ANNALES DE L'INSTITUT FOURIER

UWE FRANKEN
REINHOLD MEISE

Extension and lacunas of solutions of linear partial differential equations

Annales de l'institut Fourier, tome 46, nº 2 (1996), p. 429-464
<http://www.numdam.org/item?id=AIF_1996__46_2_429_0>

Ann. Inst. Fourier, Grenoble
46, 2 (1996), 429-464

EXTENSION AND LACUNAS OF SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

by U. FRANKEN and R. MEISE

The question whether certain zero-solutions of linear partial differential operators can be extended to larger domains, preserving certain properties, has a long history, beginning with Riemann's theorem on removable singularities of analytic functions. Various types of problems have been discussed in the literature. As typical examples we only mention Kiselman [10], Bony and Schapira [2], Kaneko [10], Liess [12] and Palamodov [21]. It seems that the extension of all C^∞-solutions of a given operator to a larger real domain has not found much attention so far. For convex, open sets it was treated as a subcase in the article of Kiselman [10] and for solutions of systems over convex sets it was investigated by Boiti and Nacinovich [1]. However, the solution of L. Schwartz's problem on the existence of continuous linear right inverses for linear partial differential operators with constant coefficients, given by Meise, Taylor and Vogt [15] indicates that this question is of interest in a different context. They show that $P(D) : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ admits a continuous linear right inverse if and only if for each relatively compact open subset ω of Ω there exists another subset $\omega' \supset \omega$ of Ω with the same properties, such that for each $f \in \mathcal{D}'(\omega')$ satisfying $P(D)f = 0$ there exists $g \in \mathcal{D}'(\Omega)$ satisfying $P(D)g = 0$ and $f|_\omega = g|_\omega$. For convex sets Ω this property is equivalent to a condition of Phragmén-Lindelöf type for plurisubharmonic functions on the zero variety $V(P) = \{z \in \mathbb{C}^n : P(-z) = 0\}$.

Key words: Whitney extension of zero-solutions – Phragmén-Lindelöf conditions for algebraic varieties – Fundamental solutions with lacunas – Continuous linear right inverses for constant coefficient partial differential operators.
Math. classification: 35E05 – 35B60 – 32F05 – 46F05.

Motivated by these results we investigate in the present paper under which conditions an analogue of Whitney's extension theorem holds for the zero-solutions of a given linear partial differential operator $P(D)$ with constant coefficients. To formulate our main result, let K be a compact, convex set in $\mathbb{R}^n$ with non-empty interior, denote by H_K its support functional and let

$$\mathcal{E}(K) := \{f \in C(K) \mid f|_{\overset{\circ}{K}} \in C^\infty(\overset{\circ}{K}) \text{ and } (f|_{\overset{\circ}{K}})^{(\alpha)} \text{ extends continuously to } K \text{ for each } \alpha \in \mathbb{N}_0^n\},$$

$$\mathcal{E}(\mathbb{R}^n, K) := \{f \in C^\infty(\mathbb{R}^n) \mid f|_K \equiv 0\},$$

both spaces being endowed with their natural Fréchet space topology. Further, let $\mathcal{E}_P(K)$ (resp. $\mathcal{E}_P(\mathbb{R}^n)$) denote the space of all zero-solutions of $P(D)$ in $\mathcal{E}(K)$ (resp. $C^\infty(\mathbb{R}^n)$). Then the main results of the present paper are stated in the following theorem.

THEOREM. — *For K and P as above, the following conditions are equivalent:*

(1) *the restriction map $\rho_K : \mathcal{E}_P(\mathbb{R}^n) \to \mathcal{E}_P(K)$, $\rho_K(f) := f|_K$ is surjective*

(2) *the map ρ_K in (1) admits a continuous linear right inverse, i.e. there exists an extension operator $E_K : \mathcal{E}_P(K) \to \mathcal{E}_P(\mathbb{R}^n)$ satisfying $\rho_K \circ E_K = \mathrm{id}_{\mathcal{E}_P(K)}$*

(3) *$P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$ is surjective*

(4) *there exists a continuous linear map $R_K : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$, such that $P(D) \circ R_K = \mathrm{id}_{\mathcal{E}(\mathbb{R}^n,K)}$*

(5) *the algebraic variety $V(P)$ satisfies the following condition* PL(K) *of Phragmén-Lindelöf type: There exist $A > \sup_{x\in K} |x|$, $k > 0$ such that each plurisubharmonic function u on $V(P)$ which satisfies (α) and (β) also satisfies (γ), where*

(α) $u(z) \leqslant H_K(\operatorname{Im} z) + O(\log(2 + |z|))$, $z \in V(P)$

(β) $u(z) \leqslant A|\operatorname{Im} z|$, $z \in V(P)$

(γ) $u(z) \leqslant H_K(\operatorname{Im} z) + k\log(2 + |z|)$, $z \in V(P)$,

and where H_K denotes the support function of K.

Note that the theorem extends a result of de Christoforis [3] who proved that (3) holding for all compact convex sets K with non-empty interior is equivalent to P being hyperbolic with respect to all non-characteristic directions. However, the latter condition is strictly stronger than those given in the theorem.

Note further that the condition $\mathrm{PL}(K)$ implies the condition $\mathrm{PL}(\overset{\circ}{K})$ which was used by Meise, Taylor and Vogt [15], sect.4, to characterize when $P(D) : C^\infty(\overset{\circ}{K}) \to C^\infty(\overset{\circ}{K})$ admits a continuous linear right inverse. For homogeneous polynomials the converse implication holds, too, however, it remains open whether it holds also for non-homogeneous polynomials.

The main steps in the proof of the theorem are the following: First we use Fourier analysis, an idea of proof from Meise and Taylor [13] and a result of Franken [4] improving a theorem of Meise, Taylor and Vogt [16], to characterize when for convex compact sets $K \subset Q \subset \mathbb{R}^n$, $\overset{\circ}{K} \neq 0$, the restriction map $\rho_{Q,K} : \mathcal{E}_P(Q) \to \mathcal{E}_P(K)$ is surjective. One of the characterizing conditions is the Phragmén-Lindelöf condition $\mathrm{PL}(K,Q)$ (see 2.9) which also characterizes the surjectivity of $\rho_{Q,K} : \mathcal{D}'_P(Q) \to \mathcal{D}'_P(K)$, where $\mathcal{D}'_P(L) = \{\mu \in \mathcal{D}'(L) \mid P(D)\mu = 0\}$. From this we obtain that (1), (3) and (5) are equivalent (see 2.11). Then we show that (1) is a local property of ∂K and use this to get "fundamental solutions" (see 3.3) having certain lacunas. Together with a particular Whitney partition of unity in $\mathbb{R}^n \setminus K$ these "fundamental solutions" allow the construction of R_K in (4). By a result of Tidten [24] on the existence of continuous linear extension operators for the functions in $\mathcal{E}(K)$, (2) is an easy consequence of (4).

In [5] the main results of the present paper are used to characterize the homogeneous differential operators $P(D)$ that admit a continuous linear right inverse on $C^\infty(\Omega)$, Ω any bounded, convex, open subset of $\mathbb{R}^n$ in terms of the existence of fundamental solutions for $P(D)$ which have support in closed half spaces.

Acknowledgement: The first named author acknowledges the support of his research at the University of Michigan in Ann Arbor by the "Deutsche Forschungsgemeinschaft".

1. Preliminaries.

In this section we introduce most of the notation that will be used subsequently and we prove some auxiliary results on plurisubharmonic functions and weighted spaces of analytic functions on algebraic varieties.

1.1. *Spaces of C^∞-functions and distributions.* — For an open set $\Omega \subset \mathbb{R}^n$ we denote by $\mathcal{E}(\Omega)$ the space of all C^∞-functions on Ω, endowed with the semi-norms

$$||f||_{L,l} := \sup_{x\in L} \sup_{|\alpha|\leqslant l} |f^{(\alpha)}(x)|, \; L \subset\subset \Omega, \; l \in \mathbb{N}_0^n.$$

For a closed set $A \subset \mathbb{R}^n$ we denote by $\mathcal{E}(A)$ the space of all C^∞-Whitney jets $f := (f^\alpha)_{\alpha\in\mathbb{N}_0^n} \in C(A)^{\mathbb{N}_0^n}$ on A, i.e. f satisfies:

$$||f||_{L,l}^w := \sup_{\substack{x,y\in L \\ x\neq y}} \sup_{|\alpha|\leqslant l} \frac{|(R_x^l f)^\alpha(y)|}{|x-y|^{l+1-|\alpha|}} < \infty,$$

where

$$(R_x^l f)^\alpha(y) := f^\alpha(x) - \sum_{|\beta|\leqslant l-|\alpha|} \frac{1}{\beta!} f^{\alpha+\beta}(x)(y-x)^\beta$$

and $L \subset\subset A$, $l \in \mathbb{N}_0$. We endow $\mathcal{E}(A)$ with the semi-norms

$$||f||_l^L := ||f||_{L,l}^w + ||f||_{L,l}, \; L \subset\subset A, \; l \in \mathbb{N}_0.$$

Moreover for a compact set $A \subset \mathbb{R}^n$ we define

$$\mathcal{D}(A) := \{f \in \mathcal{E}(\mathbb{R}^n) \mid \operatorname{Supp}(f) \subset A\},$$

endowed with the induced subspace topology. For an open set $\Omega \subset \mathbb{R}^n$ we let

$$\mathcal{D}(\Omega) := \{f \in \mathcal{E}(\mathbb{R}^n) \mid \operatorname{Supp}(f) \subset\subset \Omega\} = \underset{L\subset\subset\Omega}{\underrightarrow{\operatorname{ind}}}\ \mathcal{D}(L),$$

endowed with the inductive limit topology. If S is either open or compact then $\mathcal{E}'(S)$ resp. $\mathcal{D}'(S)$ denotes the dual of the space $\mathcal{E}(S)$ resp. $\mathcal{D}(S)$. For a compact set $L \subset S$ we define the space of C^∞-functions resp. distributions on S with lacunas in L by

$$\mathcal{E}(S,L) := \{f \in \mathcal{E}(S) \mid f|_L \equiv 0\}, \quad \mathcal{D}'(S,L) := \{\mu \in \mathcal{D}'(S) \mid \mu|_L \equiv 0\}.$$

1.2. *Remark.* — Let $A \subset \mathbb{R}^n$ be closed. By the extension theorem of Whitney [W] each C^∞-function on A can be extended to a C^∞-function on $\mathbb{R}^n$ (which is real-analytic outside A), i.e. the restriction map R_A is surjective, where

$$R_A : \mathcal{E}(\mathbb{R}^n) \longrightarrow \mathcal{E}(A),\ R_A(f) := (f^{(\alpha)}|_A)_{\alpha \in \mathbb{N}_0^n}.$$

If A is convex and $\overset{\circ}{A} \neq \varnothing$ then the definition of Whitney jets is much easier:

$$\mathcal{E}(A) = \{f \in \mathcal{E}(\overset{\circ}{A}) \mid \text{for each } \alpha \in \mathbb{N}_0^n \text{ there exists } f^\alpha \in C(A) : f^\alpha|_{\overset{\circ}{A}} = f^{(\alpha)}\}.$$

Note that in this case the extension of C^∞-functions can be done by a continuous linear operator. This is a consequence of a general result of Tidten [T], Satz 4.6.

1.3. *Partial differential operators.* — Let $\mathbb{C}[z_1, \dots, z_n]$ denote the ring of all complex polynomials in the variables $z_1, \dots, z_n$. For a polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ of degree m

$$P(z) = \sum_{|\alpha| \leqslant m} a_\alpha z^\alpha,\ z \in \mathbb{C}^n,$$

we define the partial differential operator

$$P(D) := \sum_{|\alpha| \leqslant m} a_\alpha i^{-|\alpha|} \partial^\alpha,$$

where ∂^α denotes the α-th derivative in the distribution sense. $P(D)$ is a continuous linear endomorphism on each of the spaces $\mathcal{E}(S), \mathcal{D}'(S)$, where $S \subset \mathbb{R}^n$ is either open or compact. The corresponding spaces of zero-solutions of $P(D)$ are defined as

$$\mathcal{E}_P(S) := \{f \in \mathcal{E}(S) \mid P(D)f = 0\}, \quad \mathcal{D}'_P(S) := \{\mu \in \mathcal{D}'(S) \mid P(D)\mu = 0\}.$$

A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution for $P(D)$ if it satisfies $P(D)E = \delta_0$, where δ_0 denotes the point evaluation at zero. The principal part P_m of P is defined as $P_m(z) := \sum_{|\alpha|=m} a_\alpha z^\alpha$. A vector $N \in \mathbb{R}^n$ is called characteristic for P if $P_m(N) = 0$. P or $P(D)$ is said to be hyperbolic with respect to $N \in \mathbb{R}^n \setminus \{0\}$, if N is non-characteristic for

P and if $P(D)$ admits a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ which satisfies $\mathrm{Supp}(E) \subset H^+(N)$, where we let

$$H^{\pm}(N) := \{x \in \mathbb{R}^n \mid \pm \langle x, N\rangle \geqslant 0\}$$

and where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^n$.

