Ψ^*.*

Proof. — Since $Q^{\#}\restriction_{E_k} = \mu_k^{\#}$ and $A\restriction_{E_k} = k$, we have by the spectral theorem and by (3.4) that

$$(3.5a)\quad (2A+1)Q^{\#} = \sum_{j=0}^{\infty} c_{2j}\Big(A+\frac{1}{2}\Big)^{1-2j}$$
$$+\sum_{i=1}^{N}\sum_{p=1}^{m_i-1}\sum_{j=0}^{\infty} c_{ipj}A^{n_i-1-j}\exp 2\pi i\Big(\frac{p}{m_i}A\Big) \pmod{\Psi^{-\infty}}.$$

It is clear that $Q^{\#}$ is a function of A (not only mod $\Psi^{-\infty}$).

We now observe that the second term on the right side must vanish. Indeed, $(2A+1)Q^{\#}$ and $\sum_{j=0}^{\infty} c_{2j}(A+1/2)^{1-2j}$ are ψDO's, so the second term on the right side must be a ψDO. But $\exp\Big(2\pi i\frac{p}{m_i}A\Big)$ is never a ψDO if $1 \le p \le m_i - 1$, since its underlying canonical relation is $G^{2\pi\frac{p}{m_i}} \neq \mathrm{id}$. Consequently, (3.5) simplies to:

$$(3.5b)\qquad Q^{\#} \equiv \sum_{k=0}^{\infty} c_{2k}\Big(A+\frac{1}{2}\Big)^{-2k} \pmod{\Psi^{-\infty}}.$$

This proves Lemma 3.5, since $\Delta = (A+1/2)^2 + Q^{\#}$. $\square$

We now relate maximal degeneracy to the symmetries of Δ_g on S^2.

Proof of Theorem C.

(a) By the argument of Weinstein [W], there exists a canonical transformation $\chi : T^*S^2 \to T^*S^2$ conjugating the geodesic flows G_g^t and G_{can}^t, and a unitary Fourier Integral operator U_1 quantizing χ such that

$$U_1^* \Delta_{\text{can}} U_1 = \Delta_g + Q \quad (Q \in \Psi^0). \tag{3.6}$$

We can then use the argument of Guillemin [G2] to construct a unitary pseudodifferential operator e^B $(B \in \Psi^0)$ such that $U = U_1 e^B$ satisfies:

$$U^* \Delta_{\text{can}} U = \Delta_g + Q^{\#}, \quad [Q^{\#}, \Delta] = 0. \tag{3.7}$$

Indeed, Guillemin's argument only used the maximal degeneracy of Δ_{can}.

We now claim that

$$U^* A_{\text{can}} U \equiv A_g \quad \text{(modulo finite rank operators)} \tag{3.8}$$

where $A_{\text{can}} = \sqrt{\Delta_{\text{can}} + 1/4} - 1/2$. Indeed, it is obvious from (3.7) that

$$U^* A_{\text{can}} U = A_g \quad (\text{mod}\, \Psi^{-1}).$$

Hence $U^* A_{\text{can}} U$ satisfies:

(i) $\text{Spec}(U^* A_{\text{can}} U) \subseteq \mathbb{N}$

(ii) $U^* A_{\text{can}} U \in \Psi^1$

(iii) $[U^* A_{\text{can}} U, \Delta_g] = 0$

(iv) $A_g = U^* A_{\text{can}} U \quad (\text{mod}\, \Psi^{-1})$.

We now observe that (i)–(iv) imply $A_g = U^* A_{\text{can}} U$: By (iii), $U^* A_{\text{can}} U : E_k \to E_k$. Then by (i), (ii) and (iv) $U^* A_{\text{can}} U \restriction_{E_k} = A \restriction_{E_k} = k\, \text{Id}_{E_k}$ for sufficiently large k, which implies $U^* A_{\text{can}} U = A_g$, at least off a finite-dimensional subspace; since we are free to adjust the definition of A_g on an initial finite dimensional subspace, we may assume the formula holds on all of L^2.

The same unitary Fourier Integral operator U conjugates the isometric action $\pi_{\text{can}} : SO(3) \to UF(S^2, \text{can})$ to an effective action $\pi_g = U^* \Pi_{\text{can}} U$ of $SO(3)$ on $L^2(S^2, g)$. Since U conjugates A_{can} to A_g, π_g commutes with A_g. □

(b) Let $b \in C^\infty(S^*M)$ be invariant under G^t. Using any quantization Op from symbols to operators, we set

$$B := \text{Op}^{\text{ave}}(b) := \frac{1}{2\pi} \int_0^{2\pi} U_t^* \,\text{Op}(b) U_t dt$$

where $U_t = \text{exp}it\, A_g$. In the well-known way, $[A_g, \text{Op}^{\text{ave}}(b)] = 0$. □

(c) Let γ be any closed geodesic of (S^2, g), and let $\chi(\gamma)$ be the conjugate closed geodesic of (S^2, can). There is a natural Toeplitz structure Π^0_γ associated to the cone thru $\chi(\gamma)$: indeed, the circle of rotations which preserve $\chi(\gamma)$ defines a Cartan subgroup of $S0(3)$ and hence in each eigenspace of Δ_{can} there is a unique (up to scalars) vector of highest weight for this circle action. We let Π^0_γ denote the orthogonal projection onto the span of these highest weight vectors; it is well-known to be define a Toeplitz structure on the cone thru $\chi(\gamma)$ and it evidently commutes with A_{can}. Hence its conjugate under U will define a Toeplitz structure for the cone thru γ which commutes with A_g. □

(d) For the calculation of the residue we set $P = A_g + 1/2$. By Lemma 2.1(ii), with $\ell = 2\pi$ and with $\alpha = 2$ we have

$$\text{res}\, \Pi_\gamma \sqrt{\Delta_g} = \text{res}\, \Pi_\gamma \left(A_g + \frac{1}{2}\right) + \text{res}\, \Pi_\gamma Q_{-1}.$$

Since we know the spectrum of $\Pi_\gamma(A_g + 1/2)$ we can explicitly calculate that $\text{res}\, \Pi_\gamma(A_g + 1/2) = 0$. Indeed, the zeta function $Tr\Pi_\gamma(A_g + 1/2)^{-s+1} = \sum_{k=0}^{\infty} (k + 1/2)^{-s+1}$ is a Hurwitz zeta function and has only a simple pole at $s = 2$. Hence there is no pole at $s = 0$. (See the proof of (f) for more details). On the other hand, since Q_{-1} has order -1,

$$\text{res}\, \Pi_\gamma Q_{-1} = \int_\gamma \sigma_{Q_{-1}} ds.$$

This is a special case of the general fact that for a a Toeplitz operator of order $-d$, with d half the dimension of the associated symplectic cone, the residue is given by the symplectic residue of its principal symbol. The claimed formula then follows since σ_Q is constant on γ. □

(e) A maximally degenerate Laplacian is a function of A_g. □

(f) We must improve the normal form given in Lemma 3.5. Note that it was the transmission property of Δ which was responsible for the vanishing of the odd power terms. We will now use that the non-commutative residue of Δ is zero to show that in dimension 2 the negative even power terms also vanish. At the same time, we will carry out the analogous analysis of the Schrödinger operators $-\Delta_0 + V$ on $L^2(S^2)$.

First, let us recall the basic facts about the non-commutative residue, res ([K]). For any ψDO B and any positive, elliptic first order ψDO P, we form the zeta function $\zeta(s,B) := \operatorname{Tr} BP^{-s}(\operatorname{Re} s \gg 0)$. The operator BP^{-s} is of trace class for sufficiently large $\operatorname{Re} s$, and $\zeta(s,B)$ has a meromorphic continuation to $\mathbb{C}$, with simple poles only among the points $s_j = \frac{(j-n)}{2m}$. Here, m is the order of B and $n = \dim M$. We refer to [Sh], Theorem 12.1 for the relevant background.

