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The center of a graded connected Lie algebra is a nice ideal


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THE CENTER OF A GRADED CONNECTED
LIE ALGEBRA IS A NICE IDEAL

by Yves FÉLIX

In this text graded vector spaces and graded Lie algebras are always
defined over the field $\mathbb{Q}$; $L(V)$ denotes the free graded Lie algebra on the
graded connected vector space $V$. The notation $L \square L'$ means the free
product of $L$ and $L'$ in the category of graded Lie algebras, $UL$ denotes the
enveloping algebra of the Lie algebra $L$ and $(UL)_+$ denotes the canonical
augmentation ideal of $UL$. The operator $s$ is the usual suspension operator
in the category of graded vector spaces, $(sV)_n = V_{n-1}$.

Let $(L(V), d)$ be a graded connected differential Lie algebra and $\alpha$ be a
cycle of degree $n$ in $L(V)$. An important problem in differential homological
algebra consists to compute the homology of the differential graded Lie
algebra $(L(V \oplus \mathbb{Q}x), d)$, $d(x) = \alpha$, and in particular the kernel and the
cokernel of the induced map

$$\varphi_{\alpha} : A = H(L(V), d) \longrightarrow B = H(L(V \oplus \mathbb{Q}x), d).$$

Clearly the homology of $(L(V \oplus \mathbb{Q}x), d)$ and the map $\varphi_{\alpha}$ depend only on
the class $a$ of $\alpha$, so that we can write $\varphi_a$ instead of $\varphi_{\alpha}$.

DEFINITIONS.

(i) An element $a$ in the Lie algebra $A$ is nice if the kernel of the map
$\varphi_a$ is the ideal generated by $a$.

(ii) An ideal $I$ in the Lie algebra $A$ is nice if, for every element $a$ into $I$,
the kernel of the map $\varphi_a$ is contained in $I$.

Our first result reads

Key words: Differential graded Lie algebra – Inert – Rational homotopy theory.
THEOREM 1. — Let $(\mathbb{L}(V), d)$ be a graded connected differential Lie algebra and $a$ be an element in $A_n$. If $n$ is even and $a$ is in the center, then

1. the element $a$ is nice,
2. there is a split short exact sequence of graded Lie algebras:
   \[ 0 \to \mathbb{L}(W) \to B \to A/(a) \to 0, \]
   with $W$ a graded vector space isomorphic to $s^{n+1}A/(a)_+$. 

In the case $n$ is odd, the Whitehead bracket $[a, a]$ is zero since $a$ is in the center, and thus the triple Whitehead bracket $<a, a, a>$ is well defined. We first remark that $<a, a, a>$ belongs also to the center. More generally

PROPOSITION 1. — The Whitehead triple bracket $<\alpha, \beta, \gamma>$ of three elements in the center belongs also to the center.

THEOREM 2. — Let $(\mathbb{L}(V), d)$ be a graded connected differential Lie algebra and $a$ be an element in $A_n$. If $n$ is odd, $a$ belongs to the center and $<a, a, a>=0$, then

1. the element $a$ is nice,
2. $B$ contains an ideal isomorphic to $\mathbb{L}(W)$ with $W = s^{n+1}A/(a)_+$,
3. the Lie algebra $B$ admits a filtration such that the graded associated Lie algebra $G$ is an extension
   \[ 0 \to \mathbb{L}(W) \oplus K \to G \to A/(a) \to 0, \]
   with $K = \mathbb{L}(\alpha, \beta)/([\alpha, \beta], [\alpha, \alpha]), |\alpha| = 2n + 1, |\beta| = 3n + 2$.

THEOREM 3. — Let $(\mathbb{L}(V), d)$ be a graded connected differential Lie algebra and $a$ be an element in $A_n$. If $n$ is odd, $a$ belongs to the center and $<a, a, a>\neq0$, then

1. the image of $\varphi_a$ is $A/(a, <a, a, a>)$,
2. $B$ contains an ideal isomorphic to $\mathbb{L}(W)$ with
   \[ W = s^{n+1}(A/(a, <a, a, a>))_+ , \]
3. the Lie algebra $B$ admits a filtration such that the graded associated Lie algebra $G$ is an extension
   \[ 0 \to \mathbb{L}(W) \oplus (\gamma, \rho) \to G \to A/(a, <a, a, a>) \to 0, \]
with $(\gamma, \rho)$ an abelian Lie algebra on 2 generators $\gamma$ and $\rho$; $|\rho| = 2n + 1$ and $|\gamma| = 4n + 2$.

**Corollary 1.** — When the element $a$ is in the center, then the kernel of $\varphi_a$ is always contained in the center and its dimension is at most two. In particular the center is a nice ideal.

**Corollary 2.** — If $\dim A$ is at least 3 and if $a$ is in the center, then $B$ contains a free Lie algebra on at least 2 generators.