In the present paper we are going to characterize the linear partial differential operators $P(D)$ and the convex compact sets L in $\mathbb{R}^n, \overset{\circ}{L} \neq \varnothing$, for which all elements of $\mathcal{E}_P(L)$ resp. $\mathcal{D}'_P(L)$ can be extended to elements of $\mathcal{E}_P(\mathbb{R}^n)$ resp. $\mathcal{D}'_P(\mathbb{R}^n)$. To do this we will use Fourier analysis. Therefore, we show in this section that $\mathcal{E}_P(L)$ and $\mathcal{D}'_P(L)$ are isomorphic to certain weighted spaces of holomorphic functions on the zero-variety of P. To prove this we need the following two lemmas.

1.4. LEMMA. — *Let $K \subset \mathbb{R}^n$ be compact and convex with $0 \in \overset{\circ}{K}$. Then there exists a number $C > 0$ so that for each $x \in \partial K$ there exists $A \in GL(n;\mathbb{R})$ satisfying:*

(1) $A([0\,,\,1]^n) \subset K$

(2) $A((1,\dots,1)) = x$

(3) $C^{-1}|z| \leqslant |A^t(z)| \leqslant C|z|$, *for all $z \in \mathbb{C}^n$ and the Euclidean norm $|\cdot|$.*

Proof. — For each $x \in \mathbb{R}^n \setminus \{0\}$ let $\{x/|x|, f_2(x), \dots, f_n(x)\}$ be an orthonormal basis in $\mathbb{R}^n$. Then let $e := \dfrac{1}{n}(1,\dots,1)$ and define for $0 < \delta$ the linear map $A_{x,\delta} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$A_{x,\delta}(z) := \langle z, e\rangle x + \delta\langle z, f_2(e)\rangle f_2(x) + \dots + \delta\langle z, f_n(e)\rangle f_n(x),\ \ z \in \mathbb{C}^n,$$

where $\langle z_1, z_2\rangle := \sum_{j=1}^{n} z_{1j}\overline{z}_{2j},\ z_1, z_2 \in \mathbb{C}^n$. Note that $A_{x,\delta}(\mathbb{R}^n) \subset \mathbb{R}^n$.

By our choices, $A_{x,\delta}$ satisfies (2) for each $\delta > 0$ and each $x \in \partial K$. Using a compactness argument, $0 \in \overset{\circ}{K}$ and $A_{x,\delta}((1,\dots,1)) = x$ it is easy to see that there exists $\delta_1 > 0$ such that $A_{x,\delta}$ satisfies (1) for each $0 < \delta < \delta_1$ and all $x \in \partial K$. To show (3) note that

$$A^t_{x,\delta}(z) = \langle z, x\rangle e + \delta\langle z, f_2(x)\rangle f_2(e) + \dots + \delta\langle z, f_n(x)\rangle f_n(e),\ \ z \in \mathbb{C}^n.$$

Since $\{e/|e|, f_2(e), \dots f_n(e)\}$ is an orthonormal basis of $\mathbb{C}^n$, we have

$$|A^t_{x,\delta}(z)|^2 = |e|^2|x|^2|\left\langle z, \frac{x}{|x|}\right\rangle|^2 + \delta^2 \sum_{j=2}^{n} |\langle z, f_j(x)\rangle|^2, \; z \in \mathbb{C}^n.$$

Using the fact that also $\{x/|x|, f_2(x), \dots, f_n(x)\}$ is an orthonormal basis of $\mathbb{C}^n$, the following holds for $\delta := \frac{3}{4}\delta_1$:

$$\min\left(\frac{1}{\sqrt{n}} \inf_{y\in\partial K} |y|, \frac{3}{4}\delta_1\right)|z| \leqslant |A^t_{x,\delta}(z)| \leqslant \max\left(\frac{1}{\sqrt{n}} \sup_{y\in\partial K} |y|, \frac{3}{4}\delta_1\right)|z|.$$

Hence $A := A_{x,\delta}$ satisfies (3) with a sufficiently large number $C > 0$ which depends only on δ_1 and K.

1.5. Definition. — *For a compact, convex set K in $\mathbb{R}^n$, its support function H_K is defined as*

$$H_K(y) := \sup_{x\in K} \langle x, y\rangle, \; y \in \mathbb{R}^n.$$

In the following lemma we construct certain plurisubharmonic functions. The first inequality in the lemma also follows from Langenbruch [L], 1.2. To prove the second property we use a different method for the construction.

1.6. Lemma. — *Let $K \subset \mathbb{R}^n$ be compact and convex with $\mathring{K} \neq \varnothing$. For each $k \geqslant 1$ there exist numbers $l \geqslant k$, $C > 0$ and a continuous, plurisubharmonic function $w : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that for each $z \in \mathbb{C}^n$:*

(1) $H_K(\operatorname{Im} z) - l\log(1+|z|) \leqslant w(z) \leqslant H_K(\operatorname{Im} z) - k\log(1+|z|) + C$

(2) $\sup\{|w(\xi) - w(z)| \mid \xi \in \mathbb{C}^n, \; |\xi - z| \leqslant 1\} \leqslant C.$

Proof. — Without loss of generality we may assume $0 \in \mathring{K}$. Let $k \geqslant 1$ be given. For $T \geqslant 0$ we define

$$P_T(x+iy) := \frac{y+T}{\pi} \int_{-\infty}^{\infty} \frac{\log(1+|t|)}{(x-t)^2 + (y+T)^2}\, dt, \; x, y \in \mathbb{R}, \; y \geqslant 0.$$

By the proof of Meise and Taylor [14], 2.3, there exists a number $T \geqslant 0$ such that the function $u_1 : \mathbb{C} \longrightarrow \mathbb{R}$

$$u_1(z) := \begin{cases} \frac{1}{2}|\operatorname{Im} z| - kP_T(z) & \text{if } \operatorname{Im} z \geqslant 0 \\ \frac{1}{2}|\operatorname{Im} z| - kP_T(\overline{z}) & \text{if } \operatorname{Im} z \leqslant 0 \end{cases}$$

is subharmonic on $\mathbb{C}$. Moreover, one can find $l_1 \geqslant 1$ and $C_1 > 0$ such that

$$|\operatorname{Im} z| - l_1 \log(1+|z|) - C_1 \leqslant u_1(z) \leqslant |\operatorname{Im} z| - k\log(1+|z|)$$

for all $z \in \mathbb{C}$. Next let $Q := [0,1]^n$ and

$$v(z) := \sum_{j=1}^{n} \left(u_1(z_j) + \frac{1}{2} \operatorname{Im} z_j \right) + nC_1.$$

Then for all $z \in \mathbb{C}^n$:

(3) $H_Q(\operatorname{Im} z) - nl_1 \log(1+|z|) \leqslant v(z) \leqslant H_Q(\operatorname{Im} z) - k\log(1+|z|) + nC_1.$

Now fix $x \in \partial K$ and let $A_x \in GL(n;\mathbb{R})$ be the map in Lemma 1.4. It is easy to see that the function

$$w(z) := \sup_{x \in \partial K} v(A_x^{\mathrm{t}}(z)), \ z \in \mathbb{C}^n$$

is continuous and plurisubharmonic on $\mathbb{C}^n$. Let $C_2 > 0$ be the constant in 1.4(3). By 1.4(1), 1.4(3) and (3) above we have

$$\begin{aligned} w(z) &\leqslant \sup_{x \in \partial K} (H_{A_x(Q)}(\operatorname{Im} z) - k\log(1+|A_x^{\mathrm{t}}(z)|)) + nC_1 \\ &\leqslant H_K(\operatorname{Im} z) - k\log(1+\frac{1}{C_2}|z|) + nC_1 \\ &\leqslant H_K(\operatorname{Im} z) - k\log(1+|z|) + k\log(C_2) + nC_1. \end{aligned}$$

Hence w satisfies the second inequality in (1) with $C := k\log(C_2) + nC_1$.

To show the first inequality let $z \in \mathbb{C}^n$ be given. Choose a point $x \in \partial K$ with $H_K(\operatorname{Im} z) = \langle \operatorname{Im} z\,, x \rangle$ and note that 1.4(1) implies

$$\begin{aligned} H_K(\operatorname{Im} z) - nC_2 l_1 \log(1+|z|) &\leqslant H_K(\operatorname{Im} z) - nl_1 \log(1+C_2|z|) \\ &\leqslant \langle x\,, \operatorname{Im} z \rangle - nl_1 \log(1+|A_x^{\mathrm{t}}(z)|) \\ &\leqslant H_{A_x(Q)}(\operatorname{Im} z) - nl_1 \log(1+|A_x^{\mathrm{t}}(z)|) \\ &\leqslant v(A_x^{\mathrm{t}}(z)) \leqslant w(z). \end{aligned}$$

Thus the first inequality in (1) holds with $l := nC_2l_1$. To show property (2) note that for each $\widetilde{T} \geqslant 1$ there exists $C_{\widetilde{T}} > 0$ so that for all $z_1, z_2 \in \mathbb{C}$ with $\mathrm{Im}(z_j) \geqslant 0$, $j = 1, 2$ and $|z_1 - z_2| \leqslant C_2$ we have

$$|P_{\widetilde{T}}(z_1) - P_{\widetilde{T}}(z_2)| \leqslant C_{\widetilde{T}}.$$

This implies for $z_1, z_2 \in \mathbb{C}^n$ with $|z_1 - z_2| \leqslant C_2$:

$$|v(z_1) - v(z_2)| \leqslant n(C_2 + 2kC_T).$$

To show the inequality in (2) fix $z_1, z_2 \in \mathbb{C}^n$ with $|z_1 - z_2| \leqslant 1$. Choose $x_1, x_2 \in \partial K$ such that for all $x \in \partial K$: $v(A^{\mathrm{t}}_{x_j}(z_j)) \geqslant v(A^{\mathrm{t}}_x(z_j)) - 1$, $j = 1, 2$. Without loss of generality we can assume $w(z_1) \geqslant w(z_2)$. Since $|A^{\mathrm{t}}_x(z_1) - A^{\mathrm{t}}_x(z_2)| \leqslant C_2|z_1 - z_2| \leqslant C_2$, for all $x \in \partial K$, we get:

$$w(z_1) - w(z_2) \leqslant v(A^{\mathrm{t}}_{x_1}(z_1)) + 1 - v(A^{\mathrm{t}}_{x_1}(z_2)) \leqslant n(C_2 + 2kC_T) + 1.$$

Hence (2) holds if C is sufficiently large.

1.7. DEFINITION. — *Let $L \subset \mathbb{R}^n$ be a compact, convex set with $\overset{\circ}{L} \neq \varnothing$, let $V \subset \mathbb{C}^n$ be an analytic variety and let $A(V)$ denote the space of all analytic functions on V. For $B \geqslant 1$ define*

$$A^{\pm}_{L,B} := \{f \in A(V) \mid \\ \|f\|^{\pm}_{L,B} := \sup_{z\in V} |f(z)| \exp(-H_L(\mathrm{Im}\, z) \pm B \log(1 + |z|)) < \infty\},$$

endowed with the induced Banach space topology. Moreover, define

$$A^{-}_{L}(V) := \operatorname*{ind}_{B\to\infty} A^{-}_{L,B}(V), \text{ and } A^{+}_{L}(V) := \operatorname*{proj}_{B\to\infty} A^{+}_{L,B}(V).$$

1.8. PROPOSITION. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$, let $P = P_1 \cdot \ldots \cdot P_l$, where the $P_j \in \mathbb{C}[z_1, \ldots, z_n]$ are irreducible for $1 \leqslant j \leqslant l$ and pairwise not proportional and let $V(P) := \{z \in \mathbb{C}^n \mid P(-z) = 0\}$. Then the Fourier-Laplace transform $\mathcal{F}$, defined by*

$$\mathcal{F}(\mu) : z \mapsto \langle \mu_x, \exp(-i\langle x, z\rangle)\rangle, \quad z \in V(P),$$

is a linear topological isomorphism between the following spaces:

(1) $\mathcal{F} : \mathcal{E}_P(K)'_b \longrightarrow A^{-}_{K}(V(P))$

(2) $\mathcal{F} : \mathcal{D}'_P(K)'_b \longrightarrow A^{+}_{K}(V(P))$.

Proof. — (1): This holds by Meise, Taylor and Vogt [17], 3.4(3) for $\omega(t) = \log(1+t)$.

(2): Using standard arguments from functional analysis and the Paley-Wiener theorem for C^∞-functions one gets the following topological isomorphisms

$$\mathcal{D}'_P(K)'_b \cong \mathcal{D}(K)/P(-D)\mathcal{D}(K) \cong A^+_K(\mathbb{C}^n)/\check{P}A^+_K(\mathbb{C}^n),$$

where $\check{P}(z) := P(-z)$, $z \in \mathbb{C}^n$. We claim that the map

$$\rho : A^+_K(\mathbb{C}^n)/\check{P}A^+_K(\mathbb{C}^n) \longrightarrow A^+_K(V(P)),\ \rho(f + \check{P}A^+_K(\mathbb{C}^n)) := f|_{V(P)}$$

is a topological isomorphism. Using this claim it is easy to check that the resulting isomorphism is $\mathcal{F}$.

To prove our claim, note that by the open mapping theorem it suffices to show the bijectivity of ρ. To see that ρ is injective let $f \in A^+_K(\mathbb{C}^n)$ satisfy $f|_{V(P)} = 0$. By hypothesis, P is equal to a product of irreducible polynomials, hence Hansen [Ha], 2.2, implies $f/\check{P} \in A(\mathbb{C}^n)$. By the Malgrange-Ehrenpreis lemma (see Hansen [Ha], A.1) we have $f/\check{P} \in A^+_K(\mathbb{C}^n)$. Hence ρ is injective.

