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MULTISUMMABILITY FOR SOME CLASSES OF DIFFERENCE EQUATIONS(*)

by B.L.J. BRAAKSMA and B.F. FABER

1. Introduction.

This paper concerns linear and nonlinear difference equations whose linear part involves the difference operator

$$Ty(x) := y(x+1) - \mathbf{A}(x)y(x) \tag{1.1}$$

where $x \in \mathbf{C}$, $\mathbf{A}(x)$ is an invertible $(n \times n)$-matrix meromorphic near ∞ and $y : \mathbf{C} \to \mathbf{C}^n$. We are interested in multisummability properties of formal solutions of $Ty(x) = G(x, y)$ where $G(x, y)$ is holomorphic at $(\infty, 0)$ and does not contain linear terms in y.

To formulate the results we need a formal fundamental matrix $\hat{\mathbf{Y}}(x)$ of the linear homogeneous equation $Ty(x) = 0$:

$$\hat{\mathbf{Y}}(x) = \hat{\mathbf{M}}(x) \oplus_1^m \Gamma(x)^{\lambda_j} c_j^x e^{q_j(x)} x^{\mathbf{L}_j}. \tag{1.2}$$

Here

$$\tag{1.3} \begin{cases} \hat{\mathbf{M}}(x) \in \mathrm{Gl}(n, \mathbf{C}[[x^{-1/p}]][x^{1/p}]), p \in \mathbf{N}, \\ \mathbf{L}_j \in \mathrm{End}(n_j, \mathbf{C}), n_1 + \ldots + n_m = n, \\ \lambda_j \in \dfrac{1}{p}\mathbf{Z}, \lambda_1 \leqslant \ldots \leqslant \lambda_m, c_j \in \mathbf{C}^*, \\ q_j(x) \equiv 0 \text{ or } q_j(x) \text{ is a polynomial in } x^{1/p} \text{ of degree less than } p \\ \qquad \text{with leading term } b_j x^{\mu_j}, 0 < \mu_j < 1, b_j \neq 0. \end{cases}$$

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(cf. [Tur60], [Pra83], [Duv83], [Imm84]).

In general $\hat{\mathbf{M}}(x)$ cannot be lifted to a meromorphic matrix but only on suitable sectors there exist corresponding holomorphic lifts which exhibit a Stokes phenomenon.

The **levels** of the difference operator T are said to be $k_1, ..., k_q$ where $0 < k_1 < ... < k_q = 1$ and $k_h \in (0,1)$ occurs iff there exists j such that $\lambda_j = 0, c_j = 1, q_j \not\equiv 0, k_h = \mu_j$.

A **direction** ϕ (interpreted as a real number or as a half line $\arg x = \phi$) will be called **singular** for this operator if $\exp\{x\log(c_j) + q_j(x)\}$ has maximal descent in the direction ϕ as $|x| \to \infty$ in case $\lambda_j = 0, x\log(c_j) + q_j(x) \not\equiv 0$. Here all possible determinations of the logarithm are taken into account. These directions are said to be of level 1 except if $\lambda_j = 0, c_j = 1, q_j(x) \not\equiv 0$. In the latter case it is said to be of level μ_j. In case $\lambda_j = 0, c_j = 1$, also $\phi = \pi/2 \bmod \pi$ are singular directions of level 1. Note that in case $\lambda_j = 0, |c_j| \neq 1$ there are infinitely many singular directions which have $\pi/2 \bmod \pi$ as accumulation points.

With these definitions our main result is

THEOREM 1.1. — *Let $\mathbf{A}(x)$ be as above and let $G(x,y)$ be holomorphic in $x^{-1/p}$ and y in a neighborhood of $(\infty, 0)$ and such that it does not contain a linear term in y in its Taylor expansion with respect to y. Let $\hat{y}(x) \in \mathbf{C}^n[[x^{-1/p}]]$ be a formal solution of $y(x+1) - \mathbf{A}(x)y(x) = G(x,y)$.*

Let $(I_1, ..., I_q)$ be a multi-interval such that $I_1 \supset ... \supset I_q, |I_h| > \pi/k_h$ and I_h does not contain a pair of Stokes directions $(\phi - \pi/(2k_h), \phi + \pi/(2k_h))$, where ϕ is any singular direction of level k_h.

Then $\hat{y}(x)$ is $(k_1, ..., k_q)$-summable on $(I_1, ..., I_q)$ in the following cases:

1. $\lambda_m > \lambda_1 \geqslant 0, \pi \notin I_q \bmod 2\pi$.
2. $\lambda_1 = \lambda_m = 0$.
3. $\lambda_1 < \lambda_m \leqslant 0$ *and* $0 \notin I_q \bmod 2\pi$.

Moreover, $\hat{y}(x)$ is 1-summable in upper and in lower halfplanes if $|c_j| \neq 1$ in case $\lambda_j = 0$.

For the definition of multisummability we refer to section 3. In case $q = 1$ we have 1-summability and so Borel-summability. In this case the Borel sums of the formal solutions may be represented by convergent generalized factorial series. The theorem may be extended to cases where

$\mathbf{A}(x)$ and $G(x,y)$ correspond to multisums of formal series in x as in the case of differential equations (cf. [Bra92]). Theorem 1.1. remains valid if $G(x,y(x))$ is replaced by $G(x,y(x+1))$ as can be seen by expressing $y(x+1)$ by means of the difference equation in terms of $y(x)$.

The result may be reformulated for left-difference equations $w(x-1)-\mathbf{A}(x)w(x) = G(x,w)$ by means of the substitution $w(x) = y(-x)$ and $\mathbf{A}(x) := \mathbf{A}(-x), G(x,w) := G(-x,w)$. Moreover, this implies that $Ty(x) = G(x,y)$ corresponds to $w(x+1)-\mathbf{A}^{-1}(-x-1)w(x) = -\mathbf{A}^{-1}(-x-1)G(-x-1,w(x+1))$ and thus it follows that the statement in case 3 of the theorem is a consequence of that of case 1.

Different (slightly more general) formulations of the main result are given in sections 4 and 10. In section 10 we also consider the reduction of a general nonlinear difference equation to one of the type considered in Theorem 1.1. Section 11 contains an application to normalizing transformations for the difference operator T.

The method of proof of the main result is similar to that for the multisummability for meromorphic differential equations in the style of Ecalle (cf. [Eca87], [Eca93], [Bra91] and [Bra92]) with a modification due to Malgrange (cf. [Mal]).

Multisummability of formal solutions of difference equations does not always occur and one has to apply the more general notion of accelero-summability of Ecalle in such cases (cf. [Eca87], [Eca93] and [Imm]). However, our method shows that *formal solutions* $\widehat{y}$ *always can be lifted to holomorphic solutions in upper and lower half planes with* $\widehat{y}$ *as asymptotic expansion*, the lifts need not be uniquely determined by $\widehat{y}$ (cf. Remarks 4.1 and 10.1 and [vdPS]).

The paper is arranged as follows. In section 2 we introduce some notations and recall properties of Borel and Laplace transforms and in section 3 we recall two equivalent definitions of multisummability. In section 4 we give an alternative formulation of the multisummability result for the linear difference equation. In section 5 we derive the convolution equation which corresponds through Borel transform with this difference equation and in section 6 we prove the result in the linear case by means of these convolution equations. This proof depends on two lemmas which are proved in sections 8 and 9. For this we need to study an auxiliary operator associated with the difference operator. To this study section 7 is devoted. In section 10 we consider the nonlinear difference equation.

Section 11 concerns the normalizing transformations mentioned above and a fundamental matrix of a linear difference equation in a special case.

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2. Preliminaries.

Here we present some concepts and results which we use lateron. For more details we refer to Balser [Bal94], Ramis [Ram93] and Malgrange [Mal].

By $\mathbf{C}_\infty$ we will denote the Riemann surface of the logarithm. For I an arbitrary, bounded interval in $\mathbf{R}$, we define

$$S(I) = \{x \in \mathbf{C}_\infty;\ \arg x \in I\}.$$

We call $S(I)$ a closed (open) sector, if I is closed (open). Sometimes we will write $S(a,b)$ for $S(I)$ with $I = (a,b)$, or just S if no misunderstanding is possible. Throughout this paper intervals and sectors will be open, unless stated otherwise. By $|I|$ we will denote the length of interval I.

Let $\Delta(0,r)$ be the open disc around 0 with radius $r > 0$ and $S(I,r) := S(I) \cap \Delta(0,r)$. A neighborhood of 0 in $S(I)$ is an open subset U of $S(I)$ such that for every closed subinterval I' of I there exists an $r > 0$ such that $S(I',r) \subset U$. A neighborhood of ∞ in $S(I)$ is an open subset U of $S(I)$ such that for every I' as above there exists $r > 0$ such that $\{x \in S(I') |\, |x| > r\} \subset U$.

If I is an interval then we define $\mathcal{O}(I)$ as the set of germs of functions holomorphic on a neighborhood of 0 in $S(I)$ and the corresponding sheaf on $\mathbf{R}$ is denoted by $\mathcal{O}$. Replacing 0 by ∞ we define in the same way a sheaf $\mathcal{O}_\infty$.

The subspace of $f \in \mathcal{O}(I)$ with the property that f has an asymptotic expansion $f(x) \sim \hat{f}(x) := \sum\limits_{n=0}^{\infty} a_n x^{n/p}, x \to 0, x \in S(I)$, where $\{a_n\}$ is a sequence in $\mathbf{C}$, and p is positive, is denoted by $\mathcal{A}(I)$. Replacing $x^{n/p}$ by $x^{-n/p}$ and 0 by ∞ we define in the same way $\mathcal{A}_\infty(I)$. The corresponding sheaves on $\mathbf{R}$ are denoted by $\mathcal{A}$ and $\mathcal{A}_\infty$.

From here on k **will always be a positive number.**

A function f in $\mathcal{O}(I)$ ($\mathcal{O}_\infty(I)$) is said to be exponentially small of order k at 0 (∞) if to every closed subinterval I' of I corresponds a $b > 0$ such that

$$f(x) = O(\exp(-b|x|^{-k})),\ x \to 0\ (f(x) = O(\exp(-b|x|^{k})),\ x \to \infty), \qquad x \in S(I').$$

The subspaces of such functions are denoted by $\mathcal{A}^{\leqslant -k}(I)$ and $\mathcal{A}_\infty^{\leqslant -k}(I)$ and the corresponding sheaves on $\mathbf{R}$ are denoted in the same way with deletion of the symbol (I).

If $f \in \mathcal{O}_\infty(I)$ and for every closed subinterval I' of I, there exist positive constants A, B, R such that

$$|f(x)| \leqslant A\exp(B|x|^k), \quad \text{if } |x| > R, x \in S(I'),$$

then we say that f is exponentially large of order $\leqslant k$ at ∞ on $S(I)$. By $\mathcal{E}(I,k)$ we denote the set of these functions, which, moreover, are holomorphic on the complete sector $S(I)$ and belong to $\mathcal{A}(I)$.

If I is a bounded interval in $\mathbf{R}$ and $\{I_j\}_{j\in J}$, where J is an interval of $\mathbf{Z}$, is a family of open intervals with $I_j \cap I_l = \varnothing$, if $|j-l| \geqslant 2$, and $\cup_{j\in J} I_j = I$, then we call $\{I_j\}_{j\in J}$ a good covering of I.

Now let $\{f_j\}_{j\in J}$ be a family of functions, with $f_j \in \mathcal{O}_\infty(I_j)$ and $f_{j+1} - f_j \in \mathcal{A}_\infty^{\leqslant -k}(I_j \cap I_{j+1})$, $j, j+1 \in J$. Then $\{f_j\}_{j\in J}$ is called a k-**precise quasi-function** at ∞ on I with respect to the (good) covering $\{I_j\}_{j\in J}$. Two such k-precise quasi-functions $\{f_j\}_{j\in J}$ and $\{g_i\}_{i\in \tilde{J}}$ with respect to good coverings $\{I_j\}_{j\in J}$ and $\{\tilde{I}_i\}_{i\in\tilde{J}}$ are said to be equivalent, if $f_j - g_i \in \mathcal{A}_\infty^{\leqslant -k}(I_j \cap \tilde{I}_i)$. With this equivalence relation $\simeq$ the quotient sheaf $(\mathcal{O}/\mathcal{A}^{\leqslant -k})_\infty := \mathcal{O}_\infty/\mathcal{A}_\infty^{\leqslant -k}$ is given by the equivalence classes of k-precise quasi-functions at ∞ on I.

In an analogous manner we define the quotient sheafs $(\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty$, and $(\mathcal{A}^{\leqslant -k}/\mathcal{A}^{\leqslant -l})_\infty$, for $0 < k < l$. If $\{f_j\}_{j\in J}$ form a representative for $f \in (\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty(I)$, with respect to a covering $\{I_j\}_{j\in J}$ as above, all $f_j \in \mathcal{A}(I_j)$ have the same asymptotic expansion $\hat{f}$, as exponentially small functions have asymptotic expansion 0. Therefore, we write $f \sim \hat{f}$. Similarly we may give a meaning to $f = O(x^\lambda)$ for f as above.