By definition, $\operatorname{res}(B) := \operatorname{Res}_{s=0} \zeta(S,B)$. This residue is independent of the choice of P and can be expressed in terms of the $-n$-th term b_{-n} in any complete symbol expansion for B by

$$\operatorname{res}(B) = (2\pi)^{-n} \int_{S^*M} b_{-n} \frac{dx\, d\xi}{d|\xi|}$$

(see [K]). In particular, $\operatorname{res} B = 0$ if B is a partial differential operator.

We now let $P = (A + 1/2)$ and $B = \Delta^m$ to get:

$$0 = \operatorname{res}(\Delta^m) = \operatorname{Res}_{s=0} \operatorname{Tr} \Delta^m \left(A + \frac{1}{2}\right)^{-s}. \tag{3.9}$$

On the other hand, we can express $\operatorname{Tr} \Delta^m (A + 1/2)^{-s}$ in terms of the Hurwitz zeta function, $\zeta(s,a) = \sum_{k=0}^{\infty} (k+a)^{-s}$ [I]. Indeed, we note first that

$$\begin{aligned}
\zeta\left(s, \left(A + \frac{1}{2}\right)^{\mu}\right) &= \operatorname{Tr}\left(A + \frac{1}{2}\right)^{-s+\mu} \\
&= 2\sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right)^{-s+\mu+1} \\
&= 2\zeta\left(s - \mu - 1, \frac{1}{2}\right)
\end{aligned} \tag{3.10}$$

under the usual multiplicity assumption. Using the polyhomogeneous expansion for Δ, we can express $\operatorname{Tr} \Delta^m (A + 1/2)^{-s}$ as a linear combination of Hurwitz zeta functions.

Before doing so, let us recall that $\zeta(s,a)$ has a meromorphic continuation to $\mathbb{C}$, with only a simple pole at $s = 1$ and with residue 1 there. Hence,

$$\operatorname{res}\left(A + \frac{1}{2}\right)^{\mu} = \begin{cases} 2 & \mu = -2 \\ 0 & \mu \neq -2. \end{cases} \tag{3.11}$$

Since,

$$\begin{aligned}
\operatorname{Tr} \Delta \left(A + \frac{1}{2}\right)^{-s} &= \zeta\left(s, \left(A + \frac{1}{2}\right)^{2}\right) + \sum_{j=0}^{\infty} c_{2j} \zeta\left(s, \left(A + \frac{1}{2}\right)^{-2j}\right) \\
&= \zeta\left(s - 3, \frac{1}{2}\right) + \sum_{j=0}^{\infty} c_{2j} \zeta\left(s + 2j - 1, \frac{1}{2}\right),
\end{aligned}$$

we have: $\operatorname{res}(\Delta) = 2c_2$. Hence, $c_2 = 0$.

In general we have, by induction, that $\Delta^m = \binom{m}{m-1} c_{-2m}(A+1/2)^{-2} + B$, where B is a Laurent series in $(A+1/2)$ with no term of exponent (-2). Hence, $\operatorname{res}\Delta^m = 2\binom{m}{m-1}c_{-2m} = 0$, i.e., $c_{-2m} = 0$ for $m \geq 1$. Therefore, $\Delta \equiv (A+1/2)^2 + c_0 \pmod{\Psi^{-\infty}}$, i.e. $\Delta = (A+1/2)^2 + c_0 + S$, where $S \in \Psi^{-\infty}$. S is obviously a function of Δ. Also, we can determine c_0 by using the heat trace, which in dimension 2 reads,

$$\operatorname{Tr} e^{-t\Delta} = t^{-1} + \frac{1}{3} + a_2 t + \cdots \tag{3.12}$$

(since $\operatorname{area}(S^2, g) = 4\pi$). The smoothing term S only affects terms of order $O(t)$, and $(A+1/2)^2$ is isospectral to $\Delta_0 + 1/4$, so we get:

$$\operatorname{Tr} e^{-t\Delta} = \operatorname{Tr} e^{-t(\Delta_0 + (\frac{1}{4}+c_0))} + O(t). \tag{3.13}$$

Since $\operatorname{Tr} e^{-t\Delta_0} = t^{-1} + 1/3 + O(t)$, we must have $1/4 + c_0 = 0$, completing the proof □

Remark. — The use of the non-commutative residue in the lemma above breaks down somewhat in higher dimension. For instance, in odd dimensions, the transmission property of Δ coincides with the vanishing of $\operatorname{res}(\Delta^m)$. Hence, the argument above eliminates no new terms in the polyhomogeneous expansion. In even dimensions ≥ 4, $\operatorname{res}(\Delta^m)$ is a linear combination of several coefficients c_{2j}, so its vanishing does not immediately suffice to simplify the polyhomogeneous expansion to the extent of Theorem C(f).

We now prove the analogue of Theorem C(f) in the case of Schrödinger operators $-\Delta_{\text{can}} + V$ on $L^2(S^2 \text{ can})$. Thus, we assume the eigenvalues $\lambda_j(V)$ of $-\Delta_{\text{can}} + V$ have the multiplicities, $\operatorname{mult}(\lambda_j(V)) = 2j+1$, and seek a normal form for $-\Delta_{\text{can}} + V$ in the sense above. The analogue of Lemma 2.1 is due to Guillemin [G1].

LEMMA 2.1′ ([G1], Lemma 1). — *Let $-\Delta_{\text{can}} + V$ be a Schrödinger operator on (S^2, can). Then there exists a $Q^\# \in \Psi^0$ such that $[\Delta_{\text{can}}, Q^\#] = 0$ and a unitary ψDO $F \in \Psi^0$ such that $F(-\Delta_0 + V)F^{-1} = -\Delta_{\text{can}} + Q^\#$.*

Next we give a preliminary normal form. We will use the same notations as in the case of Zoll Laplacians: $\{\mu_{k,j}\}$ will denote the eigenvalues of $Q^\# \restriction_{E_k}$, $d\mu_k$ will be the k-th eigenvalue cluster measure, etc. We have:

LEMMA 3.5′. — *Assume* $\operatorname{mult}(\lambda_j(V)) = 2j+1$ $(\forall j)$. *Then there* *exists a unitary ψDO $F \in \Psi^0$ such that*

$$F(-\Delta_{\text{can}} + V)F^{-1} \equiv -\Delta_{\text{can}} + \sum_{j=0}^{\infty} c_{2j}\Big(A + \frac{1}{2}\Big)^{-2j} \quad (\operatorname{mod} \Psi^{-\infty}),$$

where $A\restriction_{E_k} = k$.

Proof. — It suffices to show that $Q^{\#}$ has such a polyhomogeneous expansion. As above, $Q^{\#}\restriction_{E_k} = \mu_k^{\#}$ and $\mu_k^{\#}$ has precisely the expansion (3.4), except that the oscillatory terms automatically vanish since g_0 is a $C_{2\pi}$-metric. The argument then proceeds precisely as in Lemma 3.5. □

We then have

THEOREM C(f)′. — *With the same assumptions as in Lemma 4, $F(-\Delta_{\text{can}} + V)F^{-1} = -\Delta_{\text{can}} + S$, where $S \in \Psi^{-\infty}$.*

Proof. — As above, we use that $\operatorname{res}(-\Delta_{\text{can}} + V)^m = 0 \quad (\forall m \geq 0)$. Since $\operatorname{res}(AB) = \operatorname{res}(BA)$ for $A, B \in \Psi^*$, it follows that $\operatorname{res}((F(-\Delta_{\text{can}} + V)F^{-1})^m) = 0 \quad (\forall m \geq 0)$. These residues are calculated precisely as in the Zoll case, and the conclusion for the c_{2j} follows as above. □

4. Calculation of $\operatorname{res}\Pi_\gamma\sqrt{\Delta} = \sigma_Q$: proof of Theorem D.

(a) We begin with some preliminary remarks and simplifications. First, one will get a formula for σ_Q of the type described in Theorem D for any choice of quantization Op and of coordinate system. The natural choice is to adapt the coordinates and Op to γ: Thus we let $\exp : N_\gamma \to M$ denote the exponential map along the normal bundle, and choose an initial point x_0 on γ and a parallel orthonormal frame $\nu_1, \dots, \nu_n$. We let $p = \exp_{\gamma(s)}(\sum y_i \nu_i)$ define the associated Fermi normal coordinate system, with $\gamma(s)$ a unit speed parametrization of γ starting at $\gamma(0) = x_0$. We then let Op denote the Weyl calculus in these coordinates (see [HoIII] for the definitions and background).

