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FINITE GAUSS MEASURE ON THE SPACE OF INTERVAL EXCHANGE TRANSFORMATIONS. LYAPUNOV EXPONENTS

by **Anton ZORICH**

1. Introduction.

Consider an orientable measured foliation on a closed orientable surface M_g^2 of genus g with singularities of the saddle type. Throughout the paper we will assume, that the foliation has neither closed singular leaves, nor saddle connections. We will also assume, that the foliation is uniquely ergodic. A generic orientable measured foliation can be reduced to ones which obey all the indicated properties (see, say [8] or [1]), as a consequence of unique ergodicity of a generic interval exchange transformation (see [9], [16]). Recall that we can define an orientable measured foliation as a foliation of leaves of a closed 1-form ω. Any leaf of the orientable measured foliation as described above winds around the surface along one and the same cycle from the first homology group $H_1(M_g^2, \mathbb{R})$ of the surface, which is called *asymptotic cycle*, see [14]. This cycle is just Poincaré dual to the cohomology class $[\omega]$ of corresponding 1-form. In a sense asymptotic cycle gives the first term of approximation of dynamics of leaves.

Study of further terms of approximation gives the following picture (see [22] for details). Computer experiments show, that taking the next term of approximation we get a two-dimensional subspace in $H_1(M_g^2, \mathbb{R})$, i.e., with a good precision leaves deviate from the asymptotic cycle not arbitrary, but inside one and the same two-dimensional subspace $\mathcal{H}^2$ in the first homology. Taking further steps $n = 3, ..., g$ of approximation we get

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subspaces $\mathcal{H}^k$ of dimension k for the k-th step; collection of the subspaces generates a flag $\mathcal{H}^1 \subset \mathcal{H}^2 \subset \ldots \subset \mathcal{H}^g$ of subspaces in the first homology group. The largest, g-dimensional subspace, gives a Lagrangian subspace in $2g$-dimensional symplectic space $H_1(M_g^2, \mathbb{R})$, with the intersection form considered as a symplectic form. We stop at level g since deviation from corresponding Lagrangian subspace is in a sense already negligible. The main conjecture of [22] claims existence of this asymptotic Lagrangian flag for almost all orientable measured foliations on surfaces as described above.

Having an orientable measured foliation on a surface, one can consider the interval exchange transformation induced by the first return map on a piece of transversal. Taking shorter and shorter pieces of transversal we will get longer and longer pieces of leaf bounded by the point of first return. Joining the ends of the piece of leaf along transversal we get a closed cycle, representing an element of the first homology. The asymptotic behavior of this cycle is what we need to investigate. To trace modifications of our cycles we use special procedure for shortening our piece of transversal. Namely, we use iterates of Rauzy induction for the corresponding interval exchange transformation (see [13] and later expositions in [16] and [6]). The transformation operator representing modification of our cycles after k steps of Rauzy induction is the product of k elementary matrices $A_{k-1} \cdots A_0$ related to each step of Rauzy induction. Each term A_i, $0 \leqslant i \leqslant k-1$, belongs to the finite set of elementary matrices. We now need to study properties of these products of matrices.

Though the mapping $\mathcal{T} : IET \to IET$ corresponding to Rauzy induction on the space IET of interval exchange transformations is ergodic with respect to some absolutely continuous invariant measure on IET ([16]), we can not immediately use multiplicative ergodic theorem to study products of matrices $A_{k-1} \cdots A_1$ since the invariant measure is not finite.

We construct another map $\mathcal{G} : IET \to IET$, which assigns to a point $y \in IET$ some iterate $\mathcal{G}(y) = \mathcal{T}^{n(y)}(y)$ of the map $\mathcal{T}$ evaluated at y, where $n(y)$ depends on the point y. The numbers $n(y), n(\mathcal{G}(y)), \ldots$ here are analogous to the entries of continuous fraction expansion for a real number. In the simplest case of interval exchange transformation of two intervals the numbers $n(y), n(\mathcal{G}(y)), \ldots, n(\mathcal{G}^k(y)), \ldots$ are exactly the entries of the corresponding continuous fraction, and the map $\mathcal{G}$ coincides (up to duplication and conjugation) with the classical map of the unit interval to itself related to Euclidean algorithm. Morally the relation between the map $\mathcal{G}$ and Rauzy induction $\mathcal{T}$ is the same as relation between *multiplicative* and *additive* continued fraction algorithms described in [2]. We prove that the map $\mathcal{G}$ is ergodic with respect to some *finite* absolutely continuous invariant measure on IET.

Note that initial matrix-valued function $A(y)$ on IET related to Rauzy induction induces a new cocycle,

$$B(y) = A(\mathcal{T}^{n(y)-1}(y)) \cdot A(\mathcal{T}^{n(y)-2}(y)) \cdots A(y).$$

This time we are already able to apply Oseledets theorem to study products of matrices B. Consider the collection of corresponding Lyapunov exponents $\theta_1 \geqslant \ldots \geqslant \theta_m$, where m is the number of subintervals under exchange.

We prove that $\theta_{g+1} = \ldots = \theta_{m-g} = 0$, where g is the genus of the original surface. As for the remaining Lyapunov exponents, we prove, that they are grouped into pairs $\theta_i = -\theta_{m-i+1}$. We calculate explicitly the largest Lyapunov exponent θ_1. We show that general results in [18] imply $\theta_1 > \theta_2$.

We prove that Lyapunov exponents of the differential $D\mathcal{G}$ are represented by $\theta_1 + \theta_1, \theta_2 + \theta_1, \ldots, \theta_{m-1} + \theta_1$. It means in particular that all Lyapunov exponents of the map $\mathcal{G}$ are strictly positive.

Presumably Lyapunov exponents $\theta_2, \ldots, \theta_g$ are also nonzero, and hence positive, and all of them have multiplicities one. This conjecture implies existence of asymptotic *Lagrangian* flag in the first homology of the surface, responsible for approximation of the leaves.

2. Interval exchange transformations and Rauzy induction.

In this section we recall the notion of *interval exchange transformation*, and of *Rauzy induction*; see original papers [5], [6], [9], [13], [15], [16], [17], [18]. Consider an interval X, and cut it into m subintervals of lengths $\lambda_1, \ldots, \lambda_m$. Now glue the subintervals together in another order, according to some permutation $\pi \in \mathfrak{S}_m$ and preserving the orientation. We again obtain an interval X of the same length, and hence we defined a mapping $T : X \to X$, which is called interval exchange transformation. Our mapping is piecewise linear, and it preserves the orientation and Lebesgue measure. It is singular at the points of cuts, unless two consecutive intervals separated by a point of cut are mapped to consecutive intervals in the image.

Remark 1. — Note that actually there are two ways to glue the subintervals "according to permutation π". We may send the interval number k to the place $\pi(k)$, or we may have the intervals in the image to appear in the order $\pi(1), \dots, \pi(m)$. Following [16] we use the first way; under this choice the second way corresponds to permutation π^{-1}.

Given an interval exchange transformation T corresponding to a pair (λ, π), $\lambda \in \mathbb{R}^n_+$, $\pi \in \mathfrak{S}_n$, set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $X_i = [\beta_{i-1}, \beta_i[$. Define skew-symmetric $m \times m$-matrix $\Omega(\pi)$ as follows:

$$\Omega_{ij}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ and } \pi(i) > \pi(j) \\ -1 & \text{if } i > j \text{ and } \pi(i) < \pi(j) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Consider a translation vector

$$\delta = \Omega(\pi)\lambda.$$

Our interval exchange transformation T is defined as follows:

$$T(x) = x + \delta_i, \qquad \text{for } x \in X_i,\ 1 \leqslant i \leqslant m.$$

Note that, if for some $k < m$ we have $\pi\{1, \dots, k\} = \{1, \dots, k\}$, then the map T decomposes into two interval exchange transformations. We consider only the class $\mathfrak{S}^0_m$ of *irreducible* permutations — those which have no invariant subsets of the form $\{1, \dots, k\}$, where $1 \leqslant k < m$.

Having an interval exchange transformation T corresponding to the pair (λ, π) one can construct a closed orientable surface M^2_g, a closed 1-form ω on M^2_g, and a nonselfintersecting curve γ in M^2_g, such that γ would be transversal to leaves of ω, and the induced Poincaré (first return) map $\gamma \to \gamma$ would coincide with the initial interval exchange transformation T (see corresponding constructions in [16] and in [9]). The genus g of the surface is defined by combinatorics of the permutation π as follows (see [16]).

Let $\pi \in \mathfrak{S}^0_m$. Define permutation $\sigma = \sigma(\pi)$ on $\{0, 1, \dots, m\}$ (see 2.1 in [16]) by

$$\sigma(j) = \begin{cases} \pi^{-1}(1) - 1 & j = 0 \\ m & j = \pi^{-1}(m) \\ \pi^{-1}\big(\pi(j) + 1\big) - 1 & \text{otherwise.} \end{cases}$$

Let

$$S(j) = \{j, \sigma(j), \sigma^2(j), \ldots\} \subset \{0, 1, 2, \ldots, m\} \quad j = 0, 1, \ldots, m$$

be the cyclic subset for the permutation σ. To each subset S of this form assign the vector $b_S \in \mathbb{R}^m$, which is presented in components as follows (see 2.9 in [16]):

$$b_S^j = \chi_S^{j-1} - \chi_S^j, \qquad 1 \leqslant j \leqslant m \tag{2.2}$$

where

$$\chi_S^j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\Sigma(\pi) := \{\text{set of cyclic subsets for } \sigma(\pi)\}$$

$$\Sigma_0(\pi) := \Sigma(\pi) \backslash S(0) \tag{2.3}$$

$$N(\pi) := \mathrm{Card}\Sigma(\pi).$$

According to [16] the genus g of the surface M_g^2 is

$$g = \frac{m - (N(\pi) - 1)}{2}.$$

To each permutation $\pi \in \mathfrak{S}_m$ we assign $m \times m$ permutation matrix

$$P_{i,j}(\pi) = \begin{cases} 1 & \text{if } j = \pi(i), \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\tau_k \in \mathfrak{S}_m$, $1 \leqslant k < m$ the following permutation:

$$\begin{aligned} \tau_k &= \{1, 2, \ldots, k, k+2, \ldots, m, k+1\} \quad 1 \leqslant k < m-1 \\ \tau_{m-1} &= \{1, 2, \ldots, m\} = \mathrm{id}. \end{aligned}$$

Permutation τ_k cyclically moves one step forward all the elements occurring after the element k.

Define the norm $\|\lambda\|$ of $\lambda \in \mathbb{R}^m$ to be $\|\lambda\| = \sum_{i=1}^{m} |\lambda_i|$. By Δ^{m-1} we denote the standard simplex $\Delta^{m-1} = \{\lambda \,|\, \lambda \in \mathbb{R}_+^m;\ \|\lambda\| = 1\}$. Having an interval exchange transformation, defined by a pair (λ, π), where vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m$, defines the lengths of subintervals, and π is a permutation, $\pi \in \mathfrak{S}_m$, we can renormalize vector λ to $\lambda/\|\lambda\| \in \Delta^{m-1}$. The interval exchange transformation corresponding to the pair $(\lambda/\|\lambda\|\,,\ \pi)$ is obviously conjugate to the initial one.

Now we remind construction of Rauzy induction [13]. Whenever it is possible we try to use notations as in [16]. We also use some notations from [6].

Consider two maps $a, b\ :\ \mathfrak{S}^0_m \ \to\ \mathfrak{S}^0_m$ on the set of irreducible permutations (see [13]):

$$\begin{aligned} a(\pi) &= \pi \cdot \tau^{-1}_{\pi^{-1}(m)} \\ b(\pi) &= \tau_{\pi(m)} \cdot \pi \end{aligned} \tag{2.4}$$

where one should consider product of permutations as composition of operators — from right to left. Say, $b(2,3,1) = (1,3,2)\cdot(2,3,1) = (3,2,1)$. Considering permutation as a map from one ordering of $1, 2, \ldots, m$ to another, operator b corresponds to the modification of the image ordering by cyclically moving one step forward those letters occurring after the image of the last letter in the domain, i.e., after the letter m. Operation a corresponds to the modification of the ordering of the domain by cyclically moving one step forward those letters occurring after one going to the last place, i.e., after $\pi^{-1}(m)$, see [6].

Note that

$$(a(\pi))^{-1} = b(\pi^{-1}).$$

In components the maps a, b are as follows, (see [16]):

$$\begin{aligned} a(\pi)(j) &= \begin{cases} \pi(j) & j \leqslant \pi^{-1}(m) \\ \pi(m) & j = \pi^{-1}(m)+1 \\ \pi(j-1) & \text{other } j \end{cases} \\[2ex] b(\pi)(j) &= \begin{cases} \pi(j) & \pi(j) \leqslant \pi(m) \\ \pi(j)+1 & \pi(m) < \pi(j) < m \\ \pi(m)+1 & \pi(j) = m. \end{cases} \end{aligned} \tag{2.5}$$

The Rauzy class $\mathfrak{R}(\pi_0)$ of an irreducible permutation π_0 is the subset of those permutations $\pi \in \mathfrak{S}^0_m$ which can be obtained from π_0 by some composition of mappings a and b. We will also denote by the same symbol $\mathfrak{R}(\pi_0)$ the oriented graph, which vertices are indexed by elements $\pi \in \mathfrak{R}(\pi_0)$, and which directed edges are either of the type $\pi \mapsto a(\pi)$, or of the type $\pi \mapsto b(\pi)$.

Denote by E identity $m\times m$-matrix, and by $I_{i,j}$ square $m\times m$-matrix, which has only one nonzero entry, which equals one, at the (i,j) place. For any $\pi\in\mathfrak{S}^0_m$ define matrices $A(\pi,a)$, $A(\pi,b)$ as follows, (see [13]):

$$A(\pi,a)=(E+I_{\pi^{-1}(m),m})\cdot P(\tau_{\pi^{-1}(m)}) \tag{2.6}$$

$$A(\pi,b)=E+I_{m,\pi^{-1}(m)}. \tag{2.7}$$

Consider the interval exchange transformation T corresponding to a pair (λ,π), where $\lambda=(\lambda_1,\dots,\lambda_m)\in\Delta^{m-1}$, $\pi\in\mathfrak{S}^0_m$. Compare the lengths λ_m and $\lambda_{\pi^{-1}(m)}$ of the last subinterval in the domain and in the image of T. Suppose they are not equal. Let $\nu=\min(\lambda_m,\lambda_{\pi^{-1}(m)})$. Cut off an interval of the length ν from the right hand side of the initial interval and consider induction of the map T to the subinterval $[0,1-\nu[$. According to [13] the new map would be again an interval exchange transformation of m subintervals corresponding to a pair (λ',π'), where

$$(\lambda',\pi')=\begin{cases}\left(A^{-1}(\pi,a)\lambda,\ a(\pi)\right) & \lambda_m<\lambda_{\pi^{-1}(m)}\\ \left(A^{-1}(\pi,b)\lambda,\ b(\pi)\right) & \lambda_m>\lambda_{\pi^{-1}(m)}.\end{cases}$$

Rescaling the vector λ' we get the transformation

$$\mathcal{T}:\Delta^{m-1}\times\mathfrak{S}^0_m\to\Delta^{m-1}\times\mathfrak{S}^0_m$$
$$(\lambda,\pi)\mapsto\left(\frac{\lambda'}{\|\lambda'\|},\pi'\right).$$

Remark 2. — The fact that the map $\mathcal{T}$ is not defined on the "diagonals" $\lambda_m=\lambda_{\pi^{-1}(m)}$ does not lead to any trouble since we may neglect any set of zero measure in any further considerations.