LEMMA 2.1 (cf. Malgrange and Ramis, [MalR92]). — *Let $0 < k < l$, and I an open interval of length $|I| > \pi/k$. Then $(\mathcal{A}^{\leqslant -k}/\mathcal{A}^{\leqslant -l})_\infty(I) = 0$.*

Now we are able to give the definition of the **(generalized) Borel transform of order** k, denoted $\mathcal{B}_k$. Let $I = (a, b)$ be an interval in $\mathbf{R}$, with $|I| = b - a > \pi/k$, and let $f \in (\mathcal{O}/\mathcal{A}^{\leqslant -k})_\infty(I)$ with $f(x) = O(x^q)$, $x \to \infty$, for some real q. Let $I' := [a', b']$ be a compact subinterval of I with $|I'| > \pi/k$. Let $\{f_j\}_{j=1}^m$ be a representative of the restriction of f to a neighborhood of I' with respect to a good covering $\{I_j\}_{j=1}^m$ of that neighborhood.

Suppose that f_j is holomorphic on the neighborhood U_j of ∞ in $S(I_j)$. Choose $x_j \in U_j \cap U_{j+1}$, if $j = 1, \ldots, m-1$, $x_0 = Re^{ia_1} \in U_1$, $x_m = Re^{ib_1} \in U_m$, where $b_1 - a_1 > \pi/k$ and $R > 0$. Let $f_0 = f_{m+1} = 0$. Then we define

$$\begin{aligned} \phi(t) = \mathcal{B}_k f(t) :=& \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} e^{(xt)^k} f_j(x)\, d(x^k) \\ & - \frac{1}{2\pi i} \sum_{j=0}^{m} \int_{x_j}^{\infty(\arg x_j)} e^{(xt)^k} (f_{j+1} - f_j)(x)\, d(x^k). \end{aligned} \tag{2.1}$$

Here $\infty(\alpha)$ means that the path of integration ends at ∞ in the direction $\arg x = \alpha$. Thus ϕ is holomorphic on a neighborhood U of 0 in $S(\tilde{I})$, with $\tilde{I} = \left(-b_1 + \frac{\pi}{2k}, -a_1 - \frac{\pi}{2k}\right)$, and independent of the choices for the x_j. However, variation of a_1 and b_1 gives analytic continuations of ϕ. By variation of I' we obtain $\phi \in \mathcal{O}(I^*)$, with $I^* = \left(-b + \frac{\pi}{2k}, -a - \frac{\pi}{2k}\right)$. If f can be represented by a single holomorphic function on a neighborhood V of 0 in I (i.e. if all the f_j are analytic continuations of each other), then this generalized Borel transform equals the classical one :

$$\mathrm{B}_k f(t) = \int_\Gamma e^{(xt)^k} f(x)\, d(x^k),$$

where Γ is a contour in V from $\infty(a_1)$ to $\infty(b_1)$. In particular,

$$\mathrm{B}_k x^{-\lambda}(t) = t^{\lambda - k}/\Gamma(\lambda/k). \tag{2.2}$$

On the other hand, if ϕ is holomorphic on a neighborhood U of 0 in $S(I^*)$ where $I^* := (-b^*, -a^*)$ and $\phi(t) = O(t^r), t \to 0$ in U, for some $r > -k$, and for $j = 1, \ldots, m$, $t_j \in U$ with $-b_1^* = \arg t_m < \ldots < \arg t_1 = -a_1^*$, $\arg t_{j-1} - \arg t_j < \frac{\pi}{k}$ if $j > 1$, then the family $\{f_j\}_{j=1}^m$ defined by

$$f_j(x) = \int_0^{t_j} e^{-(\tau x)^k} \phi(\tau)\, d(\tau^k),\ t_j \in U, \tag{2.3}$$

is a k-precise quasi-function at ∞ on $\tilde{I} = \left(a_1^* - \frac{\pi}{2k}, b_1^* + \frac{\pi}{2k}\right)$. Variation of a_1^* and b_1^* gives continuation of $\{f_j\}$ in the sense of k-precise quasi-functions. Hence we obtain an element in $(\mathcal{O}/A^{\leqslant -k})_\infty(I)$ where $I = \left(a^* - \frac{\pi}{2k}, b^* + \frac{\pi}{2k}\right)$, which we denote by $\mathcal{L}_k\phi$, the **finite Laplace transform** of order k of ϕ. If ϕ is holomorphic on the full sector $S(I^*)$, and has exponential growth of order $\leqslant k$ at ∞ on this sector, then the classical Laplace transform of order k of ϕ :

$$\mathrm{L}_k\phi(x) = \int_0^\infty e^{-(\tau x)^k}\phi(\tau)\, d(\tau^k)$$

is equivalent (in the sense of k-precise quasi-functions) to the finite Laplace transform of order k of ϕ.

As in the classical case we have, under certain conditions, that $\mathcal{L}_k = \mathcal{B}_k^{-1}$:

THEOREM 2.1 (Malgrange [Mal91]). — *Let $m > 0, l > k > 0, \kappa = (1/k - 1/l)^{-1}, I = (a,b)$ an interval in $\mathbf{R}$, with $b - a > \pi/k$ and $I^* = (-b + \pi/(2k), -a - \pi/(2k))$.*

Then $\mathcal{B}_k$ is an isomorphism

from $x^{-m}(\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty(I)$ onto $t^{m-k}\mathcal{A}(I^*)$ and

from $x^{-m}(\mathcal{A}/\mathcal{A}^{\leqslant -l})_\infty(I)$ onto $t^{m-k}\mathcal{E}(I^*, \kappa)$,

with inverse $\mathcal{L}_k$ and $\mathcal{L}_k^l$ respectively. Here $\mathcal{L}_k^l$ can be represented by finite Laplace integrals of the form (2.3) with t_j replaced by expressions involving the independent variable x (cf. [BIS]).

Let $\hat{f}(x) = x^{-m}\sum_{n=0}^{\infty} a_n x^{-\lambda_n}$. We define **the formal Borel operator** $\hat{\mathcal{B}}_k$ of order k by applying the Borel operator termwise using (2.2) :

$$\hat{\mathcal{B}}_k\hat{f}(t) := t^{m-k}\sum_{n=0}^{\infty}\frac{a_n}{\Gamma((m+\lambda_n)/k)}t^{\lambda_n}. \tag{2.4}$$

If now $f(x) \in x^{-m}(\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty(I)$ and $f(x) \sim \hat{f}(x), x \to \infty$, then we have

$$\mathcal{B}_k f(t) \sim \hat{\mathcal{B}}_k\hat{f}(t). \tag{2.5}$$

The **formal Laplace operator** $\hat{\mathcal{L}}_k$ of order k is defined as the inverse operator of $\hat{\mathcal{B}}_k$.

We define the **k-convolution** of two functions. Let $f, g \in \mathcal{O}(I)$ and $f(t) = O(t^{\alpha-k})$, $g(t) = O(t^{\beta-k})$, $t \to 0, t \in S(I)$, for some $\alpha, \beta > 0$. Then (cf. Martinet and Ramis [MarR91])

$$(f *_k g)(t) := \int_0^t f((t^k - \tau^k)^{1/k}) g(\tau)\, d(\tau^k)$$

defines an element $f *_k g \in \mathcal{O}(I)$ with $(f *_k g)(t) = O(t^{\alpha+\beta-k}), t \to 0, t \in S(I)$. The **convolution property** of the classical Borel transform (of order k) extends to the generalized transform :

If $l \geqslant k, m > 0, |I| > \pi/k$ and $f, g \in x^{-m}(\mathcal{A}/\mathcal{A}^{\leqslant -l})_\infty(I)$ then

$$\mathcal{B}_k(fg) = (\mathcal{B}_k f) *_k (\mathcal{B}_k g) \in t^{m-k}\mathcal{A}(I^*),\ I^* \text{ as before.} \tag{2.6}$$

Finally, we define the ramification operator of order k :

$$\rho_k f(x) := f(x^{1/k}). \tag{2.7}$$

Note that, in fact, $\mathcal{B}_k = \rho_k^{-1}\mathcal{B}\rho_k$, $\mathcal{L}_k = \rho_k^{-1}\mathcal{L}\rho_k$, and $f *_k g = \rho_k^{-1}(\rho_k f * \rho_k g)$.

3. Definition of multisummability.

The following definition of multisummability is due to Malgrange and Ramis [MalR92] (for a slightly different formulation see Balser and Tovbis [BT93]).

DEFINITION 3.1. — *Let $r \in \mathbf{N}$, $0 < k_1 < \ldots < k_r < k_{r+1} := \infty, p > 0$. A formal power series $\hat{f}(x) = \sum_{n=0}^{\infty} c_n x^{-n/p}$ is said to be $(k_1, \ldots, k_r)$-summable at ∞ on a multi-interval $(I_1, \ldots, I_r)$, $I_1 \supset I_2 \supset \ldots \supset I_r$, $|I_j| > \pi/k_j$, if there exist $f_j \in (\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})_\infty(I_j)$, $j = 0, \ldots, r$ with $I_0 := \mathbf{R}$ satisfying the following conditions:*

- *$f_0(x) \sim \hat{f}(x), x \to \infty$ on $S(I_0), f_0(x)$ has period $2p\pi$ in $\arg x$.*
- *$f_{j-1}|_{I_j} = f_j \bmod \mathcal{A}_\infty^{\leqslant -k_j}$, $j = 1, \ldots, r$.*

Then by definition the multisum of $\hat{f}$ on $(I_1, \ldots, I_r)$ is $(f_1, \ldots, f_r)$.

This multisum is uniquely determined by $\hat{f}$ and $(I_1, \ldots, I_r)$. Note that $f_r \in \mathcal{A}_\infty(I_r)$ is an ordinary function on a neighborhood of ∞ in $S(I_r)$, and that

$$f_j(x) \sim \hat{f}(x),\ x \to \infty \text{ on } S(I_j),\ \forall j \in \{1, \ldots, r\}.$$

Moreover, the existence of f_0 is equivalent with the condition that $\hat{f}(x^p)$ is Gevrey of order pk_1.

There exists an equivalent definition which is closer to the original definition of Ecalle's (cf. [Eca87], [Eca93], [Eca94], [Mal]) :

DEFINITION 3.2. — *Let $I_1, \ldots, I_r$ denote open intervals satisfying the conditions in Definition 3.1, and let I_j^*, $j = 1, \ldots, r$ be the following interval: if d_j is the center of I_j then $-d_j$ is the center of I_j^*, and $|I_j^*| = |I_j| - \pi/k_j$. Let $0 < k_1 < \ldots < k_r < k_{r+1} := \infty$, $\kappa_j := (k_j^{-1} - k_{j+1}^{-1})^{-1}$, $j = 1, \ldots, r$, and $\hat{f}(x) = \sum\limits_{n=1}^{\infty} c_n x^{-n/p} \in \mathbf{C}[[x^{-1/p}]]$.*

We say that $\hat{f}$ is $(k_1, \ldots, k_r)$-summable at ∞ on multi-interval $(I_1, \ldots, I_r)$, if:

- *$\hat{\phi}(t) := \hat{\mathcal{B}}_{k_1} \hat{f}(t) = t^{-k_1} \sum\limits_{n=1}^{\infty} c_n t^{n/p} / \Gamma(n/(pk_1))$ is convergent for small positive $|t|$, and its sum can be continued analytically to a function $\phi_1 \in t^{1/p - k_1} \mathcal{E}(I_1^*, \kappa_1)$.*
- *$f_j(x) := \mathcal{L}_{k_j}^{k_{j+1}} \phi_j(x) \in x^{-1/p} (\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})_\infty (I_j)$ has the property that $\mathcal{B}_{k_{j+1}} f_j$ can be analytically continued to a function $\phi_{j+1} \in t^{1/p - k_{j+1}} \mathcal{E}(I_{j+1}^*, \kappa_{j+1})$, for j running from 1 to $r - 1$, respectively.*

Define $f_r := L_{k_r} \phi_r \in \mathcal{A}_\infty(I_r)$. Then the multisum of $\hat{f}$ on $(I_1, \ldots, I_r)$ is defined to be $(f_1, \ldots, f_r)$.

A formal series with a constant term, $\hat{g}(x) = c_0 + \hat{f}(x)$, is said to be $(k_1, \ldots, k_r)$-summable at ∞ on multi-interval $(I_1, \ldots, I_r)$ if $\hat{f}(x)$ has multisum $(f_1, \ldots, f_r)$ on this multi-interval, and then the multisum of $\hat{g}$ is $(c_0 + f_1, \ldots, c_0 + f_r)$.

The equivalence of the two definitions follows from the isomorphism Theorem 2.1 of Malgrange.