We also define Δ and A so that they operate on 1/2-densities rather than on functions. Using the natural trivialization given by $\sqrt{dv_g}$ (the volume 1/2-density), the local expression for Δ is then

$$(4.1) \qquad \Delta = J^{-1/2}\frac{\partial}{\partial s} g^{00} J \frac{\partial}{\partial s} J^{1/2} + J^{-1/2}\sum_{ij=1}^{n} \frac{\partial}{\partial y_i} g^{ij} J \frac{\partial}{\partial y_j} J^{1/2}$$

where $dv_g = J(s,y)dsdy$. Of course the usual Laplacian is unitarily equivalent to the 1/2-density Laplacian, so the condition of maximal degeneracy is the same for both. The advantage of the 1/2-density Laplacian is that its complete symbol in the Weyl calculus has the form

$$\sigma_{\text{complete}}(\Delta)(x,\xi) = |\xi|^2 + \sigma_0(x) \tag{4.2}$$

where as above $|\xi|^2$ is the length-squared of a covector and where

$$\sigma_0(x) = \Delta 1(x) \tag{4.3}$$

is a function on the base. We note –and emphasize– that $\sigma_0(x)$ depends on the choice of coordinates and of Op. The fact that the first order term vanishes is because the 1/2-density Laplacian has zero subprincipal symbol, and because in the Weyl calculus this is the first order term [HoI–IV], loc.cit.

Now let us calculate the commutator in the Weyl calculus. Let $a^w(x,D)$, resp. $b^w(x,D)$ denote the Weyl pseudodifferential operators on $\mathbb{R}^n$ with complete symbols $a(x,\xi)$ resp. $b(x,\xi)$, and let $*$ denote the composition ($*$-product) on complete symbols arising from operator composition: i.e. $(a*b)^w(x,D) = a^w(x,D)\circ b^w(x,D)$. Then one has

$$a*b - b*a \sim \{a,b\} + P_3(a,b) + \cdots \tag{4.4}$$

where $\{\ ,\ \}$ is the Poisson bracket and where P_n is the n-th order bidifferential operator given by

$$P_n(a,b) = \frac{1}{n!}(i\omega(D_x,D_\xi,D_y,D_\eta)/2)^n a(x,\xi)b(y,\eta)|_{(x,\xi)=(y,\eta)}. \tag{4.5}$$

Here, ω is the standard symplectic form. We note that only the odd P_j's occur in the commutator. See [HoIII], Theorem 18.5.4 for further discussion.

Proof of Theorem D. — We now apply this formalism to outline the calculation of σ_Q on an arbitrary Zoll manifold and complete it in dimension 2.

We first recall that, as in Theorem C(b), given any $b_0 \in C^\infty(S^*M)$ the averaging method produces an operator $B \in \Psi^0(M)$ with the properties: $[B,A]=0, \sigma_B = b_0$. From the expression in Corollary 2.1 (§2), this implies

$$[\Delta - Q, B] = 0 \tag{4.6}$$

where Q is short for $Q^\sharp$. Now denote the complete symbol of B by

$$b \sim \sum_{j=0}^{-\infty} b_{-j}$$

with b_{-j} homogeneous of degree $-j$. Since b_0 is invariant under the geodesic flow, the principal symbol (of order 1) of $[\Delta, B]$ vanishes. Moreover, since only odd P_j's occur in the complete symbol of the commutator and since $\sigma_{\text{complete}}(\Delta)$ has only even terms, the even indices in b decouple from the odd, and we find that the equation of order -1 in the complete symbol of of $[\Delta - Q, B] = 0$ is given by

$$\{\sigma_0, b_0\} - \{\sigma_Q, b_0\} + \frac{1}{i}P_3(|\xi|^2, b_0) = \{|\xi|^2, b_{-2}\}. \tag{4.7}$$

The right hand side vanishes upon integration over any closed geodesic γ. Using the invariance of σ_Q, b and the Poisson bracket, we therefore have

$$\{\sigma_Q, b_0\}(\gamma) = \int_\gamma \frac{1}{i}P_3(|\xi|^2, b_0) + \{R(\sigma_0), b_0\}(\gamma). \tag{4.8}$$

We observe that the Poisson bracket is invariantly defined and descends to the Poisson bracket on the quotient $G(M, g)$; hence the left side defines a first order differential operator applied to $b_0 \in C^\infty(G(M, g)$. On the other hand, the right side is not invariantly defined (both σ_0 and P_3 depend on the choice of operator calculus and local coordinates on M) and involves the third derivatives of b_0. To obtain a more explicit formula, we now evaluate σ_0 and P_3 in coordinates which are adapted to γ and to the symplectic geometry of $G(M, g)$. Regarding the symplectic structure on this quotient, we recall that since $G(M, g)$ is a space of geodesics, a tangent vector $X \in T_\gamma G(M, g)$ can be identified with a normal Jacobi field along γ. The symplectic structure on the vector space $\mathcal{J}_\gamma^\perp$ of such Jacobi fields is then given by the Wronskian $\omega(X, Y) = g(X, Y)' - g(X', Y)$; here X' is the covariant derivative of X along γ.

Let us first evaluate $\frac{1}{i}P_3(|\xi|^2, b_0)$ in Fermi normal coordinates along γ : Since $|\xi|^2$ is a quadratic polynomial in ξ we have, in any coordinates (x_i, ξ_i),

$$\begin{aligned}\frac{1}{i}P_3(g^{mn}\xi_m\xi_n, b_0) = -\frac{1}{2^3 3!}(&\partial^3_{x_i x_j x_k} g^{mn}\xi_m\xi_n \partial^3_{\xi_i\xi_j\xi_k} b_0) \\ &-6\partial^2_{x_i x_j} g^{mn}\xi_m \partial^3_{x_m\xi_i\xi_j} b + 6\partial_{x_i} g^{mn}\partial^3_{x_m x_n \xi_i} b_0.\end{aligned} \tag{4.9}$$

Here we have used the summation convention that repeated indices are to be summed; and of course $\partial_x := \partial/\partial x$. Now consider the Fermi normal coordinates $x_0 = s, x_j = y_j (j = 1, \ldots, n)$, with $p = \exp_{\gamma(s)}\left(\sum_{i=2}^{n} y_i e_i\right)$ with $\gamma(0) = m$ some initial point and $\{e_i\}$ some parallel normal frame along γ. In these coordinates we have

$$g^{0j} = 0 \qquad (j \geq 1)$$
$$(4.10) \qquad g^{00}(s,y) = 1 + K_{ij}^{00}(s)y_iy_j + L_{ijk}^{00}(s)y_iy_jy_k + \cdots$$
$$g^{mn}(s,y) = \delta_{mn} + K_{ij}^{mn}(s)y_jy_k + L_{ijk}^{mn}(s)y_iy_jy_k + \cdots$$

for certain expressions K, L in the curvature and its covariant derivatives [Gr], Ch. 9. Let the symplectic dual coordinates be denoted $\xi_s, \eta_1, \dots, \eta_n$. The equation of γ is given by $y_i = \eta_j = 0, \xi_s = 1$ in these coordinates. The expression for P_3 thus simplifies in Fermi normal coordinates to

$$(4.11) \qquad \frac{1}{i}P_3(g^{mn}, b_0)|_\gamma = -\frac{1}{2^3 3!}[\partial^3 x_i x_j x_k g^{00} \partial^3_{\xi_i\xi_j\xi_k} b_0 \\ -6\partial^2_{x_i x_j} g^{00} \partial^3_{s\xi_i\xi_j} b_0](s,0,1,0).$$

Note that at least two normal ∂_{y_i}-derivatives must occur in each term. We also have

$$(4.12) \qquad \sigma_0(s,0) = -\frac{1}{2}\sum_{i=1}^n \partial^2_{y_i} J(s,y)|_{y=0} = -\frac{1}{6}\sum_{i=1}^n \rho_{ii} + 2R_{0i0i} = -\frac{1}{6}\tau$$

where ρ (resp. R) is the Ricci (resp. Riemann) tensor [Gr], Theorem 9.22.

