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ON THE AXIOMATIC OF HARMONIC FUNCTIONS II

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In this paper we shall use constantly the notations and definitions from [2].

1. The fine topology of X is the least fine topology on X , which is finer than the given topology and with respect to which the superharmonic functions (non necessarily defined on the whole space X) are continuous. A set $V \ni x$ is a fine neighbourhood of x if and only if either x is an interior point of V for the initial topology or there exists a superharmonic function s , defined on a neighbourhood of x , such that

$$s(x) < \liminf_{X - V \ni y \rightarrow x} s(y).$$

THEOREM 1. — *Any point of X possesses a fundamental system of fine neighbourhoods which are compact and connected in the initial topology of X .*

Since the theorem has a local character we may suppose that there exists a harmonic positive function and a positive potential on X . Dividing the sheaf of harmonic functions by a positive harmonic function the fine topology does not change; we may suppose therefore that the constants are harmonic functions.

Let $x \in X$ and V be a fine neighbourhood of x . It is sufficient to suppose that x is not an interior point of V . There exists then a superharmonic function s on X such that

$$s(x) < \liminf_{X - V \ni y \rightarrow x} s(y).$$

Let α be a real number

$$s(x) < \alpha < \liminf_{X - V \ni y \rightarrow x} s(y)$$

and U be a regular domain containing x such that s is greater than α on $\bar{U} - V$. We denote by F the set

$$F = \{y \in \bar{U} | s(y) \leq \alpha\}$$

and by K the component of F containing x . It is sufficient to prove that K is a fine neighbourhood of x .

Let C be a component of F contained in U . There exists then an open set G , $C \subset G \subset \bar{G} \subset U$, $F \cap \partial G = \emptyset$. Since $s > \alpha$ on ∂G it follows $s > \alpha$ on G which contradicts the inequality $s \leq \alpha$ on C . Consequently any component of F has a non-empty intersection with ∂U .

Let β and ε be positive numbers such that $s + \beta > \alpha$ on ∂U and $s(x) + \varepsilon\beta < \alpha$. Let K' be a compact set in $\partial U - K$ such that

$$\omega_x^U(\partial U - (K \cup K')) < \varepsilon.$$

We denote by u the function on \bar{U} equal to

$$y \rightarrow \omega_y^U(\partial U - (K \cup K'))$$

on U equal to 1 on $\partial U - (K \cup K')$ and equal to 0 on $K \cup K'$. u is lower semicontinuous. We denote by F_ε the set

$$F_\varepsilon = \{y \in \bar{U} | s(y) + \beta u(y) \leq \alpha\}.$$

Obviously $F_\varepsilon \subset F$ and $x \in F_\varepsilon$. There exists an open set G , $K \subset G$, $K' \cap F \cap G = \emptyset$, $F \cap \partial G = \emptyset$. Let $y \in F_\varepsilon \cap G$ and C_y be the component of F_ε containing y . C_y is contained in G since $F_\varepsilon \cap \partial G = \emptyset$. $C_y \cap \partial U$ is not empty, as it was shown above; let $z \in C_y \cap \partial U$. If $z \notin K$ then

$$s(y) + \beta u(y) = s(y) + \beta > \alpha$$

which contradicts the relation $z \in F_\varepsilon$. Hence $z \in K$ and $C_y \subset K$, $y \in K$, $F_\varepsilon \cap G \subset K$. Since

$$\begin{aligned} \liminf_{U-K \ni y \rightarrow x} (s(y) + \beta u(y)) &= \liminf_{G-K \ni y \rightarrow x} (s(y) + \beta u(y)) \\ &\geq \liminf_{G-F_\varepsilon \ni y \rightarrow x} (s(y) + \beta u(y)) \geq \alpha > s(x) + \beta u(x). \end{aligned}$$

K is a fine neighbourhood of x .

2. We shall suppose in this paragraph that there exists a positive potential on X .

THEOREM 2. — *For any non-negative superharmonic function s on X and any set $E \subset X$, \hat{R}_s^E is equal to s on the fine interior of E .*

Let x be a fine interior point of E . Then [3] (pag. 435)

$$\lim_{U, \mathfrak{U}} \int_{(X-E) \cap \partial U} d\omega_x^U = 0,$$

where \mathfrak{U} denotes the filter of sections of regular neighbourhoods of x . If s is bounded in a neighbourhood of x then

$$\begin{aligned} s(x) &\geq \hat{R}_s^E(x) = \lim_{U, \mathfrak{U}} \int R_s^E d\omega_x^U \\ &\geq \lim_{U, \mathfrak{U}} \sup \int_{E \cap \partial U} R_s^E d\omega_x^U = \lim_{U, \mathfrak{U}} \sup \int_{E \cap \partial U} s d\omega_x^U \\ &\geq \lim_{U, \mathfrak{U}} \int s d\omega_x^U - \lim_{U, \mathfrak{U}} \sup \int_{(X-E) \cap \partial U} s d\omega_x^U = s(x). \end{aligned}$$

In the general case let \mathcal{G} be the set of continuous finite positive superharmonic functions dominated by s . We have

$$s(x) \geq \hat{R}_s^E(x) \geq \sup_{E \cap \mathcal{G}} \hat{R}_{s'}^E(x) = \sup_{s' \in \mathcal{G}} s'(x) = s(x).$$

COROLLARY 1. — *A polar set has no fine interior points.*

THEOREM 3. — *Let G be a fine open set and s be a non-negative superharmonic function. Then*

- a) $\hat{R}_s^G = R_s^G$;
- b) $R_{R_s^G}^E = R_s^E$ for any $E \subset G$;
- c) $R_s^G = \sup_{s' \in \mathcal{G}} R_{s'}^G$, where \mathcal{G} is an increasingly directed set of superharmonic functions with $s = \sup_{s' \in \mathcal{G}} s'$ on G ;
- d) $\hat{R}_s^G = \sup_{K \subset G} \hat{R}_s^K$ where K is compact.