In all cases, $B$ contains a free Lie algebra $\mathbb{L}(W)$ with $W$ isomorphic to $(A/(a, <a, a, a>))_+$. From those results on differential graded Lie algebras we deduce corresponding results for the rational homotopy Lie algebras of spaces.

Let $X$ denote a finite type simply connected CW complex and $Y$ the space obtained by attaching a cell to $X$ along an element $u$ in $\pi_{n+1}(X)$.

$$Y = X \bigcup_{u} e^{n+2}.$$ The graded vector space $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ together with the Whitehead product is then a graded Lie algebra. Moreover by the Milnor-Moore theorem the Hurewicz map induces an isomorphism of Hopf algebras $U\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\Omega X; \mathbb{Q})$.

Now some notations : we denote by $\Sigma$ the isomorphism $\pi_n(\Omega X) \to \pi_{n+1}(X)$ and we put $a = \Sigma^{-1}(u)$. For sake of simplicity, an element in $\pi_*(\Omega X) \otimes \mathbb{Q}$ and its image in $H_*(\Omega X; \mathbb{Q})$ will be denoted by the same letter.

Recall that the Quillen minimal model of the space $X$ ([5], [9], [1]) is a differential graded Lie algebra $(\mathbb{L}(V), d)$, unique up to isomorphism, and equipped with natural isomorphisms

1. $V \cong s^{-1}H_*(X; \mathbb{Q})$,
2. $\theta_X : H(\mathbb{L}(V), d) \cong L_X$.

The differential $d$ is an algebrization of the attaching map. More precisely, denote by $\alpha$ a cycle in $(\mathbb{L}(V), d)$ with $\theta_X([\alpha]) = a$, then the differential graded Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, $d(x) = \alpha$ is a Quillen model of $Y$ and the injection

$$\mathbb{L}(V), d \longrightarrow (\mathbb{L}(V \oplus \mathbb{Q}x), d),$$
is a Quillen model for the topological injection \( i \) of \( X \) into \( Y \). In particular \( \varphi_a \) is the induced map \( L_X \to L_Y \).

The injection \( i : X \to Y \) induces a sequence of Hopf algebra morphisms

\[
H_*(\Omega X; \mathbb{Q}) \xrightarrow{f} H_*(\Omega X; \mathbb{Q})/(a) \xrightarrow{g} H_*(\Omega Y; \mathbb{Q}).
\]

The attachment is called *inert* if \( g \circ f \) is surjective (this is equivalent to the surjectivity of \( g \)), and is called *nice* if \( g \) is injective ([7]). Clearly the attachment is nice if the element \( a \) is a nice element in \( L_X \).

The structure of the Lie algebra \( L_X = \pi_*(\Omega X) \otimes \mathbb{Q} \), for \( X \) a space with finite Lusternik-Schnirelmann, has been at the origin of a lot of recent works (cf. [3], [2], [4]). In particular the radical of \( L_X \) (union of all solvable ideals) is finite dimensional and each ideal of the form \( I_1 \times \cdots \times I_r \) satisfies \( r \leq \text{LS cat } X \).

We deduce from Theorems 1, 2 and 3 above the following results concerning attachment of a cell along an element in the center.

**Theorem 4.** — If \( n \) is even and \( a \) is in the center, then

1. the attachment is nice,
2. there is a split short exact sequence of graded Lie algebras:

\[
0 \to \mathbb{L}(W) \to L_Y \to L_X/(a) \to 0,
\]

with \( W \) a graded vector space isomorphic to \( s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a))_+ \). The splitting is given by the natural map \( L_X \to L_Y \).

**Theorem 5.** — If \( n \) is odd, \( a \) belongs to the center and \( <a, a, a> = 0 \), then

1. the attachment is nice,
2. \( L_Y \) contains an ideal isomorphic to \( \mathbb{L}(W) \) with

\[
W = s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a))_+,
\]

3. the Lie algebra \( L_Y \) admits a filtration such that the graded associated Lie algebra \( G \) is an extension

\[
0 \to \mathbb{L}(W) \amalg K \to G \to L_X/(a) \to 0,
\]

with \( K = \mathbb{L}(\alpha, \beta)/([\alpha, \beta], [\alpha, \alpha]) \), \( |\alpha| = 2n + 1 \), \( |\beta| = 3n + 2 \).
Example 1. — Let $X = P^\infty(C)$ and $u : S^2 \to P^\infty(C)$ be the canonical injection. Then $Y = X/S^2$ and its rational cohomology is $\mathbb{Q}[x_4, y_6]/(x_4^3 - y_6^2)$. In this case the Lie algebra $L_Y$ is isomorphic to $K$ and the graded vector space $W$ is zero.