We now specialize to the case $\dim(M) = 2$. Then

$$g_{11} \equiv g^{11} \equiv 1$$
$$\partial^2_{yy} g^{00} = 2\tau, \qquad \partial^3_{yys} g^{00} = 2\partial_s \tau$$
$$\partial^3_{yyy} g^{00} = 2\partial_y \tau, \qquad \partial^3_{ssy} g^{00} = 0.$$

We then have

(4.13) $(\dim M = 2)$

$$\frac{1}{i}P_3(g^{mn}, b_0)|_\gamma = -\frac{1}{2^3 3!}[6\partial_s\tau\partial^3_{\eta\eta\xi_s} b_0 + 2\partial_y\tau\partial^3_{\eta\eta\eta} b - 12\tau\partial_s\partial^2_{\eta\eta} b_0](s,0,1,0)$$
$$\sigma_0(s,0) = -\frac{1}{6}\tau(s,0).$$

The expressions simplify further if we change coordinates to polar coordinates (ρ, θ) in the cotangent planes $T^*_{(s,0)}M$, with $\xi_s = \rho\cos\theta, \eta = \rho\sin\theta$. Using the ρ−independence of b_0, and evaluating the coefficients at $y = 0, \eta = 0, \xi_s = 1$ we get (along γ)

$$(4.14) \qquad \partial^3_{\xi_s\eta\eta} b_0 = -2\partial^2_\theta b_0 \qquad \partial^3_{\eta\eta\eta} b_0 = -2\partial_\theta + \partial^3_\theta b_0 \qquad \partial^2_{\xi_s\xi_s} b_0 = \partial^2_\theta b_0.$$

Hence we have

(4.15)

$$\frac{1}{i}P_3(|\xi|^2, b_0)|_\gamma = -\frac{1}{2^3 3}[2\partial_y\tau\partial^3_\theta b_0 + (-12\tau\partial_s\partial^2_\theta b_0 - 12\partial_s\tau\partial^2_\theta b_0) - 4\partial_y\tau\partial_\theta b_0]|_\gamma.$$

Integrating over γ in s, we observe that the ∂_θ^2-term cancels. Hence

$$(4.16)\qquad \{\sigma_Q, b_0\}(\gamma) = -\frac{1}{6}\{R(\tau), b_0\}(\gamma) - \frac{1}{2^3 3!}\int_\gamma [2\partial_\theta^3 b_0\ \partial_\nu \tau \\ -4\partial_\theta b_0\ \partial_\nu \tau](s,0,1,0)ds.$$

Recalling that b_0 is invariant under the flow, and regarding it as a function on $G(S^2, g)$, we can write $b_0(s,0,1,\theta) = b_0(\gamma_{(s,\theta)})$ where $\gamma_{(s,\theta)}$ is the geodesic with initial vector $(s,\theta) \in S^*_{\gamma(s)}S^2$. More precisely, consider the maps

(4.17.1)

$$\Phi : S^1 \times S^1 \times S^1 \to S^*S^2,\ \ (t,s,\theta) \to G^t(\gamma(s), \cos\theta\ \gamma'(s) + \sin\theta\ \nu(s)),$$

the natural projection

$$q : S^*S^2 \to G(S^2, g),$$

and the composition

$$(4.17.2)\qquad \phi : S^1 \times S^1 \to G(S^2, g) \quad \phi = q\cdot\Phi\cdot i,$$

with $i : S^1 \times S^1 \to S^1 \times S^1 \times S^1$ the inclusion $(s,\theta) \to (0,s,\theta)$. We have: $\phi(s,\theta) = \gamma_{(s,\theta)}$, $b(\gamma_{(s,\theta)}) = \phi^* b_0$. Note that ϕ is a blow-down map along S^1 at $\theta = 0, \pi$ so that ∂_θ pushes forward to a circle of vectors at γ.

Now let $\overline{\nabla}$ be a connection on $TG(S^2, g)$. We will not specify it further for the moment. Under the map ϕ it pulls back to a connection $\overline{\nabla}^\phi$ on $T(S^1 \times S^1)$ (see [Besse], 1.77), and hence $\overline{\nabla}^\phi_{\partial_\theta}\partial_\theta$ is well-defined as a vector field there. To simplify the notation, we will confuse ∂_θ with the vector field $\Theta := \phi_*\partial_\theta$ along ϕ and regard $\overline{\nabla}_\Theta\Theta$ as a vector field on $G(S^2, g)$. It is of course a well defined vector field except at the point γ. We then write:

$$(4.18)\qquad \partial_\theta b(\gamma_{(s,\theta)}) = db(\Theta)$$

$$\partial_\theta^2 b(\gamma_{(s,\theta)}) = \overline{\nabla}_\Theta(db)(\Theta) + db(\overline{\nabla}_\Theta\Theta)$$

$$\partial_\theta^3 b(\gamma_{(s,\theta)}) = \overline{\nabla}^3 b(\Theta,\Theta,\Theta) + 2\overline{\nabla}^2 b(\overline{\nabla}_\Theta\Theta, \Theta) + db(\overline{\nabla}_\Theta\overline{\nabla}_\Theta\Theta).$$

Let us substitute the expressions in (4.18) back into (4.16). We then introduce a symplectic basis U, V for $\mathcal{J}_\gamma^\perp$ given by the initial conditions (at $\gamma(o) = m$)

$$\begin{pmatrix} U(0) & V(0) \\ \nabla U(0) & \nabla V(0) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

As mentioned above, at $\theta = 0, \Theta$ is a circle of vectors $\Theta_s \in T_\gamma G(S^2, g)$. For fixed s, Θ_s corresponds to the Jacobi field $\Theta_s(t) = \partial_\theta|_{\theta=0}\gamma_{(s,\theta)}$ along γ; it is the element of $\mathcal{J}_\gamma^\perp$ satisfying: $\Theta_s(s) = 0, \nabla\Theta_s(s) = \nu(s)$ where $\nu(s)$ is the unit normal field along γ. Hence we may write

$$\Theta_s(t) = \alpha(s)U(t) + \beta(s)V(t)$$

with $\alpha(s) = \omega(\Theta_s, V), \beta(s) = -\omega(\Theta_s, U)$. In dimension two we may also write $U(t) = u(t)\nu(t), V(t) = v(t)\nu(t)$. We claim:

$$\alpha(s) = -v(s) \quad \beta(s) = u(s) \tag{4.19}$$

so that

$$\Theta_s(t) = (-v(s)u(t) + u(s)v(t))\nu(t). \tag{4.19.1}$$

Indeed, let

$$P(t) := \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix}.$$

The conditions $\Theta_s(s) = 0$, $\nabla\Theta_s(s) = \nu(s)$ give

$$P(s)\begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix}$$

and hence (4.19). We thus have

$$\Theta_s = -v(s)U + u(s)V \tag{4.19.2}$$

as vectors in $T_\gamma G(S^2, g)$.

Since $\overline{\nabla}$ and b are well-defined on $G(S^2, g)$ they are independent of s in (4.16). We claim first that

$$\int_\gamma \partial_\nu \tau \overline{\nabla}^3 b(\Theta_s, \Theta_s, \Theta_s) ds = 0. \tag{4.20}$$

Indeed, using (4.19.2) we see that it contributes nine terms of which the first (and typical) one is

$$\left(\int_0^{2\pi} \partial_\nu \tau v(s)^3 ds\right) \overline{\nabla}^3 b(U, U, U). \tag{4.20.1}$$

We now observe that the integral in this term (and the other eight like it) vanish. Indeed, let $Y = y\nu$ be a Jacobi field along a geodesic γ on a Zoll surface, and let γ_r be a corresponding curve of closed geodesics. For each r, differentiation in r produces a Jacobi field Y_r along γ_r and hence solutions of

$$y''(t; r) + \tau(t; r)y(t; r) = 0.$$

Here the primes indicated t-derivatives. Differentiating in r, we get

$$y_r''(t; 0) + \partial_\nu \tau(t; 0)y^2 + \tau(t; 0)y_r = 0$$

with subscripts denoting r-derivatives. We also use that $\partial_r \tau(t;r) = y\partial_\nu \tau$. Multiplying by y and integrating the t-derivatives by parts we see that the first and third terms vanish. Hence we get

$$\int_\gamma y^3 \partial_\nu \tau dt = 0. \tag{4.21}$$

Setting $y = u + iv$ extends (4.21) to products $u^2 v$ of solutions as well, and proves that (4.20) equals zero.