COROLLARY 2. — *For any fine open set G and any measure μ with compact carrier we have*

$$\int s d\mu^G = \int \hat{R}_s^G d\mu \quad (1)$$

where s is an arbitrary non-negative superharmonic function.

(1) μ^G is the balayaged measure of μ on G [3] (p. 447).

This relation follows from Theorem 3 c) taking \mathcal{G} as the set of all continuous finite positive superharmonic functions smaller than s .

LEMMA 1. — *Let s be a positive superharmonic function on X and $F \subsetneq X$ be a closed non empty set, non polar if $X \notin \mathfrak{P}$. s is resolutive for the normed Dirichlet problem on $X - F$ and we have*

$$R_s^F = H_s^{X-F} \quad (2)$$

on $X - F$.

Since F is non-polar if $X \notin \mathfrak{P}$, there exists a locally bounded positive potential on any component of $X - F$. Let s_0 be a positive continuous superharmonic function on X . We want to prove that $X - F$ is an MP_0 -set [1]. Let $s' \in \bar{\mathcal{G}}_0^{X-F, X}$. Then $s' + \varepsilon s_0$ is non-negative outside a compact set contained in U for any $\varepsilon > 0$. From [2] (Theorem 2) it follows $s' + \varepsilon s_0 \geq 0$. ε being arbitrary we get $s' \geq 0$ and $X - F$ is an MP_0 -set. By [1] (Corollary 3) the restrictions of the functions $\min(s, ns_0)$ on ∂U are resolutive. Since $\min(s, ns_0) \uparrow s$ for $n \uparrow \infty$ it follows that the restriction of s on ∂U is resolutive.

Let $\bar{s} \in \bar{\mathcal{G}}_s^{X-F, X}$ and s' be the function on X equal to s on F and equal to $\min(s, \bar{s})$ on $X - F$. s' is superharmonic and dominates s on F . Hence $\bar{s} \geq R_s^F$, $H_s^{X-F} \geq R_s^F$ on $X - F$. The converse inequality is trivial.

THEOREM 4. — *Let s be a non-negative superharmonic function. For any $E \subset X$ such that s is finite on E*

$$R_s^E = \inf_{G \supset E} R_s^G,$$

where G is fine open.

Obviously

$$R_s^E \leq \inf_{G \supset E} R_s^G.$$

Let s' be a non-negative superharmonic function on X , $s' \geq s$ on E and θ be a real number, $0 < \theta < 1$. The set

$$G = \{x \in X | s'(x) > \theta s(x)\}$$

(2) The normed Dirichlet problem and the associated notions were introduced in [1].

is fine open and contains E . We have $\frac{s'}{\theta} \geq s$ on G and therefore

$$\frac{s'}{\theta} \geq R_s^G \geq \inf_{G \supset E} R_s^G.$$

s' , θ being arbitrary we get

$$R_s^E \geq \inf_{G \supset E} R_s^G.$$

THEOREM 5 ⁽³⁾. — *Let s_1, s_2 be non-negative superharmonic functions and E be an arbitrary set. Then*

$$R_{s_1+s_2}^E = R_{s_1}^E + R_{s_2}^E, \quad \hat{R}_{s_1+s_2}^E = \hat{R}_{s_1}^E + \hat{R}_{s_2}^E.$$

If E is compact the relation follows from lemma 1. By theorem 3 *d*) it can be extended to fine open sets and by theorem 4 to arbitrary E subjected to the condition that $s_1 + s_2$ is finite on E . In the general case we have

$$R_{s_1+s_2}^E \leq R_{s_1}^E + R_{s_2}^E$$

and

$$R_{s_1+s_2}^E = R_{s_1}^E + R_{s_2}^E$$

on E . We denote by E' the set

$$E' = \{y \in E | s_1(y) + s_2(y) < \infty\}.$$

Let $x \in X - E$. If $R_{s_1+s_2}^E(x) = \infty$ the required equality holds at x . On the contrary case there exists a non-negative superharmonic function s_0 on X finite at x and $s_0 \geq s_1 + s_2$ on E . For any non-negative superharmonic function s on X and any $\varepsilon > 0$ we have

$$R_s^{E'} \leq R_s^E \leq R_s^{E'} + \varepsilon s_0.$$

Hence

$$R_s^{E'}(x) = R_s^E(x).$$

We have therefore

$$R_{s_1+s_2}^E(x) = R_{s_1+s_2}^{E'}(x) = R_{s_1}^{E'}(x) + R_{s_2}^{E'}(x) = R_{s_1}^E(x) + R_{s_2}^E(x).$$

The second equality follows immediately from the first one.

⁽³⁾ This theorem was proved by R.-M. Hervé under the supplementary hypothesis that X has a countable basis, and either s_1, s_2 are continuous or E is closed or E is open or the axiom D is fulfilled [3].

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