4. Reformulation in the linear case.

First we consider normal forms $\tilde{T}$ for the difference operator T. From the formal fundamental matrix (1.2) of $Ty = 0$ it follows that the substitution $y(x) := \hat{\mathbf{M}}(x) y(x)$ transforms T to the normal form T^c:

$$(4.1) \quad T^c y(x) := y(x+1) - \mathbf{A}^c y(x)$$

$$\text{where } \mathbf{A}^c(x) = \bigoplus_{j=1}^{m} x^{\lambda_j} c_j e^{q_j(x+1)-q_j(x)} \left(1+\frac{1}{x}\right)^{\mathbf{L}_j}.$$

From this one may derive a related normal form $\tilde{T}^c$ (cf. [Tur60], [Pra83], [Duv83], [Imm84]):

$$(4.2) \quad \tilde{T}^c y(x) = y(x+1) - \mathbf{F}(x)y(x), \mathbf{F}(x) = \bigoplus_{j=1}^{m} x^{\lambda_j}\mathbf{F}_j(x),$$
$$\mathbf{F}_j(x) = f_j(x)\mathbf{I}_j + x^{-1}\tilde{\mathbf{L}}_j,$$

where $f_j(x) \equiv c_j$ if $q_j(x) \equiv 0$ and otherwise $f_j(x)$ is a polynomial in $x^{-1/p}$ of degree less than p, $f_j(x) \sim c_j(1+\mu_j b_j x^{\mu_j - 1})$ $(c_j \neq 0)$ as $x \to \infty$, and where $\tilde{\mathbf{L}}_j$ is a constant $(n_j \times n_j)$-matrix and $\lambda_j, \mu_j, b_j, c_j$ are the same as in (4.1). The formal normalizing matrix $\hat{\mathbf{M}}(x)$ now is modified to $\tilde{\mathbf{M}}(x)$.

Using a truncation $\tilde{\mathbf{M}}_N$ of this matrix we see that T can be meromorphically transformed by $y(x) := \tilde{\mathbf{M}}_N(x)z(x)$ to $\tilde{T}$ where

$$(4.3) \qquad \tilde{T}z(x) = \tilde{T}^c z(x) + x^{\mu}\tilde{\mathbf{F}}(x)z(x),$$

with $\mu < \min_j\{\lambda_j\} - 1$, $\tilde{\mathbf{F}}(x) \in \mathrm{End}(n, \mathbf{C}\{x^{-1/p}\})$.

In order to prove the linear case of Theorem 1.1. it is sufficient to consider the linear difference equation

$$(4.4) \qquad \tilde{T}z(x) = \tilde{c}(x),$$

where $\tilde{c}(x)$ is meromorphic in $x^{-1/p}$ at ∞. We may rewrite this equation in the following form:

$$(4.5) \quad \Delta z(x) := \left(\bigoplus_{h=0}^{r} x^{1-k_h}\mathbf{I}_h\right)(z(x+1) - z(x))$$
$$- \left(\bigoplus_{h=0}^{r} \mathbf{A}_h + x^{-1/p}\mathbf{B}(x)\right) z(x) = c(x),$$

where
(4.6)
$$\begin{cases} r \in \mathbf{N};\ 0 = k_0 < k_1 < ... < k_q = 1 < \ldots < k_r,\ 1 \leqslant q \leqslant r; \\ k_h \in p^{-1}\mathbf{N},\ h = 1,...r;\ p \in \mathbf{N}; \\ \mathbf{A}_h,\ h = 0,...,r, \text{ is an } (n_h \times n_h)\text{-matrix}; \\ n_h \text{ is a nonnegative integer },\ n_h > 0 \text{ if } k_h \neq 0,1;\ n_0 + ... + n_r = n; \\ \mathbf{A}_h \text{ is invertible if } 1 \leqslant h \leqslant r \text{ and } n_h > 0; \\ \mathbf{B}(x) \in \mathrm{End}(n, \mathbf{C}\{x^{-1/p}\}), c(x) \in \mathbf{C}^n\{x^{-1/p}\}[x^{1/p}]. \end{cases}$$

The translation of (4.4) into (4.5) runs as follows: If $\lambda_j > 0$ then there exists $k_h = \lambda_j + 1$ and $\mathbf{A}_h$ consists of blocks $c_j\mathbf{I}_j$ corresponding to all j with $k_h = \lambda_j + 1$. If $\lambda_j = 0, c_j \neq 1$ then $n_q > 0$ and $\mathbf{A}_q$ contains corresponding blocks $(c_j - 1)\mathbf{I}_j$. If $\lambda_j < 0$ then also $n_q > 0$ and $\mathbf{A}_q$ contains corresponding blocks $-\mathbf{I}_j$ whereas if none of these cases occurs we have $n_q = 0$. In case $\lambda_j = 0, c_j = 1, q_j(x) \not\equiv 0$ there exists $k_h = \mu_j$ and $\mathbf{A}_h$ consists of blocks $\mu_j b_j \mathbf{I}_j$ corresponding to all j with $k_h = \mu_j$. The case $\lambda_j = 0, c_j = 1, q_j(x) \equiv 0$ corresponds to $k_0 = 0$, and $\mathbf{A}_0$ contains corresponding blocks $\tilde{\mathbf{L}}_j$. In general the n_h's of (4.6) will not be the same as the n_j's of (1.3).

The assumptions (4.6) include also cases where in the original difference operator T the matrix $\mathbf{A} \in \mathrm{End}(n, \mathbf{C}\{x^{-1/p}\}[x^{1/p}])$ is not invertible (corresponding to a block $\mathbf{F}_j$ which is x^{-1} times a nilpotent matrix).

In agreement with the definitions of singular and Stokes directions in section 1 we now define:

DEFINITION 4.1. — *A direction θ will be called a singular direction of level k_j of the difference operator Δ (cf. (4.5)) where $j = 1, \ldots, q-1$, if $\mathbf{A}_j + k_j t^{k_j}\mathbf{I}_j$ is not invertible for some t with $\arg t = -\theta$. Moreover θ will be called a singular direction of level 1 if $n_q > 0$ and $\mathbf{A}_q + (1 - e^{-t})\mathbf{I}_q$ is not invertible for some t with $\arg t = -\theta$, or if $n_h > 0$ for some $h < q$ and $\theta = \pi/2 \bmod \pi$. If θ is a singular direction of level k_j then the pair $\left(\theta - \dfrac{\pi}{2k_j}, \theta + \dfrac{\pi}{2k_j}\right)$ is a pair of Stokes directions of level k_j.*

Then we have

THEOREM 4.1. — *Let the assumptions (4.6) concerning the difference equation (4.5) be fulfilled and let $\hat{z}(x) \in \mathbf{C}[[x^{-1/p}]]$ be a formal solution of (4.5). Let $(I_1, \ldots, I_q)$ denote a multi-interval satisfying the following conditions:*

$$(4.7)\qquad \begin{cases} I_1 \supset I_2 \supset \ldots \supset I_q,\ |I_j| > \pi/k_j \text{(cf. Definition 3.1)}; \\ I_j \text{ does not contain a pair of Stokes directions of level } k_j\ (1 \leqslant j \leqslant q). \end{cases}$$

Then $\hat{z}(x)$ is $(k_1, \ldots, k_q)$-summable on $(I_1, \ldots, I_q)$ in the following cases:

(i) *$k_r > 1, I_q \cap \{(2j+1)\pi : j \in \mathbf{Z}\} = \varnothing$ and if $n_q > 0$ then $\mathbf{A}_q + \mathbf{I}_q$ is invertible.*

(ii) *$k_r = 1$ and either $n_r > 0$ and $\mathbf{A}_r + \mathbf{I}_r$ is invertible or $n_r = 0$.*

(iii) $k_r = 1$, $n_r > 0$ *and* $I_r \cap \{2j\pi : j \in \mathbf{Z}\} = \varnothing$.

If $(z_1, ... z_q)$ *is the corresponding multisum then* z_q *is a holomorphic solution of (4.5) in a neighborhood of* ∞ *in* $S(I_q)$ *and* z_j *is a solution of (4.5) in* $(\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I_j)$ *if* $1 \leqslant j \leqslant q-1$.

Moreover, $\hat{z}(x)$ *is 1-summable in upper and in lower halfplanes if* $n_0 = 0, q = 1$, *and if* $n_1 > 0$ *then* $\mathbf{A}_q$ *does not have an eigenvalue* λ *on the circle* $|\lambda + 1| = 1$.

It is easily seen that this theorem implies Theorem 1.1. in case $G(x, y)$ is independent of y. Note that presence of a level $k_r > 1$ causes obstruction in the summability process in the left half plane, presence of level 1 with $\mathbf{A}_q + \mathbf{I}_q$ not invertible causes obstruction in the right half plane (cf. [Imm]). However, we will show that

Remarks 4.1. — The formal solution $\hat{z}(x)$ can always be lifted to analytic solutions $z_\pm(x)$ in $H_\pm := \{z \in \mathbb{C} : \pm\mathrm{Im} z > R\}$ for some $R > 0$ such that $z_\pm(x) \sim \hat{z}(x)$ as $z \to \infty$ in $H_\pm$ with $0 \leqslant \pm \arg z \leqslant \pi - \epsilon$ for any $\epsilon > 0$. Similarly there are solutions as above with the condition on $\arg z$ replaced by $\epsilon \leqslant \pm \arg z \leqslant \pi$ (cf.[vd PS]). These solutions need not be uniquely determined by $\hat{y}$.

It is sufficient to prove the theorem for the case that the formal solution $\hat{z}(x)$ and the right-hand-side $c(x)$ of (4.5) are both of order $O(x^{-N/p}), x \to \infty$ for sufficiently large $N \in \mathbf{N}$ with at least $N/p > 1$. This follows by subtracting from the formal solution $\hat{z}(x)$ a partial sum of some sufficiently high order. Then the remainder satisfies again (4.5) with $c(x)$ replaced by another function which has a zero of that order at ∞.

The proof of Theorem 4.1 will be given via convolution equations corresponding to the difference equation (4.5). These equations will be derived in section 5. In section 6 we then give the proof for case (i) of Theorem 4.1 and of Remark 4.1. The proof for case (ii) and case (iii) is similar (cf. also the correspondence between cases (i) and (iii) mentioned in sect.1).

5. The convolution equations.

We apply Borel transforms to the difference equation (4.5) with $c(x) = O(x^{-N/p})$ with N sufficiently large. Let $k \in \dfrac{1}{p}\mathbf{N} \cap (0, 1]$. Suppose that $z(x) \in x^{-N/p}(\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty(I)$ for some open interval I with length $> \pi/k$. Then also the difference $z(x+1) - z(x)$ is in this set (cf. [Imm84, §16]). Let $u(t) := \mathcal{B}_k z(t)$. Then $u(t) \in t^{\frac{N}{p}-k}\mathcal{A}(I^*)$ where I^* and I are related as in Definition 3.2.

The convolution equation associated with (4.5) involves the operator F_k defined by

$$(5.1)\quad F_k u := \mathcal{B}_k\{x^{1-k}(z(x+1)-z(x))\}, \text{ where } k \in \frac{1}{p}\mathbf{N}\cap(0,1],\ z = \mathcal{L}_k u.$$

It follows that $F_k u(t) \in t^{\frac{N+1}{p}-1}\mathcal{A}(I^*)$.

If $k = 1$ it is easy to see that $x^{1-k}(z(x+1) - z(x))$ simplifies to $\mathcal{L}_k\{(-1+e^{-t})u(t)\}(x)$, and therefore

$$(5.2)\qquad F_1 u(t) = (-1+e^{-t})u(t).$$

In case $k < 1$ we rewrite the difference $z(x+1) - z(x)$ as a perturbation of the derivative $z'(x)$ of $z(x)$:

$$(5.3)\quad z(x+1)-z(x) = z'(x) + \int_x^{x+1}(x+1-y)z''(y)dy = z'(x) + Wz(x).$$

Now, as $z(x) \in x^{-N/p}(\mathcal{A}/\mathcal{A}^{\leqslant -k})_\infty(I)$, also the derivative $z'(x)$ is a member of this set. Hence, so is $Wz(x)$. We have $z'(x) = -kx^{k-1}\mathcal{L}_k\{t^k u\}(x)$ and as a consequence we may write:

$$(5.4)\qquad F_k u(t) = -kt^k u(t) + H_k u(t), \text{ if } 0 < k < 1,$$

with

$$(5.5)\qquad H_k u(t) := \mathcal{B}_k\{x^{1-k}Wz(x)\}(t) = \mathcal{B}_k\{x^{1-k}W\mathcal{L}_k u\}(t).$$

We will investigate the operator H_k in section 7.