Consider restriction of this map to invariant subsets of the form $\Delta^{m-1}\times\mathfrak{R}(\pi)$. In [16] W.Veech proves, that Rauzy induction $\mathcal{T}$ is conservative and ergodic on each $\Delta^{m-1}\times\mathfrak{R}(\pi)$. It admits unique up to a scalar multiple absolutely continuous invariant measure, but this measure is infinite.

3. The map $\mathcal{G}$ — a "speed up" of Rauzy induction.

Fix some $\pi_0\in\mathfrak{S}^0_m$ and confine ourselves to the class $\mathfrak{R}(\pi_0)=\mathfrak{R}$. We denote

$$(\lambda^{(k)},\pi^{(k)}):=\mathcal{T}^k(\lambda,\pi)$$
$$(\lambda^{(0)},\pi^{(0)}):=(\lambda,\pi).$$

By $\Delta^{m-1} \times \pi$ we denote the standard simplex Δ^{m-1} indexed by an element $\pi \in \mathfrak{R}$ from the finite set $\mathfrak{R}$. We subdivide each simplex $\Delta^{m-1} \times \pi$ into two subsimplices

$$\Delta^{m-1} \times \pi = \big(\Delta^+(\pi) \cup \Delta^-(\pi)\big) \times \pi$$

where

$$\begin{aligned} \Delta^+(\pi) &= \{\lambda \in \Delta^{m-1} \mid \lambda_m > \lambda_{\pi^{-1}(m)}\} \\ \Delta^-(\pi) &= \{\lambda \in \Delta^{m-1} \mid \lambda_m < \lambda_{\pi^{-1}(m)}\}. \end{aligned}$$

Similarly define positive cones $\Lambda^+(\pi) \cup \Lambda^-(\pi) = \mathbb{R}^m_+$.

For almost all points on $\Delta^{m-1} \times \mathfrak{R}$ we can define the function

(3.1)
$$n(\lambda, \pi) = \min_{k=1,2,\ldots} k \text{ such that } \begin{cases} \lambda^{(k)} \in \Delta^-(\pi^{(k)}) & \text{when } \lambda \in \Delta^+(\pi) \\ \lambda^{(k)} \in \Delta^+(\pi^{(k)}) & \text{when } \lambda \in \Delta^-(\pi). \end{cases}$$

In other words we iterate Rauzy induction and count how many consecutive transformations of the same type (a or b, see (2.4), (2.5)) we can make.

DEFINITION 1. — *We define the map $\mathcal{G}$ related to Rauzy induction $\mathcal{T}$ to be*

$$\begin{gathered} \mathcal{G} : \bigsqcup_{\pi \in \mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) \to \bigsqcup_{\pi \in \mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) \\ \mathcal{G}(\lambda, \pi) := \mathcal{T}^{n(\lambda,\pi)}(\lambda, \pi). \end{gathered}$$

One should consider domain of $\mathcal{G}$ as $\bigsqcup_{\pi \in \mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi))$ forgetting that simplices $\Delta^+(\pi)$ and $\Delta^-(\pi)$ were once glued into one. Actually, defining the domain of $\mathcal{G}$ we have to take a complement to a subset of measure zero, see Remark 2. In particular the common "face" of $\Delta^+(\pi)$ and $\Delta^-(\pi)$ does not belong to the domain of $\mathcal{G}$.

Note that the map $\mathcal{G}$ maps simplices Δ^+ to Δ^- and vice versa.

Define the following matrix-valued function $B(\lambda, \pi)$ on a subset of the full measure in $\bigsqcup_{\pi \in \mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta_-(\pi))$ as

$$B(\lambda, \pi) := A(\lambda^{(0)}, \pi^{(0)}) \cdot \ldots \cdot A(\lambda^{(n(\lambda,\pi)-1)}, \pi^{(n(\lambda,\pi)-1)}) \tag{3.2}$$

where matrix-valued function $A(\lambda, \pi)$ is defined by (2.6). By definition of $B(\lambda, \pi)$ we have

$$\lambda' = \frac{B^{-1}(\lambda, \pi) \cdot \lambda}{\|B^{-1}(\lambda, \pi) \cdot \lambda\|}$$

for the image $(\lambda',\pi') = \mathcal{G}(\lambda,\pi)$ of the map $\mathcal{G}$ (see also explicit formulae (3.6) and (3.7) below). Note, that $\det B(\lambda,\pi) = \pm 1$.

We can give also a direct definition of $\mathcal{G}$ as follows. Let

$$s_n^- = \underbrace{\lambda_m + \lambda_{m-1} + \dots + \lambda_{\pi^{-1}(m)+1} + \lambda_m + \lambda_{m-1} + \dots}_{n \text{ terms}}$$

$$s_n^+ = \underbrace{\lambda_{\pi^{-1}(m)} + \lambda_{\pi^{-1}(m-1)} + \dots + \lambda_{\pi^{-1}(\pi(m)+1)} + \lambda_{\pi^{-1}(m)} + \lambda_{\pi^{-1}(m-1)} + \dots}_{n \text{ terms}}$$

We define

(3.3)
$$n(\lambda,\pi) := \begin{cases} \max\limits_{n\geqslant 1} n \ \text{ such that } s_n^- \leqslant \lambda_{\pi^{-1}(m)} & \text{when } \lambda_m < \lambda_{\pi^{-1}(m)} \\ \max\limits_{n\geqslant 1} n \ \text{ such that } s_n^+ \leqslant \lambda_m & \text{when } \lambda_m > \lambda_{\pi^{-1}(m)}. \end{cases}$$

We define

$$\nu(\lambda,\pi) := \begin{cases} s^-_{n(\lambda,\pi)} & \text{when } \lambda_m < \lambda_{\pi^{-1}(m)} \\ s^+_{n(\lambda,\pi)} & \text{when } \lambda_m > \lambda_{\pi^{-1}(m)}. \end{cases} \tag{3.4}$$

Note that definitions (3.1) and (3.3) of $n(\lambda,\pi)$ are equivalent. Consider an interval exchange transformation T corresponding to a pair (λ,π), $\lambda \in \Delta^{m-1}$, $\pi \in \mathfrak{S}_m^0$. Cut off an interval of the length $\nu(\lambda,\pi)$ from the right hand side of the initial interval and consider induction of the map T to the subinterval $[0, 1-\nu(\lambda,\pi)[$. The new map would be again an interval exchange transformation of m subintervals corresponding to the pair (λ',π'). There would be two cases.

Let

(3.5)
$$q = q(\lambda,\pi) := \begin{cases} n(\lambda,\pi) \bmod (m-\pi^{-1}(m)) & \text{when } \lambda_m < \lambda_{\pi^{-1}(m)} \\ n(\lambda,\pi) \bmod (m-\pi(m)) & \text{when } \lambda_m > \lambda_{\pi^{-1}(m)}. \end{cases}$$

Case a. $\lambda_m < \lambda_{\pi^{-1}(m)}$. In this case

$$\pi' = \pi \cdot \tau^{-q}_{\pi^{-1}(m)}$$

and

$$\lambda'_j = \begin{cases} \lambda_j & j < \pi^{-1}(m) \\ \lambda_{\pi^{-1}(m)} - \nu(\lambda,\pi) & j = \pi^{-1}(m) \\ \lambda_{m-\pi^{-1}(m)-q+j} & \pi^{-1}(m) < j \leqslant \pi^{-1}(m)+q \\ \lambda_{j-q} & \pi^{-1}(m)+q < j \leqslant m. \end{cases} \tag{3.6}$$

Case b. $\lambda_m > \lambda_{\pi^{-1}(m)}$. In this case

$$\pi' = \tau^q_{\pi(m)} \cdot \pi$$

and

$$\lambda'_j = \begin{cases} \lambda_j & j < m \\ \lambda_m - \nu(\lambda, \pi) & j = m. \end{cases} \tag{3.7}$$

One can see, that the matrix $B(\lambda, \pi)$ defined by (3.2) is the matrix of transformation (3.6) when $\lambda_m < \lambda_{\pi^{-1}(m)}$, and of transformation (3.7), when $\lambda_m > \lambda_{\pi^{-1}(m)}$.

Rescaling the vector λ' we get the transformation

$$\begin{gathered} \mathcal{G} : \bigsqcup_{\pi \in \mathfrak{R}} \left(\Delta^+(\pi) \sqcup \Delta^-(\pi)\right) \to \bigsqcup_{\pi \in \mathfrak{R}} \left(\Delta^+(\pi) \sqcup \Delta^-(\pi)\right) \\ (\lambda, \pi) \mapsto \left(\frac{\lambda'}{\|\lambda'\|}, \pi'\right). \end{gathered} \tag{3.8}$$

In other words at one step of the new induction we are shortening one and the same interval $\lambda_{\pi^{-1}(m)}$ or λ_m, whichever is larger, as much as possible, cutting cyclically from its right-hand side intervals of the lengths $\lambda_m, \lambda_{m-1}, \dots, \lambda_{\pi^{-1}(m)+1}$ in the first case, and intervals of the lengths $\lambda_{\pi^{-1}(m)}, \lambda_{\pi^{-1}(m-1)}, \dots, \lambda_{\pi^{-1}(\pi(m)+1)}$ in the second case. The lengths of the rest intervals stay unchanged (up to reenumeration in the first case).

4. Formulation of results.

THEOREM 1. — *Let $m > 1$, and let $\mathfrak{R}$ be a Rauzy class. The map $\mathcal{G}$ on the space of interval exchange transformations $\bigsqcup_{\pi \in \mathfrak{R}} \left(\Delta^+(\pi) \sqcup \Delta^-(\pi)\right)$ admits the invariant measure*

$$\mu = \sum_{\pi \in \mathfrak{R}} c(\pi) \left(f^+_\pi \omega^+(\pi) + f^-_\pi \omega^-(\pi)\right) \otimes \delta_\pi$$

where $\delta_\pi, \pi \in \mathfrak{R}$, is the unit mass at π; $c(\pi)$ are constants specified below; and $\omega^+(\pi)$ $(\omega^-(\pi))$ is the Euclidean measure on $\Delta^+(\pi)$ $(\Delta^-(\pi))$. For each $\pi \in \mathfrak{R}$ the density f^+_π (correspondingly f^-_π) is the restriction to $\Delta^+(\pi)$ (correspondingly $\Delta^-(\pi)$) of a function which is rational, positive, and homogeneous of degree $-m$ on $\mathbb{R}^m_+$.

The measure μ is finite.

We define the measure μ in section 5 following analogous definition in [16]. What is crucial for us is *finiteness* of the measure, which is proved in sections 6–7.

THEOREM 2. — *Let $m > 1$, and let $\mathfrak{R} \in \mathfrak{S}_m^0$ be a fixed Rauzy class. Then the map $\mathcal{G}$ is ergodic on*

$$\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$$

with respect to the absolutely continuous invariant probability measure μ.

Theorem 2 is proved in section 8.

We remind notation

$$\log^+(x) := \begin{cases} \log(x) & \text{when } x \geqslant 1 \\ 0 & \text{when } 0 < x < 1. \end{cases} \tag{4.1}$$

By $\|B\|$ we denote the norm of the matrix B; the particular choice of the norm is of no importance for us.

PROPOSITION 1. — *Function $\log^+ \|B(\lambda,\pi)\| = \log \|B(\lambda,\pi)\|$ is integrable over the space $\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$ with respect to the measure μ.*

$$\int\limits_{\bigsqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi))} \log \|B(\lambda,\pi)\| \, \mu(dx) < \infty.$$

COROLLARY 1. — *Cocycle $B^{-1}(\lambda,\pi)$ is measurable, i.e.*

$$\int\limits_{\bigsqcup_{\pi\in\mathfrak{R}}(\Delta^+(\pi)\sqcup\Delta^-(\pi))} \log^+ \|B^{-1}(\lambda,\pi)\| \, \mu(dx) < \infty.$$

Denote by

$$\left(B^{(k)}(\lambda,\pi)\right)^{-1} = B^{-1}(\mathcal{G}^{k-1}(\lambda,\pi)) \cdot \ldots \cdot B^{-1}(\mathcal{G}(\lambda,\pi)) \cdot B^{-1}(\lambda,\pi)$$

the product of matrices B^{-1} taken at the trajectory of a point (λ,π) under the action of the map $\mathcal{G}$. Apply multiplicative ergodic theorem to the cocycle $B^{-1}(\lambda,\pi)$. Let $\theta_1 \geqslant \ldots \geqslant \theta_m$ be the corresponding Lyapunov exponents.

THEOREM 3. — *The middle $m-2g$ Lyapunov exponents are equal to zero*

$$\theta_{g+1} = \theta_{g+2} = \ldots = \theta_{m-g} = 0.$$

The remaining $2g$ Lyapunov exponents are distributed in pairs

$$\theta_k = -\theta_{m-k+1} \qquad \text{for } k = 1, \ldots, g.$$

The first Lyapunov exponent is strictly greater then the second one

$$\theta_1 > \theta_2.$$

Differential $D_{(\lambda,\pi)}\mathcal{G}$ is also a measurable cocycle on the space of interval exchange transformations $\bigsqcup_{\pi\in\mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$. Consider the collection of corresponding Lyapunov exponents. The dimension of the space is $m-1$, so the differential has $m-1$ Lyapunov exponents.

PROPOSITION 2. — *Collection of Lyapunov exponents for the differential $D\mathcal{G}$ of the map $\mathcal{G}$ coincides with the collection*

$$\theta_1 + \theta_1 > \theta_2 + \theta_1 \geqslant \ldots \geqslant \theta_{m-1} + \theta_1.$$

In particular all Lyapunov exponents of the cocycle $D\mathcal{G}$ are strictly positive.

THEOREM 4. — *The largest Lyapunov exponent θ_1 equals*

$$\begin{aligned}
\theta_1 &= -\sum_{\pi\in\mathfrak{R}} \int_{\Delta^\pm(\pi)} \left(\log \|B^{-1}(\lambda,\pi)\cdot\lambda\| - \log\|\lambda\|\right) d\mu \\
&= \frac{1}{m} \sum_{\pi\in\mathfrak{R}} \int_{\Delta^\pm(\pi)} \log\left|\det D\mathcal{G}\right| d\mu \\
&= -\sum_{\pi\in\mathfrak{R}} \int_{\Delta^\pm(\pi)} \log(1-\nu(\lambda,\pi)) d\mu \qquad (4.2)\\
&= \sum_{\pi\in\mathfrak{R}} \int_{\Delta^\pm(\pi)} \left|\log(1-\lambda_m) - \log(1-\lambda_{\pi^{-1}(m)})\right| d\mu. \qquad (4.3)
\end{aligned}$$

CONJECTURE 1. — *The top g Lyapunov exponents are distinct and strictly positive*

$$\theta_1 > \theta_2 > \ldots > \theta_g > 0.$$

5. Construction of the invariant measure.

In this section we remind construction of the *space of zippered rectangles* presented in [16]. Then we define some particular subspace in it and an automorphism of the subspace, which projects to the map $\mathcal{G}$. Finally we define a measure on the space of interval exchange transformations invariant under $\mathcal{G}$. Since we are extensively using the technique in [16] we need to remind briefly some definitions and results from there.