Example 2. — Let $X = P^3(C)$ and $u : S^2 \to P^3(C)$ be the canonical injection. Then $Y = X/S^2 \cong S^4 \vee S^6$. In this case $W$ is the graded vector space generated by the brackets $\text{ad}^a(i_4)\langle i_6 \rangle$, $n \geq 1$, where $i_4$ and $i_6$ denote the canonical injections of the spheres $S^4$ and $S^6$ into $Y$.

Theorem 6. — If $n$ is odd, $a$ belongs to the center and $< a, a, a > \neq 0$, then

1. The image of $L_X$ in $L_Y$ is $L_X/(a, < a, a, a >)$,
2. $L_Y$ contains an ideal isomorphic to $L(W)$ with
   \[ W = s^{n+1}(H_*(\Omega X; \mathbb{Q})/(a, < a, a, a >))^+. \]
3. The Lie algebra $L_Y$ admits a filtration such that the graded associated Lie algebra $G$ is an extension
   \[ 0 \to L(W) \oplus (\gamma, \rho) \to G \to L_X/(a, < a, a, a >) \to 0, \]
   with $(\gamma, \rho)$ an abelian Lie algebra on 2 generators $\gamma$ and $\rho$; $|\rho| = 2n + 1$ and $|\gamma| = 4n + 2$.

Corollary 1'. — When the element $a$ is in the center, then the kernel of the map $L_X \to L_Y$ is always contained in the center and its dimension is at most two.

Corollary 2'. — If dim $L_X$ is at least 3 and if $a$ is in the center, then $L_Y$ contains a free Lie algebra on at least 2 generators.

Example 3. — Let $X$ be either $S^{2n+1} \times K(Z, 2n)$ or else $P^n(C)$, then the attachment of a cell of dimension $2n + 2$ along a nonzero element generates only one new rational homotopy class of degree $2n + 3$. The space $Y$ has in fact the rational homotopy type either of $S^{2n+3} \times K(Z, 2n)$ or else of $P^{n+1}(C)$. These are rationally the only situations where $L_X$ has dimension two and the attachment does not generate a free Lie algebra.

Example 4. — Let $Z$ be a simply connected finite CW complex not rationally contractible. Then for $n \geq 1$ the rational homotopy Lie algebra
Let $Y = S^{2n+1} \times Z / (S^{2n+1} \times \{\ast\})$ contains a free Lie algebra on at least two generators. It is enough to see that $Y$ is obtained by attaching a cell along the sphere $S^{2n+1}$ in $X = S^{2n+1} \times Z$.

In all cases, $L_Y$ contains a free Lie algebra $\mathbb{L}(W)$ with $W$ isomorphic to $(H_\ast(\Omega X) \otimes \mathbb{Q} / (a, < a, a, a >))_+$. The elements of $W$ have the following topological description.

Let $\beta$ be an element of degree $r-1$ in $L_X / (a, < a, a, a >)$ and $b = \Sigma \beta$. The Whitehead bracket

$$S^{n+r} [i_{n+1}, i_r] S^{n+1} \vee S^r \xrightarrow{u \vee b} X,$$

extends to $D^{n+r+1}$ because the element $a$ is in the center. On the other hand in $Y$ the map $u$ extends to $D^{n+2}$, this gives the commutative diagram

$$
\begin{array}{ccc}
D^{n+r+1} & \longrightarrow & D^{n+2} \vee S^r \\
\uparrow & & \uparrow \\
S^{n+r} & \xrightarrow{[i_{n+1}, i_r]} & S^{n+1} \vee S^r \xrightarrow{u \vee b} X
\end{array}
$$

These two extensions of the Whitehead product to $D^{r+n+1}$ define an element $\varphi(\beta)$ in $\pi_{n+r+1}(Y) \otimes \mathbb{Q}$. Now for every element $\alpha = \alpha_1 \ldots \alpha_n$ in $H_\ast(\Omega X; \mathbb{Q}) / (a, < a, a, a >)$, with $\alpha_i$ in $L_X$, we define

$$\varphi(\alpha) = [\alpha_n, [\alpha_{n-1}, \ldots, [\alpha_2, \varphi(\alpha_1)] \ldots].$$

This follows directly from the construction of $W$ given in section 1.

**Proposition 2.** — When $\{\beta_i\}$ runs along a basis of $H_\ast(\Omega X; \mathbb{Q}) / (a, < a, a, a >)$, the elements $\{\Sigma^{-1} \varphi(\beta_i)\}$ form a basis of a sub free Lie algebra of $L_Y$.

Corollary 1 means that the center of $L_X$ is a nice ideal. We conjecture:

**Conjecture.** — Let $X$ be a simply connected finite type CW complex with finite Lusternik-Schnirelmann category, then the radical of $L_X$ is a nice ideal.

We now prove this conjecture in a very particular case.

**Theorem 7.** — If the ideal $I$ generated by $a$ has dimension two and is contained in $(L_X)_{\text{even}}$, then the element $a$ is nice and $L_Y$ is an extension of a free Lie algebra $\mathbb{L}(W)$ by $L_X$ with $W \supset s^n(U(L_X / I)_+)$. 