Next, we define $(n \times n)$-matrices $\mathbf{M}_j$ as follows:

$$\mathbf{M}_j := x^{k_0-k_j}\mathbf{I}_0 \oplus \ldots \oplus x^{k_{j-1}-k_j}\mathbf{I}_{j-1} \oplus \mathbf{I}_j \oplus \ldots \oplus \mathbf{I}_r,\ 1 \leqslant j \leqslant q.$$

Then $\Delta z - c = 0$ in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_j})^n_\infty(I_j)$ is equivalent to $\mathcal{B}_{k_j}\mathbf{M}_j(\Delta z - c) = 0$ in $\mathcal{A}^n(I_j^*)$, if I_j and I_j^* are related as in Definition 3.2. The lefthand side of this equation may be expressed in terms of

$$(5.6)\qquad u := \mathcal{B}_{k_j}z, \tilde{\beta}_j := \mathcal{B}_{k_j}\{\mathbf{M}_j x^{-1/p}\mathbf{B}\}, \tilde{\gamma}_j := \mathcal{B}_{k_j}\{\mathbf{M}_j c\}.$$

Utilizing (5.1), (5.2), (5.4), (2.2) and (2.6) we get a convolution equation $Q_j u = \tilde{\gamma}_j$ where

$$Q_j := \mathcal{B}_{k_j}\mathbf{M}_j \Delta \mathcal{L}_{k_j},\ j = 1, \dots, q. \tag{5.7}$$

We can rewrite these convolution equations in the form $T_j u = u$. Denote by $v^{(h)}$ the components of n-vector v corresponding to the h^{th} block and let

$$G_j := H_{k_j}. \tag{5.8}$$

Then we get

- **if** $0 < j < q$ **and** $k := k_j$

$$\begin{cases} (T_j u)^{(h)} = (kt^k)^{-1}[(G_j u)^{(h)} - \mathbf{A}_h(\frac{t^{-k_h}}{\Gamma(1-\frac{k_h}{k})} *_k u)^{(h)} \\ \qquad -(\tilde{\beta}_j *_k u)^{(h)} - \tilde{\gamma}_j^{(h)}], \\ \qquad\qquad 0 \leqslant h < j; \\ (T_j u)^{(j)} = (\mathbf{A}_j + kt^k \mathbf{I}_j)^{-1}[(G_j u)^{(j)}(t) - (\tilde{\beta}_j *_k u)^{(j)} - \tilde{\gamma}_j^{(j)}]; \\ (T_j u)^{(h)} = -\mathbf{A}_h^{-1}[(\frac{t^{k_h-2k}}{\Gamma(\frac{k_h}{k}-1)} *_k kt^k u)^{(h)} - (\frac{t^{k_h-2k}}{\Gamma(\frac{k_h}{k}-1)} *_k G_j u)^{(h)} \\ \qquad +(\tilde{\beta}_j *_k u)^{(h)} + \tilde{\gamma}_j^{(h)}], \\ \qquad\qquad j < h \leqslant r, \end{cases} \tag{5.9}$$

- **if** $j = q$

$$\begin{cases} (T_q u)^{(h)} = -(1-e^{-t})^{-1}[\mathbf{A}_h(\frac{t^{-k_h}}{\Gamma(1-k_h)} * u)^{(h)} + (\tilde{\beta}_q * u)^{(h)} + \tilde{\gamma}_q^{(h)}], \\ \qquad\qquad 0 \leqslant h < q; \\ (T_q u)^{(q)} = -(\mathbf{A}_q + (1-e^{-t})\mathbf{I}_q)^{-1}[(\tilde{\beta}_q * u)^{(q)} + \tilde{\gamma}_q^{(q)}]; \\ (T_q u)^{(h)} = -\mathbf{A}_h^{-1}[(\frac{t^{k_h-2}}{\Gamma(k_h-1)} * (1-e^{-t})u)^{(h)} + (\tilde{\beta}_q * u)^{(h)} + \tilde{\gamma}_q^{(h)}], \\ \qquad\qquad q < h \leqslant r. \end{cases} \tag{5.10}$$

The equations with $h = 0$ and $h = q$ are deleted if $n_0 = 0$ and $n_q = 0$ respectively. We will prove Theorem 4.1 by means of the convolution equations

$$T_j u = u,\ j = 1, \dots, q.$$

6. Proof of Theorem 4.1.

The proof of Theorem 4.1, case (i), follows easily from the two lemmas below, that we will prove in sections 8 and 9, respectively.

LEMMA 6.1. — *Let $\hat{z}(x) = \sum_{m=N}^{\infty} a_m x^{-m/p}$ be a formal solution of (4.5). Let $\hat{u}_1 = \hat{\mathcal{B}}_{k_1}\hat{z}$. Then $\hat{u}_1$ is a convergent power series in $t^{1/p}$ in a full neighborhood of 0 with sum $u_1(t) \in t^{N/p-k_1}\mathbf{C}^n\{t^{1/p}\}$, which is a solution of*

$$T_1 u(t) = u(t). \tag{6.1}$$

Moreover, $\mathcal{L}_{k_1}u_1 =: z_0$ is a solution of (4.5) in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_1})^n_\infty(\mathbf{R})$. In particular $\hat{z}(x^p) \in \mathbf{C}[[x^{-1}]]_{1/(pk_1)}$.

Define

$$\kappa_j := \left(\frac{1}{k_j} - \frac{1}{k_{j+1}}\right)^{-1}, j = 1, \dots, q-1, \kappa_q := k_q = 1. \tag{6.2}$$

Let $(I_1, \dots, I_q)$ be a multi-interval satisfying the conditions in (4.7) and the additional one of Theorem 4.1, case (i). If d_j denotes the center of I_j, let I_j^* and $\tilde{I}_j$ be the intervals with center $-d_j$, and length $|I_j^*| = |I_j| - \pi/k_j$, $|\tilde{I}_j| = |I_j| - \pi/k_{j+1}$ $(j = 1, \dots, q)$; $\tilde{I}_0 := \mathbf{R}$. Since I_j does not contain any pair of Stokes directions of level k_j, and $I_q \cap \{(2j+1)\pi; j \in \mathbf{Z}\} = \varnothing$, it follows that $-I_j^*$ does not contain a singular direction of level k_j, and $I_q^* \subset (-\pi/2, \pi/2) \bmod 2\pi$. Using these conditions and notations we have

LEMMA 6.2. — *Let $j \in \{1, \dots, q\}$ be such that there exists a solution $u_j \in t^{N/p-k_j}\mathcal{A}^n(\tilde{I}_{j-1})$ of $T_j u = u$. Then u_j possesses an analytic continuation in $S(I_j^*)$, which will be denoted by u_j as well, and which has exponential growth at ∞ of order at most κ_j. Moreover, $z_j := \mathcal{L}_{k_j}^{k_{j+1}} u_j$ is a solution of (4.5) in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I_j)$, $1 \leqslant j \leqslant q-1$; $z_q := L_1 u_q$ is a holomorphic solution of (4.5) in $\mathcal{A}^n_\infty(I_q)$.*

If $j \leqslant q-1$ then $u_{j+1} := \mathcal{B}_{k_{j+1}} z_j$ satisfies $T_{j+1}u = u$ in a neighborhood of 0 in $S(\tilde{I}_j)$, $u_{j+1} \in t^{N/p-k_{j+1}}\mathcal{A}^n(\tilde{I}_j)$.

Remark 6.1. — Also if the cases (i), (ii), (iii) of Theorem 4.1 do not necessarily occur the previous lemmas remain valid except for the statement on u_q and z_q. However, if $q = 1$ or $q > 1$, $\pm\pi/2 \in \tilde{I}_{q-1}$ then $u_1(t)$ if $q = 1$

and $u_q(t) = \mathcal{B}_1 z_{q-1}(t)$ if $q > 1$ are analytic in a strip around the imaginary axis where $\pm\Im t \geqslant 0$ except for values of t where $\mathbf{A}_q + (1 - e^{-t})\mathbf{I}_q$ is not invertible or $t = 2g\pi i, g \in \mathbf{Z}^*$. Moreover, u_q has exponential growth of order at most 1 in this strip.

The proof of Theorem 4.1, case (i), now proceeds as follows: The conditions of Lemma 6.2 are satisfied for $j = 1$ because of Lemma 6.1. Then Lemma 6.2 implies a.o. that its conditions are satisfied for $j = 2$. Repeating this reasoning for $j = 2, \dots, q$ consecutively we get solutions u_j of $T_j u = u$ and solutions z_j of (4.5) in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I_j)$ for $j = 1, \dots, q$.

Since $z_j = \mathcal{L}_{k_j}^{k_{j+1}} \mathcal{B}_{k_j} z_{j-1}$ and $z_j \in (\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I_j)$ we see that $z_{j-1}|_{I_j} = z_j \bmod \mathcal{A}^{\leqslant -k_j}$. Hence $(z_1, \dots, z_q)$ is the $(k_1, \dots, k_q)$-sum of $\hat{z}$ on $(I_1, \dots, I_q)$. This proves Theorem 4.1, case (i).

The proof in the cases (ii) and (iii) is similar. Remark 4.1 follows from Remark 6.1 by choosing the intervals I_j suitably and defining $z_\mp$ as Laplace transform of u_q with path of integration in the strip around the imaginary axis. The last assertion of the theorem concerning the case $n_0 = 0, q = 1$ is obtained in the same way but now we may take the positive or negative imaginary axis as path of integration in the Laplace integral for $z_\mp$.

Remark 6.1 will be proved in section 9.

7. Properties of the operator H_k.

In this section we will investigate the operator H_k defined in section 5 for $k \in (0,1) \cap \frac{1}{p}\mathbf{N}$. The results are summarized in Lemma 7.2 at the end of this section. They will be of use, when proving Lemmas 6.1 and 6.2.

We will write H in stead of H_k in this section. H operates on functions $u \in t^{\frac{N}{p} - k}\mathcal{A}(I)$, $\frac{N}{p} > 1$ (cf. (5.5), (5.3)), where I is some open interval (in section 5 this was the interval I^*). Let $z := \mathcal{L}_k u$ with u as above. Then

$$z''(x) = -k(k-1)x^{k-2}\mathcal{L}_k\{t^k u\}(x) + k^2 x^{2(k-1)}\mathcal{L}_k\{t^{2k}u\}(x)$$

and

$$Wz(x) = \int_x^{x+1} (x+1-y)\{-k(k-1)y^{k-2}\mathcal{L}_k\{t^k u\}(y) + k^2 y^{2(k-1)}\mathcal{L}_k\{t^{2k}u\}(y)\}dy.$$

For convenience we will use the following notations:

$$v = \rho_k u, \mathcal{W} = \rho_k W \rho_k^{-1}, \mathcal{H} = \rho_k H \rho_k^{-1}, \text{ and } x^k =: \xi, t^k =: \tau,$$

where ρ_k is the ramification operator defined in (2.7). Let J be the interval defined by $\theta \in J \Leftrightarrow \theta/k \in I$. Choose M in the neighborhood of 0 in $S(J)$ where v is holomorphic, and choose $\int_0^M \exp(-\xi\tau)\tau^\alpha v(\tau)\, d\tau$ as a representative for $\mathcal{L}\{\tau^\alpha v(\tau)\}(\xi)$. Substituting this in the expression for Wz we obtain by means of a change in the order of integration if $|\arg(\xi M)| < \pi/2$:

$$x^{1-k} W z(x) = \xi^{-1+1/k} \mathcal{W}\mathcal{L}v(\xi) = \int_0^M e^{-\xi\tau} \tau v(\tau) w(\xi,\tau) d\tau,$$

with

$$\begin{aligned} w(\xi,\tau) &= w(x^k, t^k) \\ &:= k x^{1-k} \int_x^{x+1} (x+1-y) y^{k-2} \{1 - k + k y^k t^k\} e^{-t^k(y^k - x^k)}\, dy. \end{aligned}$$

Obviously $w(\xi^{pk}, \tau)$ is holomorphic in $\xi \in \mathbf{C}\backslash\{0\}$, and entire in τ.

We substitute $y = x + s$ in the last integral. As $1/2 \leqslant |1 + s/x| \leqslant 3/2$, and $|(1 + s/x)^k - 1| \leqslant c|x|^{-1}$ for some constant $c > 0$, for all $0 < s < 1$ and $x \in \mathbf{C}$ with $|x| \geqslant 2$, we obtain an estimate for w:

$$|w(\xi,\tau)| \leqslant K_1 |\xi|^{-1/k} (1 + |\xi\tau|) e^{K_2|\tau||\xi|^{1-\frac{1}{k}}}, \text{ if } |\xi|^{1/k} \geqslant 2.$$

Now, recall that $Hu(t) = \mathcal{B}_k\{x^{1-k} W z(x)\}(t)$, i.e. $\mathcal{H}v(\tau) = \mathcal{B}\{\xi^{-1+1/k} \mathcal{W}\mathcal{L}v\}$ (τ). This implies

$$\begin{aligned} \mathcal{H}v(\tau) &= \frac{1}{2\pi i} \int_C e^{\tau\xi} d\xi \int_0^M e^{-\xi\eta} \eta v(\eta) w(\xi,\eta)\, d\eta \\ &= \int_0^M \eta v(\eta) h(\tau - \eta, \eta)\, d\eta \end{aligned}$$

with

$$h(\tau,\eta) = \mathrm{B}\{w(\cdot,\eta)\}(\tau) := \frac{1}{2\pi i} \int_C e^{\tau\xi} w(\xi,\eta) d\xi.$$

So, also h is holomorphic on $\mathbf{C}_\infty$ in the first variable, and entire in the second, and $h(\tau e^{2pk\pi i}, \eta) = h(\tau,\eta)$.