For $\pi \in \mathfrak{S}_m^0$ define $H(\pi) \subset \mathbb{R}^m$ as the annulator of the system of vectors b_S, $S \in \Sigma(\pi)$, see (2.2):

$$H(\pi) = \{h \in \mathbb{R}^m \mid h \cdot b_S = 0 \text{ for all } S \in \Sigma(\pi)\}. \tag{5.1}$$

Remark 3. — There is a natural local identification of the space $H(\pi)$ with $H_1(M_g^2; \mathbb{R})$ (see Proposition 4 and Remark 4 below).

Define the parallelepiped $Z(h, \pi)$ to be the set of solutions $a \in \mathbb{R}^m$ to the following system of equations and inequalities (which are equations (2.3) and inequalities (3.1) in [16]):

$$\begin{array}{rclr}
h_i - a_i & = & h_{\sigma(i)+1} - a_{\sigma(i)} & (0 \leqslant i \leqslant m) \\
h_i & \geqslant & 0 & (1 \leqslant i \leqslant m) \\
a_i & \geqslant & 0 & (1 \leqslant i < m) \\
a_m & \geqslant & -h_{\pi^{-1}m} & \\
h_m & \geqslant & a_m & \\
h_{\pi^{-1}m+1} & \geqslant & a_{\pi^{-1}m} & \\
\min(h_i, h_{i+1}) & \geqslant & a_i & (0 < i < m,\ i \neq \pi^{-1}m)
\end{array} \tag{5.2}$$

where following [16] we use "dummy" components $h_0 = h_{m+1} = a_0 = 0$. Define the cone

$$H^+(\pi) = \{h \in H(\pi) \mid Z(h, \pi) \text{ is nonempty}\}.$$

The *zippered rectangles space of type* π is the set of triples (λ, h, a), $\lambda \in \mathbb{R}_+^m, h \in H^+(\pi), a \in Z(h, \pi)$. Parameters h and a determine the structure of the Riemann surface in the family of flat surfaces corresponding to the interval exchange transformation (λ, π) (see [16] for details).

Define $\Omega(\mathfrak{R})$ to be the set of zippered rectangles (λ, h, a, π) such that $\pi \in \mathfrak{R}$ is in a given Rauzy class $\mathfrak{R}$, and $\lambda \cdot h = 1$. Define also a codimension-one subspace $\Upsilon(\mathfrak{R}) \subset \Omega(\mathfrak{R})$ by additional constraint $\|\lambda\| = 1$.

In [16] W.Veech defines the flow $P^t(\lambda, h, a, \pi) = (e^t\lambda, e^{-t}h, e^{-t}a, \pi)$, $t \in \mathbb{R}$ on $\Omega(\mathfrak{R})$ and a one-to-one invertible (a.e.) bimeasurable transformation $\mathcal{U} : \Omega(\mathfrak{R}) \to \Omega(\mathfrak{R})$

(5.3)
$$\mathcal{U}(\lambda, h, a, \pi) = \begin{cases} (A^{-1}(\pi, a)\lambda, A^T(\pi, a)h, a', a(\pi)) & \text{when } \lambda_m < \lambda_{\pi^{-1}m} \\ (A^{-1}(\pi, b)\lambda, A^T(\pi, b)h, a'', b(\pi)) & \text{when } \lambda_m > \lambda_{\pi^{-1}m} \end{cases}$$

where matrices $A(\pi, a)$ and $A(\pi, b)$ are defined by equations (2.6); transformations $a(\pi), b(\pi)$ are defined by (2.4) and (2.5); and vectors a', a'' are defined as follows:

$$a'_j = \begin{cases} a_j & j < \pi^{-1}m \\ h_{\pi^{-1}m} + a_{m-1} & j = \pi^{-1}m \\ a_{j-1} & \pi^{-1}m < j \leqslant m \end{cases}$$
$$a''_j = \begin{cases} a_j & 0 \leqslant j < m \\ -(h_{\pi^{-1}m} - a_{\pi^{-1}m-1}) & j = m. \end{cases}$$

Define $t(x), x \in \Upsilon(\mathfrak{R})$, by $t(x) = -\log\big(1 - \min(\lambda_m, \lambda_{\pi^{-1}m})\big)$. Consider a mapping $\mathcal{S} : \Upsilon(\mathfrak{R}) \to \Upsilon(\mathfrak{R})$, $\mathcal{S}x = \mathcal{U}P^{t(x)}x$.

The following measure

$$\eta = \sum_{\pi \in \mathfrak{R}} c(\pi)\eta_\pi \tag{5.5}$$

on $\Upsilon(\mathfrak{R})$ is constructed in [16] as a measure invariant under transformation $\mathcal{S}$. Here $c(\pi)$ are the constants,

$$c(\pi) = \big(\text{Volume of fundamental domain in } \mathbb{Z}^m \cap H(\pi)\big)^{-1}$$

and

$$\eta_\pi = \xi_h(da)\,\frac{\mu_\lambda(dh)}{\|Q\lambda\|}\,\omega_{m-1}(d\lambda)$$

(see 11.4 in [16]). Here $H^+_\lambda = \{h \in H^+(\pi) \,|\, h \cdot \lambda = 1\}$; $\xi_h(da)$ is the Euclidean measure on $Z(h, \pi)$ in dimension $N(\pi) - 1$; $\mu_\lambda(dh)$ is the measure on H^+_λ induced by the Euclidean metric; $\|Q\lambda\|$ is the Euclidean norm of the orthogonal projection of the vector λ on $H(\pi)$; and $\omega_{m-1}(d\lambda)$ is Euclidean measure on Δ^{m-1}.

Finally we remind that the following diagram

$$\begin{array}{ccc} \Upsilon(\mathfrak{R}) & \xrightarrow{\mathcal{S}} & \Upsilon(\mathfrak{R}) \\ \rho\downarrow & & \downarrow\rho \\ \Delta^{m-1}\times\mathfrak{R} & \xrightarrow{\mathcal{T}} & \Delta^{m-1}\times\mathfrak{R} \end{array}$$

is commutative (see [16]), where $\rho : (\lambda, h, a, \pi) \mapsto (\lambda, \pi)$ is the natural projection. Hence the measure $\rho\eta$ is invariant under Rauzy induction $\mathcal{T}$ on the space $\Delta^{m-1} \times \mathfrak{R}$ of interval exchange transformations.

* * *

Having reminded constructions in [16] we now modify them to get a measure μ on the space of interval exchange transformations invariant under the map $\mathcal{G}$. But before we need to prove the technical lemma.

LEMMA 1. —*The subset of* $\Upsilon(\mathfrak{R})$ *determined by the equation* $a_m = 0$ *has codimension one in* $\Upsilon(\mathfrak{R})$, *and hence has measure zero.*

Proof. — If $S(m) \in \Sigma_0(\pi)$, i.e., if $0 \notin S(m)$, the statement follows easily from results of sections 2 and 3 in [16]. Suppose for some $\pi \in \mathfrak{R}$ we have $0 \in S(m)$. Consider the subset $Y \subseteq \Upsilon(\mathfrak{R})$ for which a_m vanishes, $a_m = 0$. Consider the smallest positive l such that $\sigma^l 0 = m$. Note that the definition of σ implies $l \geqslant 2$. System (5.2) implies that the following equation is valid for any point of Y:

$$h_{\sigma 0+1} - h_{\sigma 0} + h_{\sigma^2 0+1} - h_{\sigma^2 0} + \ldots + h_{\sigma^{l-1}0+1} - h_{\sigma^{l-1}0} = 0$$

(see also (2.4) in [16]). Rewrite the equation above as $h \cdot b = 0$. We need to prove that vector b is linearly independent from the system of vectors b_S, $S \in \Sigma_0(\pi)$, defined by (2.2) ((2.9) in [16]). According to section 2 in [16] the space $H(\pi)$ is defined by the system $b_S \cdot h = 0$. Hence any subset which satisfies additional independent linear relation is contained in the subspace of codimension 1, and hence has measure zero.

To prove linear independence of vector b we use an idea of Proposition 12.8 in [16] (see also Lemma 6 below). Suppose dependence holds, and replace $\Sigma_0(\pi)$ by its smallest subset Σ_b for which dependence holds. We introduce sets of indices $S_0 = \{\sigma 0, \ldots, \sigma^{l-1} 0\}$ (which is nonempty since

$l \geqslant 2$) and

$$E = S_0 \cup \bigcup_{S \in \Sigma_b} S. \tag{5.6}$$

By construction (5.6) is a disjoint union and $E \in \{1, \dots, m\}$. Linear dependence implies that every point of E is also a point of $E+1$ which is also a disjoint union. We get $E = E+1$, which is absurd. Hence vector b is linear independent from the system b_S, $S \in \Sigma_0(\pi)$. $\square$

Define the parallelepipeds

$$\begin{aligned} Z^+(h,\pi) &= \{a \in Z(h,\pi) \mid a_m \geqslant 0\} \\ Z^-(h,\pi) &= \{a \in Z(h,\pi) \mid a_m \leqslant 0\}. \end{aligned}$$

Define the subcones

$$\begin{aligned} H^{++}(\pi) &= \{h \in H^+(\pi) \mid Z^+(h,\pi) \text{ is nonempty}\} \\ H^{+-}(\pi) &= \{h \in H^+(\pi) \mid Z^-(h,\pi) \text{ is nonempty}\}. \end{aligned}$$

For a given Rauzy class $\mathfrak{R}$ define

$$\begin{aligned} \Omega^+(\mathfrak{R}) &= \{(\lambda,h,a,\pi) \in \Omega(\mathfrak{R}) \mid \lambda \in \Delta^+(\pi); h \in H^{++}(\pi); a \in Z^+(h,\pi)\} \\ \Omega^-(\mathfrak{R}) &= \{(\lambda,h,a,\pi) \in \Omega(\mathfrak{R}) \mid \lambda \in \Delta^-(\pi); h \in H^{+-}(\pi); a \in Z^-(h,\pi)\}. \end{aligned}$$

Define also

$$\begin{aligned} \Upsilon^+(\mathfrak{R}) &= \Upsilon(\mathfrak{R}) \cap \Omega^+(\mathfrak{R}) \\ \Upsilon^-(\mathfrak{R}) &= \Upsilon(\mathfrak{R}) \cap \Omega^-(\mathfrak{R}) \\ \Upsilon^\pm(\mathfrak{R}) &= \Upsilon^+(\mathfrak{R}) \cup \Upsilon^-(\mathfrak{R}). \end{aligned}$$

Consider the following map $\mathcal{F} : \Upsilon^\pm(\mathfrak{R}) \to \Upsilon(\mathfrak{R})$:

$$\mathcal{F}(\lambda,h,a,\pi) = \mathcal{S}^{n(\lambda,\pi)}(\lambda,h,a,\pi)$$

where $n(\lambda,\pi)$ is defined by (3.1).

LEMMA 2. — *The map $\mathcal{F}$ is the induction of the map $\mathcal{S}$ to the subspace $\Upsilon^\pm(\mathfrak{R}) \subset \Upsilon(\mathfrak{R})$.*

Proof. — We need to prove, that the image of $\mathcal{F}$ belongs to $\Upsilon^\pm(\mathfrak{R})$, and, that $n(\lambda,\pi)$ is the first return time, i.e., the number of the first iteration of the map $\mathcal{S}$ when the image of a point $x = (\lambda,h,a,\pi) \in \Upsilon^\pm(\mathfrak{R})$ belongs to $\Upsilon^\pm(\mathfrak{R})$. Suppose $\lambda \in \Delta^+(\pi)$. Then $x^{(1)} = (\lambda^{(1)}, h^{(1)}, a^{(1)}, \pi^{(1)}) = \mathcal{S}(x)$ is obtained by transformation "of the type b", see (5.3). Recall the remark in [16], saying that the image a' in (5.4) of the transformation of the "type a" satisfies $a'_m \geqslant 0$, and the image a'' in (5.4) of the transformation of the "type b" satisfies $a''_m \leqslant 0$. Hence, if $\lambda^{(1)} \in \Delta^+(\pi^{(1)})$ and $a_m^{(1)} \neq 0$, then the point $x^{(1)} = \mathcal{S}(x)$ does not belong to $\Upsilon^\pm(\mathfrak{R})$ since $a_m^{(1)} < 0$. The first time the iterate would get back to the space $\Upsilon^\pm(\mathfrak{R})$ is the first time vector $\lambda^{(k)} = \mathcal{T}^k\lambda$ would get to the simplex of the type Δ^- (we neglect the set of measure zero of the points $\{x = (\lambda, h, a, \pi) \in \Upsilon(\mathfrak{R}) \,|\, a_m = 0\}$, see Lemma 1). But this is exactly the definition (3.1) of the function $n(\lambda, \pi)$.

The case, when we have $\lambda \in \Delta^-(\pi)$ for the initial point is analogous to one discussed above. □

Since the map $\mathcal{S}$ is almost everywhere one-to-one, and the measure η from (5.5) is invariant under $\mathcal{S}$ we get the following obvious corollary.

COROLLARY 2. — *The map $\mathcal{F}$ is almost everywhere one-to-one map on $\Upsilon^\pm(\mathfrak{R})$. The measure η from (5.5) confined to $\Upsilon^\pm(\mathfrak{R})$ is invariant under $\mathcal{F}$.*

Proof. □

LEMMA 3. — *The following diagram is commutative:*

$$\begin{array}{ccc} \Upsilon^\pm(\mathfrak{R}) & \xrightarrow{\mathcal{F}} & \Upsilon^\pm(\mathfrak{R}) \\ \rho \downarrow & & \downarrow \rho \\ \bigsqcup\limits_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)) & \xrightarrow{\mathcal{G}} & \bigsqcup\limits_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi)). \end{array}$$

Proof. — This is just a straightforward corollary of definitions $\mathcal{F} = \mathcal{S}^{n(\lambda,\pi)}$; $\mathcal{G} = \mathcal{T}^{n(\lambda,\pi)}$; and of commutativity of the initial diagram above. □

Define measure μ on $\bigsqcup\limits_{\pi\in\mathfrak{R}} (\Delta^+(\pi) \sqcup \Delta^-(\pi))$ as $\mu = \rho\eta$.

The properties of the measure μ are described by Theorem 1. The invariance of the measure follows from its definition. The statement about the concrete form of the measure is just the original theorem 11.6 in [16] for the initial measure invariant under Rauzy induction $\mathcal{T}$. What is new (and rather essential for us) is that the measure μ is now *finite*, which would be proved in the next two sections. In other words we claim that the section $\Upsilon^\pm$ has finite "area" (while the initial section Υ chosen in [16] had infinite "area").

Morally, we claim that the fiber $\rho^{-1}(\lambda, \pi)$ is "iceberg-like", i.e., there is a huge "underwater part" specified by inequality $a_m < 0$ for $\lambda \in \Delta^+$ (and $a_m > 0$ for $\lambda \in \Delta^-$) which gives an impact to the measure leading to it infiniteness; while the rest part of the "iceberg", which is "above the water", and which volume gives us our density function, leads to the finite measure.

6. The cones $H^{++}(\pi, W)$ and $H^{+-}(\pi, W)$.

The goal of this section is to prove technical Lemmas 5 and 6 which we will use in the proof of Lemma 8 at the end of section 9.

This section is parallel to §12 in [16], but dealing with the spaces $H^{++}(\pi)$ and $H^{+-}(\pi, W)$ we are able to improve the estimate of Proposition 12.8 in [16]. Here we would not exclude the subsets containing m and $\pi^{-1}m$ anymore.