To show that ρ is surjective let $f \in A^+_K(V(P))$ be given. Then for each $k \in \mathbb{N}$ choose a plurisubharmonic function $w_k : \mathbb{C}^n \longrightarrow \mathbb{R}$ and $l(k) \geqslant k$, $C(k) > 0$ as in Lemma 1.6. By the estimates for f and 1.6(1) there exist numbers $C_1(k) \geqslant 1$ such that for each $k \in \mathbb{N}$:

$$|f(z)| \leqslant C_1(k)\exp(w_k(z)),\ z \in V(P).$$

By 1.6(2), the functions w_k, $k \in \mathbb{N}$, satisfy the hypothesis of Hansen [Ha], 2.3 (Extension theorem). By the proof of Hansen [Ha], 2.3, there exist numbers $M > 0$ and $C_2(k) > 0$, $k \in \mathbb{N}$, so that for each $k \in \mathbb{N}$ there exists a function $f_k \in A(\mathbb{C}^n)$ satisfying $f_k|_{V(P)} = f$ and

$$(3) \qquad |f_k(z)| \leqslant C_2(k)\exp(w_k(z) + M\log(1+|z|)),\ z \in \mathbb{C}^n.$$

By Hansen [Ha], 2.2, and A.1, there exist numbers $C_3(k) > 0$, $k \in \mathbb{N}$, and functions $g_k \in A(\mathbb{C}^n)$ such that $f_{k+1} - f_k = \check{P}g_k$ and

$$\begin{aligned}|g_k(z)| &\leqslant C_3(k)\exp(w_k(z) + M\log(1+|z|)) \\ &\leqslant C_3(k)e^{C(k)}\exp(H_K(\operatorname{Im} z) - (k-M)\log(1+|z|)),\ z \in \mathbb{C}^n.\end{aligned}$$

Now observe that for $k \in \mathbb{N}$ there exists a function $b_k \in A^+_K(\mathbb{C}^n)$ such that:

$$|g_k(z) - b_k(z)| \leqslant 2^{-k}\exp(H_K(\operatorname{Im} z) - (k-M)\log(1+|z|)),\quad z \in \mathbb{C}^n.$$

This implies that the following function is well-defined:

$$h_k := -\sum_{j=1}^{k-1} b_j + \sum_{j=k}^{\infty}(g_j - b_j).$$

Moreover $h_k \in A(\mathbb{C}^n)$ and there exist numbers $C_4(k) > 0,\ k \in \mathbb{N}$ such that:

$$(4)\qquad |h_k(z)| \leqslant C_4(k)\exp(H_K(\operatorname{Im} z) - (k-M)\log(1+|z|)),\quad z \in \mathbb{C}^n.$$

By definition we have $g_k = h_k - h_{k+1}$. This implies that the following function is well-defined:

$$g := f_k + \check{P} h_k,\ \text{ where } k \in \mathbb{N}.$$

From the inequalities (3) and (4) it follows that $g \in A_K^+(V(P))$. Obviously $\rho(g) = f$.

2. The $P(D)$-extension property for compact, convex sets.

In this section we introduce the $P(D)$-extension property for compact sets $K \subset \mathbb{R}^n$ with $\overset{\circ}{K} \neq \varnothing$. We show that it is equivalent to a certain condition of Phragmén-Lindelöf type holding on the zero-variety of P. Also, it is equivalent to the surjectivity of $P(D)$ on the space $\mathcal{E}(\mathbb{R}^n, K)$. Thus our results extend those of de Christoforis [Ch].

2.1. Definition. — *Let $K \subset Q \subset \mathbb{R}^n$ be closed, convex sets in $\mathbb{R}^n$ with $\overset{\circ}{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$.*

(a) *We say that (K, Q) has the $P(D)$-extension property if for each $f \in \mathcal{E}_P(K)$ there exists $g \in \mathcal{E}_P(Q)$ with $g|_K = f$. If $Q = \mathbb{R}^n$ we say that K has the $P(D)$-extension property.*

(b) *If the conditions in (a) are satisfied for $\mathcal{D}'_P$ instead of $\mathcal{E}_P$, and if K and Q are compact we say that (K, Q) (resp. K) has the $P(D)$-extension property for $\mathcal{D}'$.*

Thus, K has the $P(D)$-extension property, if the analogue of Whitney's extension theorem holds for the zero-solutions of $P(D)$. The following lemma shows that it makes no difference to extend zero-solutions or arbitrary solutions of $P(D)$.

2.2. LEMMA. — *Let $K \subset Q \subset \mathbb{R}^n$ be closed, convex sets with $\overset{\circ}{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. The following assertions are equivalent:*

(1) (K, Q) has the $P(D)$-extension property,

(2) for each $u \in \mathcal{E}(K)$ and $v \in \mathcal{E}(Q)$ satisfying $P(D)u = v|_K$ there is $w \in \mathcal{E}(Q)$ with $P(D)w = v$ and $w|_K = u$,

(3) $P(D) : \mathcal{E}(Q, K) \longrightarrow \mathcal{E}(Q, K)$ is surjective.

Proof. — (1) $\Rightarrow$ (2): Suppose (K, Q) has the $P(D)$-extension property and let u, v be given as in (2). Using Whitney's extension theorem and the fact that $P(D) : \mathcal{E}(\mathbb{R}^n) \longrightarrow \mathcal{E}(\mathbb{R}^n)$ is surjective, one can find a function $h \in \mathcal{E}(Q)$ such that $P(D)h = v$. Then we get $P(D)(u - h|_K) = 0$ in $\mathcal{E}(K)$. By (1) there exists a function $g \in \mathcal{E}(Q)$ with $P(D)g = 0$ and $g|_K = u - h|_K$. Hence $w := g + h$ has the required properties.

(2) $\Rightarrow$ (3): Let $v \in \mathcal{E}(Q, K)$ be given and define $u = 0$ on K. By hypothesis there exists $w \in \mathcal{E}(Q)$ with $P(D)w = v$ and $w|_K = u = 0$, hence $w \in \mathcal{E}(Q, K)$.

(3) $\Rightarrow$ (1): Let $f \in \mathcal{E}_P(K)$. By Whitney's extension theorem there exists $F \in \mathcal{E}(Q)$ with $F|_K = f$. Then $P(D)F \in \mathcal{E}(Q, K)$. The property (3) implies that we can solve the equation $P(D)G = P(D)F$ with $G \in \mathcal{E}(Q, K)$. Then the function $g := F - G$ is in $\mathcal{E}_P(Q)$ and satisfies $g|_K = f$.

2.3. *Remark.* — Lemma 2.2 holds too if we replace "$P(D)$-extension property" by "$P(D)$-extension property for $\mathcal{D}'$", "$\mathcal{E}$" by "$\mathcal{D}'$" and if Q is compact.

The following lemma shows that it suffices to consider irreducible polynomials in order to decide when a pair (K, Q) of compact and convex sets satisfies the $P(D)$-extension property.

2.4. LEMMA. — *Let $K \subset Q \subset \mathbb{R}^n$ be compact, convex sets with $\overset{\circ}{K} \neq \varnothing$, P_1, $P_2 \in \mathbb{C}[z_1, \ldots, z_n] \setminus \{0\}$ and let $P := P_1 \cdot P_2$. Then (K, Q) has the $P(D)$-extension property (for $\mathcal{D}'$) if and only if (K, Q) has the $P_j(D)$-extension property (for $\mathcal{D}'$), where $j = 1, 2$.*

Proof. — We prove the lemma only for the class $\mathcal{E}$.

"$\Rightarrow$" : Let $g \in \mathcal{E}(K)$ with $P_1(D)g = 0$. There exists $\tilde{g} \in \mathcal{E}(K)$ with $P_2(D)\tilde{g} = g$. This implies $P(D)\tilde{g} = 0$. By hypothesis there exists $\widetilde{G} \in \mathcal{E}(Q)$ with $P(D)\widetilde{G} = 0$ and $\widetilde{G}|_K = \tilde{g}$. Then $G := P_2(D)\widetilde{G}$ satisfies $P_1(D)\widetilde{G} = 0$ and $G|_K = g$.

"$\Leftarrow$" : Let $g \in \mathcal{E}(K)$ with $P(D)g = 0$. There exists a function $g_1 \in \mathcal{E}(Q)$ with $P_1(D)g_1 = 0$ and $g_1|_K = P_2(D)g$. Choose a function $g_2 \in \mathcal{E}(Q)$ with $P_2(D)g_2 = g_1$. Then $P_2(D)(g - g_{2|_K}) = 0$. By hypothesis there exists $f \in \mathcal{E}(Q)$ with $f|_K = g - g_2|_K$ and $P_2(D)f = 0$. We set $G := f + g_2$. Then we have $P(D)G = P_1(D)(P_2(D)f) - P_1(D)g_1 = 0$ and $G|_K = f|_K + g_2|_K = g$.

To formulate a characterization of the $P(D)$-extension property in terms of a condition on the zero-variety $V(P)$ of P we need the following definitions.

2.5. Definition. — *Let V be an analytic variety. A function $u : V \longrightarrow \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic if u is plurisubharmonic in the regular points V_{reg} of V and locally bounded on V. In order that u is upper semicontinuous on the singular points V_{sing} of V we let*

$$u(\zeta) = \limsup_{V_{\text{reg}} \ni z \longrightarrow \zeta} u(z), \ \zeta \in V_{\text{sing}}.$$

By $\mathrm{PSH}(V)$ we denote the set of all plurisubharmonic functions on V which are upper semicontinuous.

2.6. Lemma. — *Let $Q, K \subset \mathbb{R}^n$ be compact and convex sets with $K \subset Q$. Moreover let $V \subset \mathbb{C}^n$ be an algebraic variety.*

(a) *We say that V satisfies the Phragmén-Lindelöf condition $PL(K, Q)$ if for each $k \geqslant 1$ there exist $l \geqslant 1$ and $C > 0$ such that for each $u \in \mathrm{PSH}(V)$ the conditions (1) and (2) imply (3), where:*

(1) $u(z) \leqslant H_K(\operatorname{Im} z) + O(\log(1 + |z|)), \ z \in V$

(2) $u(z) \leqslant H_Q(\operatorname{Im} z) + k\log(1 + |z|), \ z \in V$

(3) $u(z) \leqslant H_K(\operatorname{Im} z) + l\log(1 + |z|) + C, \ z \in V.$

V satisfies $APL(K, Q)$ if the above implications hold for all plurisubharmonic functions $u = \log|f|$, where f is a holomorphic function on V.

(b) *We say that V satisfies $\mathrm{PL}'(K, Q)$ if for each $l \geqslant 0$ there exist $k \geqslant 1$ and $C > 0$ such that for each $u \in \mathrm{PSH}(V)$ the conditions $(1)'$ and $(2)'$ imply $(3)'$, where:*

(1)′ $u(z) \leqslant H_K(\operatorname{Im} z) - j\log(1+|z|) + O(1),\ z \in V,$ *for all* $j \geqslant 1$

(2)′ $u(z) \leqslant H_Q(\operatorname{Im} z) - k\log(1+|z|),\ z \in V$

(3)′ $u(z) \leqslant H_K(\operatorname{Im} z) - l\log(1+|z|) + C,\ z \in V.$

V satisfies APL′*(K,Q) if the above implications hold for all plurisubharmonic functions $u = \log|f|$, where f is holomorphic on V.*

(c) *We say that V satisfies* $\widetilde{\mathrm{PL}}(K,Q)$ *if there exist $l \geqslant 0$ and $C > 0$ so that for each $u \in$* PSH(V) *the conditions (1) and (2) imply (3), where:*

(1) $u(z) \leqslant H_K(\operatorname{Im} z) + O(\log(1+|z|)),\ z \in V$

(2) $u(z) \leqslant H_Q(\operatorname{Im} z),\ z \in V$

(3) $u(z) \leqslant H_K(\operatorname{Im} z) + l\log(1+|z|) + C,\ z \in V.$

Remarks. — A similar but different Phragmén-Lindelöf condition was used by Hörmander [7] to characterize the surjectivity of linear partial differential operators on $\mathcal{A}(\Omega)$, the space of all real-analytic functions on a convex open set Ω in $\mathbb{R}^n$. Hörmander was the first one who noticed that conditions of this type arise in connection with certain problems for partial differential equations.

The conditions formulated in 2.6 are close to those used by Meise, Taylor and Vogt [15] to characterize when $P(D)$ admits a continuous linear right inverse on $\mathcal{E}(\Omega)$ or $\mathcal{D}'(\Omega)$, Ω as above. For references to other PL-conditions we refer to the comprehensive article of Meise, Taylor and Vogt [18].