For C we choose a contour consisting of halfrays $(\infty, 2^k R)e^{i\theta_1}$ and $(2^k R, \infty)e^{i\theta_2}$ $(\theta_1 < \theta_2)$, such that $\cos(\arg(\tau\xi)) \leqslant -\varepsilon$ along these rays (for some $\varepsilon > 0$), connected through the arc $C_R : |\xi| = 2^k R$, $\theta_1 \leqslant \arg\xi \leqslant \theta_2$. Then, if $R \geqslant 1$, we can estimate the integrand above with $K_1|\xi|^{-1/k}(1+|\xi\eta|)\exp(|\tau\xi|\cos(\arg(\tau\xi))+K_1|\eta\xi^{1-\frac{1}{k}}|)$. The exponent takes its maximum on the arc when $\arg\xi = -\arg\tau$, and this maximum is minimal for $R \approx |\eta/\tau|^k$. Hence we choose $R = \max\{|\eta/\tau|^k, 1\}$ and obtain the following estimates:

$$\left|\frac{1}{2\pi i}\int_{C_R} e^{\tau\xi}w(\xi,\eta)d\xi\right| \leqslant L_1 R^{1-\frac{1}{k}}(1+R|\eta|)\exp(L_2R|\tau|),$$

and

$$\left|\frac{1}{2\pi i}\int_{C\setminus C_R} e^{\tau\xi}w(\xi,\eta)d\xi\right| \leqslant L_1\frac{1}{|\tau|}R^{-1/k}(1+R|\eta|)\exp(L_2R|\tau|).$$

Thus

$$|h(\tau,\eta)| \leqslant L_1R^{-1/k}(R+\frac{1}{|\tau|})(1+R|\eta|)\exp(L_2R|\tau|), \text{ if } R = \max\{|\eta/\tau|^k, 1\}.$$

But this means that $\mathrm{L}\{h(\cdot,\eta)\}(\tau)$ makes sense, and $\mathrm{L}\{h(\cdot,\eta)\}(\tau) = w(\tau,\eta)$. If we substitute this in the expression for $\xi^{-1+1/k}\mathcal{WL}v(\xi)$ we obtain:

$$\begin{aligned}\xi^{-1+1/k}\mathcal{WL}v(\xi) &= \int_0^M e^{-\xi\tau}\tau v(\tau)d\tau\int_0^\infty e^{-\xi\sigma}h(\sigma,\tau)d\sigma\\ &= \int_0^M e^{-\xi t}dt\int_0^t \tau v(\tau)h(t-\tau,\tau)d\tau + \int_M^\infty e^{-\xi t}dt\int_0^M \tau v(\tau)h(t-\tau,\tau)d\tau.\end{aligned}$$

As the second term in this last expression is exponentially small for $\Re(M\xi)$ large positive, we see that $\xi^{-1+1/k}\mathcal{WL}v$ and $\mathcal{L}\{\int_0^t \tau v(\tau)h(t-\tau,\tau)d\tau\}$ define the same element in $(\mathcal{A}/\mathcal{A}^{\leqslant -1})(\tilde{J})$ where $\phi \in \tilde{J}$ iff $|\phi+\theta| < \pi/2$ for some $\theta \in J$. That is,

$$\mathcal{H}v(t) = (\mathcal{B}\xi^{-1+1/k}\mathcal{WL}v)(t) = \int_0^t \tau v(\tau)h(t-\tau,\tau)d\tau.$$

However, this last expression still has meaning for functions $v(t)$ that belong to the set of holomorphic functions on $S(J,r)$ (for some open interval J, and some $r > 0$), that are bounded on $S(J',r')$ for any closed subinterval J' of J, and any $r' \in (0,r)$. Moreover, $\mathcal{H}$ maps this set into itself.

Hence we have proved the following lemmas:

LEMMA 7.1. — *Let $0<k<1$ and*

$$w_k(\xi,\tau) := \xi^{\frac{1}{k}-1}\int_{\xi}^{(\xi^{1/k}+1)^k}(\xi^{1/k}+1-\eta^{1/k})\eta^{-1/k}\{1-k+k\eta\tau\}e^{-\tau(\eta-\xi)}d\eta,$$

and

$$h_k(\tau,\eta) := \mathrm{B}\{w_k(\cdot,\eta)\}(\tau).$$

Then $h_k(\tau^{pk},\eta)$ is holomorphic in τ, $\tau\in\mathbf{C}\backslash\{0\}$, entire in η, and there exist positive constants K_1 and K_2 such that

$$|h_k(\tau,\eta)| \leqslant K_1R^{-1/k}\left(R+\frac{1}{|\tau|}\right)(1+R|\eta|)\exp(K_2R|\tau|), \tag{7.1}$$
$$\text{if } R=\max\{|\eta/\tau|^k,1\}.$$

LEMMA 7.2. — *Let J be an open interval. We define $\mathcal{H}_k=\rho_kH_k\rho_k^{-1}$, where H_k is defined by (5.5). Then $\mathcal{H}_k$ can be extended to a mapping of the space of holomorphic functions on $S(J,r)$, that are bounded on $S(J',r')$ for any closed subsector of J' of J and any $r'\in(0,r)$, into itself. If $v(t^{pk})$ is holomorphic in a (full) neighborhood U of 0, and continuous on the closure $\overline{U}$, then also $\mathcal{H}_kv(t^{pk})$ has this property.*

Moreover, if v is an element of the function space described above, then

$$\mathcal{H}_kv(t)=\int_0^t\eta v(\eta)h_k(t-\eta,\eta)d\eta, \tag{7.2}$$

where h_k is the function defined in the previous lemma.

8. Proof of Lemma 6.1.

For convenience we rewrite the system $T_ju=u$ to $\mathcal{T}_jv=v$, where $\mathcal{T}_j=\rho_{k_j}T_j\rho_{k_j}^{-1}$, $v=\rho_{k_j}u$. Writing

$$\mathcal{G}_j=\rho_{k_j}G_j\rho_{k_j}^{-1}=\rho_{k_j}H_{k_j}\rho_{k_j}^{-1} \tag{8.1}$$

(cf. (5.8), (5.5),(5.3)), $\beta_j=\rho_{k_j}\tilde{\beta}_j$, $\gamma_j=\rho_{k_j}\tilde{\gamma}_j$, so (cf. (5.6))

$$\beta_j=\mathcal{B}\{\rho_{k_j}\mathbf{M}_jx^{-1/p}\mathbf{B}\},\gamma_j=\mathcal{B}\{\rho_{k_j}\mathbf{M}_jc\}, \tag{8.2}$$

the new system looks as follows:

- **if** $0 < j < q$ **and** $k = k_j$

(8.3)
$$\begin{cases} (\mathcal{T}_j v)^{(h)} &= (kt)^{-1}[(\mathcal{G}_j v)^{(h)} - \mathbf{A}_h(\frac{t^{-k_h/k}}{\Gamma(1-\frac{k_h}{k})} * v)^{(h)} \\ & \quad -(\beta_j * v)^{(h)} - \gamma_j^{(h)}], \\ & \qquad\qquad 0 \leqslant h < j; \\ (\mathcal{T}_j v)^{(j)} &= (\mathbf{A}_j + kt\mathbf{I}_j)^{-1}[(\mathcal{G}_j v)^{(j)}(t) - (\beta_j * v)^{(j)} - \gamma_j^{(j)}]; \\ (\mathcal{T}_j v)^{(h)} &= -\mathbf{A}_h^{-1}[(\frac{t^{k_h/k-2}}{\Gamma(\frac{k_h}{k}-1)} * ktv)^{(h)} - (\frac{t^{k_h/k-2}}{\Gamma(\frac{k_h}{k}-1)} * \mathcal{G}_j v)^{(h)} \\ & \quad +(\beta_j * v)^{(h)} + \gamma_j^{(h)}], \\ & \qquad\qquad j < h \leqslant r, \end{cases}$$

- **if** $j = q$ then $\mathcal{T}_q = T_q$ since $k_q = 1$ (cf. 5.10).

Let us define $k := k_1$, and $\hat{v}_1 = \rho_{k_1}\hat{u}_1$. We may choose N so large that the formal solution $\hat{z}(x) \in \mathbf{C}^n[[x^{-1/p}]]$ is uniquely determined. So $\hat{v}_1$ is the unique formal solution of $\mathcal{T}_1 v(t) = v(t)$. We will prove that it is convergent.

If w is an l-vector, $w = (w_1, \ldots, w_l)$, then by $|w|$ we denote its 1-norm, $|w| := |w_1| + \ldots + |w_l|$.

We will frequently use the property below of the convolution operator : Let $\alpha, \beta > 0$, and f, g holomorphic functions on a sectorial neighborhood $S = S(I, \delta)$ of 0 with $|f(t)| \leqslant K|t|^{\alpha-1}$, $|g(t)| \leqslant L|t|^{\beta-1}$, if $t \in S$, for some constants $K, L, \delta > 0$. Then $(f * g)(t)$ is holomorphic on S, and

$$|(f * g)(t)| \leqslant KL\,B(\alpha, \beta)|t|^{\alpha+\beta-1},\ t \in S.$$

Here B is the beta function. From Stirling's formula it follows that, given $w > 0, \alpha > 0$, there exists $C > 0$ such that for all z with $\Re z > 0$

$$|B(w, z+\alpha)| \leqslant C|z|^{-w}.$$

From (8.2) it follows that

$$\begin{cases} \beta_1^{(h)}(t) = O(t^{\frac{1}{pk}}),\ t \to 0 & (h = 0) \\ \beta_1^{(h)}(t) = O(t^{\frac{1}{pk}-1}),\ t \to 0 & (1 \leqslant h \leqslant r). \end{cases}$$

As $c(x) = O(x^{-N/p})$, we have $t^{-1}\gamma_1^{(0)}(t) = O(t^{N/(pk)-1})$, and $\gamma_1^{(h)}(t) = O(t^{N/(pk)-1})$, $1 \leqslant h \leqslant r$.

Choose ρ such that $0 < \rho < 2\pi$ and such that for $0 < |t| \leqslant \rho$ the matrices $\mathbf{A}_1 + k_1 t\mathbf{I}_1$ and $\mathbf{A}_1 + (1-e^{-t})\mathbf{I}_1$ are invertible in case $q > 1$ and $q = 1, n_1 > 0$ respectively. Define $\rho_1 := \rho^{1/(pk)}$, and let $W_{\rho,N}$ be the Banach space of functions f, for which $t^{pk-N} f(t^{pk})$ is a holomorphic function from $\Delta(0,\rho_1)$ to $\mathbf{C}^n$ with continuous extension to $\overline{\Delta(0,\rho_1)}$, provided with the norm

$$\|f\|_{\rho,N} = \max\{|t^{pk-N} f(t^{pk})|;\ t \in \overline{\Delta(0,\rho_1)}\}.$$

We will show that $\mathcal{T}_1$ maps $W_{\rho,N}$ into itself, and is a contraction if N is large enough. Therefore, from now on v will denote a function in $W_{\rho,N}$.

Let us first assume $q > 1$, i.e. $k_1 < 1$.

Let w be a function such that $t^{\frac{N}{pk}-\beta} w \in W_{\rho,N}$ for some $\beta \in \dfrac{1}{pk}\mathbf{N}$. If $\alpha + \dfrac{N}{pk} \in \dfrac{1}{pk}\mathbf{N}$, then $(w * t^\alpha v)(t) \in W_{\rho,N}$, and

$$|(w * t^\alpha v)(t)| \leqslant B \|t^{\frac{N}{pk}-\beta} w\|_{\rho,N} \|v\|_{\rho,N} N^{-\beta} |t|^{\frac{N}{pk}+\alpha+\beta-1},\ |t| \leqslant \rho.$$

Hence

$$(8.4)\quad \begin{cases} |t^{-1}(\beta_1 * v)^{(h)}(t)| & \leqslant & B\|v\|_{\rho,N} N^{-\frac{1}{pk}-1} |t|^{\frac{N+1}{pk}-1}, & (h=0) \\ |(\beta_1 * v)^{(h)}(t)| & \leqslant & B\|v\|_{\rho,N} N^{-\frac{1}{pk}} |t|^{\frac{N+1}{pk}-1}, & (1 \leqslant h \leqslant r) \end{cases}$$

$$(8.5)\quad \begin{cases} |(1 * v)^{(h)}(t)| & \leqslant & B\|v\|_{\rho,N} N^{-1} |t|^{\frac{N}{pk}}, & (h=0) \\ |(t^{k_h/k-2} * tv)^{(h)}(t)| & \leqslant & B\|v\|_{\rho,N} N^{-\frac{k_h-k}{k}} |t|^{\frac{N}{pk}+\frac{k_h}{k}-1}, & (1 \leqslant h \leqslant r) \end{cases}$$

for all $t \in \overline{\Delta(0,\rho)}$, and some $B > 0$. Hence, $t^{-1}(\beta_1 * v)^{(0)}$, $t^{-1}(\beta_1 * v)^{(h)}$ $(1 \leqslant h \leqslant r)$, etc. are elements of $W_{\rho,N}$.