Let $m > 1$, and fix $\pi \in \mathfrak{S}^0_m$. Consider $W \subset \{1, 2, \ldots, m\}$ such that $W \neq \varnothing$; $W \neq \{1, 2, \ldots, m\}$. Define $\Sigma_0(W)$ (cf. 2.3) to be the set of $S \in \Sigma_0(\pi)$ such that

$$\big(S \cup \{S+1\}\big)\backslash\{m+1\} \subseteq W.$$

Here $\{S+1\} = \{j+1 | j \in S\}$.

Next define $H^{++}(\pi, W)$ $(H^{+-}(\pi, W))$ to be the subset of those $h \in H^{++}(\pi)$ (correspondingly , $H^{+-}(\pi)$) which are supported on W i.e., $h_j = 0, j \notin W, h \in H^{++}(\pi)$ (correspondingly $h_j = 0, j \notin W, h \in H^{+-}(\pi)$). We use the same definition for $H^+(\pi, W)$ as in [16], except that we do not assume $\pi^{-1}m$, $m \notin W$ anymore, unless it is specially indicated.

We will need the following statement to prove Lemma 5:

Lemma 4. — *In both of the following cases*

(1) *$m \in W$, $\pi^{-1}m \notin W$ and $h \in H^{+-}(\pi, W)$, $a \in Z^-(h, \pi)$;*

(2) *$m \notin W$, $\pi^{-1}m \in W$; and $h \in H^{++}(\pi, W)$, $a \in Z^+(h, \pi)$;*

the following equality is valid:

$$0 \leqslant a_j \leqslant h_j, h_{j+1} \qquad (0 \leqslant j \leqslant m).$$

Proof. — Case (1): $h \in H^{+-}(\pi, W)$, $a \in Z^{-}(h,\pi)$, and $m \in W$, $\pi^{-1}m \notin W$. In this case $a_m \leqslant 0$. Since $\pi^{-1}m \notin W$, we get $h_{\pi^{-1}m} = 0$. Since by definition $h_{m+1} = 0$ we may combine equation

$$h_{\pi^{-1}m} - a_{\pi^{-1}m} = h_{m+1} - a_m$$

from (5.2) with inequality $a_{\pi^{-1}(m)} \geqslant 0$ to obtain

$$0 \leqslant a_{\pi^{-1}m} = a_m \leqslant 0.$$

Combining this with inequalities from (5.2) we prove the lemma for this case.

Case (2): $h \in H^{++}(\pi, W)$, $a \in Z^{+}(h,\pi)$, and $m \notin W$, $\pi^{-1}m \in W$. In this case $a_m \geqslant 0$. Since $m \notin W$, we get $h_m = 0$. Since from (5.2) $a_m \leqslant h_m$, we obtain $0 \leqslant a_m \leqslant h_m = 0$, and hence

$$a_m = 0. \tag{6.1}$$

Using the following equation from (5.2)

$$h_{\pi^{-1}m} - a_{\pi^{-1}m} = h_{m+1} - a_m$$

we get

$$a_{\pi^{-1}m} = h_{\pi^{-1}m}.$$

Combining this equation and equation (6.1) with inequalities (5.2) we complete the proof of Lemma 4. □

LEMMA 5. — *In both of the following cases*

(1) $m \in W$, $\pi^{-1}m \notin W$ *and* $h \in H^{+-}(\pi, W)$, $a \in Z^{-}(h,\pi)$;

(2) $m \notin W$, $\pi^{-1}m \in W$; *and* $h \in H^{++}(\pi, W)$, $a \in Z^{+}(h,\pi)$;

the following strict inequality is valid:

$$\dim H^{+\pm}(\pi, W) + \operatorname{Card} \Sigma_0(W) < \operatorname{Card} W.$$

Proof. — The proof is the same as the proof of Proposition 12.8 in [16], except that Lemma 12.3 from [16] used in the proof should be replaced by Lemma 4. □

Now we consider one more case. We stress that the statement below is formulated for the subcone $H^{+}(\pi, W) \subset H^{+}(\pi)$ in the "old" cone from [16].

LEMMA 6. — *Let $\pi^{-1}m$, $m \in W$, where W is as above. Then*

$$\dim H^+(\pi, W) + \operatorname{Card} \Sigma_0(W) < \operatorname{Card} W.$$

Proof. — If $H^+(\pi, W) = \{0\}$ we can apply the same arguments as in Proposition 12.8 in [16]. Suppose now that $H^+(\pi, W) \neq \{0\}$. We have to consider two cases separately.

Case (i). $W \neq \{k, k+1, \dots, m\}$, $1 \leqslant k \leqslant m$. In this case we can apply the arguments similar to those in Proposition 12.8 in [16].

Since W is nonempty, we are able to find $i \in W$, $i \neq m$, so that $i+1 \notin W$. Define $l \geqslant 1$ to be the first integer such that $\sigma^l i \neq m$ and at least one of $\sigma^l i$, $\sigma^l i + 1$ fails to belong to W. Since $i+1 \notin W$, we have $h_{i+1} = 0$ for any $h \in H^+(\pi, W)$. Since $i < m$ equations (3.1) in [16] (see also equations (5.2)) imply for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ the relation $0 \leqslant a_i \leqslant h_{i+1} = 0$. By construction $\sigma^l i \neq m$, and at least one of $h_{\sigma^l i}, h_{\sigma^l i+1}$ is equal to zero for any $h \in H^+(\pi, W)$. If $\sigma^l i \neq \pi^{-1}m$, then for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ equations (3.1) in [16] (see also equations (5.2)) provide us with $0 \leqslant a_{\sigma^l i} \leqslant \min(h_{\sigma^l i}, h_{\sigma^l i+1}) = 0$. Note that $\pi^{-1}m \in W$, hence if $\sigma^l i = \pi^{-1}m$, then $\sigma^l i + 1 = \pi^{-1}m + 1 \notin W$. In this case equations (3.1) in [16] (see also equations (5.2)) provide us with $0 \leqslant a_{\pi^{-1}m} \leqslant h_{\pi^{-1}m+1} = 0$, which is valid for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$.

Since for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ we have $a_i = a_{\sigma^l i} = 0$ equations (3.1) in [16] (see also equations (5.2)) provide us with additional equation

$$h_i - h_{\sigma i+1} + h_{\sigma i} - \ldots + h_{\sigma^{l-1} i} - h_{\sigma^l i+1} = 0 \qquad (h \in H^+(\pi, W)),$$

which is valid for any $h \in H^+(\pi, W)$ (see (12.6) in [16]). We rewrite it as $h \cdot b = 0$, $h \in H^+(\pi, W)$. We have to prove that vector b and vectors b_S, $S \in \Sigma_0(W)$ are linearly independent. Note that by Lemma 2.12 in [16] the collection $\{b_S | S \in \Sigma_0(\pi)\}$ is linearly independent. Hence the collection $\{b_S | S \in \Sigma_0(W)\}$ is linearly independent on W.

If $S(m) \notin \Sigma_0(W)$ and for any $1 \leqslant j < l$ we have $\sigma^j i \neq m$, then we can apply the same arguments as in Proposition 12.8 in [16] which complete the proof in this case.

Suppose $S(m) \in \Sigma_0(W)$, or $\sigma^j i = m$ for some $1 \leqslant j < l$. Suppose the collection b, b_S, $S \in \Sigma_0(W)$ is not linearly independent. Consider the smallest subset $\Sigma_b \subseteq \Sigma_0(W)$ for which dependence holds. We introduce sets $S_0 = \{\sigma i, \dots, \sigma^{l-1} i\}(= \varnothing \text{ if } l = 1)$ and

$$E = \{i\} \cup S_0 \cup \bigcup_{S \in \Sigma_b} S. \tag{6.2}$$

By construction $E \subseteq W$, and (6.2) is a disjoint union. If there is to be dependence then every point of E is also a point of

$$E' = (S_0 + 1) \cup \bigcup_{S \in \Sigma_b} (S + 1) \cup \{\sigma^l i + 1\}. \tag{6.3}$$

Now (6.3) is also a disjoint union, and cardinality considerations imply $E = E'$. But if $m \notin E$, then E is invariant under the map $j \mapsto \sigma j + 1$, which contradicts Lemma 12.7 in [16]. On the other hand, inclusion $m + 1 \in E'$, contradicts $E \subseteq W$. Hence we proved linear independence in this case. As b_S, $S \in \Sigma_0(W)$, and b (restricted to W) are orthogonal to $H^+(\pi, W)$ we get the desired inequality.

Case (ii). $W = \{k, k+1, \dots, m\}$, $1 \leqslant k \leqslant m$. First note that $k > 1$ by assumptions on the set W. By assumptions of Lemma 6 $\pi^{-1} m$, $m \in W$. Since permutation π is irreducible, we have $\pi^{-1} m \neq m$ and hence $k < m$.

Define $l \geqslant 1$ to be the first integer such that $\sigma^{-l}(k-1) \neq m$ and at least one of $\sigma^{-l}(k-1)$, $\sigma^{-l}(k-1)+1$ fails to belong to W. Since $k - 1 \neq \pi^{-1} m, m$, we conclude that for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ the equation $a_{k-1} = 0$ is valid. If $\sigma^{-l}(k-1)+1 \in W$, then $\sigma^{-l}(k-1) = k-1$ and for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ the equation $a_{\sigma^{-l}(k-1)} = 0$ is valid. If $\sigma^{-l}(k-1)+1 \notin W$, then $\sigma^{-l}(k-1) < k-1$ and hence both $\sigma^{-l}(k-1), \sigma^{-l}(k-1)+1$ do not belong to W, which implies that for any $h \in H^+(\pi, W)$ and $a \in Z(h, \pi)$ the equation $a_{\sigma^{-l}(k-1)} = 0$ is valid. Hence any $h \in H^+(\pi, W)$ satisfy the equation

$$h_k - h_{\sigma^{-1}(k-1)} + h_{\sigma^{-1}(k-1)+1} - \dots - h_{\sigma^{-l+1}(k-1)} + h_{\sigma^{-l+1}(k-1)+1} = 0$$

which we rewrite as $(h \cdot b) = 0$. We need to prove that vector b and the vectors $\{b_S | S \in \Sigma_0(W)\}$ are linear independent. For suppose not, and replace $\Sigma_0(W)$ by its smallest subset $\Sigma_b \subseteq \Sigma_0(W)$ for which dependence holds. Consider sets $S_0 = \{\sigma^{-l+1}(k-1), \dots, \sigma^{-1}(k-1)\}$ (we let $S_0 = \varnothing$ if $l = 1$) and

$$E = S_0 \cup \bigcup_{S \in \Sigma_b} S$$

which is a disjoint union. Note that $E \subseteq W$. If dependence holds, then every point of E is also a point of

$$E' = \big((S_0+1) \cup \bigcup_{S \in \Sigma_b} (S+1) \cup \{k\}\big) \backslash \{m+1\}$$

which is again a disjoint union. By the same reasons every point of E' is a point of E, so $E = E'$. Note that if dependence holds, then $m \in E$, otherwise cardinalities of the sets E and E' differ by one, which leeds to contradiction.

Note that $k \in E'$, and hence $k \in E$, which implies $k+1 \in E'$, and then $k+1 \in E$, etc. Hence $E = W$.

Consider the map $U : j \mapsto \sigma j + 1$ on $\{0, 1, \dots, m\}$. is easy to see that $U(E) = (E \cup \{m+1\}) \backslash \{\sigma^{-l+1}(k-1)+1\}$. By the assumptions on the set W we have $1 \notin W$. Hence for any $j \in W$, and for any $q \geqslant 1$ such that $U^q(j)$ is well-defined, we get inequality $U^q(j) \neq 1$. In particular the set W does not contain any closed orbits of the map U, i.e., for any $j \in W$, and for any $q \geqslant 1$ such that $U^q(j)$ is well-defined, we get inequality $U^q(j) \neq j$ (see the proof of Lemma 12.7 in [16])). Hence the whole set W is represented as a single orbit of some $j_0 \in W$ under the action of the map U:

$$j_1 \mapsto j_2 \mapsto \dots \mapsto j_{m-k+1}$$

where $j_{m-k+1} = \pi^{-1}m$, $U(\pi^{-1}m) = m+1$. Note that $U^p j_1$ never equals 0. Note that $U^p j_1$ never equals $\pi^{-1}(\pi 1 - 1)$, since $U(\pi^{-1}(\pi 1 - 1)) = 1$, and $1 \notin W$. Hence for any $1 \leqslant p < m-k+1$ we have

$$\sigma U^p j_1 = \pi^{-1}(\pi U^p j_1 + 1) - 1. \tag{6.4}$$

Let $j_1 = \pi^{-1}(r)$. Due to (6.4) we get $j_2 = \pi^{-1}(r+1), \dots, j_{m-k+1} = \pi^{-1}(r+m-k)$. On the other hand $j_{m-k+1} = \pi^{-1}(m)$. Hence $r+m-k = m$, and $r = k$. Hence $W = \{\pi^{-1}(k), \pi^{-1}(k+1), \dots, \pi^{-1}(m)\}$. By assumption $W = \{k, k+1, \dots, m\}$, $1 < k < m$, which means that permutation π^{-1}, and hence π is not irreducible. We proved that assumption on linear dependence of b and $\{b_S | S \in \Sigma_0(W)\}$ leads to contradiction, so this collection is linearly independent. As b_S, $S \in \Sigma_0(W)$, and b (restricted to W) are orthogonal to $H^+(\pi, W)$ and linearly independent we get the desired inequality, and prove case (ii).

Lemma 6 is proved. □

7. Finiteness of the measure.

In this section we will prove that the integrals of the density functions

$$\begin{aligned} f^+_\pi(\lambda) &= \int_{H^{++}_\lambda} \text{Volume } \left(Z^+(h,\pi)\right) dh \\ f^-_\pi(\lambda) &= \int_{H^{+-}_\lambda} \text{Volume } \left(Z^-(h,\pi)\right) dh \end{aligned} \tag{7.1}$$

of the measure μ in Theorem 1 over corresponding simplices $\Delta^\pm(\pi)$ are finite. We use the scheme similar to one in §13 in [16]. In particular we use the following bound (4.12) from there:

$$\text{Volume } Z(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} B(\lambda,S)^{-1} \quad \text{for } h\in H^+_\lambda(\pi) \tag{7.2}$$

where

$$B(\lambda,S) = \sum_{j\in S\cup\{S+1\}} \lambda_j \qquad (m\notin S\in\Sigma_0(\pi)) \tag{7.3}$$

and

$$B(\lambda,S(m)) = \text{Min}\,(\lambda_{\pi^{-1}m},\lambda_m) + \sum_{\substack{j\in S(m)\cup\{S(m)+1\}\\ j\neq\pi^{-1}m,\ m\ m+1}} \lambda_j \qquad \text{if } S(m)\in\Sigma_0(\pi)$$

(see (4.7), (4.8) in [16]). We apologize for using the busy notation $B(\lambda,S)$ — we don't want to change the original notation in [16]. Since we would not use the matrices (3.2) in this section, and since $B(\lambda,S)$ has different arguments we hope that it would not lead to any confusion.