2.7. Proposition. — *Let $K \subset Q \subset \mathbb{R}^n$ be compact and convex sets with $\overset{\circ}{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. The following assertions are equivalent:*

(1) *(K,Q) satisfies the $P(D)$-extension property*

(2) *$V(P)$ satisfies* APL*(K,Q)*

(3) *$V(P)$ satisfies* PL*(K,Q).*

Proof. — (1) $\Leftrightarrow$ (2): By Lemma 2.4 we may assume that P is a product of irreducible polynomials which are pairwise not proportional. Then the Fourier transforms $\mathcal{F}_K : \mathcal{E}_P(K)'_b \longrightarrow A_K^-(V(P))$ and $\mathcal{F}_Q : \mathcal{E}_P(Q)'_b \longrightarrow A_Q^-(V(P))$ in 1.8 are topological isomorphisms. By definition, the pair (K,Q) has the $P(D)$-extension property if and only if the restriction map:

$$R : \mathcal{E}_P(Q) \longrightarrow \mathcal{E}_P(K),\ f \mapsto f|_K,$$

is surjective. Obviously, the map $\iota := \mathcal{F}_Q \circ R^{\mathrm{t}} \circ \mathcal{F}_K^{-1}$ is equal to the inclusion map $A_K^-(V(P)) \hookrightarrow A_Q^-(V(P))$. Hence R is surjective if and only if $A_K^-(V(P))$ is a topological subspace of $A_Q^-(V(P))$. Since both spaces $A_K^-(V(P))$ and $A_Q^-(V(P))$ are (DFS)-spaces Baernstein's lemma (see e.g. Meise-Vogt [20], 26.26) implies that this is equivalent to

(4) for each bounded set $B \subset A_Q^-(V(P))$ the set $B \cap A_K^-(V(P))$ is bounded in $A_K^-(V(P))$.

Since for each convex, compact set $L \subset \mathbb{R}^n$ with $\overset{\circ}{L} \neq 0$ the sets

$$B_{m,L} := \{f \in A_L^-(V(P)) \mid \sup_{z\in V(P)} |f(z)| \exp(-H_L(\operatorname{Im} z) - m\log(1+|z|)) \leqslant 1\}, m \in \mathbb{N},$$

form a fundamental sequence of bounded sets in $A_L^-(V(P))$, (4) is equivalent to

(5) for each $k \geqslant 1$ there exists $l \geqslant 1$ and $C > 0$ such that $B_{k,Q} \cap A_K^-(V(P)) \subset CB_{l,K}$.

Obviously, property (5) is equivalent to the Phragmén-Lindelöf condition $\mathrm{APL}(K,Q)$.

(2) $\Leftrightarrow$ (3): This follows from Franken [F2], Thm. 10.

Remark. — The equivalence of the conditions 2.7(1) and 2.7(2) also follows from Thm. 3.2 of Boiti and Nacinovich [BN] who investigated when solutions of systems can be extended.

2.8. PROPOSITION. — *Let $K \subset Q \subset \mathbb{R}^n$ be compact and convex sets with $\overset{\circ}{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. The following assertions are equivalent:*

(1) *(K,Q) satisfies the $P(D)$-extension property for $\mathcal{D}'$*

(2) *$V(P)$ satisfies* APL'*(K,Q)*

(3) *$V(P)$ satisfies* PL'*(K,Q).*

Proof. — (1) $\Leftrightarrow$ (2): As in the proof of 2.7 one can show that (1) is equivalent to

(4) for each zero-neighborhood $U \subset A_K^+(V(P))$ there exists a zero-neighborhood $V \subset A_Q^+(V(P))$ satisfying $V \cap A_K^+(V(P)) \subset U$.

Since for each compact, convex set $L \subset \mathbb{R}^n$ with $\mathring{L} \neq \varnothing$ the sets

$$U_{L,k} := \{f \in A_L^+(V(P)) \mid \sup_{z \in V(P)} |f(z)| \\ \exp(-H_L(\operatorname{Im} z) + k \log(1 + |z|)) < \infty\}, \quad k \in \mathbb{N},$$

are a fundamental sequence of zero-neighborhoods in $A_L^+(V(P))$, (4) is equivalent to

(5) for each $l \geqslant 1$ there exist $k \geqslant 1$ and $C > 0$ such that $CU_{Q,k} \cap A_K^+(V(P)) \subset U_{K,l}$.

It is easy to check that (5) is equivalent to the Phragmén-Lindelöf condition $\mathrm{APL}'(K, Q)$.

(2) $\Leftrightarrow$ (3): This follows from Franken [F2], Thm. 10.

2.9. THEOREM. — *Let $K \subset Q \subset \mathbb{R}^n$ be compact, convex sets with $\mathring{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. Then the following assertions are equivalent:*

(1) *(K, Q) has the $P(D)$-extension property*

(2) *(K, Q) has the $P(D)$-extension property for $\mathcal{D}'$*

(3) *$V(P)$ satisfies* $\mathrm{PL}(K, Q)$

(4) *$V(P)$ satisfies* $\mathrm{PL}'(K, Q)$

(5) *$V(P)$ satisfies* $\widetilde{\mathrm{PL}}(K, Q)$.

Proof. — (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) hold by Propositions 2.7 and 2.8. Hence the proof is complete, if we show that (3), (4) and (5) are equivalent. In doing this, we let $V := V(P)$.

(3) $\Rightarrow$ (4): Let $l' \geqslant 1$ be given and choose $l_0 \geqslant 1$, $C_0 > 0$ according to $\mathrm{PL}(K, Q)$ for $k = 0$. Then define $k' := l' + l_0$ and let $u \in \mathrm{PSH}(V)$ satisfy the inequalities $2.6(1)'$ and $2.6(2)'$ with k'. Then the function

$$v(z) := u(z) + k' \log(1 + |z|), \ z \in V$$

satisfies the inequalities 2.6(a)(1) and 2.6(a)(2) for $k = 0$. By the property 2.6(a)(3) we have:

$$u(z) + (l' + l_0) \log(1 + |z|) \leqslant H_K(\operatorname{Im} z) + l_0 \log(1 + |z|) + C_0, \ z \in V,$$

which implies $2.6(3)'$ with the numbers l' and C_0.

(4) $\Rightarrow$ (5): It is easy to see that (2) implies the existence of $k_0' \geqslant 1$, and $C_0' > 0$ such that for each number $\lambda \in [1,2]$ and for each $u \in \mathrm{PSH}(V)$ the conditions (i)$'$ and (ii)$'$ imply (iii)$'$, where:

(i)$'$ $u(z) \leqslant H_{\lambda K}(\operatorname{Im} z) - j\log(1+|z|) + O(1),\ z \in V$

(ii)$'$ $u(z) \leqslant H_{\lambda Q}(\operatorname{Im} z) - k_0' \log(1+|z|),\ z \in V$

(iii)$'$ $u(z) \leqslant H_{\lambda K}(\operatorname{Im} z) + C_0',\ \ z \in V.$

Now let u be a plurisubharmonic function on V satisfying (1) and (2) of $\widetilde{\mathrm{PL}}(K,Q)$ and fix $z_0 \in V$. Choose a function $\varphi \in \mathcal{D}([-1,1]^n)$, $\widehat{\varphi}(z_0) \neq 0$ so that:

$$\log|\widehat{\varphi}(z)| \leqslant |\operatorname{Im} z| - j\log(1+|z|) + C'(j),\ C'(0) = 0.$$

By 1.6 there exist $w \in \mathrm{PSH}(\mathbb{C}^n)$ and numbers $l_1 \geqslant 2k_0'$, $C_1 > 0$ such that for all $z \in \mathbb{C}^n$:

$$H_K(\operatorname{Im} z) - l_1 \log(1+|z|) \leqslant w(z) \leqslant H_K(\operatorname{Im} z) - 2k_0' \log(1+|z|) + C_1.$$

For $\varepsilon > 0$ let

$$v_\varepsilon(z) := \frac{1}{2}(u(z) + w(z)) + \varepsilon \log|\widehat{\varphi}(z)|,\ z \in V.$$

By the properties of u and w, the function v_ε satisfies

(5) $v_\varepsilon(z) \leqslant H_K(\operatorname{Im} z) + \varepsilon|\operatorname{Im} z| - j\log(1+|z|) + O(1),\ z \in V\quad \text{all } j \in \mathbb{N}$

and

(6) $\quad v_\varepsilon(z) \leqslant H_Q(\operatorname{Im} z) + \varepsilon|\operatorname{Im} z| - k_0' \log(1+|z|) + \frac{1}{2}C_1,\ z \in \mathbb{C}^n.$

Now fix $\lambda \in \,]1,2]$. By Dini's theorem there exists $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$:

$$\begin{aligned} H_K(y) + \delta|y| &\leqslant H_{\lambda K}(y),\ y \in \mathbb{R}^n \\ H_Q(y) + \delta|y| &\leqslant H_{\lambda Q}(y),\ y \in \mathbb{R}^n. \end{aligned}$$

(6) and (7) imply that the function $v_\varepsilon - \frac{1}{2}C_1$ satisfies (i)$'$ and (ii)$'$ for the sets λK and λQ. From (iii)$'$ we get for all $0 < \delta < \varepsilon$:

$$\frac{1}{2}(u(z) + w(z)) + \delta \log|\widehat{\varphi}(z)| \leqslant H_{\lambda K}(\operatorname{Im} z) + C_0' + \frac{1}{2}C_1.$$

Passing to the limit $\delta = 0$ this implies

$$\frac{1}{2}(u(z_0) + w(z_0)) \leqslant H_{\lambda K}(\operatorname{Im} z_0) + C_0' + \frac{1}{2}C_1.$$

Since z_0 was arbitrarily given and the above inequality holds for all $\lambda \in]1,2]$ we get

$$\begin{aligned} u(z) &\leqslant 2H_K(\operatorname{Im} z) - w(z) + 2C_0' + C_1 \\ &\leqslant H_K(\operatorname{Im} z) + l_1 \log(1+|z|) + 2C_0' + C_1. \end{aligned}$$

Hence u satisfies (5) of $\widetilde{\mathrm{PL}}(K,Q)$ with l_1 and $C := 2C_0' + C_1$.

(5) $\Rightarrow$ (3): Let $k \geqslant 1$ be given and fix $u \in \mathrm{PSH}(V)$ satisfying 2.6(a)(1) and 2.6(a)(2) for the number k. By 1.6 there exist $w \in \mathrm{PSH}(\mathbb{C}^n)$ and $l' \geqslant k$, $C' > 0$ such that

$$H_K(\operatorname{Im} z) - l' \log(1+|z|) \leqslant w(z) \leqslant H_K(\operatorname{Im} z) - k\log(1+|z|) + C', \ z \in \mathbb{C}^n.$$

Next let

$$v(z) := \frac{1}{2}(u(z) + w(z) - C'), \ z \in V.$$

Obviously, v satisfies (1) and (2) of $\widetilde{\mathrm{PL}}(K,Q)$, hence it satisfies condition (3) of $\widetilde{\mathrm{PL}}(K,Q)$ by hypothesis. This implies for $z \in \mathbb{C}^n$

$$\begin{aligned} u(z) &\leqslant 2v(z) - w(z) + C' \\ &\leqslant H_K(\operatorname{Im} z) + (2l + l') \log(1+|z|) + 2C + C'. \end{aligned}$$

Thus we have shown that V satisfies $\mathrm{PL}(K,Q)$.

Remark. — As the proof of Theorem 2.9 shows, the conditions (3), (4) and (5) in 2.9 are equivalent for any algebraic variety V and not only for $V(P)$.

Recall that an algebraic variety V is called homogeneous, if for each $z \in V$ and each $\lambda \in \mathbb{C}$, also λz belongs to V.

2.10. Proposition. — *Let $K \dot{\subset} Q \subset \mathbb{R}^n$ be compact, convex sets with $\mathring{K} \neq \varnothing$ and let V be an algebraic variety.*

(a) *If V is homogeneous then V satisfies $\widetilde{\mathrm{PL}}(K,Q)$ if, and only if each $u \in \mathrm{PSH}(V)$ satisfying condition (1) and (2) in 2.6(c) also satisfies*

$$u(z) \leqslant H_K(\operatorname{Im} z), \quad z \in V.$$

(b) *If V satisfies $\widetilde{\mathrm{PL}}(K,Q)$ then V also satisfies $\widetilde{\mathrm{PL}}(\lambda K, \lambda Q)$ for each $\lambda > 0$.*

(c) *The condition* $\widetilde{\mathrm{PL}}(K,Q)$ *is not changed if condition (1) in 2.6(c) is replaced by*

$$u(z) \leqslant H_K(\operatorname{Im} z) + O(1).$$

Proof. — (a) Choose l and C according to 2.6(c) and let $u \in \mathrm{PSH}(V)$ satisfy (1) and (2) in 2.6(c). Since V is a homogeneous variety, for each $R > 0$ the function $u_R(z) := Ru(z/R)$, $z \in V$, is plurisubharmonic on V and satisfies (1) and (2) in 2.6(c). By 2.6(c)(3) we get

$$u(z) \leqslant \frac{1}{R} u_R(zR) \leqslant H_K(\operatorname{Im} z) + \frac{l}{R}\log(1+|zR|) + \frac{C}{R} \quad \text{for} \quad z \in V.$$

Since the right hand side tends to $H_K(\operatorname{Im} z)$ as R tends to infinity, u satisfies (3).

(b) This is easy to check.

(c) This can be shown as in [MTV1], 2.8(a).