Although the ∂_θ^2-term in (4.15) cancelled, we record that also

$$\int_\gamma \partial_s \tau \overline{\nabla}^2 b(\Theta_s, \Theta_s) ds = 0. \tag{4.22}$$

Indeed, from the Jacobi equation we get

$$y_s'' + \partial_s \tau y + \tau \partial_s y = 0.$$

Multiplying by y and integrating by parts twice again kills the outer terms, leaving

$$\int_\gamma \partial_s \tau y^2 ds = 0.$$

Then (4.22) follows from (14.9.2) and from the previous argument.

Thus we have: for any connection $\overline{\nabla}$ on $TG(S^2, g)$,

$$\begin{aligned}\left\{\sigma_Q + \frac{1}{6}R(\tau), b_0\right\} = -\frac{1}{2^3 3!}\int_\gamma &\left[2\partial_\nu \tau [2\overline{\nabla}^2 b_0(\overline{\nabla}_{\Theta_s}\Theta_s, \Theta_s)\right. \\ &\left. + db_0(\overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s)] + -4\partial_\nu \tau db_0(\Theta_s)\right] ds.\end{aligned} \tag{4.23}$$

We now observe that for any torsion-free connection $\overline{\nabla}$

$$\int_\gamma \overline{\nabla}^2 b_0(\overline{\nabla}_{\Theta_s}\Theta_s, \Theta_s) ds = 0 \tag{4.24}$$

for any b_0. Indeed, the left side of (4.23) defines a derivation on $C^\infty(G(S^2, g))$. Hence if we set $b_0 = b_1 b_2$, the cross term involving $db_1 \otimes db_2 + db_2 \otimes db_1$ must vanish. Evaluating at γ, $db_1 \otimes db_2 + db_2 \otimes db_1$ can be any symmetric tensor of the form $\xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1$. Since any symmetric tensor is a sum of such terms, and since $\overline{\nabla} db_0$ is symmetric if $\overline{\nabla}$ is torsion-free, we get (4.24).

Therefore (4.23) simplifies to

$$\left\{\sigma_Q + \frac{1}{6}R(\tau), b_0\right\} = -\frac{1}{2^3 3!}\int_\gamma \left[2\partial_\nu \tau db_0(\overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s) - 4\partial_\nu \tau db_0(\Theta_s)\right] ds. \tag{4.25}$$

Let us again write $\Theta_s = -v(s)U + u(s)V$ so that

$$\int_\gamma \partial_\nu \tau \Theta_s ds = \Big(\int_\gamma \partial_\nu \tau(-v(s))ds\Big)U + \Big(\int_\gamma \partial_\nu \tau(u(s))ds\Big)V$$
$$-d_\gamma R(\tau)(V)\cdot U + d_\gamma R(\tau)(U)\cdot V.$$

Here we use that for any Jacobi field $Y = y\nu$ along γ,

$$d_\gamma R(\tau)(Y) = \int_\gamma y(s)\partial_\nu \tau ds.$$

The second term on the right side of (4.25) is then $\frac{1}{12}\{R(\tau), b_0\}$. Moving this term to the left side, we get

$$\Big\{\sigma_Q - \frac{1}{12}R(\tau), b_0\Big\} = -\frac{1}{2^3 3!}db\Big(\int_\gamma \overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s ds\Big). \tag{4.26}$$

□

Remark. — It follows from (4.26) that the vector field

$$V_\gamma := \int_\gamma \overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s ds$$

is a Hamilton vector field on $G(S^2, g)$, and that

$$H_{\sigma_Q} - \frac{1}{12}H_{R(\tau)} = V.$$

Here, H_f denotes the Hamiltonian vector of f with respect to the symplectic form ω. The Hamiltonian f for V is given by

$$df(X) := \omega(H_f, X) = 2\int_\gamma g\Big(\frac{D}{dt}V(t), X(t)\Big)\ dt$$
$$= 2\int_\gamma\int_\gamma g(\frac{D}{dt}(\overline{\nabla}_{\Theta_s}\overline{\nabla}_{\Theta_s}\Theta_s)(t), X(t))\ ds\ dt. \tag{4.27}$$

Examples of $\overline{\nabla}$. — A nice connection $\overline{\nabla}$ on $TG(S^2, g)$ is defined in [Besse], 2.41: namely, the Riemannian connection for the metric $\overline{g}$ on $G(S^2, g)$ given by

$$\overline{g}_\gamma(\overline{X}, \overline{Y}) = \int_\gamma g(X_t, Y_t)dt$$

where X_t, Y_t are the Jacobi fields along γ corresponding to $\overline{X}, \overline{Y}$. The connection is given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} := P^\gamma(\nabla_X Y)$$

where P^γ is the L^2-orthogonal projection onto the space of normal Jacobi fields along γ. Note that if γ_s is a curve in $G(S^2, g)$ with initial tangent vector $\overline{X}$, then both X, Y are vector fields along the map $(s,t) \to \gamma_s(t)$. The expression $\nabla_X Y$ means to hold t fixed and to differentiate in s.

Also, if one uses the natural (Kaluza-Klein) connection on S^*S^2 obtained from g and from the Riemannian connection of g, one obtains very explicit formulae for σ_Q in terms of the Jacobi fields and curvature along γ. The expression is a good deal lengthier than the above and we omit the details.

Proof of (b). — Let us denote by R the operator $\frac{2\pi}{\ell}\left(A+\frac{\alpha}{4}\right)$ ($= A+\frac{1}{2}$ in dimension 2). We easily find that

(4.28.1) $$\sigma_R = |\xi|, \quad \sigma_{\mathrm{sub}}(R) = 0.$$

We then observe that

(4.28.2) $$\sigma_{\mathrm{sub}}(\mathrm{Op}(b)^{\mathrm{ave}}) = \frac{1}{2\pi}\int_0^{2\pi} G^t\ {}^*\sigma_{\mathrm{sub}}(\mathrm{Op}(b))dt.$$

Indeed, we can replace U_t in the definition of $\mathrm{Op}(b)^{\mathrm{ave}}$ with $\exp it\,R$. Following [G3], we then set

$$B_t = \int_0^t e^{-\,\mathrm{it}\,R} B e^{\mathrm{it}\,R} dt$$

with $B = \mathrm{Op}(b)$. We note that

$$B_t'' = [R, B_t']$$

where the primes denote t-derivatives. It follows that

$$\sigma_{\mathrm{sub}}(B_t'') = \{|\xi|, \sigma_{\mathrm{sub}}(B_t')\}$$

hence

$$\sigma_{\mathrm{sub}}(B_t') = G^t\ {}^*\sigma_{\mathrm{sub}}(B).$$

Integrating we get (4.28.2).

Since the complete Weyl symbol b of $\mathrm{Op}(b)$ is homogeneous of degree zero, we see that

$$\sigma_{\mathrm{sub}}(\mathrm{Op}(b)^{\mathrm{ave}}) = 0.$$

Hence the complete Weyl symbol of $\mathrm{Op}(b)^{\mathrm{ave}}$ has the form

(4.28.3) $$b + b_{-2} + \cdots$$

Writing out the equation $[\Delta, Q] = 0$ symbolically we also find that $\{q_{-1}, |\xi|\} = 0$, i.e. that $q_{-1} := \sigma_{\mathrm{sub}}(Q)$ is invariant under the geodesic flow. Hence it may be viewed as a function on $G(S^2, g)$.