We need to improve slightly bound (7.2). In our situation the equation $-h_{\pi^{-1}m}\leqslant a_m\leqslant h_m$ is replaced by one of the equations $-h_{\pi^{-1}m}\leqslant a_m\leqslant 0$ or $0\leqslant a_m\leqslant h_m$ depending on whether $\lambda\in\Delta^-$ or $\lambda\in\Delta^-$. Recall the bound (4.4) in [16]

$$\text{Volume } Z(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} J(h,S) \quad \text{for } h\in H^+_\lambda(\pi)$$

where

$$J(h,S) = \text{Min}\,\{h_i,h_{i+1} \,|\, i\in S\} \text{ for } S\in\Sigma_0(\pi), S\neq S(m)$$

and

$$J(h,S(m)) = \text{Min}\,\left[h_{\pi^{-1}m+1}, h_{\pi^{-1}m}+h_m, \{h_i,h_{i+1}|i\in S(m), i\neq\pi^{-1}m, m\}\right]$$

assuming $S(m) \in \Sigma_0(\pi)$.

Let $J^+(h,S) = J^-(h,S) = J(h,S)$ for $S \neq S(m)$. Let

$$J^+(h,S(m)) = \text{Min}\left[h_{\pi^{-1}m+1}, h_m, \{h_i, h_{i+1} | i \in S(m), i \neq \pi^{-1}m, m\}\right]$$

and

$$J^-(h,S(m)) = \text{Min}\left[h_{\pi^{-1}m+1}, h_{\pi^{-1}m}, \{h_i, h_{i+1} | i \in S(m), i \neq \pi^{-1}m, m\}\right].$$

It is easy to see that

$$\text{Volume } Z^+(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} J^+(h,S) \quad \text{for } h \in H^+_\lambda(\pi)$$

$$\text{Volume } Z^-(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} J^-(h,S) \quad \text{for } h \in H^-_\lambda(\pi)$$

(cf. (4.6) in [16]).

Next define $B^+(\lambda,S) = B^-(\lambda,S) = B(\lambda,S)$ for $S \in \Sigma_0(\pi), S \neq S(m)$. For those $S(m)$ which obey $S(m) \in \Sigma_0(\pi)$ define

$$B^+(\lambda,S(m)) = \lambda_m + \sum_{\substack{j\in S(m)\cup\{S(m)+1\}\\ j\neq\pi^{-1}m,\, m\, m+1}} \lambda_j \qquad \text{if } S(m) \in \Sigma_0(\pi)$$

(7.4)

$$B^-(\lambda,S(m)) = \lambda_{\pi^{-1}m} + \sum_{\substack{j\in S(m)\cup\{S(m)+1\}\\ j\neq\pi^{-1}m,\, m\, m+1}} \lambda_j \qquad \text{if } S(m) \in \Sigma_0(\pi).$$

Note that if $\lambda \in \Delta^+(\lambda,\pi)$, $h \in H^{++}(\pi)$, $S(m) \in \Sigma_0(\pi)$, then

$$B^+(\lambda,S(m))J^+(h,S(m)) \leqslant h \cdot \lambda$$

and if $\lambda \in \Delta^-(\lambda,\pi)$, $h \in H^{+-}(\pi)$, $S(m) \in \Sigma_0(\pi)$, then

$$B^-(\lambda,S(m))J^-(h,S(m)) \leqslant h \cdot \lambda$$

(cf. (4.9) in [16]). Now we are ready to modify bound (7.2).

LEMMA 7. — *Suppose h lies in the $H(\pi)$ interior of $H^{++}(\pi)$, and let $\lambda \in \Delta^+(\pi)$ be such that $h \cdot \lambda = 1$. Then*

$$\text{Volume } Z^+(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} \left(B^+(\lambda,S)\right)^{-1}. \tag{7.5}$$

Suppose h lies in the $H(\pi)$ interior of $H^{+-}(\pi)$, and let $\lambda \in \Delta^-(\pi)$ be such that $h \cdot \lambda = 1$. Then

$$\text{Volume } Z^-(h,\pi) \leqslant \prod_{S\in\Sigma_0(\pi)} \left(B^-(\lambda,S)\right)^{-1} \tag{7.6}$$

(cf. Proposition 4.10 in [16]).

From now on we fix the permutation π and one of the subsimplices $\Delta^+(\pi) = \{\lambda \in \Delta(\pi) \mid \lambda_m \geqslant \lambda_{\pi^{-1}m}\}$ or $\Delta^-(\pi) = \{\lambda \in \Delta(\pi) \mid \lambda_m \leqslant \lambda_{\pi^{-1}m}\}$. Corresponding cone $H^{++}(\pi)$ in the first case and $H^{+-}(\pi)$ in the second case can be subdivided to a finite union of cones with simplex base. Note that this subdivision is not canonical unless $H^{++}(\pi)$ (correspondingly $H^{+-}(\pi)$) is itself a cone with a simplex base. Fix some subdivision. Each cone $\mathcal{C}$ intersects with the hyperplane $(h \cdot \lambda) = 1$ by simplex Δ_λ. Integral (7.1) decomposes to the sum of integrals like

$$\int_{\Delta_\lambda} \text{Volume}\left(Z^\pm(h,\pi)\right) dh. \tag{7.7}$$

According to bounds (7.5) and (7.6) each of integrals (7.7) is bounded by

$$\int_{\Delta_\lambda} \text{Volume}\left(Z^+(h,\pi)\right) dh \leqslant \text{Volume}\left(\Delta_\lambda\right) \cdot \prod_{S\in\Sigma_0(\pi)} \left(B^+(\lambda,S)\right)^{-1}$$

$$\int_{\Delta_\lambda} \text{Volume}\left(Z^-(h,\pi)\right) dh \leqslant \text{Volume}\left(\Delta_\lambda\right) \cdot \prod_{S\in\Sigma_0(\pi)} \left(B^-(\lambda,S)\right)^{-1}.$$

Let $v_1,\dots,v_{2g}$, where $2g = \dim H^{+\pm}(\pi) = \dim \mathcal{C} = m - N(\pi) + 1$ be extremals which span $\mathcal{C}$. We can choose vectors v_j to be positive. They are defined up to multiplication by positive scalars, and do not depend on λ. Fix collection of v_j. The vertices of the simplex Δ_λ are given by points $v_j/(v_j \cdot \lambda)$, where v_j does not depend on λ. Hence

$$\text{Volume}\left(\Delta_\lambda\right) = \text{const} \cdot \prod_{j=1}^{2g} \frac{1}{v_j \cdot \lambda}$$

(cf. 13.5 in [16]) where const is a constant that does not depend on λ.

PROPOSITION 3. — *For each subsimplex $\Delta^+(\pi)$, $\Delta^-(\pi)$, $\pi \in \mathfrak{R} \subset \mathfrak{S}^0_m$, and for each cone $\mathcal{C}$ in the corresponding space $H^{+\pm}(\pi)$ function*

$$f^\pm(\lambda) = \left(\prod_{j=1}^{2g} \frac{1}{v_j \cdot \lambda}\right) \prod_{S\in\Sigma_0(\pi)} \left(B^\pm(\lambda,S)\right)^{-1} \tag{7.8}$$

is integrable over the corresponding subsimplex $\Delta^\pm(\pi)$.

Proof. — Consider one of the simplices $\Delta^+(\pi)$ or $\Delta^-(\pi)$ in the standard simplex $\Delta^{m-1} = \{\lambda \in \mathbb{R}^m_+ \mid \sum_{1 \leqslant i \leqslant m} \lambda_i = 1\}$. We use the following change of coordinates to replace domain of f by standard simplex Δ^{m-1}. For $\Delta^+(\pi)$, that is for subsimplex $\lambda_m \geqslant \lambda_{\pi^{-1}m}$ we define

$$\begin{cases} \lambda_{\pi^{-1}m} & = \frac{1}{2}\lambda'_{\pi^{-1}m} \\ \lambda_m & = \lambda'_m + \frac{1}{2}\lambda'_{\pi^{-1}m} \\ \lambda_j & = \lambda'_j \quad \text{for } j \neq \pi^{-1}m,\ m. \end{cases} \tag{7.9}$$

For $\Delta^-(\pi)$ we define

$$\begin{cases} \lambda_{\pi^{-1}m} & = \lambda'_{\pi^{-1}m} + \frac{1}{2}\lambda'_m \\ \lambda_m & = \frac{1}{2}\lambda'_m \\ \lambda_j & = \lambda'_j \quad \text{for } j \neq \pi^{-1}m,\ m. \end{cases} \tag{7.10}$$

Consider vectors $v_1{}', \ldots, v_{2g}{}'$ such that

$$v_j \cdot \lambda(\lambda') = v_j{}' \cdot \lambda' \qquad 1 \leqslant j \leqslant 2g.$$

Denote

$$B'(\lambda', S) := \begin{cases} B^+(\lambda(\lambda'), S) & \text{if } \lambda \in \Delta^+(\pi) \\ B^-(\lambda(\lambda'), S) & \text{if } \lambda \in \Delta^-(\pi). \end{cases}$$

Consider induced function

$$f(\lambda') = f^{\pm}(\lambda(\lambda')) = \left(\prod_{j=1}^{2g} \frac{1}{v'_j \cdot \lambda'}\right) \prod_{S \in \Sigma_0(\pi)} B'(\lambda', S)^{-1} \tag{7.11}$$

on Δ^{m-1}.

LEMMA 8. — *Consider a subset $W \subset \{1, 2, \ldots, m\}$, $0 < \operatorname{Card} W < m$. Then number of factors $N'(W)$ in (7.11) which depend only on the variables with subscripts in W is strictly less than* Card W

$$N'(W) < \operatorname{Card} W$$

(cf. the statement following 13.6 in [16]).

Proof. — Note that components of the vectors v_j, $1 \leqslant j \leqslant$, are nonnegative. All components of (co)vectors $B^{\pm}(\lambda, S)$, $S \in \Sigma_0(\pi)$, are nonnegative as well. Coordinates are modified in (7.9) and (7.10) by nonnegative matrices. It means that if some factor in (7.11) depends only on the variables λ'_j with subscripts in W, then the corresponding factor in (7.8) also depends only on the variables with subscripts in W. Hence $N'(W) \leqslant N(W)$, where $N(W)$ is the number $N(W)$ of those factors in (7.8) which depend only on the variables with subscripts in W.

By construction all the vectors $v_j \in H^{+\pm}(\pi) \subset H^+(\pi)$, $1 \leqslant j \leqslant 2g = \dim H^+(\pi)$, are linearly independent. Due to definition (7.3) for every factor $B(\lambda, S)^{-1}$, $S \neq S(m)$ with subscripts in W corresponding b_S also has subscripts in W. Note that if $S(m) \in \Sigma_0(\pi)$ and $B'(\lambda, S(m))$ has subscripts in W, then due to definitions (7.4) and to the form of corresponding changes of coordinates (7.9) and (7.10) we get $\pi^{-1}m, m \in W$. Hence if $S(m) \in \Sigma_0(\pi)$ and $B'(\lambda, S(m))$ has subscripts in W, then $b_{S(m)}$ is supported on W.

For those W, such that $\pi^{-1}m$, $m \notin W$ the statement of the lemma follows from Proposition 12.8 in [16].

For those W, which contain both $\pi^{-1}m$, $m \in W$, the statement of the lemma follows from Lemma 6.

Now we have to consider cases of subsimplex $\Delta^+(\pi)$ and $\Delta^-(\pi)$ separately. Suppose we started with the subsimplex $\Delta^+(\pi)$. Then the case $\pi^{-1}m \in W$, $m \notin W$ follows from Lemma 5. Consider the rest case, when $\pi^{-1}m \notin W$, $m \in W$. Due to our change of coordinates (7.9), each factor in (7.11) containing variable λ'_m would necessarily contain $\lambda'_{\pi^{-1}m}$. Hence none of them would be counted towards $N'(W)$ for the W like ours. Hence

$$N'(W) = N'(W \setminus m) \leqslant N(W \setminus m) \leqslant \operatorname{Card} W - 1,$$

and we obtain desired strict inequality.

We use similar arguments for the subsimplex $\Delta^-(\pi)$ to complete the proof of Lemma 8. □

To complete the proof of Proposition 3 we apply Proposition 13.2 in [16] to function $f(\lambda')$. For every subset $W \subset \{1, 2, \ldots, m\}$, $0 < \operatorname{Card} W < m$ we define $f_W(\lambda')$ to be the product of all factors in (7.11) which have subscripts in W, and we define $D(W, \lambda')$ to be the product of the rest factors. Functions $f_W(\lambda')$, and $D(W, \lambda')$ obey all conditions of Proposition 13.2 in [16], except that $f(\lambda')$ is homogeneous of degree $-m$ on $\mathbb{R}^m_+$, which does not affect the proof of this proposition. Proposition 3, and hence Theorem 1 are proved. □

8. Ergodicity of the map $\mathcal{G}$.

Now we can prove ergodicity of the map $\mathcal{G}$. In fact, since we have already proved that the section $\Upsilon^{\pm}$ has finite "area" ergodicity of $\mathcal{G}$ follows from the ergodicity of the Teichmüller geodesic flow on the corresponding connected component of the corresponding stratum in the space of quadratic differentials, see [9], [7], [19], [20]. We prefer to present an independent direct proof. The proof is similar to the proof of ergodicity of Rauzy induction $\mathcal{T}$ (c.f. Theorem 13.8 in [16], and Theorem 1.11 in [6]).

Proof. — Let A be a matrix such that $\det A = 1$, with some of the entries possibly negative. Consider projective linear map $T_A : \lambda \mapsto \dfrac{A\lambda}{\|A\lambda\|}$ and suppose T_A maps some compact subset $K \subset \Delta^{m-1}$ into Δ^{m-1}, $\mathrm{Im}\,(K) \subseteq \Delta^{m-1}$. Let J_A be Jacobian of T_A. Then according to (7.1) and (7.2) in [15]

$$\sup_{\lambda,\lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \leqslant \sup_{\substack{\lambda,\lambda' \in K \\ 1 \leqslant i \leqslant m}} \left(\frac{\lambda_i}{\lambda'_i}\right)^m .$$

Consider a subset $\Delta_\epsilon = \{\lambda \mid \lambda_i \geqslant \epsilon, \quad i = 1, \dots, m; \ \sum \lambda_i = 1\}$ Then for any $K \subseteq \Delta_\epsilon$ and any matrix $A \in \mathrm{SL}\,(m)$ such that $A(K) \subseteq \Delta^{m-1}$ we get from the estimate above, that

$$\sup_{\lambda,\lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \leqslant \left(\frac{1}{\epsilon}\right)^m .$$

Note that this estimate does not depend neither on A nor on the subset K anymore. We remind that

$$\mathcal{G}^k(\lambda, \pi_0) = \left(\frac{A\lambda}{\|A\lambda\|}, \pi\right), \quad \det A = 1.$$

Consider the set $\Delta_{\mathcal{G}}(\lambda, \pi_0, k) \in \Delta^{m-1}$ of (λ', π_0) for which $\mathcal{G}^k$ uses the same matrix A. Then $\mathcal{G}^k(\lambda, \pi_0)$ maps $\Delta_G(\lambda, \pi_0, k)$ onto one of the $(\Delta^+(\pi), \pi)$, $(\Delta^-(\pi), \pi)$.