2.11. THEOREM. — *Let $K \subset \mathbb{R}^n$ be a compact, convex set with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1,\dots,z_n]$. Then the following assertions are equivalent:*

(1) *K satisfies the $P(D)$-extension property*

(2) *(K,Q) satisfies the $P(D)$-extension property for each compact, convex set Q in $\mathbb{R}^n$ with $K \subset Q$*

(3) *(K,Q) satisfies the $P(D)$-extension property for some compact, convex set Q in $\mathbb{R}^n$ with $K \subset \overset{\circ}{Q}$*

(4) *$P(D) : \mathcal{E}(\mathbb{R}^n, K) \longrightarrow \mathcal{E}(\mathbb{R}^n, K)$ is surjective*

(5) *K satisfies the $P(D)$-extension property for $\mathcal{D}'$*

(6) *(K,Q) satisfies the $P(D)$-extension property for $\mathcal{D}'$ for each compact, convex set Q in $\mathbb{R}^n$ with $K \subset Q$*

(7) *(K,Q) satisfies the $P(D)$-extension property for $\mathcal{D}'$ for some compact, convex set Q in $\mathbb{R}^n$ with $K \subset \overset{\circ}{Q}$*

(8) *$P(D) : \mathcal{D}'(\mathbb{R}^n, K) \longrightarrow \mathcal{D}'(\mathbb{R}^n, K)$ is surjective*

(9) *$V(P)$ satisfies $\widetilde{\mathrm{PL}}(K, B_A)$ for some $A > 0$ with $K \subset \overset{\circ}{B}_A$, for $B_A := \{x \in \mathbb{R}^n \mid |x| \leqslant A\}$.*

Proof. — (1) $\Rightarrow$ (2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Without loss of generality we may assume that $0 \in \overset{\circ}{K}$. Choose $\mu > 1$ such that $\mu K \subset Q$. By hypothesis $(K, \mu K)$ satisfies the $P(D)$-extension property, hence $V(P)$ satisfies $\mathrm{PL}(K, \mu K)$ by 2.7. Hence 2.10(b) implies that $V(P)$ satisfies $\mathrm{PL}(\mu^{n-1}K, \mu^n K)$ for each $n \in \mathbb{N}$. Consequently, by 2.7 and 2.2, the pair $(\mu^{n-1}K, \mu^n K)$ satisfies the condition 2.2(2) for each $n \in \mathbb{N}$. Let now $g \in \mathcal{E}(\mathbb{R}^n, K)$ be given. Set $f_0 := 0 \in \mathcal{E}(K)$. By 2.2(2) we can find recursively functions $f_n \in \mathcal{E}(\mu^n K)$, $n \in \mathbb{N}$ such that $f_n|_{\mu^{n-1}K} = f_{n-1}$ and $P(D)f_n = g|_{\mu^n K}$, $n \in \mathbb{N}_0$. Obviously the function $f \in \mathcal{E}(\mathbb{R}^n, K)$ defined by $f|_{\mu^n K} := f_n$, $n \in \mathbb{N}$ satisfies $P(D)f = g$.

(4) $\Rightarrow$ (1): By Whitney's extension theorem, (4) implies that $P(D)$ is surjective on $\mathcal{E}(Q, K)$. Hence (1) holds by Lemma 2.2.

The proof of the equivalence of the properties (5) – (8) is the same as the one of the equivalence of the properties (1) – (4).

(3) $\Leftrightarrow$ (7): This follows from Theorem 2.9.

(2) $\Rightarrow$ (9) $\Rightarrow$ (3): This follows from Theorem 2.9.

Remark. — Theorem 2.11 extends a result of de Christoforis [Ch], who proved that condition 2.11(4) holding for all convex, compact sets K with non-empty interior is equivalent to P being hyperbolic with respect to each non-characteristic vector. As we show in Example 3.13, there exist operators $P(D)$ which are not hyperbolic at all and convex, compact sets K in $\mathbb{R}^n$, $\overset{\circ}{K} \neq \varnothing$, which have the $P(D)$-extension property.

From Theorem 2.11 and Proposition 2.10(b) we get the following corollary.

2.12. Corollary. — *If a convex, compact set $K \subset \mathbb{R}^n$ with non-empty interior has the $P(D)$-extension property, then λK has the $P(D)$-extension property for each $\lambda > 0$.*

2.13. Corollary. — *Let $K \subset \mathbb{R}^n$ be a convex, compact set with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \dots, z_n,]$.*

(a) *If K has the $P(D)$-extension property then $P(D) : \mathcal{E}(\overset{\circ}{K}) \to \mathcal{E}(\overset{\circ}{K})$ and $P(D) : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ admit a continuous linear right inverse.*

(b) *If P is homogeneous and if $P(D) : \mathcal{E}(\overset{\circ}{K}) \to \mathcal{E}(\overset{\circ}{K})$ admits a continuous linear right inverse then K has the $P(D)$-extension property.*

Proof. — (a) Without restriction one may assume $0 \in \overset{\circ}{K}$. Then it follows easily from Corollary 2.12 and Theorem 2.11(6) that $\Omega = \overset{\circ}{K}$ and $\Omega = \mathbb{R}^n$ satisfy condition 2.1(3) of Meise, Taylor and Vogt [MTV]. Hence (a) and (b) follow from [MTV], Thm. 2.7.

(b) By Meise, Taylor and Vogt [MTV], the hypothesis implies that $V(P)$ satisfies the condition $\mathrm{PL}(\Omega)$, stated there. Since P is homogeneous, it follows from Meise, Taylor and Vogt [MTV1] that $V(P)$ satisfies the condition $\widetilde{\mathrm{PL}}(K, Q)$ for each convex set Q with $\overset{\circ}{Q} \supset K$. By Theorem 2.11 this implies (b).

2.14. Corollary. — *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a non-constant polynomial and let P_m denote its principal part. If the convex, compact set $K \subset \mathbb{R}^n$ satisfies $\overset{\circ}{K} \neq \varnothing$ and has the $P(D)$-extension property then K also has the $P_m(D)$-extension property.*

Proof. — By Corollary 2.13(a), the operator $P(D) : \mathcal{E}(\overset{\circ}{K}) \to \mathcal{E}(\overset{\circ}{K})$ admits a continuous linear right inverse. Hence it follows from Meise, Taylor and Vogt [MTV], 4.5 and [MTV1], 4.1, that $P_m(D)$ has the same property. Thus the result follows from Corollary 2.13(b).

3. Local and linear $P(D)$-extension property.

In this section we show that the $P(D)$-extension property for a compact convex set K is equivalent to some local $P(D)$-extension property, to the existence of "fundamental solutions" with certain lacunas and to the existence of continuous extension operators for the zero-solutions of $P(D)$ on K.

Notation. — For $\varepsilon > 0$ and $x \in \mathbb{R}^n$ let $B_\varepsilon(x) := \{y \in \mathbb{R}^n \mid |x-y| \leqslant \varepsilon\}$.

3.1. Lemma. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$, $x \in \partial K$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. Then the following assertions are equivalent:*

(1) *there exists $\varepsilon > 0$ such that for each $f \in \mathcal{E}_P(B_\varepsilon(x) \cap K)$ there are $0 < \delta \leqslant \varepsilon$ and $g \in \mathcal{E}_P(B_\delta(x))$ satisfying $g|_{B_\delta(x)\cap K} = f|_{B_\delta(x)\cap K}$.*

(2) *there exists $\varepsilon > 0$ such that $(B_\varepsilon(x) \cap K, B_\varepsilon(x))$ has the $P(D)$-extension property.*

Proof. — Obviously, it suffices to show that (1) implies (2). To do this, choose $\varepsilon > 0$ according to (1) and consider the Fréchet spaces

$$E := \mathcal{E}_P\left(B_\varepsilon(x) \cap K\right) \ , \quad F_n := \mathcal{E}_P\left((B_\varepsilon(x) \cap K) \cup B_{\frac{\varepsilon}{n}}(x)\right) , \ n \in \mathbb{N}.$$

Further define

$$r_n : F_n \to E \ , \quad r_n(f) := f|_{B_\varepsilon(x) \cap K}.$$

To show that $E = \bigcup_{n \in \mathbb{N}} r_n(F_n)$, fix $f \in E$. By (1) there exists $n \in \mathbb{N}$ and $g \in \mathcal{E}_P\left(B_{\frac{\varepsilon}{n}}(x)\right)$ satisfying $g|_{B_{\frac{\varepsilon}{n}}(x) \cap K} = f|_{B_{\frac{\varepsilon}{n}}(x) \cap K}$. Hence g can be extended to $\tilde{g} \in F_n$ satisfying $r_n(\tilde{g}) = f$. By Grothendieck's factorization theorem (see Meise and Vogt [MV], 24.33), $E = \bigcup_{n \in \mathbb{N}} r_n(F_n)$ implies the existence of some $m \in \mathbb{N}$ satisfying $r_m(F_m) = E$. To prove that this implies (2), let $\eta := \dfrac{\varepsilon}{m}$ and fix $f \in \mathcal{E}\left(B_\eta(x), B_\eta(x) \cap K\right)$. By Whitney's extension theorem there exists $F \in \mathcal{E}\left(B_\varepsilon(x), B_\varepsilon(x) \cap K\right)$ such that $F|_{B_\eta(x)} = f$. Next choose $G \in \mathcal{E}\left(B_\varepsilon(x)\right)$ satisfying $P(D)G = F$ and note that $G|_{B_\varepsilon(x) \cap K}$ is in $\mathcal{E}_P\left(B_\varepsilon(x) \cap K\right)$. Hence there exists $h \in \mathcal{E}_P\left((B_\varepsilon(x) \cap K) \cup B_{\frac{\varepsilon}{m}}(x)\right)$ satisfying $h|_{B_\varepsilon(x) \cap K} = G|_{B_\varepsilon(x) \cap K}$. Consequently, $g := G|_{B_\eta(x)} - h|_{B_\eta(x)}$ is in $\mathcal{E}\left(B_\eta(x), B_\eta(x) \cap K\right)$ and satisfies

$$P(D)g = P(D)G|_{B_\eta(x)} = F|_{B_\eta(x)} = f.$$

Thus, $P(D) : \mathcal{E}\left(B_\eta(x), B_\eta(x) \cap K\right) \to \mathcal{E}(B_\eta(x), B_\eta(x) \cap K)$ is surjective. By Lemma 2.2 this proves (2) for $\varepsilon = \eta$.

3.2. Definition. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant.*

(a) *We say that K has the local $P(D)$-extension property at $x \in \partial K$ if one of the equivalent conditions of 3.1 holds.*

(b) *We say that K has the local $P(D)$-extension property if each point of ∂K has this property.*

3.3. Lemma. — *Let K and P be as in 3.2. If K has the $P(D)$-extension property then it has the local $P(D)$-extension property, too.*

Proof. — Fix $x \in \partial K$ and $\varepsilon > 0$. By Lemma 2.2 it suffices to show that

$$P(D) : \mathcal{E}(B_\varepsilon(x),\ B_\varepsilon(x) \cap K) \to \mathcal{E}(B_\varepsilon(x), B_\varepsilon(x) \cap K)$$

is surjective. To prove this let $f \in \mathcal{E}(B_\varepsilon(x),\ B_\varepsilon(x) \cap K)$ be given. Then $F \in \mathcal{E}(B_\varepsilon(x) \cup K, K)$, defined by $F|_{B_\varepsilon(x)} \equiv f$ and 0 otherwise, extends f. By Whitney's extension theorem there exists $F_1 \in \mathcal{E}(\mathbb{R}^n, K)$ such that F is the restriction of F_1 to $B_\varepsilon(x) \cup K$. Since K has the $P(D)$-extension property, Theorem 2.11(4) implies the existence of $G \in \mathcal{E}(\mathbb{R}^n, K)$ satisfying $P(D)G = F_1$. Obviously $g := G|_{B_\varepsilon(x)}$ is in $\mathcal{E}(B_\varepsilon(x),\ B_\varepsilon(x) \cap K)$ and satisfies $P(D)g = f$.