We now return to (4.6). Using (4.28, 1-2) we find that the order (-2) part of the equation reads

$$\{q_{-1}, b_0\} = 0 \tag{4.29}$$

for all $b_0 \in C^\infty(G(S^2, g))$. It follows that $\sigma_{\rm sub}(Q)$ is a constant. Since Q has the transmission property, the term q_{-1} is an odd function on S^*S^2 hence odd with respect to the involution $\gamma \to \gamma^{-1}$ on $G(S^2, g)$. It follows that the $\sigma_{\rm sub}(Q) = 0$.

The above applies to any Zoll metric on any manifold. If we now assume that $\sigma_Q = 0$ we see that Q is of order -2.

Now suppose that the equation in the statement of Theorem D(b) is true. Then σ_Q is constant, so $\Delta = [(A + 1/2)^2 + C] \bmod \Psi^{-2}$. (The constant C could of course be determined). Hence the clusters have widths $O(k^{-2})$. Conversely, if the widths are of order $O(k^{-2})$ then $\sigma_Q = C$ for some constant C, and by (a) the equation in (b) will hold. □

Remarks. — As mentioned in the introduction, the condition for the widths to be $O(k^{-3})$ could be determined by carrying out the same sort of calculations one step further. We only wish to observe here that the stronger results of [G1], [G2] in the case of Schrodinger operators on CROSSes have no easy analogues in the Zoll case. The difference is that Schrodinger operators are perturbations of known operators (Laplacians) by known operators (potentials), whereas in the Zoll the Laplacian is a perturbation of the unknown operator R^2 by the unknown operator $Q^\#$.

5. Schrödinger operators and surfaces of revolution: proof of Theorem E.

Proof of Theorem E(a). — If $-\Delta_{\rm can} + V$ has the multiplicities $\mathrm{mult}(\lambda_j(V)) = 2j+1$, then the eigenvalues must be of the form $j(j+1)+\mu_j^\#$, where $\mu_j^\# = O(j^{-N})$ $(\forall N > 0)$. However, in the case of Schrödinger operators, the symbol and sub-principal symbol of $Q^\#$ are calculable, and one gets a contradiction if even $\mu_j^\# = o(j^{-2})$. Indeed, $\sigma_{Q^\#} = \hat{V}$ (the Radon transform of σ_Q), so $\hat{V} = 0$ if $\mu_j^\# = O(j^{-2})$. This implies V is odd. By studying the "first return operator" $W = \exp(2\pi i(-\Delta_{\rm can} + V)^{1/2}) - \mathrm{Id}$, Guillemin [G2] and Widom [Widon] have shown that the clusters c_k have widths $o(k^{-2})$ if and only if $V = 0$. This immediately implies the theorem. □

Proof of Theorem E(b). — By a Zoll surface of revolution we mean a metric having S^1 as an effective isometry group. Such an action has precisely two fixed points N and S. There is a natural parametrization of $S^2 - \{N, S\}$ in which

$$g = dr^2 + c^2(r)d\omega^2.$$

One has:

$$c^{(2p)}(0) = c^{(2p)}(1) = 0\,, c'(0) = -c(1) = 1 \quad (\forall p \geq 0) \tag{5.1}$$

(see [Besse], Ch. IV). Here, we have normalized things so that the distance from N to S is 1.

Each $\phi \in C^\infty(S^2)$ can then be expanded as

$$\phi(r, \omega) = \sum_{n=-\infty}^{\infty} \phi_n(r) e^{in\,\omega}$$

and one has

$$\Delta\phi = \sum_n (\Delta_n \phi_n) e^{in\,\omega}$$

where

$$\Delta_n \phi_n = \left(-\frac{d^2}{dr^2} - \frac{\theta'}{\theta}\frac{d}{dr} + n^2\theta^{-2} \right) \phi_n$$

where $d\,\mathrm{vol} = \theta(r, \omega)dr\; d\omega = c(r)dr\; d\omega$. Δ_n is unitarily equivalent to

$$D_n = -\frac{d^2}{dr^2} + [\beta(r) + n^2\theta^{-2}] \tag{5.2}$$

where

$$\beta(r) = \frac{2c(r)c''(r) - c'(r)^2}{4c(r)^2}. \tag{5.3}$$

More precisely, Δ_n is unitarily equivalent to the Friedrichs extension of $D_n \restriction_{C_0^\infty[0,1]}$.

Following [Bru], p. 174, we write D_n in the form

$$D_n = -\frac{d^2}{dr^2} + \frac{a_n(r)}{r^2(1-r)^2}.$$

By (5.1) we have

$$\frac{a_n(r)}{r^2(1-r)^2} = \frac{2c(r)c''(r) - c'(r)^2 + 4n^2}{4c(r)^2} = \frac{-1 + 4n^2}{4(r-i)^2} + \frac{b_i(i-r)}{i-r} \tag{5.4}$$

as $r \to i$, where $i = 0, 1$ and where b_i is smooth and odd:

$$b_i^{(2k)}(0) = 0 \quad (k \geq 0). \tag{5.5}$$

Now, let us calculate the trace of the heat kernel generated by D_n in two different ways. The first is to use our knowledge of the spectrum, $\mathrm{spec}(D_n)$. We have

$$\mathrm{Spec}(D_n) = \{\ell(\ell+1) + \mu_\ell^\# : \ell \geq |n|\}. \tag{5.6a}$$

Hence the heat trace, $\Theta_n(t) = \mathrm{Tr}\, e^{-tD_n}$, is given by

$$\Theta_{|n|}(t) = \sum_{\ell \geq |n|} e^{-t\ell(\ell+1)} e^{-t\mu_\ell^\#} = \Theta_{\mathrm{can}}(t) + r_n(t), \tag{5.6b}$$

where

$$\Theta_{\mathrm{can}}(t) = \sum_{\ell=0}^{\infty} e^{-t\ell(\ell+1)},$$

and where $r_n(t) \in C^\infty(\mathbb{R})$. Here, we use that $\mu_\ell^\# = O(\ell^{-N})$ for all N.

On the other hand, the asymptotics of $\Theta_{|n|}(t)$ as $t \to O^+$ has been calculated by Brüning [Bru]. Specializing his results to our situation, we have ([Bru], Main Theorem):

$$\Theta_{|n|}(t) \sim (4\pi t)^{-1/2} \sum_{j \geq 0} t^{j/2}(A_j + B_j) \tag{5.7}$$

where $A_0 = 1$, $A_{2j+1} = 0 = B_{2j+1}$.

Let us pause to discuss (5.7), and especially its relation to Brüning's calculations.

In general, the heat trace $\mathrm{tr}\, e^{-tL}$ of a self-adjoint singular Sturm-Liouville operator on $L^2[0,1]$ has two types of additional terms: (i) $B_j t^{j/2}$ terms, where B_j is a universal polynomial in the boundary values $a_n^{(k)}(0)$, $a_n^{(k)}(1)$ of $a_n(r)$; and (ii) $c_j(\log t)t^{j/2}$ terms. As Brüning has pointed out [Bru], p. 174, no $\log t$ terms occur in the case where L arises from separating variables in S^1-symmetric Laplacians. Moreover, we can also check that no $B_j t^{j/2}$ terms arise either if j is odd. This can be dug out of the proof of the Main Theorem [Bru], pp. 194–195. First, we observe that the coefficient B_j in the statement of the Main Theorem is the coefficient A_j^1 in [Bru], (6.4); see p. 195. This coefficient has the form:

$$A_j^1 = \sum_{\substack{\ell \geq 1,\ m \geq 0 \\ \ell+m+1=j,\ |\beta|=m}} b^{(\beta_1)}(0) \cdots b^{(\beta_{\ell-1})}(0) C_{\ell,m,\beta}. \tag{5.8}$$

More precisely, we get one such term from each endpoint i; and $b = b_i$ from (5.4).

We now observe that (at either endpoint) $A_j^1 = 0$ if j is odd. Indeed, by (5.7), only terms where all β_i are odd contribute to A_j^1. Hence $m = |\beta| =$

is a sum of $\ell - 1$ odd numbers, and thus has the parity of $\ell - 1$. Then $j = \ell + m + 1$ must be even, as claimed.