Finally, we need to take care of the expressions in $\mathcal{T}_1 v$ containing $\mathcal{G}_1 v$. To do this we use Lemma 7.2. From this lemma we know that $\mathcal{G}_1 v(t^{pk})$ is continuous on $\overline{\Delta(0,\rho_1)}$ and holomorphic in its interior. Moreover, that lemma implies the following estimate for $\mathcal{G}_1 v$: If $g(\tau,\eta) := h_{k_1}(\tau,\eta)$ then

$$
\begin{aligned}
|\mathcal{G}_1 v(t)| &= |\int_0^t \eta v(\eta) g(t-\eta,\eta)\, d\eta| \\
&\leqslant \|v\|_{\rho,N} \int_0^t |\eta|^{N/(pk)} |g(t-\eta,\eta)\, d\eta| \\
&\leqslant \|v\|_{\rho,N} \int_0^{|t|/2} \eta^{N/(pk)} \left(1+\frac{1}{|t|-\eta}\right)(1+\eta) e^{K_2(|t|-\eta)}\, d\eta \\
&\quad + \|v\|_{\rho,N} \int_{|t|/2}^{|t|} \eta^{N/(pk)-1} (|t|-\eta) \left(\left(\frac{\eta}{|t|-\eta}\right)^k + \frac{1}{|t|-\eta} \right) \\
&\qquad \times (1+\eta^{1+k}(|t|-\eta)^{-k}) e^{K_2 \eta^k (|t|-\eta)^{1-k}}\, d\eta \\
&\leqslant L_1 \|v\|_{\rho,N} |t|^{-1} \int_0^{|t|/2} \eta^{N/(pk)}\, d\eta \\
&\quad + L_2 \|v\|_{\rho,N} \int_{|t|/2}^{|t|} \eta^{N/(pk)-1} (1+\eta^{1+k}(|t|-\eta)^{-k})\, d\eta,
\end{aligned}
$$

for some constants L_1, L_2, independent of N and t. The last expression can be estimated by utilizing the estimates for a convolution product and the beta function as given above. Thus we obtain:

$$
|\mathcal{G}_1 v(t)| \leqslant L \|v\|_{\rho,N} N^{-(1-k)} |t|^{\frac{N}{pk}}, \forall t \in \overline{\Delta(0,\rho)},
$$

and it follows that $t^{-1}\mathcal{G}_1$ maps $W_{\rho,N}$ into itself if N is sufficiently large. From the estimates for a convolution product mentioned in the beginning of the proof it follows that also $t^{-2+k_h/k_1} * \mathcal{G}_1 v$ belongs to $W_{\rho,N}$ if v belongs to that space and $h > 1$.

If $q = 1$ we have $\mathcal{T}_1 = \mathcal{T}_q = T_q$. As $1 - e^{-t} = Ot$, $t \to 0$, all expressions occurring in T_q are essentially of the same order as the corresponding expressions in $\mathcal{T}_1$ in the case $q > 1$.

So by choosing N large enough we can make $\mathcal{T}_1$ into a contraction on $W_{\rho,N}$. Hence equation $\mathcal{T}_1 v = v$ has a unique solution v_1 in $W_{\rho,N}$.

Its Taylor expansion in powers of $t^{1/(pk)}$ is its asymptotic expansion and is therefore a formal solution of this equation. As the equation has only one formal solution in $\mathbf{C}^n[[t^{1/(pk)}]]$, we must have that v_1 is the sum of $\hat{v}_1$.

Moreover, we have the following relationships (cf. section 5): $\mathcal{T}_1 v_1 = v_1 \Leftrightarrow T_1 u_1 = u_1 \Leftrightarrow Q_1 u_1 = \tilde{\gamma}_1 \Leftrightarrow \Delta z_0 = c$ in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_1})^n_\infty(\mathbf{R})$.

This proves Lemma 6.1.

9. Proof of Lemma 6.2.

We will prove this lemma in terms of the 'ramified' operators $\mathcal{T}_j$, $\mathcal{G}_j$, etc.. Let J_j (J_j^*, $\tilde{J}_j$) be the interval defined by the relation '$\theta \in J_j$ (J_j^*, $\tilde{J}_j$) if and only if $\theta/k_j \in I_j$ (I_j^*, $\tilde{I}_j$)'. As I_j does not contain any pair of Stokes directions of level k_j, and $I_q \cap \{(2j+1)\pi; j \in \mathbf{Z}\} = \varnothing$, $\mathcal{T}_j$ is regular in $S(J_j^*)$, and $J_q^* = I_q^* \subset (-\pi/2, \pi/2) \bmod 2\pi$. Define (cf. (6.2))

$$\mu_j := \kappa_j/k_j = \frac{k_{j+1}}{k_{j+1} - k_j}, j = 1, \ldots, q-1, \; \mu_q := 1. \tag{9.1}$$

We have a solution $v_j = \rho_{k_j} u_j$ of the singular Volterra integral equation $\mathcal{T}_j v = v$ on a neighborhood of 0 in $S(\tilde{J}_{j-1})$ (say U_{j-1}). Fix a $\tilde{t} \in U_{j-1} \cap S(J_j^*)$. Then

$$\mathcal{T}_j v(t) - \mathcal{T}_j v(\tilde{t}) = \int_{\tilde{t}}^{t} K(t,\tau) v(\tau) d\tau$$

for a certain holomorphic kernel $K(t,\tau)$. Hence $\mathcal{T}_j v = v$ is equivalent to the regular Volterra integral equation

$$v(t) = \mathcal{T}_j v_j(\tilde{t}) + \int_{\tilde{t}}^{t} K(t,\tau) v(\tau) d\tau,$$

which has a unique holomorphic solution $\tilde{v}_j$ on $S(J_j^*)$. However, for $t \in U_j$ this solution corresponds with v_j, i.e. $\tilde{v}_j$ is an analytic continuation of v_j. We will write v_j in stead of $\tilde{v}_j$.

Remains to prove that v_j has the right growth rate. Let's first consider the case $j < q$. We will write $S_j = S(J_j^*)$ and $k = k_j$. Let $\widetilde{S}$ denote a closed subsector of S_j. With the pair $(v_j, \widetilde{S})$ we associate the continuous function

$$\vartheta_j(s) = \sup\{|v_j(t)|; t \in \widetilde{S}, |t| = s\}, \; s \in (0, \infty); \tag{9.2}$$

ϑ_j becomes continuous on $[0, \infty)$ by putting $\vartheta_j(0) = 0$, since $v_j(t) \sim \rho_{k_j} \hat{\mathcal{B}}_{k_j} \hat{z}(t)$.

Next with $\mathcal{T}_j$ and $\mathcal{G}_j$, and $\widetilde{S}$ we will associate dominating operators $\overline{\mathcal{T}}_j$ and $\overline{\mathcal{G}}_j$ as follows. Let v be holomorphic on S_j, bounded on closed, bounded subsectors of S_j. Define ψ by

$$\psi(s) := \sup\{|v(t)|; t \in \widetilde{S}, |t| = s\}, \; s \in (0, \infty). \tag{9.3}$$

We now consider the expressions occurring in the definition of $\mathcal{T}_j$ (cf. section 8). From (8.2) it follows that there exist positive constants M_1 and B_1 such that

$$|\beta_j(t)| + |\gamma_j(t)| \leqslant M_1 \exp(B_1|t|), \text{ if } t \in \widetilde{S}. \tag{9.4}$$

Expressions in $\mathcal{T}_j v$ of the form $(f * v)(t)$, with f holomorphic on S_j and $|f(t)| \leqslant Me^{B|t|}, \forall t \in \widetilde{S}$, are estimated by

$$|(f * v)(t)| \leqslant (Me^{Bs} * \psi)(s), \forall t \in \widetilde{S}, \text{ with } s = |t|. \tag{9.5}$$

From Lemmas 7.2 and 7.1 and the definition of $\mathcal{G}_j$ in (8.1) we easily derive the following dominating operator for $\mathcal{G}_j$ on $\widetilde{S}$:

$$|\mathcal{G}_j v(t)| \leqslant \overline{\mathcal{G}_j}\psi(s) := K_1 \int_0^s \psi(\xi)\xi R^{-1/k}\left(R + \frac{1}{s-\xi}\right)(1+R\xi) \exp(K_2 R(s-\xi))d\xi, \tag{9.6}$$

where $R = \max\left\{\left(\frac{\xi}{s-\xi}\right)^k, 1\right\}$.

Let $M > 0$ and $B > 0$ which will be chosen later on. Then we define

(9.7)

$$\overline{\mathcal{T}_j}\,\psi(s) := M\left[e^{Bs} + (e^{Bs} * \psi)(s) + \sum_{h=0}^{j-1}(s^{-\frac{k_h}{k}} * \psi)(s) + \sum_{h=j+1}^{r}(s^{\frac{k_h}{k}-2} * s\psi)(s) + s^{-1}\overline{\mathcal{G}_j}\psi(s) + \sum_{h=j+1}^{r}(s^{\frac{k_h}{k}-2} * \overline{\mathcal{G}_j}\psi)(s)\right].$$

From (8.3), (9.2) and the estimates above it follows, that $\vartheta_j(s) < \overline{\mathcal{T}_j}\vartheta_j(s)$ if $s > 1$, if we choose M and B large enough. Since $\vartheta_j(s)$ is bounded on $[0,1]$ we may choose M and B such that this inequality holds for all $s \geqslant 0$.

From here on we will omit the indices j if no confusion is possible, i.e. we will write $\mu = \mu_j$ (cf. (9.1)), $\overline{\mathcal{T}} = \overline{\mathcal{T}_j}$, etc.. Note that $\overline{\mathcal{G}}$ and $\overline{\mathcal{T}}$ are monotone operators.

Let

$$\psi_0(s) = M_0 e^{cs^\mu} \tag{9.8}$$

for some positive constants c and M_0. We will first show that for sufficiently large $c > 0$ and all $M_0 > 0$

$$\overline{\mathcal{T}}\psi_0(s) \leqslant \psi_0(s), \forall s \in [1, \infty). \tag{9.9}$$

Finally we will prove that we may choose M_0 such that

$$\vartheta(s) \leqslant \psi_0(s), \forall s \in (0,\infty) \tag{9.10}$$

where ϑ is defined in (9.2). This implies that v_j has the right growth rate.

For the proof of (9.9) we use two inequalities from [Bra91]:

If $a, b, c, \mu > 0$, and $0 \leqslant a+b-1 \leqslant a\mu$, then

$$\text{(I)} \qquad \int_0^s (s-\sigma)^{a-1}\sigma^{b-1}e^{c\sigma^\mu}d\sigma \leqslant Kc^{-(a+b-1)/\mu}e^{cs^\mu} \text{ for } s>0,$$

and, if $c \geqslant c_0+1, c_0>0, \mu \geqslant 1$, then

$$\text{(II)} \qquad \int_0^s e^{c_0(s-\sigma)^\mu + c\sigma^\mu}d\sigma \leqslant Kc^{-1/\mu}e^{cs^\mu}, \text{ for } s>0,$$

where K is a constant independent of s and c.

From these inequalities it follows that all terms in the expression for $\overline{\mathcal{T}}\psi_0(s)$ involving ψ_0, but not $\overline{\mathcal{G}}\psi_0$, can be estimated by $Kc^{-\alpha}\psi_0(s)$, $\forall s>0$, for some constants $K,\alpha>0$ independent of c and s, provided $c \geqslant B+1$.

Next we consider the contribution from $\overline{\mathcal{G}}$ (as defined in (9.6)) to $\overline{\mathcal{T}}$. By splitting the path of integration in the definition of $\overline{\mathcal{G}}$ in a part from 0 to $s/2$ and a part from $s/2$ to s, we may write $\overline{\mathcal{G}}$ as a sum of two operators : $\overline{\mathcal{G}} = \overline{\mathcal{G}}_1 + \overline{\mathcal{G}}_2$, with

$$\overline{\mathcal{G}}_1\psi(s) := K_1\int_0^{s/2}\psi(\xi)\xi(1+\frac{1}{s-\xi})(1+\xi)\exp(K_2(s-\xi))d\xi,$$

and

$$\overline{\mathcal{G}}_2\psi(s) := K_1\int_{s/2}^s \psi(\xi)\{\xi^{1+2k}(s-\xi)^{1-2k} + s\xi^k(s-\xi)^{-k}+1\}$$
$$\exp\{K_2\xi^k(s-\xi)^{1-k}\}d\xi.$$

As $\xi\left(1+\dfrac{1}{s-\xi}\right)(1+\xi)\exp(K_2(s-\xi)) \leqslant B_1\exp(B_2(s-\xi)^\mu)$, $\forall s>0$, $\forall\xi \in (0,s/2)$, for some positive constants B_1 and B_2, we obtain the following estimate by applying inequality (II):

$$\overline{\mathcal{G}}_1\psi_0(s) \leqslant Lc^{-1/\mu}\psi_0(s), \forall s>0, \text{ provided } c \geqslant B_2+1, \tag{9.11}$$

for some constant $L>0$ independent of c and s.