3.4. Lemma. — *Let K and P be as in 3.2. If K has the local $P(D)$-extension property then the following holds:*

(∗) *for each $A, \lambda > 0$ there exists an equicontinuous set $B \subset \mathcal{D}'([-A, A]^n)$, so that for each $x \in \partial K$ there exists $E_x \in B$ satisfying $P(D)E_x = \delta_0$ and $E_x\ |_{\mathcal{D}([-A,A]^n \cap \lambda(K-x))} = 0$.*

Proof. — By hypothesis, for each $x \in \partial K$ there exists $\varepsilon(x) > 0$ such that $(B_{\varepsilon(x)}(x) \cap K, B_{\varepsilon(x)}(x))$ has the $P(D)$-extension property. Obviously, there exist points $x_1, \ldots, x_m \in \partial K$ such that $\partial K = \bigcup_{j=1}^{m} B_{\varepsilon(x_j)}(x_j) \cap \partial K$. Let $B_j := B_{\varepsilon(x_j)}(x_j)$ and choose a number $\delta > 0$ so that for each $x \in \partial K$ there is $1 \leqslant j \leqslant m$ with $B_\delta(x) \subset B_j$. Next fix $A, \lambda > 0$ and find $\lambda_1 \geqslant \lambda$ so that for all $x \in \partial K$:

$$\lambda_1(B_\delta(x) \cap K - x) \supset \lambda(K - x), \quad B_{\lambda_1\delta}(0) \supset [-A, A]^n. \tag{1}$$

By Hörmander [H2], 10.7.10, the operator $P(D)$ has a fundamental solution $\tilde{E} \in \mathcal{D}'(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ define the shift operator

$$\tau_x\ :\ \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n),\ \tau_x(\mu)[\varphi] := \mu(\varphi(\,\cdot\, + x))$$

and let $\tilde{E}_x := \tau_x(\tilde{E})$. Then for each $x \in \mathbb{R}^n$ we have $P(D)\tilde{E}_x = \delta_x$. It is easy to see that for each $1 \leqslant j \leqslant m$ the map

$$\tau_j\ :\ \partial K \cap B_j \longrightarrow \mathcal{D}'_P(\lambda_1(B_j \cap K)),\ x \longmapsto \tilde{E}_{\lambda_1 x}|_{\lambda_1(B_j \cap K)}$$

is continuous. This implies that

$$D_j := \{\tilde{E}_{\lambda_1 x}|_{\lambda_1(B_j \cap K)}\ |\ x \in \partial K \cap B_j\}$$

is compact in $\mathcal{D}'_P(\lambda_1(B_j \cap K))$. Since $(B_j \cap K, B_j)$ has the $P(D)$-extension property, 2.11(6) and 2.10(b) imply that the restriction map $\rho_j : \mathcal{D}'_P(\lambda_1 B_j) \to \mathcal{D}'_P(\lambda_1(B_j \cap K))$ is surjective. Hence ρ_j lifts compact sets. Therefore, there exists a compact set C_j in $\mathcal{D}'_P(\lambda_1 B_j)$ satisfying $\rho_j(C_j) = D_j$. Now fix $x \in \partial K$, choose $j = j(x)$ such that $B_\delta(x) \subset B_j$ and find $F_x \in C_j$ such that $\rho_j(F_x) = \tilde{E}_{\lambda_1 x}|_{\lambda_1(B_j \cap K)}$. Then note that (1) implies

$$\lambda_1(B_j - x) \supset \lambda_1(B_\delta(x) - x) = B_{\lambda_1\delta}(0) \supset [-A, A]^n$$

and

$$\lambda_1(B_j \cap K - x) \supset \lambda_1(B_\delta(x) \cap K - x) \supset \lambda(K - x).$$

Therefore $E_x : \mathcal{D}(\lambda_1(B_j - x)) \to \mathbb{C}$, defined as

$$E_x(\varphi) := \left(\tilde{E}_{\lambda_1 x}|_{\lambda_1 B_j} - F_x\right)(\varphi(\cdot - \lambda_1 x))$$

is in $\mathcal{D}'(\lambda_1(B_j - x), \lambda_1(B_j \cap K - x))$ and satisfies $P(D)E_x = \delta_0$. From the construction it is obvious that

$$B := \{E_y|_{[-A,A]^n} \mid y \in \partial K\}$$

is compact, hence equicontinuous in $\mathcal{D}'([-A, A]^n)$. Therefore B has all the required properties.

3.5. LEMMA. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$ and $A > 0$ such that $K \subset [-A, A]^n$. There exists a collection of closed cubes $(Q_j)_{j \in \mathbb{N}}$ in $\mathbb{R}^n$ and functions $(\varphi_j)_{j \in \mathbb{N}}$ in $\mathcal{E}(\mathbb{R}^n)$ such that:*

(1) $\mathbb{R}^n \setminus K = \bigcup_{j=1}^{\infty} Q_j$

(2) *there exist $0 < m_0 < 1 < M_0 < \infty$ such that for all $x \in Q_j$:*

$$m_0 \operatorname{diam} Q_j \leqslant \operatorname{dist}(x, K) \leqslant M_0 \operatorname{diam} Q_j$$

(3) *for all $j \in \mathbb{N}$ with $Q_j \cap [-A, A]^n \neq \varnothing$ we have* $\operatorname{diam} Q_j \leqslant 1$

(4) *there exists $C > 0$ such that for all $j \in \mathbb{N}$:*

$$\sharp\{i \in \mathbb{N} | Q_i \cap Q_j \neq \varnothing\} \leqslant C$$

(5) *there exists $\lambda > 0$ such that for each $j \in \mathbb{N}$ with $Q_j \cap [-A, A]^n \neq \varnothing$ there exists $x_j \in \partial K$ satisfying*

$$K - Q_j \subset \lambda(K - x_j)$$

(6) $\mathrm{Supp}(\varphi_j) \subset Q_j$, *for each* $j \in \mathbb{N}$

(7) *for each* $m \in \mathbb{N}$ *there exists* $C_m > 0$, *so that for all* $j \in \mathbb{N}$, $x \in \mathbb{R}^n$ *:*

$$|\varphi_j^{(\alpha)}(x)| \leqslant C_m (\operatorname{diam} Q_j)^{-|\alpha|}, \ |\alpha| \leqslant m$$

(8) $1 = \sum_{j=1}^{\infty} \varphi_j(x)$, *for each* $x \in \mathbb{R}^n \setminus K$.

Proof. — Without loss of generality we may assume that $0 \in \overset{\circ}{K}$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset K$. For $l \in \mathbb{Z}$ and $r \in (2^{-l}\mathbb{Z})^n$ let $Q(l,r) := [-2^{-l}, \ 2^{-l}]^n + r$. Then $\operatorname{diam} Q(l,r) = \sqrt{n}2^{-l+1}$. We define $M_l := \{Q(l,r) | r \in (2^{-l}\mathbb{Z})^n\}$. For $B \geqslant 1$ we denote by $M_{l,B}$ the set of all cubes $Q(l,r) \in M_l$ satisfying

$$\frac{B}{2} \operatorname{diam} Q(l,r) \leqslant \operatorname{dist}(Q(l,r), K) \leqslant 2B \operatorname{diam} Q(l,r)$$

and we let $M^B := \bigcup\{M_{l,B} | l \in \mathbb{Z}\}$. For a sufficiently large number $B \geqslant 1$, which will be fixed later let $(Q(l_j, r_j))_{j \in \mathbb{N}}$ be some collection of the cubes in M^B. For $j \in \mathbb{N}$ define $Q_j := Q(l_j - 1, r_j)$ and note that there exist functions $(\varphi_j)_{j \in \mathbb{N}}$ in $\mathcal{E}(\mathbb{R}^n)$ so that the properties (1), (2), (4), (6), (7) and (8) hold for arbitrarily given compact sets K whenever $B \geqslant 1$ is sufficiently large (see e.g. Stein [S], Chap. 6).

To show the properties (3) and (5) observe that for each $j \in \mathbb{N}$ the following holds:

$$\begin{aligned} \left(\frac{B}{4} - 1\right) \operatorname{diam} Q_j &= \frac{B}{2} \operatorname{diam} Q(l_j, \ r_j) - \operatorname{diam} Q_j \\ &\leqslant \operatorname{dist}(Q(l_j, \ r_j), \ K) - \operatorname{diam} Q_j \leqslant \operatorname{dist}(Q_j, K) \\ &\leqslant \operatorname{dist}(Q(l_j, \ r_j), K) \leqslant 2B \operatorname{diam} Q(l_j, \ r_j) \\ &\leqslant B \operatorname{diam} Q_j. \end{aligned}$$

Hence (3) holds whenever $4(\sqrt{n}A + 1) \leqslant B$. To prove (5), note that the convexity of K and the choice of $\varepsilon > 0$ implies

$$B_{(1-\lambda)\varepsilon}(\lambda x) \subset K \ \text{ for each } \ x \in K, \ \lambda \in [0,1].$$

For $j \in \mathbb{N}$ there exists a unique $x_j \in \partial K$ with $x_j \in \,]0, \ r_j]$ (where $]a,b] = \{(1-\lambda)a + \lambda b \mid \lambda \in \,]0,1]\}$). Now fix $x \in K \setminus (\mathbb{R} r_j)$ arbitrarily. Then there exists a unique $x'_j \in \,]0,x]$ such that $x'_j - x_j \in \mathbb{R}(x - r_j)$. Then

$$B_{(1-|x'_j|/|x|)\varepsilon}(x'_j) \subset K.$$

Obviously we have: $|x_j'|/|x| = |x_j|/|r_j|$. Since $\operatorname{dist}(r_j, K) + |x_j| \leqslant |r_j|$ we get

$$\left(1 - \frac{|x_j'|}{|x|}\right)\varepsilon = \left(1 - \frac{|x_j|}{|r_j|}\right)\varepsilon \geqslant \frac{\varepsilon}{|r_j|}\operatorname{dist}(r_j, K)$$
$$\geqslant \frac{\varepsilon}{|r_j|}\operatorname{dist}(Q_j, K) \geqslant \frac{\varepsilon}{|r_j|}\left(\frac{B}{4} - 1\right)\operatorname{diam} Q_j.$$

For all $j \in \mathbb{N}$ with $Q_j \cap [-A, A]^n \neq \varnothing$ we get from (3):

$$\left(1 - \frac{|x_j'|}{|x|}\right)\varepsilon \geqslant \frac{\varepsilon}{\sqrt{n}A + 1}\left(\frac{B}{4} - 1\right)\operatorname{diam} Q_j =: \delta_B \operatorname{diam} Q_j.$$

By the definition of x_j', for each $j \in \mathbb{N}$ there exists $\lambda_j > 0$ with $x - r_j = \lambda_j(x_j' - x_j)$. Further, there exists $\lambda \geqslant 1$ so that

$$\lambda_j = \frac{|x - r_j|}{|x_j' - x_j|} = \frac{|r_j|}{|x_j|} \in [\frac{1}{\lambda}, \lambda] \text{ for each } j \in \mathbb{N} \text{ with } Q_j \cap [-A, A]^n \neq \varnothing.$$

Now fix $B \geqslant 1$ so large that $\delta_B > \max(\lambda, 4(\sqrt{n}A + 1))$, let $\tilde{s}_j \in Q_j$ be arbitrarily given, define $s_j := \tilde{s}_j - r_j$ and note that

$$x - \tilde{s}_j = x - r_j - s_j = \lambda_j\left(x_j' - \frac{s_j}{\lambda_j} - x_j\right).$$

For $j \in \mathbb{N}$ with $Q_j \cap [-A, A]^n \neq \varnothing$ we now have

$$\frac{|s_j|}{\lambda_j} \leqslant \lambda|s_j| \leqslant \lambda \operatorname{diam} Q_j \leqslant \delta_B \operatorname{diam} Q_j \leqslant \left(1 - \frac{|x_j'|}{|x|}\right)\varepsilon,$$

hence

$$x_j' - \frac{s_j}{\lambda_j} \in B_{(1-|x_j'|/|x|)\varepsilon}(x_j') \subset K$$

and consequently

$$x - \tilde{s}_j \in \lambda_j(K - x_j) \subset \lambda(K - x_j).$$

Thus we have shown

$$(K \setminus (\mathbb{R}r_j)) - Q_j \subset \lambda(K - x_j),$$

for all $j \in \mathbb{N}$ with $Q_j \cap [-A, A]^n \neq \varnothing$, which implies (5).

3.6. LEMMA. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \dots, z_n]$. If 3.4 $(*)$ holds then $P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$ admits a continuous linear right inverse.*

Proof. — (a) First we show that for each compact, convex set $L \subset \mathbb{R}^n$ with $K \subset \overset{\circ}{L}$ there exists $R_{L,K} \in L(\mathcal{E}(\mathbb{R}^n, K))$, satisfying

$$P(D) \circ R_{L,K}(f) \,|_L = f \,|_L\,, \quad f \in \mathcal{E}(\mathbb{R}^n, K). \tag{1}$$

To prove (1) choose $A > 0$ such that $L + [-1,1]^n \subset [-A, A]^n$. For A and K let $(Q_j)_{j\in\mathbb{N}}$ and $(\varphi_j)_{j\in\mathbb{N}}$ resp. $\lambda \geqslant 1$ be as in 3.5 resp. 3.5(5). By hypothesis, there exists an equicontinuous set $B \subset \mathcal{D}'([-3A, 3A]^n)$ so that for each $j \in \mathbb{N}$ there exists $E_j \in B$ with $P(D)E_j = \delta_0$ and $\mathrm{Supp}(E_j) \subset [-3A, 3A]^n \setminus \lambda(\overset{\circ}{K} - x_j)$, where $x_j \in \partial K$ is as in 3.5(5). Choose a function $\chi \in \mathcal{D}([-3A, 3A]^n)$ with $\chi \mid [-2A,\ 2A]^n \equiv 1$ and define the operator $R_{L,K}$ by

$$R_{L,K}(f) := \sum_{\substack{j=1 \\ Q_j\cap L\neq\varnothing}}^{\infty} (\chi E_j) * (\varphi_j f)\,, \quad f \in \mathcal{E}(\mathbb{R}^n, K). \tag{2}$$

To show that this formula is well-defined let $r \in \mathbb{N}$ and $L' \subset\subset \mathbb{R}^n$ be arbitrarily given. Since $B \subset \mathcal{D}'([-3A, 3A]^n)$ is equicontinuous there exist $L_1 \subset\subset \mathbb{R}^n$, $C_1 > 0$ and $l \geqslant r$ such that for all $j \in \mathbb{N}$:

$$\|(\chi E_j) * h\|_{L',r} \leqslant C_1 \|h\|_{L_1,l}\,, \quad h \in \mathcal{E}(\mathbb{R}^n).$$