It follows therefore that all powers of t occuring in $\Theta_n(t)$ are half-integral and never integral. But the remainder term $r_n(t)$ in (5.6) is smooth, and hence has only integral powers of t. We conclude:

$$r_n(t) \sim O(t^N) \text{ as } t \to 0 \quad (\forall N)(\forall n). \tag{5.9}$$

On the other hand,

$$\begin{aligned} r_n(t) &= \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \sum_{\ell \geq |n|} e^{-t(\ell(\ell+1))} (\mu_\ell^\#)^k \\ &= -t \sum_{\ell \geq |n|} e^{-t\ell(\ell+1)} \mu_\ell^\# + O(t^2) \\ &= -t \sum_{\ell \geq |n|} \mu_\ell^\# + O(t^2) \end{aligned}$$

where we use repeatedly the rapid decay of the $\mu_\ell^\#$'s.

We therefore have,

$$\sum_{\ell \geq |n|} \mu_\ell^\# = 0 \text{ for all } |n|, \tag{5.10}$$

i.e.

$$\mu_\ell^\# = 0 \quad (\forall \ell). \tag{5.10.1}$$

Hence, Δ_g is isospectral to Δ_{can}. But it is well-known (and easy to see) that Δ_{can} is spectrally determined. Let us recall the simple proof [BGM], p. 227, since it will be relevant to the general case. First, the heat trace $\Theta(t, g) = e^{-t\Delta_g}$ has the following small t expansion on any surface (M, g):

$$\Theta(t, g) = (4\pi t)^{-1}(a_0 + a_1 t + a_2 t^2 + \cdots) \tag{0.1}$$

where

$$\begin{aligned} a_0 &= \text{area}(M, g) \\ a_1 &= \frac{1}{3} \int_M \tau_g d\nu_g \\ a_2 &= \frac{1}{60} \int_M \tau_g^2 d\nu_g. \end{aligned} \tag{5.11}$$

Here, τ_g is the scalar curvature and $d\nu_g$ is the volume form. On a sphere of area 4π, we have

$$\Theta(t, g) = t^{-1}\Big(1 + \frac{1}{3}t + a_2 t^2 + \cdots\Big).$$

By the Schwartz inequality

$$\Big(\int_M \tau_g d\nu_g\Big)^2 \leq \operatorname{area}(M,g)\int_M \tau_g^2 d\nu_g$$

with equality if and only if τ_g is constant. Hence, if (S^2, g) is isospectral to (S^2, can), τ_g must be constant. $\square$

Let us carry this reasoning a little further, in view of its potential applications to the general case.

PROPOSITION 5.12. — *Suppose g is a metric on S^2, such that Δ_g is isospectral to $\Delta_{\mathrm{can}} + S$, where S is a smoothing operator which commutes with Δ_{can}. Then: $\Theta(t,g) = \Theta(t,g_0) - t\operatorname{Tr} S + O(t^2)$. Hence g is isometric to* can *if and only if* $\operatorname{Tr} S = 0$.

Proof. — We have

$$\begin{aligned}
\Theta(t,g) &= \operatorname{Tr} e^{-t(\Delta_{\mathrm{can}}+S)} \\
&= \operatorname{Tr} e^{-t\Delta_{\mathrm{can}}} e^{-tS} \\
&= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \operatorname{Tr} S^k e^{-t\Delta_{\mathrm{can}}} \\
&= \Theta(t,g_0) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k} \operatorname{Tr} S^k e^{-t\Delta_{\mathrm{can}}}.
\end{aligned}$$

Since

$$\operatorname{Tr} S^k e^{-t\Delta_{\mathrm{can}}} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \operatorname{Tr} S^k \Delta_{\mathrm{can}}^n$$

(which converges since $S^k \in \Psi^{-\infty}$), we see that

$$\Theta(t,g) = \Theta(t,g_0) - t\operatorname{Tr} S + t^2 \operatorname{Tr}\Big(S\Delta_{\mathrm{can}} + \frac{1}{2}S^2\Big) + \cdots.$$

$\square$

Now, we have

$$\operatorname{Tr} S = \sum_{\ell=0}^{\infty} (2\ell+1)\mu_\ell^{\#}. \tag{5.13}$$

In the case of a metric of revolution, the equivariant traces of S are given by

$$\operatorname{Tr} \Pi_n S \Pi_n = \sum_{\ell \geq |n|} \mu_\ell^{\#} \tag{5.14}$$

where Π_n is the orthogonal projection onto the eigenspace of $\left(\frac{1}{i}\frac{\partial}{\partial\theta}\right)^2$ of eigenvalue n^2. Hence $\operatorname{Tr} S = 0$ if $\operatorname{Tr} \Pi_n S \Pi_n = 0$ for all n; and we proved the latter above.

6. Maximally degenerate Laplacians on $\mathbb{RP}^d$: proof of Theorem F.

Let us first recall the facts of life on $\mathbb{RP}^d$. The eigenvalues, multiplicities, lengths and Morse indices of primitive closed geodesics of the canonical metric are given by:

(6.1)

$\lambda_k(\text{can})$	$m_k(\text{can})$	$\ell(\text{can})$	$\alpha(\text{can})$	$\frac{2\pi}{\ell}\left(k+\frac{\alpha}{4}\right)$
$2k(d+2k-1)$	$\frac{(d+2k-2)!}{(2k)!(d-1)!}(d+4k-1)$	π	$d-1$	$2\left(k+\frac{d-1}{4}\right)$

In this table, can is normalized so that $\mathbb{RP}^d = S^d/\pm 1$, where $S^d \subset \mathbb{R}^{d+1}$ is the unit sphere. The formulae in this table can be found in [BGM], C.II.1 and [Besse], 8.8.

Further, we will need to use the

WIEDERSEHENRAUM THEOREM 6.2 (Green-Berger-Kazdan). — *The only SC_ℓ-metrics on $\mathbb{RP}^d$ are the (constant) multiples of* can.

We can now give the

Proof of Theorem F.

(a) Suppose that g is a metric on $\mathbb{RP}^d$ with $m_k(g) = m_k(\text{can})$. Then g is a P_ℓ-metric. In view of the Wiedersehenraum Theorem, it will suffice to show that g is a SC_ℓ-metric.

As in §2, we will begin by comparing eigenvalue multiplicities m_k and cluster multiplicities d_k.

In dimension d, one has [CV], Theorem 1.43:

$$d_k = R\left(k+\frac{\alpha}{4}\right) + \sum_{j=1}^{N}\sum_{p=1}^{m_j-1} (\omega_{j,p})^k R_{j,p}(k) \tag{6.3}$$

where $\omega_{j,p} = \exp\left(2\pi i \frac{p}{m_j}\right)$, and where

$$R(t) = b_1 t^{d-1} + b_3 t^{d-3} + \cdots + b_d \tag{6.4}$$

is a polynomial of degree $d-1$ with the parity of $d-1$. (We refer to [CV] for details.)

From Weyl's law, we also have

$$b_1(g) = (2\pi)^{-d} \operatorname{vol}(S^{d-1} \operatorname{can}) \operatorname{vol}(\mathbb{RP}^d, g)\left(\frac{2\pi}{\ell(g)}\right)^d. \tag{6.5}$$

See, for instance, [CV], p. 517; keep in mind that g is normalized there so that $\ell = 2\pi$. It follows that $b_1(g) = b_1(\text{can})$ if $(\ell(g), \operatorname{vol}(\mathbb{RP}^d, g)) = (\ell(\text{can}), \operatorname{vol}(\mathbb{RP}^d, \text{can}))$. Hence, $b_1(g) = b_1(\text{can})$ under the assumptions of Theorem F.