**Example 5.** — Let $X$ be the geometric realization of the commutative differential graded algebra $(\wedge(x, c, y, z, t), d)$ with $d(x) = d(c) = 0$, $d(y) =$
Denote by $(L(V), d)$ and $(L(V \oplus \mathbb{Q}x), d)$ free differential graded connected Lie algebras, $d(x) = a$, $[a] = a$.

By putting $V$ in gradation 0 and $x$ in gradation 1, we make $(L(V \oplus \mathbb{Q}x), d)$ into a filtered differential graded algebra. The term $(E^1, d^1)$ of the associated spectral sequence has the form

$$(E^1, d^1) = (A \coprod L(x), d), \quad A = H_*(L(V), d) = L_X, \quad d(x) = a.$$

The ideal $I$ generated by $x$ is the free Lie algebra on the elements $[x, \beta_i]$, with $\{\beta_i\}$ a graded basis of $UA$ and where by definition, we have

$$[x, 1] = x$$
$$[x, \alpha_1 \alpha_2 \ldots \alpha_n] = \ldots [x, \alpha_1], \alpha_2, \ldots, \alpha_n], \quad \alpha_i \in L_X.$$

Since $a$ is in the center, if $\beta_i \in (UA)_+$, then $d([x, \beta_i]) = 0$. Therefore the ideal $J$ generated by $[x, x]$ and the $[x, \beta_i]$, $\beta_i \in (UA)_+$ is a subdifferential graded Lie algebra. The ideal $J$ is in fact the free Lie algebra on $[x, x]$ and the elements $[x, \beta_i]$ and $[x, [x, \beta_i]]$, with $\{\beta_i\}$ a basis of $(UA)_+$. A simple computation shows that

$$d[x, x] = 2[a, x]$$
$$d[x, \beta_i] = 0$$
$$d[x, [x, \beta_i]] = -[x, \beta_i a].$$

This shows that $H(J, d)$ is isomorphic to the free graded Lie algebra $L(W)$ where $W$ is the vector space formed by the elements $[x, \beta_i]$ with $\beta_i \in (UA/(a))_+.$

The short exact sequence $0 \to J \to A \coprod L(x) \to A \oplus (x) \to 0$ yields thus a short exact sequence in homology

$$0 \to L(W) \to H(A \coprod L(x)) \to A/(a) \to 0.$$
This implies that the term $E^2$ is generated in degrees 0 and 1. The spectral sequence degenerates thus at the $E^2$ level: $E^2 = E^\infty$.

Denote now by $u_\beta_i$ a class in $H(L(V \oplus \mathbb{Q}x), d)$ whose representative in $E^\infty$ is $[x, \beta_i]$. The Lie algebra generated by the $u_\beta_i$ admits a filtration such that the graded associated Lie algebra is free. This Lie algebra is therefore free, and its quotient is $A/(a)$. This proves the theorem. □

2. Proof of Proposition 1.

The elements $\alpha$, $\beta$ and $\gamma$ are represented by cycles $x$, $y$ and $z$ in $(L(V), d)$. Since the elements $\alpha$, $\beta$ and $\gamma$ are in the center, there exist elements $a$, $b$ and $c$ in $L(V)$ such that

$$d(a) = [x, y], \quad d(b) = [y, z], \quad d(c) = [z, x].$$

The triple Whitehead product is then represented by the element

$$\omega = (-1)^{|xy|}[c, y] + (-1)^{|xz|}[b, x] + (-1)^{|zx|}[a, z].$$

We will show that the class of $\omega$ is central, i.e. for every cycle $t$ the bracket $[\omega, t]$ is a boundary. First of all, since $\alpha$, $\beta$ and $\gamma$ are in the center there exists elements $x_1$, $y_1$ and $z_1$ such that

$$d(x_1) = [x, t], \quad d(y_1) = [y, t], \quad d(z_1) = [z, t].$$

We now easily check that the three following elements $\alpha_1$, $\alpha_2$ and $\alpha_3$ are cycles:

$$\alpha_1 = (-1)^{|xz|+|t|}[t, c] + (-1)^{|xz|+|z|+1+|xt|}[x, z_1] + (-1)^{|zt|+|z|}[z, x_1]$$
$$\alpha_2 = (-1)^{|x|+|t|}[t, b] + (-1)^{|z|+|xy|+1+|xt|}[z, y_1] + (-1)^{|yt|+|y|}[y, z_1]$$
$$\alpha_3 = (-1)^{|t|+|ty|}[t, a] + (-1)^{|y|+|ty|+1+|xz|}[y, x_1] + (-1)^{|x|+|xt|}[x, y_1].$$