Consider analogous subsimplices $\Delta_{\mathcal{T}}(\lambda, \pi_0, k)$ corresponding to Rauzy induction $\mathcal{T}$. It is known that diameters of subsimplices $\Delta_{\mathcal{T}}(\lambda, \pi_0, k)$ tend to zero as $k \to \infty$ for almost all λ (see [16] and [6]). (Actually this set of full measure is exactly the set of uniquely ergodic transformations.) Since $\Delta_{\mathcal{G}}(\lambda, \pi_0, k) = \Delta_{\mathcal{T}}(\lambda, \pi_0, l(k))$ for some $l(k)$ we conclude, that diameters of the subsimplices $\Delta_G(\lambda, \pi_0, k)$ tend to zero for almost all λ as well. Hence up to a set of measure zero we can subdivide Δ_ϵ to subsimplices $\Delta_{\mathcal{G}}(\lambda_j, \pi_0, k)$, $\lambda_j \in \Delta_\epsilon$ Suppose now E is an invariant subset under the mapping $\mathcal{G}$. If for some $\epsilon > 0$ we have $\mu(E \cap \Delta_\epsilon) < \mu(\Delta_\epsilon)$, then, probably refining our subdivision, for any $\delta > 0$ we will find a subsimplex $\Delta_0 = \Delta_G(\lambda_0, \pi_0, k_0)$ from our subdivision such that $\mu(E \cap \Delta_0)/\mu(\Delta_0) < \delta$. Let $(\Delta^{\pm}(\pi), \pi) = \mathcal{G}^{k_0}(\Delta_G(\lambda_0, \pi_0, k_0)$. Then $\mu(E \cap (\Delta^{\pm}, \pi)) \leqslant \delta/\epsilon^m$. Since δ is arbitrary small, we can find some π, such that $\mu(E \cap (\Delta^{\pm}, \pi)) = 0$. Combining this with the following lemma we complete the proof of ergodicity of $\mathcal{G}$. □

LEMMA 9. — *The only invariant subcollections of simplices of the form $(\Delta^{\pm}, \pi)$, $\pi \in \mathfrak{R}(\pi_0)$ are $\varnothing$ and $(\Delta^+ \cup \Delta^-) \times \mathfrak{R}(\pi_0)$.*

Proof. — Consider the oriented graph representing Rauzy class $\mathfrak{R}(\pi_0)$. Any ordered pair of vertices of this graph can be joined by an oriented path (see [16]).

Consider the following oriented graph, responsible for the map $\mathcal{G}$. We enumerate the set of vertices of the new graph by duplicated set $\mathfrak{R}(\pi_0)$, providing each $\pi \in \mathfrak{R}(\pi_0)$ with additional superscript "+" or "−". We join π_1^+ with π_2^-, $\pi_1, \pi_2 \in \mathfrak{R}(\pi_0)$, by an arrow, if there is some $(\lambda, \pi_1) \in (\Delta^+(\pi_1), \pi_1)$ which is mapped by $\mathcal{G}$ to $(\Delta^-(\pi_2), \pi_2)$. Similarly we join π_1^- with π_2^+, $\pi_1, \pi_2 \in \mathfrak{R}(\pi_0)$, by an arrow, if there is some $(\lambda, \pi_1) \in (\Delta^-(\pi_1), \pi_1)$ which is mapped by $\mathcal{G}$ to $(\Delta^+(pi_2), \pi_2)$. (Note that points of $\Delta^{\pm}$ are always mapped to points of $\Delta^{\mp}$.) To prove the lemma we need to prove that any ordered pair of vertices of the graph just constructed can be connected by an oriented path.

First note that for each $\pi \in \mathfrak{R}(\pi_0)$ there is a pair of arrows going in opposite directions joining π^+ and π^-. This arrows come from the points determining $q(\lambda, \pi) = 0$, see (3.5). Next note that for each edge of the graph, corresponding to Rauzy induction, which goes from the vertex π_1 to vertex π_2, there is corresponding edge of the new graph, which joins either edges π_1^+ and π_2^- or edges π_1^- and π_2^+, depending on whether the initial edge of the Rauzy graph was of type "a" or "b" correspondingly (see (2.4)). Note also that there is a natural orientation preserving projection of the new graph to Rauzy graph, which sends each pair of vertices π^+ and π^- to vertex π, and each edge of the new graph to the oriented chain of the edges of Rauzy graph.

Now having an arbitrary pair of vertices π_1^+ and π_2^- we construct an oriented path in the Rauzy graph joining π_1 and π_2. Taking into consideration remarks above it is easy to "lift this path up" to the new graph. Lemma is proved, and hence Theorem 2 is proved as well. □

9. Lyapunov exponents.

We will need several facts concerning *quotient* cocycles.

Consider a map $g : Y \to Y$ preserving a probability measure on the space Y. Consider a cocycle $A(y)$, $y \in Y$, with the values in the group $GL(m)$, i.e., a $GL(m)$-valued measurable function on Y. We remind that cocycle C is called *measurable* if the function $\log^+ \|C\|$ is integrable. Here $\|C\|$ is some norm of the matrix, and $\log^+$ is defined by (4.1). Suppose that corresponding fiberwise linear mapping on the total space of the trivialized linear bundle has measurable invariant subbundle $K(y) \subset \mathbb{R}^m$, i.e., $A(y) : K(y) \to K(g(y))$ almost everywhere. We can consider restriction $A|_K$ of A to K. Subspace $K(y)$ has the natural induced norm; *restricted cocycle* is obviously measurable, and collection of its Lyapunov exponents at a point y coincides with corresponding subcollection of Lyapunov exponents of the initial cocycle A.

Consider now one more measurable subbundle $L(y) \subset \mathbb{R}^m$ and assume that $L(y)$ is transversal to $K(y)$ almost everywhere, and that $\dim L + \dim K = m$. Assume for simplicity that we choose the Euclidian norm as a norm in $\mathbb{R}^m$. One can easily generalize the condition below to the case of other norms. Consider the angle $\widehat{K(y), L(y)}$ between linear subspaces $K(y)$ and $L(y)$. We will assume that

$$\log \sin(\widehat{K(y), L(y)}) \in L^1(Y, \mu)$$

which is equivalent to

$$\log(\widehat{K(y), L(y)}) \in L^1(Y, \mu). \tag{9.1}$$

This obviously works, say, if we choose $L = K^\perp$, or if, say, for almost all $y \in Y$ the angle is separated from zero by some constant. We define the *quotient cocycle* C on the subbundle L as follows: let

$$A(y) : v \mapsto v^{(1)} + w^{(1)}$$

where $v \in L(y)$ and $v^{(1)} \in L(g(y))$, $w^{(1)} \in K(g(y))$. We define $C(y)$ as

$$C(y) : v \to v^{(1)}.$$

In other words $C(y) := Pr(g(y)) \circ A(y)|_{L(y)}$ where $Pr : \mathbb{R}^m_{(y)} \to L(y)$ is the operator of projection to L along K. Note that since K is invariant we get

$$C^{(k)}(y) = Pr\big(g^{(k)}(y)\big) \circ A^{(k)}(y)|_{L(y)}.$$

LEMMA 10. — *Under assumptions of condition (9.1) quotient cocycle is measurable.*

Proof. — Since

$$\|C(y)\| \leqslant \frac{\|A(y)\|}{\sin(\widehat{K(g(y)), L}(g(y)))}$$

we get

$$\log^+ \|C(y)\| \leqslant \log^+ \|A(y)\| - \log\sin(\widehat{K(g(y)), L}((y)))$$

and both functions in the right-hand side of the inequality are integrable.

□

Let

$$\theta(A, y, v) := \lim_{k\to+\infty} \frac{1}{k} \log \|A^{(k)}(y) \cdot v\|. \tag{9.2}$$

Define similarly $\theta(C, y, v)$.

LEMMA 11. — *For almost any point $y \in Y$ and for any $v \in L(y)$ the limits $\theta(A, y, v)$ and $\theta(C, y, v)$ exist and*

$$\theta(A, y, v) \geqslant \theta(C, y, v)$$

(note that we do not assume ergodicity of g).

Proof. — Due to multiplicative ergodic theorem (Oseledets theorem [12]) the limits above exist for both of our measurable cocycles for the set of full measure in Y. Take the intersection of this two sets of full measure. We have

$$\|C^{(k)}(y) \cdot v\| \leqslant \frac{\|A^{(k)}(y) \cdot v\|}{\sin\big(\widehat{K(g^{(k)}(y)), L}(g^{(k)}(y))\big)}.$$

Hence

$$\lim_{k\to+\infty}\frac{1}{k}\log^+\|C^{(k)}(y)\| \leqslant \lim_{k\to+\infty}\frac{1}{k}\log^+\|A^{(k)}(y)\| - \lim_{k\to+\infty}\frac{1}{k}\log\sin\big(\widehat{K(g^{(k)}(y)),L(g^{(k)}(y))}\big).$$

Due to Ergodic theorem assumption (9.1) implies that the last limit in expression above is equal to zero for almost all $y \in Y$. □

Let us prove Proposition 1.

Proof. — Choose the norm

$$\|B\| := m\cdot\max_{i,j}|B_{ij}|.$$

Recalling the definitions (3.2)–(3.7) of the nonnegative integer matrix-valued function $B(\lambda,\pi)$ we see, that the following inequalities for the entries of matrix B are valid:

$$B_{i,j}(\lambda,\pi) \leqslant \frac{\lambda_{\pi^{-1}(m)}}{\lambda_m} \leqslant \frac{1}{\lambda_m} \quad \text{when } \lambda\in\Delta^-(\pi)$$

and

$$B_{i,j}(\lambda,\pi) \leqslant \frac{\lambda_m}{\lambda_{\pi^{-1}(m)}} \leqslant \frac{1}{\lambda_{\pi^{-1}(m)}} \quad \text{when } \lambda\in\Delta^+(\pi).$$

To prove integrability of the function $\log\|B(\lambda,\pi)\|$ over $\bigsqcup_{\pi\in\mathfrak{R}}\Delta^+(\pi)\sqcup\Delta^-(\pi)$ with respect to the measure μ, it is sufficient to prove integrability of the function

$$h(\lambda,\pi)=\begin{cases}\log\lambda_m & \text{if } \lambda\in\Delta^+(\pi)\\ \log\lambda_{\pi^{-1}(m)} & \text{if } \lambda\in\Delta^-(\pi).\end{cases}$$

To prove integrability of h, it is sufficient to prove for each $\pi\in\mathfrak{R}$ integrability of the product $h(\lambda')f(\lambda')$ of h with the function f in (7.11), bounding the density of μ, over the standard simplex Δ^{m-1}, now already with respect to Lebesgue measure. We do it the same way, as we proved integrability of f in section 7. We use Lemma 8 and then trivially modify the proof of Proposition 13.2 in [16] to fit our case. Proposition 1 is proved.

□

To prove Corollary 1 note that $|\det B(\lambda,\pi)|=1$. Hence

$$\|B^{-1}\| = m\cdot\max_{ij}|B^{-1}_{ij}| \leqslant m\cdot(m-1)!\cdot\left(\max_{ij}|B_{ij}|\right)^{m-1} \leqslant \|B\|^{m-1}.$$

Hence

$$\log^+ \|B^{-1}\| \leqslant (m-1)\log\|B\|$$

which implies that the function $\log^+ \|B^{-1}(\lambda,\pi)\|$ is integrable. Corollary 1 is proved.

Since the function $\log^+ \|B^{-1}(\lambda,\pi)\|$ is integrable, we can use multiplicative ergodic theorem to study products $B^{-1}(\mathcal{G}^k(\lambda,\pi))\cdot\ldots\cdot B^{-1}(\lambda,\pi)$. To prove Theorem 3 let us first prove the following

LEMMA 12. — *At least $m-2g$ Lyapunov exponents are equal to zero, i.e., there is some j, $1 \leqslant j \leqslant 2g+1$ such that*

$$\theta_j = \theta_{j+1} = \ldots = \theta_{j+m-2g-1} = 0.$$

Proof. — We need to consider only nontrivial case when $m > 2g$. Consider the $(m-2g)$-dimensional subspace $K(\pi) = \operatorname{Ker}(\Omega(\pi))$ — the kernel of the "degenerate symplectic form" $\Omega(\pi)$, see (2.1). Let

$$(\lambda^{(k)},\pi^{(k)}) = \mathcal{G}^{(k)}(\lambda,\pi).$$

According to [17] $K(\pi^{(1)}) = B^{-1}(\lambda,\pi)K(\pi)$, i.e., the kernel is preserved by our cocycle. The collection of vectors b_S, $S \in \Sigma_0(\pi)$, (see (2.2) and (2.3)) provides the canonical basis in $K(\pi)$, see [17]. Moreover, our cocycle maps the canonical basis in $K(\pi)$ to the canonical basis in $K(\pi^{(1)})$, see Lemma 5.6 in [17]. For any vector $b_S(\pi)$ from the canonical basis one has $1 \leqslant \|b_S(\pi)\| \leqslant m$ for any π. Since $\left(B^{(k)}(\lambda,\pi)\right)^{-1}$ maps this basis to the corresponding canonical basis in $K(\pi^{(k)})$ one has

$$1 \leqslant \left\|\left(B^{(k)}(\lambda,\pi)\right)^{-1}\cdot b(\pi)\right\| \leqslant m.$$

For every point (λ,π) having infinite orbit under iterations of the map $\mathcal{G}$ we have presented $m-2g$ linearly independent vectors $b_S(\pi)$, $S \in \Sigma_0(\pi)$, such that

$$\lim_{k\to\infty}\frac{1}{k}\log\left\|\left(B^{(k)}(\lambda,\pi)\right)^{-1}\cdot b(\pi)\right\| = 0.$$

Hence at least $m-2g$ Lyapunov exponents of the cocycle $B^{-1}(\lambda,\pi)$ are equal to zero. Lemma 12 is proved. □

Recall, that there is natural local identification between the space $\mathbb{R}_+^m$ of interval exchange transformations with fixed permutation $\pi \in \mathfrak{S}_m^0$ and the first relative cohomology $H^1(M_g^2,\{\text{saddles}\};\mathbb{R})$ of corresponding surface M_g^2 with respect to the set of saddles of corresponding foliation (see [4]). Recall that the saddles are enumerated by the classes $S \in \Sigma(\pi)$ (see section 6 in [16]). Consider the following terms of the exact sequence of the pair $\{set\ of\ saddles\} \subset M_g^2$

$$\begin{array}{cccccccc} 0 = & H^0(M_g^2, \{\text{saddles}\}; \mathbb{R}) & \to & H^0(M_g^2; \mathbb{R}) = \mathbb{Z} & \to & H^0(\text{saddles}; \mathbb{R}) & \to \\ \to & H^1(M_g^2, \{\text{saddles}\}; \mathbb{R}) & \to & H^1(M_g^2; \mathbb{R}) & \to & H^1(\text{saddles}; \mathbb{R}) = 0. \end{array}$$

We present now two statements based on the results in [22].

LEMMA 13. — *Under identification with cohomology, vector b_S represents the image of the element in H^0(saddles; $\mathbb{R}$) dual to the saddle corresponding to the class S. In particular the $(m-2g)$-dimensional image of H^0(saddles; $\mathbb{R}$) is spanned by vectors b_S, $S \in \Sigma(\pi)$ and hence coincides with Ker $(\Omega(\pi))$.*

PROPOSITION 4. — *Under local identification of the space of interval exchange transformations with relative cohomology $H^1(M_g^2$, saddles; $\mathbb{R})$ the quotient space over the subspace spanned by vectors b_S, $S \in \Sigma(\pi)$, coincides with the absolute cohomology $H^1(M_g^2; \mathbb{R})$. The symplectic structure induced by $\Omega(\pi)$ on the quotient space coincides with the intersection form on cohomology.*

Remark 4. — Recalling the definition (5.1) of the space $H(\pi)$ as the annulator of the subspace spanned by vectors b_S, $S \in \Sigma(\pi)$, we see, that $H(\pi)$ is locally identified in our setting with the absolute homology $H_1(M_g^2; \mathbb{R})$.