Now let us consider the relation between $m_k(g)$ and $d_k(g)$ if $m_k(g) = m_k(\text{can})$. By an argument similar to that in Lemma 2.2, we can see that the number d_k^* of distinct eigenvalues in C_k eventually becomes a constant, day d_+^*. We have

$$\begin{aligned} d_k &= \sum_{r=1}^{d_k^*} \operatorname{mult}(\lambda_{k,r}) \\ &= \sum_{r=1}^{d_k^*} \frac{(2n(k,r)+d-2)!}{(2n(k,r))!(d-1)!}(4n(k,r)+d-1) \\ &= \sum_{r=1}^{d_k^*} \frac{(2n(k,r))^{d-2}}{(d-1)!} 4n(k,r)\left[1 + O\left(\frac{1}{n(k,r)}\right)\right] \end{aligned}$$

where (as in §2) $C_k = \{\lambda_{k,r}\}$ and $\lambda_{k,r}$ is the $n(k,r)^{tn}$ eigenvalue in increasing order. The leading order behaviour of d_k can be determined from this, as in §2:

$$\begin{aligned} d_k &\sim \frac{4 \cdot 2^{d-2}}{(d-1)!} \sum_{r=1}^{d_k^*} n(k,r)^{d-1} \\ &\sim \frac{4 \cdot 2^{d-2}}{(d-1)!} d_k^* n(k,r)^{d-1} \\ &\sim \frac{4 \cdot 2^{d-2}}{(d-1)!} d_k^* \Big(\sum_{j \le k-1} d_j^*\Big)^{d-1}. \end{aligned} \tag{6.6}$$

Since $d_j^* = d_+^*$ for sufficiently large j, we have

$$d_k \sim \frac{4 \cdot 2^{d-2}}{(d-1)!}(d_+^*)^2 k^{d-1} + O(k^{d-2}). \tag{6.7}$$

In the above, $\sim$ denotes equality modulo terms of order $O(k^{d-2})$.

Comparing (6.7) and (6.5) as in §2, we see that

$$(6.8)\quad b_1(g) = \frac{4 \cdot 2^{d-2}}{(d-1)!}(d_+^*)^2 = (2\pi)^{-d} \operatorname{vol}(S^{d-1}, \operatorname{can}) \operatorname{vol}(\mathbb{RP}^d, g)\Big(\frac{2\pi}{\ell(g)}\Big)^d.$$

Since $(d-1)! \operatorname{vol}(S^{d-1}, \operatorname{can}) = \dfrac{2 \cdot (2\pi)^d}{\operatorname{vol}(s^d, \operatorname{can})}$, we get

$$(6.9)\qquad (d_+^*)^2 = 2^{-(d-1)} \frac{\operatorname{vol}(\mathbb{RP}^d, g)}{\operatorname{vol}(S^d, \operatorname{can})}\Big(\frac{2\pi}{\ell(g)}\Big)^d.$$

With no loss of generality, we may rescale g so that $\operatorname{vol}(\mathbb{RP}^d, g) = \operatorname{vol}(\mathbb{RP}^d, \operatorname{can}) = 1/2 \operatorname{vol}(S^d, \operatorname{can})$. Then we have the unconditional formula,

$$(6.10)\qquad (d_+^*)^2 = \Big(\frac{\pi}{\ell(g)}\Big)^d.$$

In the case of a general P_ℓ-metric g on a manifold M, Weinstein has proved the integrality theorem

$$(6.11)\qquad m^{d-1} \frac{\operatorname{vol}(M, g)}{\operatorname{vol}(S^d)}\Big(\frac{2\pi}{\ell(g)}\Big)^{d-1} \in \mathbb{N}$$

where m is a certain integer (the least common multiple of the orders of the isotropy groups of the exceptional orbits, see [Besse], p. 61). In the case at hand, this implies

$$(6.12)\qquad \frac{1}{2} m^{d-1}\Big(\frac{2\pi}{\ell(g)}\Big)^{d-1} \in \mathbb{N}.$$

Hence by (6.10)–(6.12) there exits $M \in \mathbb{N}$ so that

$$(6.13)\qquad \frac{\pi}{\ell(g)} = \Big(\frac{\ell(g)}{\pi}\Big)^{d-1} (d_+^*)^2 = \frac{(2m)^{d-1}}{2M} (d_+^*)^2,$$

hence $\dfrac{\pi}{\ell(g)}$ is rational. But by (6.10), $\dfrac{\pi}{\ell(g)} = (d_+^*)^{2/d}$. It follows that $(d_+^*)^{2/d}$ is rational, hence integral. Thus, $\ell(g) = \dfrac{1}{N_d}\pi$ where $N_d \in \mathbb{N}$. It seems almost obvious that N_d should equal one, since $\operatorname{vol}(\mathbb{RP}^d, g) = \operatorname{vol}(\mathbb{RP}^d, \operatorname{can})$. Below we will prove that $N_d = 1$ when $d = 2$. Unfortunately, we do not see how to prove $N_d = 1$ for $d > 2$, so we have added this assumption to the hypotheses of Theorem F(b). We conjecture that it follows from the other hypotheses by some modification of Weinstein's integrality theorem.

Let us prove $N_2 = 1$. Just as in the proof of Corollary 2.2, we first observe that the multiplicity assumption forces d_k to be a polynomial. Hence, the oscillatory term in (6.3) must vanish. But it is a linear combination of principal symbols, so it vanishes only if complete cancellation takes place between the terms coming from each period of G_g^t. In particular, there must be at least two exceptional closed geodesics with the minimal period. Now the flow G_g^t lifts under the composition of covers, $S^3 \to S^*(S^2) \to S^*(\mathbb{RP}^2)$, to an S^1 action on S^3, which has at most two exceptionally short orbits, and (as mentioned above) these two orbits would have to have unequal primitive periods. Since all orbits are cut in half under the projection to $S^*(S^2)$, the same description applies there. Thus, g lifts under the natural projection $S^2 \to S^2/\sigma$ to a metric $\tilde{g}$ whose geodesic flow $\widetilde{G}^t$ has least common period 2π and (possibly) two exceptional geodesics say α and β, of lengths $< 2\pi$. Here, σ is the antipodal map. Since $\sigma(\alpha) \neq \beta$ and $\sigma(\alpha)$ cannot be a third exceptional geodesic, we must have $\sigma(\alpha) = \alpha$. Similarly $\sigma(\beta) = \beta$. hence, both α and β get cut in half under the projection $S^2 \to \mathbb{RP}^2$. Since the unexceptional geodesics must also get cut in half, we see that $\mathbb{RP}^2$ has at most two exceptional geodesics of unequal primitive periods. If follows as in §2 that the contributions to d_k from the exceptional geodesics cannot cancel unless they are absent. We conclude that they are absent, so that g is C_π. Let $\tilde{g}$ be its lift as a metric on S^2. Then for each $x \in S^2$, all the geodesics starting from x meet again at time π. To apply Green's Wiedersehenraum theorem, we also need that the distance to the first conjugate locus is the constant π at each x. But we have $\alpha_g = \alpha_{\text{can}} = 1$, so the rendezvous point at time π is the first conjugate point along each geodesic. Hence $g = \text{can}$.□

(b) Let us now prove $g = \text{can}$ when $d > 3$, under the additional assumption $\ell(g) = \pi$. To this end, we first observe that the oscillatory term in (5.3) must vanish since $d_k - R(k + \alpha/4)$ is a polynomial. Hence, $d_k = R(k + \alpha/4) = b_1(k + \alpha/4)^{d-1} + O((d + d/4)^{d-3})$. Since $b_1(g) = b_1(\text{can})$ if $\ell(g) = \pi$ and since $m_k(g) = d_k(g)$ if $d_+^* = 1$, we see that $m_k(g) = b_1(\text{can})(k + \alpha/4)^{d-1} + O((k + \alpha/4)^{d-2})$. Since $m_k(g) = m_k(\text{can})$ by assumption, we have $\alpha(g) = \alpha(\text{can})$. Again this implies that, for any closed geodesic of length π, there are no conjugate points in the interval $(0, \pi)$ (cf. [Besse], 8.8). Hence, $(S^d, \tilde{g})$ is a Wiedersehenraum in the strict sense of the Berger-Kazdan theorem: that is, the first conjugate time along any geodesic is a constant. Hence $\tilde{g} = \text{can}$, so $g = \text{can}$. □

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