We deduce elements $\beta_1$, $\beta_2$ and $\beta_3$ satisfying

$$d(\beta_1) = [\alpha_1, y], \quad d(\beta_2) = [\alpha_2, x], \quad d(\beta_3) = [\alpha_3, z].$$

Now we verify that $[\omega, t]$ is the boundary of

$$(-1)^{|xy|+|ct|+|yc|+1}[t, c] + (-1)^{|yc|+|bt|+|bz|+1}[x_1, b]$$
$$+ (-1)^{|xz|+|az|+|ax|+1}[z_1, a] + (-1)^{|xz|+|xy|+|zx|}[x, y_1, z]$$
$$+ (-1)^{|ty|+|yz|+|ty|+|xz|}[b_1, z] + (-1)^{|tx|+|tx|+|xz|+|xz|}[b_3].$$
3. Proof of Theorems 2 and 3.

We filter the Lie algebra \((\mathbb{L}(V \oplus \mathbb{Q}x), d)\) by putting \(V\) in gradation 0 and \(x\) in gradation 1. We obtain a spectral sequence \(E_{\ast, \ast}^{\ast}\). We will first compute the term \((E_{1, \ast}^{1}, d^{1})\) and its homology \(E^{2}\). We will see that \(E^{2}\) is generated only in filtration degrees 0, 1 and 2. The differential \(d^{2}\) is thus defined by its value on \(E_{2, \ast}^{2}\). Under the hypothesis of Theorem 2, we show that \(d^{2} = 0\) and that the spectral sequence collapses at the \(E^{2}\)-level. Under the hypothesis of Theorem 3, we show that the image of \(d^{2} : E_{2, \ast}^{2} \rightarrow E_{0, \ast}^{0}\) has dimension 1 and that a basis is given by the class of the Whitehead bracket \(\langle a, a, a \rangle\). We will compute explicitly the term \(E_{3, \ast}^{3}\). This term is generated only in filtration degrees 0 and 1, so that the spectral sequence collapses at the \(E^{3}\)-level in that case.

3.1. Description of \((E^{1}, d^{1})\).

The term \((E^{1}, d^{1})\) has the form

\[
(E^{1}, d^{1}) = (A \prod \mathbb{L}(x), d), \quad A = H_{\ast}(\mathbb{L}(V), d) = L_{\chi}, \quad d(x) = a.
\]

We denote by \(I\) the ideal generated by \(x\),

\[
I = \mathbb{L}([x, \beta_{i}], \text{ with } \{\beta_{i}\} \text{ a basis of } U A),
\]

and by \(J\) the ideal of \(I\) generated by the \([x, \beta_{i}]\) with \(\beta_{i}\) in \((UA)_{+}\).

\[
J = \mathbb{L}([x, [x, \cdots [x, \beta_{i}]], \text{ with } \{\beta_{i}\} \text{ a basis of } (UA)_{+}).
\]

For sake of simplicity we introduce the notation

\[
\varphi_{1}(\beta) = [x, \beta], \quad \varphi_{n}(\beta) = [x, \varphi_{n-1}(\beta)], n \geq 2.
\]

Lemma 1.

1. \(d(\varphi_{1}(\beta)) = 0\) for \(\beta \in (UA)_{+}\).

2. \([a, \varphi_{n}(\beta)] =_{(J^{2})} -\varphi_{n}(a\beta), n \geq 1.\)

3. \(d(\varphi_{n}(\beta)) =_{(J^{2})} -(n - 1)\varphi_{n-1}(a\beta), n \geq 2.\)

In the above formulas \(=_{(J^{2})}\) means equality modulo decomposable elements in the Lie algebra \(J\).
Proof.

(1) \( d\varphi_1(\beta) = d[x, \beta] = [a, \beta] = 0. \)

(2) The Jacobi identity shows that \([a, \varphi_1(\beta)] = [a, [x, \beta]] = -[[x, a], \beta] = -[x, a\beta].\) By induction we deduce \([a, \varphi_n(\beta)] = (\beta) [x, [a, \varphi_{n-1}(\beta)]] = -\varphi_n(a\beta). \)

(3) \( d\varphi_n(\beta) = [a, \varphi_{n-1}(\beta)] + [x, d\varphi_{n-1}(\beta)] = (\beta) - (n - 1)\varphi_{n-1}(a\beta). \)

\[ \]

**Lemma 2.**

(1) \( d\varphi_1(a) = 0, d\varphi_2(a) = 0 \)

(2) \( d\varphi_n(a) \in \mathbb{L}(\varphi_1(a), \ldots, \varphi_{n-2}(a)) \)

(3) \( d\varphi_n(a) + \alpha_n[\varphi_1(a), \varphi_{n-2}(a)] \) is a decomposable element in \( \mathbb{L}(\varphi_2(a), \ldots, \varphi_{n-3}(a)), \) for \( n \geq 3, \) with \( \alpha_3 = 1 \) and for \( n \geq 4, \alpha_n = 5 + \frac{(n+3)(n-4)}{2}. \)