Choose a constant $C_2 > 0$ so that for all $|\beta| \leqslant l$ and $x \in L_1$:

$$|h^{(\beta)}(x)| \leqslant C_2 \|h\|_{L_2,2l+n+1} \operatorname{dist}(x, K)^{l+n+1}, \quad h \in \mathcal{E}(\mathbb{R}^n, K),$$

where $L_2 \subset\subset \mathbb{R}^n$ is sufficiently large. By 3.5(3) for each $j \in \mathbb{N}$ with $L\cap Q_j \neq \varnothing$ we have $\operatorname{diam} Q_j \leqslant 1$. This implies for all $j \in \mathbb{N}$ with $Q_j\cap L \neq \varnothing$ and $f \in \mathcal{E}(\mathbb{R}^n, K)$:

$$\begin{aligned}
\|(\chi E_j) * (\varphi_j f)\|_{L_1',r} &\leqslant C_1 \sup_{|\beta|\leqslant l}\ \sup_{x\in L_1} |(\varphi_j f)^{(\beta)}(x)| \\
&\leqslant C_1 \sup_{|\beta|\leqslant l}\ \sup_{x\in L_1} \Big|\sum_{\gamma\leqslant\beta} \binom{\beta}{\gamma} \varphi^{(\beta-\gamma)}(x) f^{(\gamma)}(x)\Big| \\
&\leqslant C_1 2^l \sup_{\substack{|\beta|\leqslant l \\ \gamma\leqslant\beta}}\ \sup_{x\in L_1} \left|\varphi^{(\beta-\gamma)}(x)\right| \left|f^{(\gamma)}(x)\right| \\
&\leqslant C_1 2^l C_2 C_l' \|f\|_{L_2,2l+n+1} (\operatorname{diam} Q_j)^{-l} \left(M_0 \operatorname{diam} Q_j\right)^{l+n+1} \\
&= C_3 \|f\|_{L_2,2l+n+1} (\operatorname{diam} Q_j)^{n+1},
\end{aligned}$$

where C_l' is the number in 3.5(7) and $C_3 := C_1 2^l C_2 C_l'$. Let C be the constant in 3.5(4). Then for all $f \in \mathcal{E}(\mathbb{R}^n, K)$ the following holds:

$$\begin{aligned}
\|R_{L,K}(f)\|_{L',r} &\leqslant C_3 \|f\|_{L_2,2l+n+1} \sum_{\substack{j=1 \\ Q_j \cap L \neq \varnothing}}^{\infty} (\operatorname{diam} Q_j)^{n+1} \\
&\leqslant C_3 \|f\|_{L_2,2l+n+1} \sum_{k=0}^{\infty} \sum_{\substack{2^{-k-1} \leqslant \operatorname{diam} Q_j \leqslant 2^{-k} \\ Q_j \cap L \neq \varnothing}} (\operatorname{diam} Q_j)^{n+1} \\
&\leqslant C_3 \|f\|_{L_2,2l+n+1} \sum_{k=0}^{\infty} \sum_{\substack{2^{-k-1} \leqslant \operatorname{diam} Q_j \leqslant 2^{-k} \\ Q_j \cap L \neq \varnothing}} \left(\frac{1}{2^k}\right)^{n+1} \\
&\leqslant C_3 \|f\|_{L_2,2l+n+1} C \left(4\sqrt{n}A\right)^n \sum_{k=0}^{\infty} 2^{kn-k(n+1)} \\
&= 2C_3 C \left(4\sqrt{n}A\right)^n \|f\|_{L_2,2l+n+1}.
\end{aligned}$$

Thus the formula in (2) defines a continuous linear operator $R_{L,K} : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n)$. To show $R_{L,K}(\mathcal{E}(\mathbb{R}^n, K)) \subset \mathcal{E}(\mathbb{R}^n, K)$, note that for all $j \in \mathbb{N}$ with $Q_j \cap L \neq \varnothing$ and $x \in K$ the following holds:

$$\operatorname{Supp}((\varphi_j f)(x - \cdot)) \subset x - Q_j \subset K - Q_j \subset \lambda(K - x_j).$$

This implies

$$(\chi E_j) * (\varphi_j f)(x) = \langle \chi E_j, (\varphi_j f)(x - \cdot)\rangle = 0 \quad \text{for all} \quad x \in K.$$

Hence we have shown that $R_{L,K}$ maps into $\mathcal{E}(\mathbb{R}^n, K)$. Moreover, for all $j \in \mathbb{N}$ with $Q_j \cap L \neq \varnothing$ and $x \in L$ we have $\operatorname{Supp}((\varphi_j f)(x - \cdot)) \subset [-2A, 2A]^n$. Then for all $f \in \mathcal{E}(\mathbb{R}^n, K)$ and $x \in L$:

$$(\chi E_j) * (\varphi_j f)(x) = E_j * (\varphi_j f)(x).$$

This implies

$$\begin{aligned}
P(D) \circ R_{L,K}(f)\,|_L &= \sum_{\substack{j=1 \\ Q_j \cap L \neq \varnothing}}^{\infty} P(D) E_j * (\varphi_j f)\,|_L \\
&= \sum_{\substack{j=1 \\ Q_j \cap L \neq \varnothing}}^{\infty} \delta_0 * (\varphi_j f)\,|_L = \left(\sum_{\substack{j=1 \\ Q_j \cap L \neq \varnothing}}^{\infty} \varphi_j \right) f\,|_L = f\,|_L.
\end{aligned}$$

Hence we have shown property (1).

(b) To construct a continuous linear right inverse for $P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$ assume $0 \in \overset{\circ}{K}$. Then note that for each $\mu > 0$ the set μK also satisfies the condition 3.4 $(*)$. Therefore we get from part (a) the existence of $R_{(j+1)K,jK} \in L(\mathcal{E}(\mathbb{R}^n,\ jK))$ satisfying

$$P(D) \circ R_{(j+1)K,jK}(f) \mid_{(j+1)K} = f \mid_{(j+1)K}, \quad f \in \mathcal{E}(\mathbb{R}^n, jK).$$

Next define the sequence $(\iota_j)_{j\in\mathbb{N}}$ in $L(\mathcal{E}(\mathbb{R}^n, K))$ recursively by

$$\text{(3)} \quad \iota_1 := R_{2K,K}, \ \iota_{j+1} := \iota_j + R_{(j+2)K,(j+1)K} \circ (\mathrm{id}_{\mathcal{E}(\mathbb{R}^n,K)} - P(D) \circ \iota_j).$$

To show that the operators ι_j are well-defined we claim that for each $f \in \mathcal{E}(\mathbb{R}^n, K)$ the following holds:

$$\text{(4)} \qquad P(D)\iota_j(f) \mid_{(j+1)K} = f \mid_{(j+1)K}$$

$$\text{(5)} \qquad \iota_{j+1}(f) \mid_{(j+1)K} = \iota_j(f) \mid_{(j+1)K}.$$

Obviously $\iota_1 = R_{2K,K}$ has property (4). If ι_j satisfies (4) then

$$(\mathrm{id}_{\mathcal{E}(\mathbb{R}^n,K)} - P(D)\iota_j)(f) \mid_{(j+1)K} = 0 \mid_{(j+1)K}.$$

Hence the operator

$$F_j := R_{(j+2)K,(j+1)K} \circ (\mathrm{id}_{\mathcal{E}(\mathbb{R}^n,K)} - P(D) \circ \iota_j) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, (j+1)K)$$

is well-defined. Moreover for $f \in \mathcal{E}(\mathbb{R}^n, K)$ we get

$$\begin{aligned} P(D)\iota_{j+1}(f) \mid_{(j+2)K} &= P(D)\iota_j(f) \mid_{(j+2)K} + (f - P(D)\iota_j(f)) \mid_{(j+2)K} \\ &= f \mid_{(j+2)K}. \end{aligned}$$

Hence ι_{j+1} satisfies (4). Property (5) follows from the fact that the operator F_j maps into $\mathcal{E}(\mathbb{R}^n, (j+1)K)$. Now (5) implies that the sequence $\iota_1, \iota_2, \ldots$ converges to a continuous linear operator $R_K : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$. By (4) this operator is a right inverse for $P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$.

3.7. Definition. — *Let $K \subset \mathbb{R}^n$ be compact, convex with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \ldots, z_n]$. We say that K has the linear $P(D)$-extension property if the restriction map $\rho_K : \mathcal{E}_P(\mathbb{R}^n) \to \mathcal{E}_P(K)$ admits a continuous linear right inverse.*

3.8. LEMMA. — *Let $K \subset \mathbb{R}^n$ be a compact, convex set with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \dots, z_n]$. Then the following assertions are equivalent:*

(1) *K has the linear $P(D)$-extension property.*

(2) *Let $H_K := \{(u,v) \in \mathcal{E}(K) \times \mathcal{E}(\mathbb{R}^n) \mid P(D)u = v|_K\}$. There exists $\iota_K \in L(H_K, \mathcal{E}(\mathbb{R}^n))$ such that*

$$P(D)\iota_K = \pi_2, \quad \rho_K \circ \iota_K = \pi_1,$$

where $\pi_i, i = 1, 2$, denotes the projection map to the i-th factor.

(3) *$P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$ admits a continuous linear right inverse.*

Proof. — (1) $\Rightarrow$ (2): Let $\sigma_K : \mathcal{E}_P(K) \to \mathcal{E}_P(\mathbb{R}^n)$ be a continuous linear extension operator for the zero-solutions of $P(D)$ on K. Since (1) implies trivially that K has the $P(D)$-extension property, we get from Corollary 2.13 that $P(D) : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ admits a continuous linear right inverse R. If $(u,v) \in H_K$ is given then

$$P(D)\left(u - R(v)|_K\right) = P(D)u - P(D)R(v)|_K = v|_K - v|_K = 0.$$

Hence $u - R(v)|_K \in \mathcal{E}_P(K)$. This implies that the following map is well-defined:

$$\iota_K : H_K \to \mathcal{E}(\mathbb{R}^n), \quad \iota_K(u,v) := R(v) + \sigma_K(u - R(v)|_K).$$

Note that ι_K is continuous, linear and has the following properties:

$$P(D)\iota_K(u,v) = P(D)R(v) + P(D)\sigma_K(u - R(v)|_K) = v,$$

$$\iota_K(u,v)|_K = R(v)|_K + \sigma_K(u - R(v)|_K)|_K = R(v)|_K + u - R(v)|_K = u.$$

This implies (2).

(2) $\Rightarrow$ (3): Obviously the map

$$j_K : \mathcal{E}(\mathbb{R}^n, K) \to H_K, \quad j_K(v) = (0, v)$$

is well-defined, continuous and linear. By (2) the map $\iota_K \circ j_K : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n)$ is continuous, linear and has the following property:

$$\iota_K \circ j_K(v)|_K = \iota_K(0,v)|_K = \pi_1(0,v) = 0, \quad v \in \mathcal{E}(\mathbb{R}^n, K).$$

Hence $\iota_K \circ j_K(\mathcal{E}(\mathbb{R}^n, K)) \subset \mathcal{E}(\mathbb{R}^n, K)$. Moreover

$$P(D)\iota_K \circ j_K(v) = P(D)\iota_K(0, v) = v \quad \text{for} \quad v \in \mathcal{E}(\mathbb{R}^n, K).$$

This implies that $\iota_K \circ j_K$ is a right inverse for $P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$.

(3) $\Rightarrow$ (1): By 1.2, the restriction map $\rho_K : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(K)$ admits a continuous linear right inverse E_K. Let now $f \in \mathcal{E}_P(K)$ be given. Then $P(D)E_K(f)|_K = P(D)f = 0$, hence $P(D)E_K(f) \in \mathcal{E}(\mathbb{R}^n, K)$. By (3), there exists a continuous linear right inverse $\mu_K \in L(\mathcal{E}(\mathbb{R}^n, K))$ for $P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$. Therefore, the following map is well-defined, continuous and linear

$$\sigma_K : \mathcal{E}_P(K) \to \mathcal{E}(\mathbb{R}^n), \quad \sigma_K(f) := E_K(f) - \mu_K\left(P(D)E_K(f)\right)$$

and satisfies

$$P(D)\sigma_K(f) = P(D)E_K(f) - P(D)\mu_K\left(P(D)E_K(f)\right) = 0 \quad \text{for } f \in \mathcal{E}_P(K).$$

This implies $\sigma_K\left(\mathcal{E}_P(K)\right) \subset \mathcal{E}_P(\mathbb{R}^n)$. Since $\mu_K\left(P(D)E_K(f)\right)|_K = 0$ the map σ_K is a continuous linear right inverse for ρ_K.

Now we can formulate the main theorem of this section:

3.9. THEOREM. — *Let $K \subset \mathbb{R}^n$ be a compact, convex set with $\overset{\circ}{K} \neq \varnothing$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. Then the following assertions are equivalent:*

(1) *K has the $P(D)$-extension property*

(2) *K has the local $P(D)$-extension property*

(3) *property 3.4 $(*)$ holds*

(4) *$P(D) : \mathcal{E}(\mathbb{R}^n, K) \to \mathcal{E}(\mathbb{R}^n, K)$ admits a continuous linear right inverse*

(5) *K has the linear $P(D)$-extension property.*

Proof. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold by Lemma 3.3, Lemma 3.4, Lemma 3.6 and Lemma 3.8, while the implication (5) $\Rightarrow$ (1) holds trivially.

To show that condition 3.4 $(*)$ becomes more explicit if ∂K contains flat pieces of dimension $n-1$, we prove the following lemma, which is also used in [FM].

3.10. Lemma. — *Let $Q := [-1,1]^n$, $K := \{x \in Q \,|\, x_n \leqslant 0\}$ and $P \in \mathbb{C}[z_1, \ldots, z_n]$. If (K, Q) satisfies the $P(D)$-extension property then $P(D)$ admits a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ satisfying* $\mathrm{Supp}(E) \subset H^+ := \{x \in \mathbb{R}^n \,|\, x_n \geqslant 0\}$.