Remark 5. — Suppose $\mathcal{G}^k(\lambda, \pi)$ preserves the permutation π. In [16] W. Veech constructs the pseudo Anosov diffeomorphism determined by the matrix $B^k(\lambda, \pi)$. We note that the automorphism in cohomology $H^1(M_g^2; \mathbb{R})$ defined by $B^{(k)}(\lambda, \pi)$ above, coincides with the automorphism in cohomology, induced by the corresponding pseudo Anosov transformation.

Now let us prove the relation

$$\theta_k = -\theta_{m-k+1} \qquad \text{for } k = 1, \ldots, g.$$

Proof. — Matrix $\Omega(\pi)$ defined by (2.1) provides us with the "degenerate symplectic form" in the fibers of $(\Delta^{m-1} \times \mathfrak{R}) \times \mathbb{R}^m$. This form is preserved by the cocycle $\left(B^{(k)}\right)^{-1}$ (see, say, [10]):

$$\Omega(\pi)=\left(\left(B^{(k)}(\lambda,\pi)\right)^{-1}\right)^{T}\cdot\Omega(\pi^{(k)})\cdot\left(B^{(k)}(\lambda,\pi)\right)^{-1}.$$

In those rare cases, when Rauzy class $\mathfrak{R}$ determines nondegenerate form Ω the statement follows directly from result in [3], where it is proved that if a cocycle has values in symplectic matrices then the Lyapunov exponents appear in pairs θ, $-\theta$. The fact, that our symplectic structure is different over different simplices $\Delta^{m-1}\times\pi$, where $\pi\in\mathfrak{R}$, does not affect the statement. Indeed, we can induce our map to some simplex $\Delta^{m-1}\times\pi$. The induced cocycle would now preserve the fixed symplectic form $\Omega(\pi)$, and hence the result in [3] would be directly applicable. But Lyapunov exponents of the induced cocycle are proportional to the ones of the initial cocycle with coefficient of proportionality equal to the inverse of $\mu\left(\Delta^{m-1}\times\pi\right)$ (see [21]).

In general $\Omega(\pi)$ has a kernel. This kernel determines $(m-2g)$-dimensional subbundle $K(\lambda,\pi)=K(\pi)\subset\mathbb{R}^m$ in our trivialized vector bundle over $(\Delta^{m-1}\times\mathfrak{R})$. This subbundle is invariant under the action of the cocycle $B^{-1}(\lambda,\pi)$. According to Lemma 12 all Lyapunov exponents of the cocycle $B^{-1}(\lambda,\pi)$ restricted to the subbundle K are equal to zero.

Consider the quotient cocycle on the quotient bundle $\mathbb{R}^m/K$. Since we are taking the quotient over the kernel of $\Omega(\pi)$, we can induce the form $\Omega(\pi)$ to the quotient bundle to get there nondegenerate symplectic form. This symplectic form is obviously preserved by the quotient cocycle. Applying the arguments mentioned above to the quotient cocycle we see that the Lyapunov exponents of the quotient cocycle are distributed into pairs $\tilde{\theta}_i$, $-\tilde{\theta}_i$, $1\leqslant i\leqslant g$. Applying Lemma 11 to our case we see that the whole collection of Lyapunov exponents of the cocycle $B^{-1}(\lambda,\pi)$ is obtained by joining $(m-2g)$ zero ones (corresponding to the cocycle restricted to K), with the rest ones, which are in the one-to-one correspondence with ones of the quotient cocycle. Moreover, for every pair we have inequality

$$\theta_i\geqslant\tilde{\theta}_i. \tag{9.3}$$

Now note that the sum of all Lyapunov exponents of the quotient cocycle equals zero, since it is symplectic. The sum of all Lyapunov exponents of the cocycle $B^{-1}(\lambda,\pi)$ is also equal to zero since $\det B(\lambda,\pi)=1$. Hence in (9.3) we have equalities for all pairs of exponents. □

To complete the proof of Theorem 3 we have to show that $\theta_1>\theta_2$.

Proof. — It is easy to find a point (λ_0, π_0) and a positive k such that the matrix $B := B^k(\lambda_0, \pi_0)$ is strictly positive. (In fact, almost all points have this property.) There is a whole neighborhood $O(\lambda, \pi, k)$ of the points sharing the same matrix $B^k(\lambda, \pi) = B^k(\lambda_0, \pi_0) = B$; this neighborhood has nontrivial measure. The following strict inequality is valid for the matrix B:

$$\min_{i,j,r,s} \frac{B_{ir}B_{js}}{B_{is}B_{jr}} > 0.$$

The map $\mathcal{G}$ is ergodic with respect to finite measure, and hence the general results in section 6 of [18] imply the desired inequality $\theta_1 > \theta_2$. Theorem 3 is proved. $\square$

Let us prove now Theorem 4.

Proof. — Denote

$$(\lambda^{(k)}, \pi^{(k)}) := \mathcal{G}^k(\lambda, \pi).$$

We see, that vectors $\left(B^{(k)}(\lambda, \pi)\right)^{-1} \cdot \lambda$ and $\lambda^{(k)}$ are proportional. Let

$$r(\lambda, \pi) := \frac{\|B^{-1}(\lambda, \pi) \cdot \lambda\|}{\|\lambda\|}$$

be coefficient of contraction for one iteration. To prove that some number θ belongs to the collection of Lyapunov exponents it is sufficient to present for a set of points (λ, π) of nonzero measure a vector $v(\lambda, \pi) \in \mathbb{R}^m$ such that

$$\lim_{k\to+\infty} \frac{1}{k} \log \frac{\|\left(B^{(k)}(\lambda, \pi)\right)^{-1} \cdot v(\lambda, \pi)\|}{\|v(\lambda, \pi)\|} = \theta.$$

Let

$$v(\lambda, \pi) := \lambda.$$

Then

$$\begin{aligned}
&\frac{1}{k} \log \frac{\|\left(B^{(k)}(\lambda, \pi)\right)^{-1} \cdot \lambda\|}{\|\lambda\|} \\
&\quad = \frac{1}{k} \log\bigl(r(\mathcal{G}^{k-1}(\lambda, \pi)) \cdot \ldots r(\mathcal{G}(\lambda, \pi)) \cdot r(\lambda, \pi)\bigr) \\
&\quad = \frac{1}{k} \bigl(\log r(\lambda, \pi) + \log r(\mathcal{G}(\lambda, \pi)) + \ldots + \log r(\mathcal{G}^{k-1}(\lambda, \pi))\bigr).
\end{aligned}$$

Applying Ergodic theorem to the sum above we prove that the following number θ is present in the collection of Lyapunov exponents

$$\theta = \sum_{\pi \in \mathfrak{R}} \int_{\Delta^{\pm}(\pi)} \left(\log \|B^{-1}(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\|\right) d\mu. \tag{9.4}$$

Note that in fact we have absolute freedom in choosing the norm $\| \ \|$. Choosing the norm $\|v\| := |v_1| + \ldots + |v_m|$ we will get for $\lambda \in \Delta^{\pm}$

$$\log \|B^{-1}(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\| = \log(1 - \nu(\lambda, \pi)) - \log 1$$

where $\nu(\lambda, \pi)$ is defined by (3.6). Choosing for $v \in \mathbb{R}^m \times \pi$ another norm

$$\|v\| = |v_1| + \ldots + |v_{\pi^{-1}(m)-1}| + |v_{\pi^{-1}(m)+1}| + \ldots + |v_{m-1}| + \max(|v_m|, |v_{\pi^{-1}(m)}|)$$

we get

$$\log \|B(\lambda, \pi) \cdot \lambda\| - \log \|\lambda\| = \begin{cases} \log(1 - \lambda_{\pi^{-1}(m)}) - \log(1 - \lambda_m) & \text{for } \lambda \in \Delta^-(\pi) \\ \log(1 - \lambda_m) - \log(1 - \lambda_{\pi^{-1}(m)}) & \text{for } \lambda \in \Delta^+(\pi). \end{cases}$$

Remark 6. — Note that the second norm is different for the spaces $\mathbb{R}^m$ corresponding to different $\Delta^{\pm}(\pi)$. In fact, we should consider $\mathbb{R}^m$ as a fiber of a trivialized vector bundle over $\bigsqcup_{\pi \in \mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$, and we can even choose the norm, which would differ (continuously) from fiber to fiber. It is easy to see, that the integral (9.4) would be the same anyway.

Note that expressions (4.2) and (4.3) for θ_1 in the statement of Theorem 4 differ from the corresponding expressions for θ above only by a sign. Since we already proved, that $\theta_1 = -\theta_m$, to complete the proof of Theorem 4 we just need to prove, that Lyapunov exponent θ computed above is the smallest one, i.e., that $\theta = \theta_m$. This is true since for almost every point $(\lambda, \pi) \in \Delta^{\pm}(\pi)$ and for every $w \in \Delta^{\pm}(\pi)$

$$\lim_{k \to \infty} \frac{B^{(k)}(\lambda, \pi) w}{\|B^{(k)}(\lambda, \pi) w\|} = \lambda.$$

Hence the whole space $\mathbb{R}^m$ is asymptotically contracted by $B^{(k)}(\lambda, \pi)$ to the one-dimensional subspace spanned by λ as k tends to infinity. Theorem 4 is proved. □

We complete this section by proving Proposition 2.

Proof. — We remind that the cocycle $B^{-1}(\lambda, \pi)$ has a nice invariant one-dimensional subbundle corresponding to the smallest Lyapunov exponent $-\theta_1$. The fiber of this subbundle over a point (λ, π) is just $\langle \lambda \rangle_{\mathbb{R}}$, i.e., it is spanned by the vector λ. We will call this subbundle as *tautological* bundle. Consider the quotient vector bundle and the induced quotient cocycle on it.

As a representative of the cohomology class of the quotient cocycle we can choose the quotient cocycle $C^{-1}(\lambda,\pi)$ on the hyperplane $L = \{v \in \mathbb{R}^m \mid v_1+\ldots+v_m = 0\}$, or any other hyperplane transversal to any $\lambda \in \mathbb{R}^m_+$. We can identify the quotient bundle $\mathbb{R}^m/\langle\lambda\rangle_{\mathbb{R}}$ with the trivialized vector bundle with the fiber L. Since $\min_{\lambda\in\Delta^{m-1}} \sin(\widehat{\lambda, L}) > 0$ by the choice of the hyperplane L, we conclude that condition (9.1) is valid and Lemma 10 is applicable to our case. Hence cocycle $C^{-1}(\lambda,\pi)$ is measurable with respect to the measure μ.

Choose a point (λ,π) and consider the limits $\theta(B^{-1},(\lambda,\pi),v)$ and $\theta(C^{-1},(\lambda,\pi),v)$ defined by (9.2) We will show that in our case inequalities from Lemma 11 become equalities:

LEMMA 14. — *For almost any point $(\lambda,\pi) \in \Delta^{m-1}\times\mathfrak{R}$ and for any $v \in L$ the limits $\theta(B^{-1},(\lambda,\pi),v)$ and $\theta(C^{-1},(\lambda,\pi),v)$ exist and coincide :*

$$\theta(C^{-1}(\lambda,\pi),v) = \theta(B^{-1}(\lambda,\pi),v).$$

Proof. — Due to multiplicative ergodic theorem the limits above exist for both of our measurable cocycles for the set of full measure in $\Delta^{m-1}\times\mathfrak{R}$. Take the intersection Z of this two sets of full measure. Obviously $\mu(Z) = 1$.

Choose some small neighborhood $O_\varepsilon \in \Delta^{m-1}$ of the point $(1/m,\ldots, 1/m)$. Let

$$\alpha := \inf_{\lambda\in O_\varepsilon}\ \inf_{0\neq v\in\mathbb{R}^m\setminus(\mathbb{R}^m_+\cup\mathbb{R}^m_-)} (\widehat{\lambda, v})$$

where $\mathbb{R}^m\setminus(\mathbb{R}^m_+\cup\mathbb{R}^m_-)$ is the complement to the union of positive and negative cones, and $(\widehat{\lambda, v})$ is the angle between two vectors. By construction of O_ε we have $\alpha > 0$.

Note that since $B^{(k)}(\lambda,\pi)$ is nonnegative matrix for any (λ,π) we have

$$B^{(k)}(\lambda,\pi) : (\mathbb{R}^m_+\cup\mathbb{R}^m_-) \to (\mathbb{R}^m_+\cup\mathbb{R}^m_-).$$

Since $L\bigcap(\mathbb{R}^m_+\cup\mathbb{R}^m_-) = 0$ we conclude that for any $0\neq v\in L$ we have

$$\left(B^{(k)}(\lambda,\pi)\right)^{-1}\cdot v \notin (\mathbb{R}^m_+\cup\mathbb{R}^m_-).$$

For a set of points (λ, π) of full measure trajectory (λ, π), $\mathcal{G}(\lambda, \pi)$, $\mathcal{G}(\mathcal{G}(\lambda, \pi)), \ldots$ will visit $O_\varepsilon \times \pi$ infinitely many times. Let $\tilde{Z}$, $\mu(\tilde{Z}) = 1$, be intersection of this set with the set Z. Then for any $(\lambda, \pi) \in \tilde{Z}$ both limits $\theta(B^{-1}(\lambda, \pi), v)$ and $\theta(C^{-1}(\lambda, \pi), v)$ exist for any $v \in L$. On the other hand whenever $\mathcal{G}^{(k)}(\lambda, \pi) \in O_\varepsilon$ the angle between $\left(B^{(k)}(\lambda, \pi)\right)^{-1} \cdot v$ and λ is greater than or equal to α. Hence for such values k the norms

$$\|(B^{(k)})^{-1}(\lambda, \pi)v\| \leqslant a_1\|(C^{(k)})^{-1}(\lambda, \pi)v\|$$
$$\|(C^{(k)})^{-1}(\lambda, \pi)v\| \leqslant a_2\|(B^{(k)})^{-1}(\lambda, \pi)v\|$$

are mutually bounded by means of positive constants a_1 and a_2 depending only on the choice of L and O_ε. Since by construction for any point $(\lambda, \pi) \in \tilde{Z}$ we have infinitely many values k for which the relations above are valid we proved coincidence of our limits. □

Since we already know the Lyapunov exponents of the cocycle B^{-1}, and we know that the Lyapunov exponent corresponding to the tautological subbundle is equal to $\theta_m(B^{-1})$ we get the following obvious corollary of Lemma 14:

Corollary 3. — *The collection of Lyapunov exponents of the cocycle $C^{-1}(\lambda, \pi)$ coincides with $\theta_1(B^{-1}), \theta_2(B^{-1}), \ldots, \theta_{m-1}(B^{-1})$ (i.e., the collection is obtained by omitting the least Lyapunov exponent of the cocycle $B^{-1}(\lambda, \pi)$).* □

Consider the trivialized vector bundle with the base $\bigsqcup_{\pi \in \mathfrak{R}} \Delta^+(\pi) \sqcup \Delta^-(\pi)$ and a fiber $\mathbb{R}^m$. The map $\mathcal{G}$ and the cocycle $B^{-1}(\lambda, \pi)$ define the map on the total space of this bundle $(x, v) \mapsto (\mathcal{G}(x), B^{-1}(x)v)$, where $x = (\lambda, \pi)$ is a point in the base, and v is a vector in the fiber. Note, that the quotient of the trivialized bundle over tautological bundle is isomorphic to the tangent bundle over our base. Moreover, it is easy to see, that the composition of the induced action in the total space of the quotient bundle with fiberwise homothety with coefficient $\|B^{-1}\lambda\|^{-1}$ coincides with the action of the differential $D\mathcal{G}$ under suggested identification. In fact, we just use canonical isomorphism $TG_1(m) \cong \mathrm{Hom}(\gamma, \gamma^\perp)$, where γ is the tautological, and $\gamma^\perp$ is the normal bundle to the Grassmann manifold $G_1(m) = \mathbb{R}P^{m-1}$. The impact of the homothety can be easily computed since homothety commutes with our induced cocycle in the quotient bundle (and, actually with any fiberwise linear mapping). This impact is just a shift of all Lyapunov exponents by θ_1. Proposition 2 is proved. □

10. Appendix. Examples of Gauss measures. Values of Lyapunov exponents for $m = 2$ and 3.

For the interval exchange transformations of two and three subintervals we know all Lyapunov exponents.