**Proof.** — We first check that \([a, \varphi_1(a)] = 0 \) and that

\( [a, \varphi_2(a)] = -[\varphi_1(a), \varphi_1(a)]. \)

Now, using Jacobi identity we find that

\[ [a, \varphi_n(a)] = -\sum_{p=1}^{n-1} \left( \begin{array}{c} n-1 \\ p \end{array} \right) [\varphi_p(a), \varphi_{n-p}(a)]. \]

Using once again Jacobi identity we find

\[ [x, [\varphi_n(a), \varphi_m(a)]] = [\varphi_{n+1}(a), \varphi_m(a)] + [\varphi_n(a), \varphi_{m+1}(a)]. \]

The derivation formula valid for \( n \geq 2 \)

\( d\varphi_{n+1}(a) = [a, \varphi_n(a)] + [x, d\varphi_n(a)], \)

gives now point (3) of the lemma.

\[ \]

**3.2. Computation of** \( E^2 = H(E^1, d^1). \)

**Lemma 3.** — The homology \( H(\mathbb{L}(\varphi_n(a), n \geq 1), d) \) is the quotient of the free Lie algebra \( \mathbb{L}(\varphi_1(a), \varphi_2(a)) \) by the ideal generated by the relations \([\varphi_1(a), \varphi_1(a)]\) and \([\varphi_1(a), \varphi_2(a)].\)
Proof. — Since the differential $d$ is purely quadratic, the graded Lie algebra $(L((\varphi_n(a), n \geq 1), d)$ represents a formal space $Z$ with rational cohomology $H^*(Z; \mathbb{Q})$ isomorphic to the dual of the suspension of the graded vector space generated by the $\varphi_n(a), n \geq 1$.

The rational cup product in $H^*(Z; \mathbb{Q})$ is given by the dual of the differential. This means that $H^*(Z; \mathbb{Q})$ is generated by elements $u_1$ and $u_2$ defined by $\langle u_i, s\varphi_j(a) \rangle = 1$ if $i = j$ and 0 otherwise. The description of $d(\varphi_5(a))$ yields the relation $u_1^3 = \frac{9}{8} u_2^2$. Now since the Poincaré series of $H^*(Z; \mathbb{Q})$ and $\mathbb{Q}[u_1, u_2]/\left( u_1^3 - \frac{9}{8} u_2^2 \right)$ are both equal to $\frac{1}{1 - t^{n+1} - t^{n+1}}$, there is no other relation. Therefore

$$H^*(Z; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2]/\left( u_1^3 - \frac{9}{8} u_2^2 \right).$$

The Lie algebra $H(L((\varphi_n(a), n \geq 1), d)$ is thus isomorphic to the rational homotopy Lie algebra $L_Z$; its dimension is three and a basis is given by the elements $\varphi_1(a), \varphi_2(a)$ and $[\varphi_2(a), \varphi_2(a)]$. This implies the result. \qed

The differential ideal $J$ is thus the free product of two differential ideals

$$J = L((\varphi_n(a)), n \geq 1) \prod L((\varphi_n(\beta_i), \varphi_n(a\beta_i), n \geq 1),$$

with $\{\beta_i\}$ a basis of $(UA/a)_+$. Each factor is stable for the differential. Therefore

$$H(J) = \frac{L(\varphi_1(a), \varphi_2(a))}{([\varphi_1(a), \varphi_1(a)], [\varphi_1(a), \varphi_2(a)])} \prod L((\varphi_1(\beta_i)),$$

with $\{\beta_i\}$ a basis of $(UA/a)_+$. The short exact sequence of Lie algebras

$$0 \to J \to A \prod L(x) \to A \oplus \mathbb{Q} x \to 0,$$

closes the description of the term $E^2$ of the spectral sequence

Corollary. — The term $E^2$ satisfies $E^2_{0,*} = A/(a)$, and $E^2_{+,*} = H(J)$. In particular $E^2$ is generated in filtration degrees 0, 1 and 2.
3.3. Description of the differential $d^2$.

Recall that $a$ is in the center. The element $[\alpha, \alpha]$ is thus a boundary: there exists some element $b$ with $d(b) = [\alpha, \alpha]$. Then the element $[b, \alpha]$ is also a cycle and its homology class is the triple Whitehead bracket $\langle a, a, a \rangle$.

**Lemma 4.** Denote by $[\varphi_2(\alpha)]$ the class of $\varphi_2(\alpha)$ in the $E^2$-term of the spectral sequence. We then have

$$d^2([\varphi_2(\alpha)]) = -\frac{3}{2} \langle [\alpha, b] \rangle,$$

where $\langle \cdots \rangle$ means the class of a cycle in the $E^2$ term.