Next consider $\overline{\mathcal{G}}_2$. Since $\xi^{1+k}(s-\xi)^{1-k}$ is uniformly bounded for $\xi \in (s/2, s)$, $s \in (0,1]$, we may deduce from inequality (I) with $a = 1-k$ and $b = 1+k$ the following bound:

$$\overline{\mathcal{G}}_2\psi_0(s) \leqslant M_1 c^{-1/\mu}\psi_0(s), \forall s \in (0,1], \tag{9.12}$$

for some $M_1 > 0$ independent of c and s. If $s \geqslant 1$ we have

$$\overline{\mathcal{G}}_2\psi(s) \leqslant K_1 \int_{s/2}^{s} \psi(\xi) \exp\{K_2 s(1-\xi/s)^{1-k}\} \\ \{s^2(1-\xi/s)^{1-2k} + s(1-\xi/s)^{-k} + 1\}\, d\xi.$$

Next, we substitute $\psi = \psi_0$ and $\xi = s(1-\sigma)^{1/\mu}$ in this integral. Then $\sigma \in (0, \delta)$ with $\delta := 1 - 2^{-\mu} < 1$, hence $(1-\sigma)^{\alpha}$ is bounded for any α, and $m_1\sigma \leqslant 1 - (1-\sigma)^{1/\mu} \leqslant m_2\sigma$, $\forall \sigma \in (0,\delta)$ for certain $m_1, m_2 > 0$ (dependent on μ). Furthermore, $\psi_0(\xi) = \psi_0(s)\exp(-cs^{\mu}\sigma)$. All this yields us the estimate

$$\overline{\mathcal{G}}_2\psi_0(s) \leqslant M' s\psi_0(s) \int_0^{\delta} \exp(-cs^{\mu}\sigma + K_1' s\sigma^{1-k})(s^2\sigma^{1-2k} + s\sigma^{-k} + 1)\, d\sigma.$$

Observing that $\mu \geqslant 1/(1-k)$ (cf.(9.1)), hence $s\sigma^{1-k} \leqslant (s^{\mu}\sigma)^{1-k}$ for $s \geqslant 1$, we see that $\exp(-cs^{\mu}\sigma + K_1' s\sigma^{1-k}) \leqslant \exp\left(-\dfrac{1}{2}cs^{\mu}\sigma\right)$ if c is sufficiently large. Utilizing this in the last integral and substituting $w = cs^{\mu}\sigma$ we obtain a final estimate for $\overline{\mathcal{G}}_2\psi_0(s)$:

$$\overline{\mathcal{G}}_2\psi_0(s) \leqslant M_2 s c^{-(1-k)}\psi_0(s),\ \forall s \in [1, \infty), \tag{9.13}$$

if c is sufficiently large.

From (9.11), (9.12), (9.13), and $1/\mu \leqslant 1-k$ we may conclude that

$$\overline{\mathcal{G}}\psi_0(s) \leqslant C_1 c^{-1/\mu}(s+1)\psi_0(s),\ \forall s \in (0,\infty),$$

and, utilizing inequality (I) twice, with $a = -1 + k_h/k, b = 1$ and $b = 2$, that

$$(s^{k_h/k-2} * \overline{\mathcal{G}}\psi_0)(s) \leqslant C_2 c^{-1/\mu}\psi_0(s),\ \forall s \in (0,\infty),\ \text{if } h \geqslant j+1,$$

for some positive constants C_1, C_2 independent of c and s.

The discussion above implies that there exist positive constants K, α such that $\overline{\mathcal{T}}\psi_0(s) \leqslant M(e^{Bs^{\mu}} + Kc^{-\alpha}\psi_0(s))$, for all $s > 1$ if c is large enough. So (9.9) holds if c is sufficiently large positive.

To obtain (9.10) we choose $M_0 > \max_{0\leqslant s\leqslant 1}\vartheta(s)$ (cf. (9.2) and (9.8)). Let $s_1 := \sup\{s \in (1,\infty]; \vartheta(t) < \psi_0(t), \text{ if } 0 \leqslant t \leqslant s\}$. Thus $1 < s_1 \leqslant \infty$. If $s_1 < \infty$ we deduce from the monotonicity of $\overline{T}$ that

$$\vartheta(s_1) < \overline{T}\vartheta(s_1) < \overline{T}\psi_0(s_1) \leqslant \psi_0(s_1)$$

which is impossible by the definition of s_1. Hence $s_1 = \infty$ and (9.10) follows which completes the proof of the lemma in case $j < q$.

This proof also applies to the case $j = q$, with

$$\begin{aligned}\overline{T}_q\psi(s) := M\Big[e^{Bs} + (e^{Bs} * \psi)(s) &+ \sum_{h=0}^{q-1}(s^{-k_h} * \psi)(s)\\ &+ \sum_{h=q+1}^{r}(s^{k_h-2} * \psi)(s)\Big].\end{aligned} \tag{9.14}$$

Here we have used the fact, that in the right hand side of $\mathcal{T}_q = T_q$ (cf. (5.10)) $(1-e^{-t})^{-1}$ and if $n_q > 1$ also $(\mathbf{A}_q + (1-e^{-t})\mathbf{I}_q)^{-1}$ are bounded on $S(I_q^*)$ for $|t| \geqslant 1$. Moreover, we have used that the factor $1 - e^{-t}$ occurring in (5.10) for $q < h \leqslant r$ is bounded since $I_q^* \subset (-\pi/2, \pi/2) \bmod 2\pi$. In fact, operator $\mathcal{T}_q$ does not contain $\mathcal{G}_q$ and we could have estimated the growth rate of the terms in $\mathcal{T}_q v$ immediately, utilizing inequalities (I) and (II).

Now assume $1 \leqslant j \leqslant q-1$, and the existence of a solution u_j of $T_j u = u$, i.e. $Q_j u_j = \tilde{\gamma}_j$ on S_j. We have proved the statement in the lemma about the growth rate of u_j at ∞ on this sector. Thus we may restrict $z_j = \mathcal{L}_{k_j} u_j$ to $z_j = \mathcal{L}_{k_j}^{k_{j+1}} u_j$. Since $\Delta z_j - c = 0$ in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_j})^n_\infty(I)$, and $\Delta z_j - c \in (\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I)$ we deduce from the relative Watson Lemma 2.1 that $\Delta z_j - c = 0$ in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_{j+1}})^n_\infty(I)$. In particular, z_q is a holomorphic solution of $\Delta z = c$ on a neighborhood of ∞ in $S(I_q)$.

The last statement of Lemma 6.2 now easily follows and the proof is completed. The proof of Remark 6.1 is an obvious modification of the previous proof, since (9.14) also holds on vertical strips away from the singularities.

10. Nonlinear difference equations.

Let $p, \nu \in \mathbf{N}$. Consider the following nonlinear difference equation:

$$x^{-\nu/p} y(x+1) = F(x^{1/p}, y(x)),\ x \in \mathbf{C},\ y(x) \in \mathbf{C}^n, \tag{10.1}$$

with $F(x,y)$ holomorphic in a neighborhood of (∞, a).

Suppose that (10.1) has a formal solution

$$\hat{y}(x) = \sum_{m=0}^{\infty} a_m x^{-m/p},\ a_m \in \mathbf{C}^n,\ a_0 = a.$$

We will rewrite (10.1) to a linear, inhomogeneous difference equation of the form (4.5) plus a perturbation which is nonlinear in y. Let $y(x) = P(x) + x^{-\mu/p}\tilde{y}(x)$, with $P(x) = \sum_{m=0}^{M+\mu-1} a_m x^{-m/p}$, where $M, \mu \in \mathbf{N}$ are to be chosen later in order to control the orders of the nonlinear part, while bringing the equation to a suitable normalized form (cf. transformation $\tilde{y}(x) = \mathbf{S}(x)y(x)$ below, and equation (10.3)). This yields the following difference equation for $\tilde{y}$:

(10.2)

$$x^{-\nu/p}\tilde{y}(x+1) = x^{-M/p}\widetilde{F}_0(x^{1/p}) + \widetilde{\mathbf{A}}(x^{1/p})\tilde{y}(x) + x^{-\mu/p}\widetilde{F}_2(x^{1/p}, \tilde{y}(x)),$$

where

$$\widetilde{F}_0(x^{1/p}) = x^{(M+\mu)/p}(1+1/x)^{\mu/p}\left\{F(x^{1/p}, P(x)) - x^{-\nu/p}P(x+1)\right\};$$

$\widetilde{\mathbf{A}}(x^{1/p}) = (1+1/x)^{\mu/p} D_y F(x^{1/p}, P(x))$, i.e. $\widetilde{\mathbf{A}}(x)$ is holomorphic in $x = \infty$;

$$\widetilde{F}_2(x^{1/p}, \tilde{y}) = x^{2\mu/p}(1+1/x)^{\mu/p}\left\{F(x^{1/p}, P(x) + x^{-\mu/p}\tilde{y}) - F(x^{1/p}, P(x)) - D_y F(x^{1/p}, P(x))x^{-\mu/p}\tilde{y}\right\}.$$

Hence $\widetilde{F}_2(x, \tilde{y})$ is holomorphic at $(\infty, 0)$, and

$$\widetilde{F}_2(x, \tilde{y}) = O(|\tilde{y}|^2), \quad \tilde{y} \to 0, \text{ uniformly in } x.$$

Equation (10.2) has of course $\hat{\tilde{y}}(x) = \sum_{m=M}^{\infty} a_{m+\mu} x^{-m/p}$ as formal power series solution. By substituting this series in (10.2) we see, that $\widetilde{F}_0(x)$, too, is holomorphic at $x = \infty$.

The homogeneous linear part $x^{-\nu/p}\tilde{y}(x+1) - \widetilde{\mathbf{A}}(x^{1/p})\tilde{y}(x)$ is of the form $x^{-\nu/p}Ty(x)$, T defined by (1.1), and can be transformed to a normalized form $x^{-\nu/p}\tilde{T}y$ (cf. (4.3)) by a transformation $\tilde{y}(x) = \mathbf{S}(x)z(x)$ with $\mathbf{S}(x) \in \mathrm{Gl}(n, \mathbf{C}\{x^{-1/pq}\}[x^{1/pq}])$ for some positive integer q. This transformation can be obtained by applying a method of Turrittin (cf.[Tur60]). It consists of block-diagonalizations up to some order in the series expansion and shearing transformations. There exists a bound for the number of these transformations which only depends on n and ν/p. The block-diagonalizations only involve transformation matrices $\mathbf{S}(x) \sim I$, but the shearing transformations involve matrices $\mathbf{S}(x)$ which are regular at ∞ but with $\mathbf{S}^{-1}(x) = O(x^{\lambda})$ where λ has an upper bound which only depends on $(n-1)/p$ (cf. [Tur60], sect. 7). Hence $\mathbf{S}^{-1}(x) = O(x^{\mu_0/pq})$, $x \to \infty$ for some $\mu_0 \in \mathbf{Z}$ which is only depending on n, p and ν. Then also $\mathbf{S}^{-1}(x+1) = O(x^{\mu_0/pq})$, $x \to \infty$.