In dimension two there is only one Rauzy class containing the only one permutation $(2, 1)$

$$\mathfrak{R}_2 := \mathfrak{R}((2,1)) = \{(2,1)\}.$$

Corresponding map $\mathcal{G}$ is conjugate to the duplication of the classical map $x \mapsto \{1/x\}$ related to Euclidean algorithm and to continuous fraction expansion. The highest Lyapunov exponent is equal to

$$\theta_1(\mathfrak{R}_2) = \frac{\pi^2}{12\log 2} = \frac{\mathrm{Li}_2(-1)}{\mathrm{Li}_1(-1)} \approx 1.18656911041562545 28...$$

where $\mathrm{Li}_n(x)$ is the n-polylogarithm. Note that $\theta_1(\mathfrak{R}_2)$ is exactly the Lévy constant responsible for the growth rate of denominator of continued fraction. The second Lyapunov exponent equals the first one taken with the opposite sign:

$$\theta_2(\mathfrak{R}_2) = -\theta_1(\mathfrak{R}_2).$$

In dimension three there is again only one Rauzy class containing three permutations:

$$\mathfrak{R}_3 := \mathfrak{R}((3,2,1)) = \{(3,2,1),(2,3,1),(3,1,2)\}.$$

The highest Lyapunov exponent is equal to

$$\begin{aligned}\theta_1(\mathfrak{R}_3) &= \frac{\pi^2}{6(1+2\log 2)} = \frac{\mathrm{Li}_2(1)}{1-2\mathrm{Li}_1(-1)}\\ &= \frac{2\mathrm{Li}_2(-1)}{2\mathrm{Li}_1(-1)-1} \approx 0.68932571507073294...\end{aligned}$$

The second Lyapunov exponent vanishes, and the third one is equal to the first one taken with the opposite sign:

$$\theta_2(\mathfrak{R}_3) = 0 \qquad \theta_3(\mathfrak{R}_3) = -\theta_1(\mathfrak{R}_3).$$

The first completely nontrivial case is interval exchange transformations of four subintervals. There are two Rauzy classes here; the following one is interesting for us:

$$
\begin{aligned}
\mathfrak{R}_4 &:= \mathfrak{R}((4,3,2,1)) \\
&= \{(4,3,2,1),(4,1,3,2),(2,4,3,1),(3,1,4,2),(2,4,1,3), \\
&\qquad (4,2,1,3),(3,2,4,1)\}.
\end{aligned}
$$

Corresponding surface has genus 2; corresponding measured foliation on this surface has single 6-prongs saddle. The other Rauzy class for $m = 4$ corresponds to surface of genus 1.

The measures of corresponding simplices $\Delta^+(\pi)$, $\pi \in \mathfrak{R}_4$ are as follows. (We do not normalize the total measure to 1 to avoid fractional expressions.)

$$
\begin{aligned}
\mu(\Delta^+(4,3,2,1)) &&= \frac{\pi^2}{2}\log 2 - \frac{5}{4}\zeta(3) &\approx 1.91797 \\
\mu(\Delta^+(4,1,3,2)) &= \mu(\Delta^+(2,4,3,1)) &= \frac{3}{4}\zeta(3) &\approx 0.901543 \\
\mu(\Delta^+(3,1,4,2)) &= \mu(\Delta^+(2,4,1,3)) &= \frac{5}{8}\zeta(3) &\approx 0.751286 \\
\mu(\Delta^+(4,2,1,3)) &= \mu(\Delta^+(3,2,4,1)) &= \frac{\pi^2}{6}\log 2 &\approx 1.14018.
\end{aligned}
$$

Rauzy class $\mathfrak{R}_4$ is invariant under operation of taking inverse permutation. The measures of the simplices $\Delta^-(\pi)$, $\pi \in \mathfrak{R}_4$ satisfy the following relation:

$$
\mu(\Delta^-(\pi)) = \mu(\Delta^+(\pi^{-1})).
$$

In this normalization the total measure is equal to

$$
\begin{aligned}
\mu\left(\bigsqcup_{\pi\in\mathfrak{R}} \Delta^{\pm}(\pi)\right) &= 3\,\zeta(3) + \frac{5}{3}\pi^2\log 2 \\
&= 3\mathrm{Li}_3(1) - 10\mathrm{Li}_2(1)\mathrm{Li}_1(-1) \approx 15.00798.
\end{aligned}
\tag{10.1}
$$

Here $\zeta(3) = \mathrm{Li}_3(1) \approx 1.20206$ is the value of the Riemann zeta function at 3.

The densities of the measure μ on the simplices $\Delta^+(\pi)$ in our normalization are as follows (compare with analogous densities in [16]):

Density on $\Delta^+(4,3,2,1)$, $\lambda_4 \geqslant \lambda_1$ is equal to:

$$\begin{aligned} f_{4321}(\lambda) &= \frac{1}{\lambda_1+\lambda_2}\,\frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_3+\lambda_4}\,\frac{1}{\lambda_2+\lambda_3+\lambda_4} \\ &= \frac{1}{\lambda_1+\lambda_2}\,\frac{1}{\lambda_2+\lambda_3}\,\frac{1}{1-\lambda_1} + \frac{1}{1-(\lambda_1+\lambda_2)}\,\frac{1}{\lambda_2+\lambda_3}\,\frac{1}{1-\lambda_1}. \end{aligned}$$

Density on $\Delta^+(4,1,3,2)$, $\lambda_4 \geqslant \lambda_1$ is equal to:

$$\begin{aligned} f_{4132}(\lambda) &= \frac{1}{\lambda_2+\lambda_4}\,\frac{1}{\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3}\left(\frac{1}{\lambda_2+\lambda_3+\lambda_4}\right. \\ &\qquad\left. + \frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\right) \\ &= \frac{1}{\lambda_2+\lambda_4}\,\frac{1}{1-\lambda_1}\,\frac{1}{1-\lambda_4} + \frac{1}{\lambda_3+\lambda_4}\,\frac{1}{1-\lambda_1}\,\frac{1}{1-\lambda_4} \\ &\qquad + \frac{1}{\lambda_2+\lambda_4}\,\frac{1}{\lambda_3+\lambda_4}\,\frac{1}{1-\lambda_1}. \end{aligned}$$

Density on $\Delta^+(2,4,3,1)$, $\lambda_4 \geqslant \lambda_2$ is equal to:

$$\begin{aligned} f_{2431}(\lambda) &= \frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\ &= \frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_3+\lambda_4}\,\frac{1}{1-\lambda_2}. \end{aligned}$$

Density on $\Delta^+(3,1,4,2)$, $\lambda_4 \geqslant \lambda_3$ is equal to:

$$\begin{aligned} f_{3142}(\lambda) &= \frac{1}{\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_4}\,\frac{1}{\lambda_1+\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\ &= \frac{1}{\lambda_4}\,\frac{1}{1-\lambda_2}\,\frac{1}{1-\lambda_3}. \end{aligned}$$

Density on $\Delta^+(2,4,1,3)$, $\lambda_4 \geqslant \lambda_2$ is equal to:

$$\begin{aligned} f_{2413}(\lambda) &= \frac{1}{\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\left(\frac{1}{\lambda_1+\lambda_3+\lambda_4}\right. \\ &\qquad\left. + \frac{1}{\lambda_2+\lambda_3+\lambda_4}\right) \\ &= \frac{1}{\lambda_3+\lambda_4}\,\frac{1}{1-\lambda_1}\,\frac{1}{1-\lambda_4} + \frac{1}{\lambda_3+\lambda_4}\,\frac{1}{1-\lambda_2}\,\frac{1}{1-\lambda_4}. \end{aligned}$$

Density on $\Delta^+(4,2,1,3)$, $\lambda_4 \geqslant \lambda_1$ is equal to:

$$\begin{aligned} f_{4213}(\lambda) &= \frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_2+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\left(\frac{1}{\lambda_1+\lambda_2+\lambda_4}\right.\\ &\qquad\left.+\frac{1}{\lambda_2+\lambda_3+\lambda_4}\right)\\ &= \frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_2+\lambda_4}\,\frac{1}{1-\lambda_1}+\frac{1}{\lambda_2+\lambda_3}\,\frac{1}{\lambda_2+\lambda_4}\,\frac{1}{1-\lambda_3}. \end{aligned}$$

Density on $\Delta^+(3,2,4,1)$, $\lambda_4 \geqslant \lambda_3$ is equal to:

$$\begin{aligned} f_{3241}(\lambda) &= \frac{1}{\lambda_4}\,\frac{1}{\lambda_1+\lambda_2}\,\frac{1}{\lambda_2+\lambda_3+\lambda_4}\,\frac{1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\\ &= \frac{1}{\lambda_4}\,\frac{1}{\lambda_1+\lambda_2}\,\frac{1}{1-\lambda_1}. \end{aligned}$$

The densities on the simplices $\Delta^-(\pi)$, $\pi \in \mathfrak{R}_4$ are obtained from ones on the corresponding simplices $\Delta^+(\pi^{-1})$ by means of the change of coordinates dictated by the corresponding permutation.

The highest Lyapunov exponent $\theta_1(\mathfrak{R}_4)$ is equal to the ratio

$$\begin{aligned} \theta_1(\mathfrak{R}_4) &= \frac{1}{\frac{3}{2}\zeta(3)+\frac{5}{6}\pi^2\log 2}\left(\int_{\Delta^+(4321)} f_{4321}(\lambda)\big(\log(1-\lambda_1)-\log(1-\lambda_4)\big)\,d\lambda\right.\\ &\quad+\int_{\Delta^+(4132)} f_{4132}(\lambda)\big(\log(1-\lambda_1)-\log(1-\lambda_4)\big)\,d\lambda\\ &\quad+\int_{\Delta^+(2431)} f_{2431}(\lambda)\big(\log(1-\lambda_2)-\log(1-\lambda_4)\big)\,d\lambda\\ &\quad+\int_{\Delta^+(3142)} f_{3142}(\lambda)\big(\log(1-\lambda_3)-\log(1-\lambda_4)\big)\,d\lambda\\ &\quad+\int_{\Delta^+(2413)} f_{2413}(\lambda)\big(\log(1-\lambda_2)-\log(1-\lambda_4)\big)\,d\lambda\\ &\quad+\int_{\Delta^+(4213)} f_{4213}(\lambda)\big(\log(1-\lambda_1)-\log(1-\lambda_4)\big)\,d\lambda\\ &\qquad\left.+\int_{\Delta^+(3241)} f_{3241}(\lambda)\big(\log(1-\lambda_3)-\log(1-\lambda_4)\big)\,d\lambda\right). \end{aligned}$$

The approximate value of this exponent is as follows : $\theta_1(\mathfrak{R}_4) \approx 0.48679$. Presumably

$$\theta_1 \stackrel{?}{=} \frac{\frac{24}{7}\xi(4)}{3\xi(3) + \frac{5}{3}\pi^2 \log 2}.$$

Actually we have the approximate values for Lyapunov exponents for all Rauzy classes up to $m = 10$. These values are obtained by computer experiments; presumably precision is four significant digits. We will discuss these experiments in another paper.

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BIBLIOGRAPHY

[1] P. ARNOUX, G. LEVITT, Sur l'unique ergodicité des 1-formes fermées singulières, Inventiones Math., 85 (1986), 141–156 & 645–664.

[2] P. ARNOUX, A. NOGUEIRA, Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles, Ann. scient. Éc. Norm. Sup., 4^e série, 26 (1993), 645–664.

[3] G. BENETTIN, I. GALGANI, A. GIORGILLI, J.-M. STRELCYN, Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1 : theory. Meccanica (1980), 9–20.

[4] A.B. KATOK, Invariant measures of flows on oriented surfaces, Soviet Math. Dokl., 14 (1973), 1104–1108.

[5] M. KEANE, Interval exchange transformations, Math. Z., 141, (1975), 25–31.

[6] S.P. KERCKHOFF, Simplicial systems for interval exchange maps and measured foliations, Ergod. Th. & Dynam. Sys., 5 (1985), 257–271.

[7] S. KERCKHOFF, H. MASUR, J. SMILLIE, Ergodicity of billiard flows and quadratic differentials, Annals of Math., 124 (1986), 293–311.

[8] A. MAIER, On trajectories on closed orientable surfaces, Mat. Sbornik, 12 (1943), 71–84.

[9] H. MASUR, Interval exchange transformations and measured foliations, Annals of Math., 115-1 (1982), 169–200.

[10] A. NOGUEIRA, D. RUDOLPH, Topological weakly mixing of interval exchange maps, to appear.

[11] A. NOGUEIRA, The 3-dimensional Poincaré continued fraction algorithm, preprint ENSL, 93 (1993), 1–25.

[12] V.I. OSELEDETS, A Multiplicative Ergodic Theorem. Ljapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc., 19 (1968), 197–231.

[13] G. RAUZY, Echanges d'intervalles et transformations induites, Acta Arith., 34 (1979), 315–328.

[14] S. SCHWARTZMAN, Asymptotic cycles, Annals of Mathematics, 66 (1957), 270–284

[15] W.A. VEECH, Projective swiss cheeses and uniquely ergodic interval exchange transformations, Ergodic Theory and Dynamical Systems, Vol. I, in Progress in Mathematics, Birkhauser, Boston, 1981, 113–193.

[16] W.A. VEECH, Gauss measures for transformations on the space of interval exchange maps, Annals of Mathematics, 115 (1982), 201–242

[17] W.A. VEECH, The metric theory of interval exchange transformations I. Generic spectral properties, Amer. Journal of Math., 106 (1984), 1331–1359.

[18] W.A. VEECH, The metric theory of interval exchange transformations II. Approximation by primitive interval exchanges, Amer. Journal of Math., 106 (1984), 1361–1387.

[19] W.A. VEECH, The Teichmüller geodesic flow, Annals of Mathematics, 124 (1986), 441–530.

[20] W.A. VEECH, Moduli spaces of quadratic differentials, Journal d'Analyse Mathématique, 55 (1990), 117–171.

[21] M. WOJTKOWSKI, Invariant families of cones and Lyapunov exponents, Ergod. Th. & Dynam. Sys., 5 (1985), 145–161.

[22] A. ZORICH, Asymptotic flag of an orientable measured foliation on a surface, in "Geometric Study of Foliations", World Sci., 1994, 479–498.

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