**Proof.** We easily verify that in the differential Lie algebra $(\mathbb{L}(V \oplus \mathbb{Q}x), d)$, we have

$$d \left( \varphi_2(\alpha) - \frac{3}{2} [x, b] \right) = -\frac{3}{2} [\alpha, b].$$

Since $[x, b]$ is in filtration degree 1, and $[\alpha, b]$ in filtration degree 0, this gives the result by definition of the differential $d^2$. □

3.4. End of the proof of Theorem 2.

If $\langle [b, \alpha] \rangle = 0$, then $d^2 = 0$, the spectral sequence degenerates at the $E^2$ level and Theorem 2 is proved. □

3.5. Computation of the term $E^3_{*,*}$.

Henceforth, we suppose $\langle [b, \alpha] \rangle \neq 0$. A simple computation using Jacobi identity gives the following identity.

**Lemma 5.** $d^2(\varphi_2(\alpha)) = 3\varphi_1(\alpha), \varphi_1(\langle [\alpha, b] \rangle)$.

Let $\{\beta_i\}, i \in I$, denote a basis of $(U A/a)_{+}$ such that $\langle [\alpha, b] \rangle = \beta_i$ for some index $i_0$. The elements $\varphi_1(\alpha)$ and $[\varphi_2(\alpha), \varphi_2(\alpha)]$ together with the elements $\varphi_1(\beta_i)$ generate an ideal $M$ in the graded Lie algebra $E^3_{*,*}$. The Lie algebra $M$ is generated by the elements $\varphi_1(\beta_i), [\varphi_2(\alpha), \varphi_1(\beta_i)], [\varphi_2(\alpha), \varphi_2(\alpha)]$ and $\varphi_1(\alpha)$ and satisfies the two relations $[\varphi_1(\alpha), \varphi_1(\alpha)] = 0$ and $[[\varphi_2(\alpha), \varphi_2(\alpha)], \varphi_1(\alpha)] = 0$. Denote by $N$ the graded
Lie algebra

\[ N = L(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \prod_{i} \frac{L([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}. \]

Since \( M \) and \( N \) have the same Poincaré series they coincide. Therefore as a Lie algebra, \( M \) can be written

\[ M = L(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)]) \prod_{i} \frac{L([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([\varphi_1(a), \varphi_1(a)], [[\varphi_2(a), \varphi_2(a)], \varphi_1(a)])}. \]

The equation

\[ d^{2}[\varphi_2(a), \varphi_1(\beta_i)] = \frac{3}{2} \varphi_1(\beta_i[a, b]). \]

shows that the Lie algebra \( M \) decomposes into the free product of three differential graded Lie algebras, the first one being acyclic:

\[ M = L(\varphi_1(\beta_i), [\varphi_2(a), \varphi_1(\beta_i)], i \in I) \prod_{i \in I \setminus \{i_0\}} L(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \prod K, \]

\[ K = L(\varphi_1([\alpha, b])) \prod_{i} \frac{L([\varphi_2(a), \varphi_2(a)], \varphi_1(a))}{([[\varphi_2(a), \varphi_2(a)], \varphi_1(a)], [\varphi_1(a), \varphi_1(a)])}. \]

We thus have

\[ H(M) = L(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \prod H(K). \]

To compute the homology of \( K \) we put \( t = \varphi_1([\alpha, b]), y = \varphi_1(a) \) and \( z = [\varphi_2(a), \varphi_2(a)] \).

**Lemma 6.** — Let \((\mathcal{L}, d) = (L(y, z, t)/[[y, z], [y, y]], d)\) be a differential graded Lie algebra with \( t \) and \( z \) in \( \mathcal{L}_{\text{even}} \), \( y \) in \( \mathcal{L}_{\text{odd}} \), and where the differential \( d \) is defined by \( d(t) = d(y) = 0 \) and \( d(z) = [t, y] \). Then \( H(\mathcal{L}, d) \) is a \( \mathbb{Q} \)-vector space of dimension two generated by the classes of \( t \) and \( y \).

**Proof.** — Denote by \( R \) the ideal generated by \( t \). As a Lie algebra \( R \) is the free Lie algebra generated by \( t \), \( w = [t, y] \), the elements \( u_n = ad^n(z)(t) \), for \( n \geq 1 \) and the elements \( w_n = ad^n(z)[t, y] \), for \( n \geq 1 \).
Using the Jacobi identity, we get the following sequence of identities:

\[
\begin{align*}
    d(t) &= 0 \\
    d(w) &= 0 \\
    d(u_1) &= [t, w] \\
    d(u_2) &= 2[u_1, w] - [w_1, t] \\
    &\vdots \\
    d(u_n) &= n[u_{n-1}, w], \text{ modulo } L(u_1, \ldots u_{n-2}, t, w_i) \\
    d(w_1) &= -[w, w] \\
    d(w_2) &= -3[w_1, w] \\
    &\vdots \\
    d(w_n) &= -(n+1)[w_{n-1}, w], \text{ modulo } L(w_1, \ldots w_{n-2}).
\end{align*}
\]

This shows that the cohomology of the cochain algebra on \( R \) is \( \mathbb{Q}[w^\circ] \otimes \wedge (t^\circ) \), with \( w^\circ \) and \( t^\circ \) 1-cochains satisfying \( \langle w^\circ, w \rangle = 1 \) and \( \langle t^\circ, t \rangle = 1 \). The interpretation of \( H(R, d) \) as the dual of the vector space of indecomposable elements of the Sullivan minimal model of \( C^\ast(R) \) shows that \( H(R, d) \cong \mathbb{Q}w \oplus \mathbb{Q}t \).