So, writing p in stead of pq, we have transformed (10.2) into a difference equation for $z(x)$ of the following form:

$$\Delta z(x) = c(x) + E(x, z(x)), \tag{10.3}$$

with Δ the difference operator as defined by (4.5), and

$$c(x) = \left(\bigoplus_{k=0}^{r} x^{1-k_h} I_h\right) x^{(\nu-M)/p} \mathbf{S}(x+1)^{-1} \widetilde{F}_0(x^{1/p}),$$

$$E(x,z) = \left(\bigoplus_{k=0}^{r} x^{1-k_h} I_h\right) x^{(\nu-\mu)/p} \mathbf{S}(x+1)^{-1} \widetilde{F}_2(x, \mathbf{S}(x)z),$$

and we choose $\mu, M > \nu + \mu_0 + p$. Then we know the following of c and E:

$c(x)$ is holomorphic in $x^{-1/p}$ at ∞, and $c(x) = O(x^{-N/p})$, $x \to \infty$, where $N = M - (\nu + \mu_0 + p) > 0$ (cf. remark at the beginning of section 5); $E(x,z)$ is holomorphic in $x^{-1/p}$ and z in a neighborhood U of $(\infty, 0)$ and $E(\infty, z) = 0$ if $z \in U$. We choose $U = \Delta_1(\infty, \rho_0') \times \Delta_n(0, \rho_0)$, $\Delta_1(\infty, \rho_0') = \{x \in \mathbf{C}_\infty;\, |x| > \rho_0'\}$, $\Delta_n(0, \rho_0) = \{(y_1, \dots, y_n) \in \mathbf{C}^n;\, |y| < \rho_0\}$. In particular we may expand $E(x,z)$ in powers of z:

$$E(x,z) = \sum_{m \in \mathbf{N}^n, |m| \geqslant 2} E_m(x) z^m,\ (x,z) \in U.$$

The levels of equation (10.3) are defined to be the levels of the linear part $\Delta z(x)$. As before we will associate a system of convolution equations with each level of (10.3). Let $z(x) \in x^{-1/p}(\mathcal{A}/\mathcal{A}^{\leqslant -k})^n_\infty(I)$, and $u(t) := \mathcal{B}_k z(t) \in t^{-k+1/p}\mathcal{A}^n(I^*)$, I and I^* related as in Definition 3.2. We define

$$\mathcal{E}_{j*}(t, u(t)) := \mathcal{B}_{k_j} E(\cdot, \mathcal{L}_{k_j} u(\cdot))(t) = \sum_{m \in \mathbf{N}^n, |m| \geqslant 2} (\mathcal{E}_{m,j} *_{k_j} u_{j*m})(t), \tag{10.4}$$

where $\mathcal{E}_{m,j} := \mathcal{B}_{k_j} E_m$, and $u_{j*m}(t) := \mathcal{B}_{k_j}(\mathcal{L}_{k_j} u)^m$, the '$m$-fold' k_j-convolution of u. Let matrices $\mathbf{M}_j$ and convolution operators Q_j be defined as in section 5. Then

$$\Delta z(x) = c(x) + E(x, z(x))$$

in $(\mathcal{A}/\mathcal{A}^{\leqslant -k_j})^n_\infty(I_j)$ is equivalent to

$$Q_j u(t) = \mathcal{B}_{k_j}\{\mathbf{M}_j c\}(t) + (\mathcal{B}_{k_j}\mathbf{M}_j *_{k_j} \mathcal{E}_{j*}(t, u(t))(t)$$

in $\mathcal{A}^n(I_j^*)$. Rewriting these convolution equations in the form $X_j u = u$ thus yields the following convolution equations where $k = k_j$:

- **if** $1 \leqslant j \leqslant q-1$**:**

$$(10.5)\quad \begin{cases} X_j u^{(h)} &= T_j u^{(h)} - (kt^k)^{-1}\left(\frac{t^{-k_h}}{\Gamma(1-k_h/k)} *_k \mathcal{E}_{j*}(t,u)\right)^{(h)}, \\ & \qquad 0 \leqslant h < j; \\ X_j u^{(j)} &= T_j u^{(j)} - (\mathbf{A}_j + kt^k\mathbf{I}_j)^{-1}\mathcal{E}_{j*}(t,u)^{(j)}; \\ X_j u^{(h)} &= T_j u^{(h)} - \mathbf{A}_h^{-1}\mathcal{E}_{j*}(t,u)^{(h)}, \\ & \qquad j < h \leqslant r, \end{cases}$$

- **if** $j = q$**:**

$$(10.6)\quad \begin{cases} X_q u^{(h)} &= T_q u^{(h)} - (1-e^{-t})^{-1}\left(\frac{t^{-k_h}}{\Gamma(1-k_h)} * \mathcal{E}_{q*}(t,u)\right)^{(h)}, \\ & \qquad 0 \leqslant h < q; \\ X_q u^{(j)} &= T_q u^{(q)} - (\mathbf{A}_q + (1-e^{-t})\mathbf{I}_q)^{-1}\mathcal{E}_{q*}(t,u)^{(q)}; \\ X_q u^{(h)} &= T_q u^{(h)} - \mathbf{A}_h^{-1}\mathcal{E}_{q*}(t,u)^{(h)}, \\ & \qquad q < h \leqslant r, \end{cases}$$

and we define $\mathcal{X}_j := \rho_{k_j} X_j \rho_{k_j}^{-1}$. Analogous to Lemma 1 in [Bra92] we now have the following lemma which can be proved in a similar manner.

LEMMA 10.1. — *If $f \in W_{\rho,N}$ (cf. sect. 8), and $m \in \mathbf{N}^n$, $|m| \geqslant 2$, then*

$$|f_{*m}(t)| \leqslant \frac{\{\Gamma(N/pk_1)\|f\|_{\rho,N}|t|^{N/pk_1}\}^{|m|}}{\Gamma(|m|N/pk_1)}|t|^{-1},$$

where f_{*m} is the m-fold convolution of f.

Moreover,

$$|\mathcal{E}_{m,j}(t)| \leqslant Kb^{|m|}|t|^{-k_j+1/p}\exp|c_0 t^{k_j}|,\ j = 1, \ldots, q,$$

where $K, b,$and c_0 are certain positive constants.

Utilizing this lemma we may show as in [Bra92] that $\mathcal{X}_1 - \mathcal{T}_1$ is a contraction in $W_{\rho,N}$ with norm tending to 0 if $N \to \infty$. In sect. 8 we saw that $\mathcal{T}_1$ is a contraction with norm remaining away from 1 as $N \to \infty$

and therefore the same holds for $\mathcal{X}_1$. Hence Lemma 6.1 remains valid for a formal solution $\hat{z}(x)$ of (10.3).

Next we may prove an analogue of Lemma 6.2. First we may show that a solution $u_j \in t^{N/p-k_j}\mathcal{A}^n(\tilde{I}_{j-1})$ of $X_j u = u$ can be analytically continued in $S(I_j^*)$ (cf. Lemma 3 in [Bra92] and its proof in sect.5 in that paper). That this solution u_j has exponential growth at ∞ of order at most κ_j can be shown using the majorant method as in sect. 9 and [Bra92], sect 6. We now use a dominating operator $\overline{\mathcal{X}}_j$ of $\mathcal{X}_j$ which is an extension of the dominating operator $\overline{\mathcal{T}}_j$ of $\mathcal{T}_j$ as defined in (9.7) and we may deduce from Lemma 10.1

$$(10.7) \quad \overline{\mathcal{X}}_j\psi(s) = \overline{\mathcal{T}}_j\psi(s) + L(\sum_{h=0}^{j} s^{(-k_h+1/p)/k_j} e^{cos} * \sum_{m=2}^{\infty} c^m \psi_{1*m})(s),$$

where L and c are positive constants. From this the order of exponential growth of u_j can be deduced as in sect. 9.

Thus we may show

THEOREM 10.1 — *Theorem 4.1 also holds if the linear difference equation (4.5) is replaced by the nonlinear difference equation (10.3) under the assumptions that $E(x,z)$ is holomorphic in $x^{-1/p}$ and z in a neighborhood of $(\infty,0), E(\infty,z) = 0$ if $|z|$ is sufficiently small and $E(x,z) = O(|z|^2)$ as $z \to 0$ for x in a fixed neighborhood of ∞.*

It is clear that this result implies Theorem 1.1. Moreover we see

Remark 10.1 — Remark 4.1 is also valid for the nonlinear equation (10.3).

11. Multisummability of a 'normalizing' transformation.

Consider the difference operator

$$(11.1) \qquad y(x+1) - \mathbf{A}(x)y(x),$$

under the assumptions, that $\mathbf{A}(x) = \mathbf{F}(x) + x^\mu\tilde{\mathbf{F}}(x)$ where $\mathbf{F}(x)$ is of the form as in (4.2) with all $\lambda_j = \lambda$, $\mu \leqslant \lambda - 1 - 1/p$ and $\tilde{\mathbf{F}}(x)$ is holomorphic at infinity in $x^{-1/p}$.

Let $\mathbf{T}(x) = \mathbf{I} + x^{-1/p}\widetilde{\mathbf{T}}(x) \in \mathrm{Gl}(n, \mathbf{C}[[x^{-1/p}]])$ be a 'normalizing' transformation matrix for (11.1), i.e. the transformation $y = \mathbf{T}(x)w$ conjugates (11.1) with the operator $w(x+1) - \mathbf{F}(x)w(x)$. Hence

$$\mathbf{T}(x+1)^{-1}\mathbf{A}(x)\mathbf{T}(x) = \mathbf{F}(x). \tag{11.2}$$

Then

THEOREM 11.1. — *The normalizing transformation matrix* $\mathbf{T}(x)$ *is multisummable in all directions, except at most countably many. In any case* $\pi/2 \bmod \pi$ *will be singular directions of level 1.*

For general results concerning normalizing transformations see [Imm84, §18], [Imm91], and [GLS].

Proof. — From (11.2) we obtain a matrix difference equation for $\mathbf{T}(x)$:

$$\mathbf{T}(x+1) = \mathbf{A}(x)\mathbf{T}(x)\mathbf{F}(x)^{-1},$$

where

$$\mathbf{F}(x)^{-1} = x^{-\lambda}\bigoplus_{j=1}^{m}(g_j(x)\mathbf{I}_j + x^{-1}\mathbf{H}_j(x)), \tag{11.3}$$

$$g_j(x) = \sum_{l=0}^{p-1} g_{jl}x^{-l/p},\ g_{j0} \neq 0,\ \mathbf{H}_j(x) \in \mathrm{End}(n_j, \mathbf{C}\{x^{-1/p}\}).$$

If $\mathbf{X}$ denotes a $(n \times n)$-matrix, we will write $\mathbf{X}_{\bullet v}$ for the $(n \times n_v)$-matrix consisting of the n_v columns corresponding with the v^{th} block in the partition of $\mathbf{F}(x)$ above, (so $1 \leqslant v \leqslant m$), and $\mathbf{X}_v$ will denote the v^{th} diagonal block. Using (11.3) we obtain a difference equation for $\mathbf{T}_{\bullet v}(x)$ that we can write as follows:

$$\mathbf{T}_{\bullet v}(x+1) = \left((\bigoplus_{j=1}^{m} h_{jv}(x)\mathbf{I}_j) + x^{-1}\mathbf{R}(x) \right) \mathbf{T}_{\bullet v}(x) + x^{-1}\mathbf{A}(x)\mathbf{T}_{\bullet v}(x)\mathbf{H}_v(x),$$

where $h_{jv}(x)$ is the polynomial in $x^{-1/p}$ of degree $\leqslant p-1$ given by

$$f_j(x)g_v(x) = h_{jv}(x) + O(x^{-1}),\ x \to \infty;$$

in particular $h_{jv}(\infty) \neq 0$ and $h_{vv}(x) = 1$.

So $\mathbf{T}_{\bullet v}(x)$, considered as a $(n \times n_v)$-vector, formally satisfies a difference equation of the form as in Theorem 4.1, case (ii), with a non-degenerate block of level 0, which causes $\pi/2 \bmod \pi$ to be singular directions of level 1 of this difference equation. The result now follows from that theorem.

THEOREM 11.2. — *Under the assumptions made above the difference equation* $y(x+1) = \mathbf{A}(x)y(x)$ *possesses a formal fundamental matrix*

$$\Gamma(x)^{\lambda}\hat{G}(x)e^{Q(x)}x^{\Lambda},$$

where $Q(x) = \bigoplus_{j=1}^{m} q_j(x)\mathbf{I}_j$, $q_j(x)$ *a polynomial in* $x^{-1/p}$ *of degree at most* p,

$\Lambda = \bigoplus_{j=1}^{m} \Lambda_j$, Λ_j *an* $(n_j \times n_j)$*-matrix, and*

$\hat{G}(x) \in \mathbf{C}[[x^{-1/p}]]$ *is multisummable in all directions except at most countably many.*

To show that $\hat{G}(x)$ is multisummable we first derive a difference equation of which $\hat{G}(x)$ is a formal solution and then show that Theorem 4.1, case (ii) is applicable, similarly as in the proof of Theorem 11.1.

Remark 11.1. — This theorem implies that $y(x) = G(x)w(x)$ where $G(x)$ is the multisum of $\hat{G}(x)$, conjugates the difference operator (11.1) with the normal form $w(x+1) - C(x)w(x)$ where $C(x) := x^{\lambda}\left(1+\dfrac{1}{x}\right)^{\Lambda} \exp\{Q(x+1) - Q(x)\}$.

Remark 11.2. — In case $\mathbf{A}(x)$ is as in (11.1), but with not all λ_j equal (and $\mu \leqslant \min_j\{\lambda_j\} - 1 - 1/p$), the normalizing transformation matrix $\mathbf{T}(x)$ of Theorem 11.1 (considered as a n^2-vector) satisfies a difference equation

$$\mathbf{T}(x+1) = (\bigoplus x^{d_i}\mathbf{F}_i(x) + x^{d}\tilde{\mathbf{F}}(x))\mathbf{T}(x),$$

where $\mathbf{F}_i(x)$ is as in (4.2), $\tilde{\mathbf{F}}(x)$ is holomorphic in $x^{-1/p}$, d_i runs through the set of differences of λ_j's, and $d < \min_i\{d_i\} - 1$. Hence positive and negative numbers d_j occur and therefore Theorem 4.1 is not applicable (cf. section 4). However, Remark 4.1 is applicable and thus formal normalizing transformations $\hat{\mathbf{T}}$ can be lifted to analytic transformations in lower and upper half planes.

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