Proof. — By Hörmander [H2], 10.7.10, there exists $E_1 \in \mathcal{D}'(\mathbb{R}^n)$ with $P(D)E_1 = \delta_0$. By Theorem 2.9 the pair (K, Q) has the $P(D)$-extension property for $\mathcal{D}'$. Since $\delta_0|_K = 0$, there exists $F \in \mathcal{D}'(Q)$ with $P(D)F = 0$ and $F|_K = E_1|_K$. Then $E_2 := E_1|_Q - F$ satisfies $P(D)E_2 = \delta_0$ and $\mathrm{Supp}(E_2) \subset Q \cap H^+$. Choose a function $\varphi \in \mathcal{D}(Q)$ with $\varphi|_{[-1/2,1/2]^n} \equiv 1$ and define $E_3 := \varphi E_2$. As in the proof of Hörmander [H2], 12.8.1(i) $\Rightarrow$ (ii), we get $C > 1$ such that the inequality (12.8.3) in [H2] is satisfied. By [H2], 12.8.1, this implies the existence of the required fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ with support in H^+.

3.11. Theorem. — *Let $K \subset \mathbb{R}^n$ be a compact, convex set with $\overset{\circ}{K} \neq \varnothing$ and let $P \in \mathbb{C}[z_1, \ldots, z_n]$. Assume that for some $N \in S^{n-1}$ and $\alpha \in \mathbb{R}$ the set K is contained in $\{x \in \mathbb{R}^n \mid \langle x, N\rangle \leqslant \alpha\}$ and that $D := \partial K \cap \{x \in \mathbb{R}^n \mid \langle x, N\rangle = \alpha\}$ contains a point ξ which is in the interior of D relative to the hyperplane $\{x \in \mathbb{R}^n \mid \langle x, N\rangle = \alpha\}$ and that K has the local extension property at ξ. Then $P(D)$ admits a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ satisfying* $\mathrm{Supp}(E) \subset H^+(N)$.

Proof. — Without restriction we can assume $N = (0, \ldots, 0, 1), \alpha = 0$ and $\xi = 0$. By hypothesis, there exists $\varepsilon > 0$ such that $(B_\varepsilon \cap K, B_\varepsilon)$ has the $P(D)$-extension property for $B_\varepsilon := B_\varepsilon(0)$. Then let $Q_\delta := [-\delta, \delta]^n$ and $Q_\delta^- := [-\delta, \delta]^{n-1} \times [-\delta, 0]$ for $\delta > 0$ and choose $\delta < \varepsilon$ so small that $Q_\delta \subset B_\varepsilon$ and $\overset{\circ}{Q}{}_\delta^- \subset B_\varepsilon \cap \overset{\circ}{K}$. Next fix $f \in \mathcal{E}(Q_\delta, Q_\delta^-)$ and define $\tilde{F} \in \mathcal{E}(B_\varepsilon, B_\varepsilon \cap K)$ by $\tilde{F} = 0$ on $B_\varepsilon \cap K$ and $\tilde{F} = f$ on $Q_\delta \backslash B_\varepsilon \cap K$. By Whitney's extension theorem [W] there exists $F \in \mathcal{E}(B_\varepsilon, B_\varepsilon \cap K)$ satisfying $F\,|_{Q_\delta \cup (B_\varepsilon \cap K)} = f\,|_{Q_\delta \cup (B_\varepsilon \cap K)}$. By Lemma 2.2 the hypothesis implies that there exists $G \in \mathcal{E}(B_\varepsilon, B_\varepsilon \cap K)$ such that $P(D)G = F$. Hence $\tilde{g} := G\,|_{Q_\delta}$ satisfies $P(D)\tilde{g} = f$ on $Q_\delta \setminus \overset{\circ}{Q}{}_\delta^-$ and $\tilde{g}$ vanishes on $]-\delta, \delta[^{n-1} \times [-\eta, 0[$ for some $0 < \eta < \delta$. Therefore, there exists $g \in \mathcal{E}(Q_\delta, Q_\delta^-)$ satisfying $P(D)g = f$. By Lemma 2.2, this implies that (Q_δ, Q_δ^-) has the $P(D)$-extension property. Hence the result follows from Lemma 3.10.

3.12. PROPOSITION. — *Let K be a polyhedron and $P \in \mathbb{C}[z_1, \dots, z_n]$. The following assertions are equivalent:*

(1) *K has the $P(D)$-extension property*

(2) *for each $N \in S^{n-1}$ which is an outer normal to any $(n-1)$-dimensional face of K there exists a fundamental solution $E_N \in \mathcal{D}'(\mathbb{R}^n)$ of $P(D)$ with $\mathrm{Supp}(E_N) \subset H^+(N)$*

(3) *K has the linear $P(D)$-extension property.*

Proof. — (1) $\Rightarrow$ (2): By Theorem 2.11 there exists a compact, convex set Q satisfying $\mathring{Q} \supset K$ such that (K, Q) has the $P(D)$-extension property. Hence (2) follows from Lemma 3.3 and 3.11.

(2) $\Rightarrow$ (3): Obviously, (2) implies condition 3.4($*$). Hence (3) holds by Theorem 3.9.

(3) $\Rightarrow$ (1): This is obvious.

As a direct consequence of Proposition 3.12 we get the following corollary.

3.13. COROLLARY. — *Let $K_1, \dots, K_m \in \mathbb{R}^n$ be polyhydra such that $K = \bigcap\{K_j \mid 1 \leqslant j \leqslant m\}$ has non-empty interior. Let $P \in \mathbb{C}[z_1, \dots, z_n]$. Suppose that K_j has the $P(D)$-extension property for each $1 \leqslant j \leqslant m$. Then K has the $P(D)$-extension property, too.*

3.14. *Example.* — (1) In Meise, Taylor and Vogt [MTV3] it is shown that for each polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ of the form

$$P(z_1, \dots, z_n) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{n} z_j^2 \text{ resp. } P(z_1, \dots, z_n) = \sum_{j=1}^{r} z_j^2 - \sum_{j=r+1}^{n-1} z_j^2 + \lambda z_n$$

where $1 < r < n$ resp. $1 < r < n-1$ and $\lambda \in \mathbb{R} \setminus \{0\}$, the operator $P(D)$ admits a fundamental solution E_N with $\mathrm{Supp}\, E_N \subset H^+(N)$ for each $N \in S^{n-1}$ which is characteristic for P. From this and Proposition 3.12 it follows that each polyhedron K in $\mathbb{R}^n$ for which all $(n-1)$-dimensional faces are characteristic for P, has the $P(D)$-extension property.

Particular examples are

$$Q(x, y, z) = x^2 - y^2 + z \quad \text{and} \quad R(z_1, \dots, z_4) = z_1^2 + z_2^2 - z_3^2 - z_4^2,$$

which both are not hyperbolic with respect to any direction. Note that the boxes

$$K_Q(\alpha,\beta,\gamma) := \{(x,y,z) \in \mathbb{R}^3 \mid |x-y| \leqslant \alpha,\ |x+y| \leqslant \beta,\ |z| \leqslant \gamma\}$$

resp.

$$K_R(\alpha,\beta,\gamma,\delta) := \{x \in \mathbb{R}^4 \mid |x_1 - x_3| \leqslant \alpha,\\ |x_1+x_3| \leqslant \beta, |x_2 - x_4| \leqslant \gamma,\ |x_2 + x_4| \leqslant \delta\}$$

satisfy the $Q(D)$- resp. the $R(D)$-extension property whenever $\alpha, \beta, \gamma, \delta > 0$.

(2) Whenever $P(D)$ is hyperbolic with respect to $N \in \mathbb{R}^n$, then there exist compact, convex sets K which have the $P(D)$-extension property. To show this, assume without restriction that $N = (0, \dots, 0, 1) \in \mathbb{R}^n$. For $0 < \alpha < 1$ let

$$B_\pm(\alpha) := \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n-1}^2 + (x_n \pm \alpha)^2 \leqslant 1\},$$

$$K(\alpha) := B_+(\alpha) \cap B_-(\alpha).$$

Since $P(D)$ is hyperbolic with respect to N there exist fundamental solutions $E_\pm \in \mathcal{D}'(\mathbb{R}^n)$ and closed, convex cones $\Gamma_\pm$ in $\mathbb{R}^n$ with $\mathrm{Supp}(E_\pm) \subset \Gamma_\pm$, where

$$\Gamma_\pm \setminus \{0\} \subset \{x \in \mathbb{R}^n \mid \pm x_n > 0\}.$$

Now it is easy to check that there exists $0 < \alpha_0 < 1$, depending on Γ_+ and Γ_- so that $K(\alpha)$ satisfies condition 3.4 $(*)$ for each $0 < \alpha \leqslant \alpha_0$. Hence $K(\alpha)$ has the $P(D)$-extension property for these α, by Theorem 3.9.

BIBLIOGRAPHY

[1] C. BOITI and M. NACINOVICH, Evolution and hyperbolic pairs, preprint.

[2] J.-M. BONY and M. SCHAPIRA, Existence et prolongement des solutions holomorphes des équations aux derivées partielles, Inventiones Math., 17 (1972), 95–105.

[3] M.L. de CHRISTOFORIS, Soluzioni con lacune di certi operatori differenziali lineari, Rendiconti Accademia Nazionale delle Scienze detta dei XL, Memorie di Matematica 102, vol. VIII (1984), 137–142.

[4] U. FRANKEN, On the equivalence of holomorphic and plurisubharmonic Phragmén-Lindelöf principles, Michigan Math. J., 42 (1995), 163–173.

[5] U. FRANKEN and R. MEISE, Continuous linear right inverses for homogeneous linear partial differential operators on bounded convex open sets and extension of zero-solutions, Proceedings of the Trier work shop on "Functional Analysis", S. Dierolf, S. Dineen, and P. Domanski (Eds.) de Gruyter (1996), to appear.

[6] S. HANSEN, On the fundamental principle of L. Ehrenpreis, Banach Center Publ., 10 (1983), 185–201.

[7] L. HÖRMANDER, On the existence of real analytic solutions of partial differential equations with constant coefficients, Invent. Math., 21 (1973), 151–183.

[8] L. HÖRMANDER, The Analysis of Linear Partial Differential Operators I and II, Springer 1983.

[9] A. KANEKO, Hartogs type extension theorem of real analytic solutions of linear partial differential equations with constant coefficients, Advances in the theory of Fréchet Spaces (T. Terzioglu, e.d.) NATO Adv. Sci. Inst., Ser. C : Math. Phys. Sci., 289 (1989), 63–72.

[10] C.O. KISELMAN, Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants, Bull. Soc. Math. France, 97 (1969), 329–356.

[11] M. LANGENBRUCH, Extension of ultradifferentiable functions, Manuscripta Math., 83 (1994), 123–143.

[12] O. LIESS, Extension of zero solutions of linear partial differential operators, Darmstadt 1983, preprint.

[13] R. MEISE and B.A. TAYLOR, Whitney's extension theorem for ultradifferentiable functions of Beurling type, Ark. Mat., 26 (1988), 265–287.

[14] R. MEISE and B.A. TAYLOR, Linear extension operators for ultradifferentiable functions of Beurling type on compact sets, Amer. J. Math., 111 (1989), 309–337.

[15] R. MEISE, B.A. TAYLOR and D. VOGT, Characterization of linear partial differential operators with constant coefficients that admit a continuous linear right inverse, Ann. Inst. Fourier, Grenoble, 40-3 (1990), 619–655.

[16] R. MEISE, B.A. TAYLOR and D. VOGT, Equivalence of analytic and plurisubharmonic Phragmén-Lindelöf principles on algebraic varieties, Proceedings of Symposia in Pure Mathematics, 52 (1991), 287–308.

[17] R. MEISE, B.A. TAYLOR and D. VOGT, Continuous linear right inverses for partial differential operators with constant coefficients and Phragmén-Lindelöf conditions, in "Functional Analysis", K. D. Bierstedt, A. Pietsch, W. M. Ruess, and D. Vogt (Eds.) Lecture Notes in Pure and Applied Math., Vol. 150 Marcel Dekker, (1994), pp. 357–389.

[18] R. MEISE, B.A. TAYLOR and D. VOGT, Phragmén-Lindelöf principles on algebraic varieties, J. Amer. Math. Soc., to appear.

[19] R. MEISE, B.A. TAYLOR and D. VOGT, Continuous linear right inverses for partial differential operators of order 2 and fundamental solutions in half spaces, preprint.

[20] R. MEISE, D. VOGT, Einführung in die Funktionalanalysis, Vieweg, 1992.

[21] V. P. PALAMODOV, Linear Differential Operators with constant Coefficients, Springer, 1970.

[22] E.M. STEIN, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

[23] R. SCHNEIDER, Convex Bodies : the Minkowski Theory, Cambridge University Press, 1993.

[24] M. TIDTEN, Fortsetzung von C^{∞}-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind, Manuscripta Math., 27 (1979), 291–312.

[25] H. WHITNEY, Analytic extension of differentiable functions, defined on closed sets, Trans. Am. Math. Soc., 36 (1934), 63–89.

Manuscrit reçu le 6 septembre 1995,
accepté le 20 décembre 1995.

U. FRANKEN & R. MEISE,
Mathematisches Institut
Heinrich–Heine–Universität
Universitätsstr. 1
D–40225 Düsseldorf.