The examination of the short exact sequence of differential complexes

\[ 0 \rightarrow (R, d) \rightarrow (L, d) \rightarrow (\mathbb{Q}y \oplus \mathbb{Q}z, 0) \rightarrow 0, \]

shows that \( H(L, d) \cong \mathbb{Q}t \oplus \mathbb{Q}y. \)

This shows that \( H(M) \) is isomorphic to the free product of \( L(\varphi_1(\beta_i), i \in I \setminus \{i_0\}) \) with the abelian Lie algebra on the two elements \( \varphi_1(a) \) and \( \varphi_1(\langle [a, b] \rangle) \).

### 3.6. End of the proof of Theorem 3.

From the short exact sequence of chain complexes

\[ 0 \rightarrow M \rightarrow E^2 \rightarrow E^2_{0,\ast} \oplus \varphi_2(a)\mathbb{Q} \rightarrow 0, \]

we deduce the isomorphism of graded vector spaces

\[ E^3 = H(E^2, d^2) \cong H(M) \oplus A/(a, \langle [a, b] \rangle). \]

Since \( E^3 \) is generated by elements in gradation 0 and 1, the spectral sequence degenerates at the term \( E^3 \), \( E^3 = E^\infty \). This closes the proof of Theorem 3. \( \square \)

We suppose that the ideal generated by \( a \) is composed of \( a \) and \( b = [a, c] \). We choose an ordered basis \( \{u_i\} \), \( i = 1, \ldots \) of \( L_X \) with \( u_1 = c \), \( u_2 = a \) and \( u_3 = b \). We consider the set of monomials of \( UL_X \) of the form \( \beta_i = u_{i_1} u_{i_2} \ldots u_{i_n} \) with \( i_n \leq i_{n-1} \leq \ldots \leq i_2 \leq i_1 \) and \( i_j \neq i_{j+1} \) when the degree of \( u_j \) is odd. This set of monomials forms a basis of \( UL_X \).

The ideal generated by \( x \) in \( L_X \] \( \mathbb{L}(x) \) is then the free Lie algebra on the elements \([x, \beta_i]\). For sake of simplicity, we denote \( x' = [x, c] \).

In particular the ideal \( J \) generated by the elements \([x, x], [x', x']\) and the \([x, \beta_i]\) for \( \beta_i \notin \{1, c\} \) is a differential sub Lie algebra that is a free Lie algebra on two types of elements:

- First type: \([x, x], [x, x'], [x', x'], [x, [x, x]], [x', [x, x]], \) \( \beta_i \neq 1, c \).
- Second type: \([x, \beta_i], [x, [x, \beta_i]], [x', [x, \beta_i]], [x', [x, [x, \beta_i]]], \) \( \beta_i \neq 1, c \).

We have

\[
\begin{align*}
d([x, x]) &= -2[x, a] \\
d([x', x']) &= -2[x, cb] \\
d([x, x']) &= -[x, ca] - [x, b] \\
d([x', [x, x]]) &= -2[x, [x, b]] + 2[x', [x, a]] \\
d([x, \beta_i]) &= 0 \\
d([x, [x, \beta_i]]) &= -[x, \beta_i a] \\
d([x', [x, \beta_i]]) &= -[x, \beta_i b] \\
d([x', [x, [x, \beta_i]]]) &= -[x, [x, \beta_i b]] + [x', [x, \beta_i a]] \text{ modulo decomposable elements.}
\end{align*}
\]

Looking at the linear part of the differential we see directly that \( H(J) \) is isomorphic to the free Lie algebra on the element \([x, ca]\) and the elements \([x, \beta_i]\) with \( \beta_i \) a non empty word in the variables \( u_j \) different of \( a \) and \( b \).

**Example 6.** — Let \( X \) be the total space of the fibration with fibre \( S^7 \) and base \( S^3 \times S^5 \) whose Sullivan minimal model is \((\wedge(x, y, z), d)\), \( d(x) = d(y) = 0, d(z) = xy, |x| = 3, |y| = 5 \) and \( |z| = 7 \). If we attach a cell along the sphere \( S^3 \) we obtain the space \( Y = (S^5 \times S^{10}) \vee S^{12} \).
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